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## EQUIVARIANT K-RING OF G-MANIFOLD $(U(n), ad_\rho)$ II

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### Introduction

Let  $G$  be a compact Lie group. Throughout this paper  $K_{\mathbb{C}}^*$  will denote the complex equivariant  $K$ -theory associated with the group  $G$  and  $R(G)$  the ring of virtual complex representations of  $G$ .

Let  $V$  be a  $G$ -module over the field of the complex numbers and  $U(V)$  the group of isometries of  $V$  with the action of  $G$  defined by conjugation. In [2], Hodgkin has announced the  $K_G$ -ring structure of  $U(V)$  without proof. So we have proved a special case of Hodgkin's theorem in [4]. The purpose of this paper is to give a proof of the general case.

### 1. Statement of the theorems

Let  $G$  be a compact Lie group and  $\rho$  a unitary representation of  $G$  of dimension  $n$ . That is,  $\rho$  is a continuous homomorphism of  $G$  into a unitary group  $U(n)$ .

We consider  $U(n)$  a differentiable  $G$ -manifold together with the adjoint operation  $ad_\rho: G \times U(n) \rightarrow U(n)$ , defined by

$$ad_\rho(g, u) = \rho(g)u\rho(g)^{-1} \quad g \in G, u \in U(n)$$

and then we denote the  $G$ -manifold  $U(n)$  by  $(U(n), ad_\rho)$ .

We denote by  $V$  the representation space of  $\rho$  over the field of the complex numbers  $\mathbb{C}$ , by  $\underline{V}$  the product  $G$ -vector bundle with a fibre  $V$  over  $U(n)$  and by  $\lambda^k(\underline{V}) = \underline{\lambda^k(V)}$  the  $k$ -th exterior power of  $\underline{V}$  for  $1 \leq k \leq n$ . Then we can define an automorphism  $\theta_k^G$  of  $\lambda^k(\underline{V})$  by

$$\theta_k^G(u, z) = (u, \lambda^k(u)(z)) \quad n \in U(n), z \in \lambda^k(V).$$

Hence  $\theta_k^G$  determines an element  $[\lambda^k(\underline{V}), \theta_k^G]$  of  $K_{\mathbb{C}}^1(U(n), ad_\rho)$ . Afterwards we shall use the same symbol  $\theta_k^G$  in writing this induced element. Our main theorem is:

**Theorem 1.1.** *Let  $G$  be a compact Lie group and  $\rho$  a unitary representation of  $G$  of dimension  $n$ . Then*

$$K_{\mathbb{C}}^*(U(n), ad_\rho) = \Lambda_{R(G)}(\theta_1^G, \dots, \theta_n^G)$$

as an algebra over  $R(G)$ .

Theorem 1.1 has the following corollaries.

**Corollary 1.2.** *Let  $\rho$  be as in Theorem 1.1 and  $X$  a compact locally  $G$ -contractible  $G$ -space whose orbit space  $X/G$  has a finite covering dimension. Then the external tensor product homomorphism*

$$\mu: K_G^*(U(n), ad_\rho) \otimes_{R(G)} K_G^*(X) \rightarrow K_G^*((U(n), ad_\rho) \times X)$$

is an isomorphism.

*Proof.* Put  $U=U(n)$  for the simplicity.  $K_G^*(U, ad_\rho) \otimes_{R(G)} K_G^*(X)$  is an equivariant cohomology theory because  $K_G^*(U, ad_\rho)$  is a free module over  $R(G)$  and also  $K_G^*((U, ad_\rho) \times X)$  is an equivariant cohomology theory. As easily checked, we can construct spectral sequences of Segal's type for these equivariant cohomology theories [5].

Let  $\bar{X}$  denote the orbit space of  $X$  by  $G$ . There are two sheaves over  $\bar{X}$ ,  $\varphi_*$  and  $\tau_*$  whose stalks are respectively

$$\varphi_*(\bar{x}) = K_G^*(U, ad_\rho) \otimes_{R(G)} K_G^*(Gx)$$

and

$$\tau_*(\bar{x}) = K_G^*((U, ad_\rho) \times Gx)$$

where  $\bar{x} \in \bar{X}$  and  $Gx \subset X$  is the orbit of  $x \in X$  lying over  $\bar{x}$ .

The external tensor product homomorphism  $\mu$  induces a map of the spectral sequence

$$E_2^{p,q} = H^p(\bar{X}, \varphi_q) \Rightarrow K_G^*(U, ad_\rho) \otimes_{R(G)} K_G^*(X)$$

to the spectral sequence

$$E_2^{p,q} = H^p(\bar{X}, \tau_q) \Rightarrow K_G^*((U, ad_\rho) \times X).$$

Let  $G_x$  denote the isotropy group at  $x$ . Since  $Gx$  is homeomorphic to  $G/G_x$  as a  $G$ -space we have

$$\begin{aligned} \varphi_*(\bar{x}) &\cong K_G^*(U, ad_\rho) \otimes_{R(G)} K_G^*(G/G_x) \\ &\cong K_G^*(U, ad_\rho) \otimes_{R(G)} R(G_x) \end{aligned}$$

and

$$\begin{aligned} \tau_*(\bar{x}) &\cong K_G^*((U, ad_\rho) \times G/G_x) \\ &\cong K_{G_x}^*(U, ad_{\rho'}) \end{aligned}$$

where  $\rho'$  is the restriction of  $\rho$  onto  $G_x$ . Therefore, from Theorem 1.1 we see

$$\varphi_*(x) \cong \Lambda_{R(G_x)}(\theta_1^G, \dots, \theta_n^G), \tau_*(x) \cong \Lambda_{R(G_x)}(\theta_1^{G_x}, \dots, \theta_n^{G_x})$$

and so  $\mu$  induces an isomorphism on the  $E_2$ -level. This permits the corollary.

Let  $X$  be a  $G$ -space as in Corollary 1.2 and  $E$  an  $n$ -dimensional complex  $G$ -vector bundle over  $X$ . Here we consider the unitary bundle  $\pi: U(E) \rightarrow X$  of  $E$  (See [2], §3). For  $1 \leq k \leq n$  we can define also an automorphism  $\theta_k^E$  of the  $G$ -vector bundle  $\pi^*(\lambda^k(E)) = \lambda^k(\pi^*(E))$  over  $U(E)$  by

$$\theta_k^E(u, z) = (u, \lambda^k(u)(z)) \quad u \in U(E_x), z \in \lambda^k(E_x)$$

and we write  $\theta_k^E$  for an element of  $K_G^1(U(E))$  determined by  $\theta_k^E$ . Then we have the following

**Corollary 1.3.**

$$K_G^*(U(E)) = \Lambda_{K_{G \times X}^*}(\theta_1^E, \dots, \theta_n^E)$$

as an algebra over  $K_G^*(X)$ .

Proof. For the sake of simplicity, put  $U = U(n)$  and  $ad = ad_{U(n)}$ , the adjoint operation of the identity representation of  $U(n)$ .

Let  $P$  be the associated principal bundle to  $E$ . Then  $P$  is a  $G \times U$ -space on which  $U$  acts freely:  $P/U = X$  and

$$U(E) = P \times_U (U, ad).$$

We can regard  $(U, ad)$  as a  $G \times U$ -space where  $G$  acts on  $(U, ad)$  trivially. Then we have

$$K_{G \times U}^*(U, ad) \cong R(G) \otimes K_U^*(U, ad)$$

by a parallel proof to that of [5], Proposition (2.2).

From Corollary 1.2 we obtain

$$K_{G \times U}^*(P) \otimes_{R(G \times U)} K_{G \times U}^*(U, ad) \cong K_{G \times U}^*(P \times (U, ad)).$$

Hence we get

$$K_G^*(X) \otimes_{R(U)} K_U^*(U, ad) \cong K_G^*(U(E))$$

by [5], Proposition (2.1). This shows the corollary from Theorem 1.1.

In the following sections we shall give a proof of Theorem 1.1.

**2. Proof when  $G$  is connected**

The proof consists of two steps.

*Step 1. Proof when  $G$  is a compact abelian Lie group.*

For the sake of simplicity we write  $U(\rho)$  for the  $G$ -manifold  $(U(n), ad_\rho)$ .

Since  $G$  is abelian, there exist 1-dimensional representations of  $G$ ,  $\rho_k: G \rightarrow U(1)$   $1 \leq k \leq n$ , such that  $\rho$  is equivalent to the sum  $\rho_1 \oplus \dots \oplus \rho_n$ . Then

$$U(\rho) \cong U(\rho_1 \oplus \dots \oplus \rho_n)$$

as a  $G$ -manifold. So it suffices to show the theorem for  $U(\rho)$ ,  $\rho = \rho_1 \oplus \dots \oplus \rho_n$ .

Before beginning the proof of the theorem we prepare an elementary lemma. Let  $W$  be the representation space over  $\mathbb{C}$  of the representation  $1 \oplus \rho_1^{-1} \rho_2 \oplus \dots \oplus \rho_1^{-1} \rho_n$ . Then the unit sphere  $S(W)$  in  $W$  is homeomorphic to the homogeneous space  $U(\rho)/U(\rho_2 \oplus \dots \oplus \rho_n)$  as a  $G$ -space where  $U(\rho_2 \oplus \dots \oplus \rho_n) = 1 \times U(\rho_2 \oplus \dots \oplus \rho_n)$  and also  $S(W)$  has a fixed point  $p = (1, 0, \dots, 0)$ .

**Lemma 2.1.** *For each point  $q = (z_1, \dots, z_n)$  of  $S(W)$  there exists a continuous map  $f: [0, 1] \rightarrow U(n)$  such that  $f(0)(p) = q$ ,  $f(1) = 1$  and  $\rho(g)f(t)\rho(g)^{-1} = f(t)$  for  $g \in G_q$  and  $t \in [0, 1]$  where  $G_q$  is the isotropy group at  $q$ .*

*Proof.* We shall prove Lemma 2.1 by induction on  $n$ . For the case of  $n = 1$  we have nothing to do. Assume that the assertion is true for  $n < l$ . In case of  $n = l$  we consider two types of  $q$  as follows.

(i) If  $z_2 \dots z_n \neq 0$ , then

$$\rho_1(g) = \dots = \rho_n(g) \quad g \in G_q.$$

Namely  $\rho(g)$  is a diagonal matrix for any  $g \in G_q$ . So it is sufficient to show the existence of a continuous map  $f: [0, 1] \rightarrow U(n)$  such that  $f(0)(p) = q$  and  $f(1) = 1$ . But this is clear because  $U(n)$  acts on  $S^{2n-1}$  transitively and  $U(n)$  is arcwise connected.

(ii) If there is an integer  $k \geq 2$  such that  $z_k = 0$ , then we consider a subgroup,  $U'(n-1)$ , of  $U(n)$  consisting of  $(n-1)$ -dimensional minors of which the  $(k, k)$ -component is 1, i.e.

$$k \begin{pmatrix} & & k & & \\ & & 0 & & \\ & * & \vdots & & * \\ & & 0 & & \\ k & 0 \dots 0 & 1 & 0 \dots 0 & \\ & & 0 & & \\ & * & \vdots & & * \\ & & 0 & & \end{pmatrix} \in U(n).$$

Let  $\rho'$  be a continuous homomorphism of  $G$  into  $U'(n-1)$  defined by

$$\rho' = \rho_1 \oplus \dots \oplus \rho_{k-1} \oplus 1 \oplus \rho_{k+1} \oplus \dots \oplus \rho_n.$$

In virtue of the inductive hypothesis there is a map  $f': [0, 1] \rightarrow U'(n-1)$  satisfying the assertion mentioned in Lemma 2.1. Then we have

$$\rho'(g)f'(t)\rho'(g)^{-1} = \rho(g)f'(t)\rho(g)^{-1} \quad g \in G, t \in [0, 1].$$

Therefore when we put

$$f = if'$$

where  $i: U'(n-1) \rightarrow U(n)$  is the inclusion of  $U'(n-1)$ ,  $f: G \rightarrow U(n)$  is a map which we require. q.e.d.

Now we proceed by induction on  $n$  to complete the step 1. In case of  $n=1$ , since  $G$  acts on  $U(\rho_1)$  trivially we have

$$K_G^*(U(\rho_1)) \cong R(G) \otimes K^*(U(1))$$

by [5], Proposition (2,2).  $K^*(U(1))$  is an exterior algebra with one generator  $\theta$  and by the above isomorphism  $\theta_1^G$  corresponds to  $\rho_1 \otimes \theta$ . Hence  $K_G^*(U(\rho_1)) = \Lambda_{R(G)}(\theta_1^G)$  is valid for any compact abelian Lie group  $G$  and any 1-dimensional representation  $\rho = \rho_1$  of  $G$ .

Let  $\pi: U(\rho) \rightarrow S(W) (= U(\rho)/U(\rho_2 \oplus \dots \oplus \rho_n))$  be the projection. From [4], Lemma 1 we get

**Lemma 2.2.** *There exists an element  $g$  in  $K_G^1(S(W))$  such that*

$$K_G^*(S(W)) = \Lambda_{R(G)}(g)$$

as an algebra over  $R(G)$  and

$$\pi^*(g) = \sum_{k=1}^n (-1)^k \rho_1^{-k} \theta_k^G \quad \theta_k^G \in K_G^1(U(\rho)).$$

Proof. We observe the exact sequence of the pair  $(D(W), S(W))$  where  $D(W)$  is the unit disk in  $W$ . Then we see that

$$\tilde{K}_G^0(S(W)) = 0$$

and the coboundary homomorphism

$$\delta: K_G^1(S(W)) \rightarrow K_G^0(W)$$

is an isomorphism.

When we denote by  $\lambda_W$  the Thom class of the vector bundle  $W \rightarrow P$  ( $=$  a point),  $K_G^0(W)$  is a free module over  $R(G)$  generated by  $\lambda_W$ . So if we put  $g = \delta^{-1}(\lambda_W)$ , then

$$K_G^*(S(W)) = \Lambda_{R(G)}(g).$$

Next we consider the following diagram

$$\begin{array}{ccc}
 K_T^*(U(n), ad_i) & \xrightarrow{\rho^*} & K_G^*(U(\rho)) \\
 \pi'^* \uparrow & & \uparrow \pi^* \\
 K_T^*(S(W')) & \xrightarrow{\rho^*} & K_G^*(S(W)) \\
 \delta \downarrow & & \downarrow \delta \\
 K_T^*(W') & \xrightarrow{\rho^*} & K_G^*(W)
 \end{array}$$

where  $i: T \rightarrow U(n)$  is the inclusion map of the standard maximal torus  $T$  of  $U(n)$ ,  $\rho^*$  the homomorphism induced by the continuous homomorphism  $\rho = \rho_1 \oplus \dots \oplus \rho_n: G \rightarrow T$  and  $\pi': U(n) \rightarrow S(W')$  represents the map  $\pi: U(n) \rightarrow S(C \oplus W)$  in [4], §2. Then this diagram commutes and  $\rho^*(\lambda_{W'}) = \lambda_W$ . Therefore we get

$$\pi^*(g) = \sum_{k=1}^n (-1)^k \rho_1^{-k} \theta_k^G$$

by [4], Lemma 1. q.e.d.

Let  $\mathfrak{M}$  be an exterior algebra over  $R(G)$  generated by  $\theta_1^G, \dots, \theta_{n-1}^G$  where  $\theta_k^G \in K_G^1(U(\rho))$  for  $1 \leq k \leq n-1$ . Then we have a homomorphism

$$\kappa_1: \mathfrak{M} \rightarrow K_G^*(U(\rho))$$

of algebras, defined by  $\kappa_1(\theta_k^G) = \theta_k^G$ . Because, when we observe the homomorphism  $\rho^*: K_T^*(U(n), ad_i) \rightarrow K_G^*(U(\rho))$  mentioned in the proof of Lemma 2.2 we get

$$(\theta_k^G)^2 = \rho^*((\theta_k^T)^2) = 0 \quad \text{for } 1 \leq k \leq n$$

since  $(\theta_k^T)^2 = 0$  in  $K_T^*(U(n), ad_i)$  by [4], Theorem 1 and also we get the relations  $\theta_k^G \theta_l^G + \theta_l^G \theta_k^G = 0$  for  $1 \leq k, l \leq n$  obviously since  $\theta_k^G$  are the elements of  $K_G^1(U(\rho))$ . Moreover, for each closed invariant subspace  $X$  of  $S(W)$  we can define a homomorphism

$$\lambda: K_G^*(X) \otimes_{R(G)} \mathfrak{M} \rightarrow K_G^*(\pi^{-1}(X))$$

by

$$\lambda(x \otimes y) = \pi^*(x) j^* \kappa_1(y) \quad x \in K_G^*(X), y \in \mathfrak{M}$$

where  $j: \pi^{-1}(X) \rightarrow U(\rho)$  is the inclusion of  $\pi^{-1}(X)$ .

Under the assumption that the assertion of Theorem 1.1 in the step 1 is true for  $n < l$  the following lemma is proved.

**Lemma 2.3.** *The homomorphism*

$$\lambda: K_G^*(S(W)) \otimes_{R(G)} \mathfrak{M} \rightarrow K_G^*(U(\rho))$$

*is an isomorphism.*

Proof. Let  $\overline{S(W)}$  denote the orbit space of  $S(W)$  by  $G$ . We have two

sheaves over  $\overline{S(W)}$ ,  $\varphi_*$  and  $\tau_*$  whose stalks are respectively

$$\varphi_*(\bar{q}) = K_G^*(Gq) \otimes_{R(G)} \mathfrak{M}$$

and

$$\tau_*(\bar{q}) = K_G^*(\pi^{-1}(Gq))$$

where  $q \in S(W)$ ,  $\bar{q} \in \overline{S(W)}$  and  $Gq = \pi^{-1}(\bar{q})$ .

Since  $\mathfrak{M}$  is a free module over  $R(G)$ ,  $K_G^*(X) \otimes_{R(G)} \mathfrak{M}$  is an equivariant cohomology theory. Then  $\lambda$  induces a map of the spectral sequence [5]

$$E_2^{p,q} = H^p(\overline{S(W)}, \varphi_q) \Rightarrow K_G^*(S(W)) \otimes_{R(G)} \mathfrak{M}$$

to the spectral sequence

$$E_2^{p,q} = H^p(\overline{S(W)}, \tau_q) \Rightarrow K_G^*(U(\rho)).$$

We shall prove that  $\lambda$  induces an isomorphism on the  $E_2$ -level. Clearly we have

$$\begin{aligned} \varphi_*(\bar{q}) &= K_G^*(Gq) \otimes_{R(G)} \mathfrak{M} \\ &\cong K_G^*(G/G_q) \otimes_{R(G)} \mathfrak{M} \\ &\cong \Lambda_{R(G_q)}(\theta_1^G, \dots, \theta_{n-1}^G). \end{aligned}$$

Next we observe the stalks  $\tau_*(\bar{q})$ . Let  $f: [0, 1] \rightarrow U(n)$  is a continuous map in Lemma 2.1. Then we have

$$\pi^{-1}(Gq) = \bigcup_{g \in G} (\rho(g)f(0)\rho(g)^{-1})U(n-1)$$

and so we can define a  $G$ -map

$$\phi: G/G_q \times U(\rho_2 \oplus \dots \oplus \rho_n) \rightarrow \pi^{-1}(Gq)$$

by

$$\phi(gG_q, u) = (\rho(g)f(0)\rho(g)^{-1})u \quad g \in G, u \in U(\rho_2 \oplus \dots \oplus \rho_n)$$

because  $\rho(g)f(0)\rho(g)^{-1} = f(0)$  for any  $g \in G_q$ . Further we can easily check that  $\phi$  is an isomorphism. Therefore

$$\begin{aligned} \tau_*(\bar{q}) &= K_G^*(\pi^{-1}(Gq)) \\ &\cong K_G^*(G/G_q \times U(\rho_2 \oplus \dots \oplus \rho_n)) \\ &\cong K_{G_q}^*(U(\rho_2 \oplus \dots \oplus \rho_n)). \end{aligned}$$

Thus we obtain

$$\tau_*(\bar{q}) \cong \Lambda_{R(G_q)}(\theta'_1, \dots, \theta'_{n-1})$$

by the inductive hypothesis where  $\theta'_k = \theta_k^G$  for  $1 \leq k \leq n-1$ .



Here we consider the homomorphism

$$\lambda': \Lambda_{R(G_q)}(\theta_1^G, \dots, \theta_{n-1}^G) \rightarrow K_{G_q}^*(U(\rho_2 \oplus \dots \oplus \rho_n))$$

induced by the homomorphism

$$\lambda: \varphi_*(\bar{q}) \rightarrow \tau_*(\bar{q}).$$

From the definition of  $\theta_k^G$  we obtain easily

$$\lambda'(\theta_k^G) = [U(n-1) \times \lambda^k(V), \rho_k] \quad (1 \leq k \leq n-1)$$

where  $\xi_k$  is an automorphism of the product  $G_q$ -bundle  $U(n-1) \times \lambda^k(V)$  given by

$$\xi_k(u, z) = (u, \lambda^k(f(0)u)(z)) \quad u \in U(n-1), z \in \lambda^k(V).$$

Since  $f$  is a homotopy from  $f(0)$  to the identity element of  $U(n)$  satisfying  $\rho(g)f(t) = f(t)\rho(g)$  for any  $g \in G_q$  and  $t \in [0, 1]$ , we get

$$\lambda'(\theta_k^G) = \begin{cases} \theta_1^T & (k = 1) \\ \theta_k^T + \rho_1 \theta_{k-1}^T & (2 \leq k \leq n-1). \end{cases}$$

Hence we see that  $\lambda'$  is an isomorphism. This shows that  $\lambda$  induces an isomorphism on the  $E_2$ -level. Consequently we obtain Lemma 2.3. q.e.d.

Lemma 2.2 and lemma 2.3 show that the assertion in the case of  $n=l$  is also true. This completes the step 1.

*Step 2. Proof when  $G$  is connected.*

Let  $T$  be a maximal torus of  $G$  and  $i: T \rightarrow G$  the inclusion of  $T$ . Then from the step 1 we get

$$K_T^*(U(n), ad_{\rho_T}) = \Lambda_{R(T)}(\theta_1^T, \dots, \theta_n^T)$$

where  $\rho_T$  is the restriction of  $\rho$  onto  $T$  and therefore, from [5], Proposition (3.8) and [4], Lemma 2 we get

$$K_G^*(U(n), ad_\rho) \cong K_T^*(U(n), ad_{\rho_T})^{W(G)}$$

where  $W(G)$  is the Weyl group of  $G$ . This shows

$$K_G^*(U(n), ad_\rho) = \Lambda_{R(G)}(\theta_1^G, \dots, \theta_n^G).$$

### 3. Proof when $G$ is not connected

We recall

**Theorem 3.1.** (Segal [6]) *Let  $G$  be a compact Lie group. Then the restric-*

tion  $R(G) \rightarrow \sum_S R(S)$  is injective where  $S$  runs through the representatives of conjugacy classes of Cartan subgroups of  $G$ .

Then we have

**Lemma 3.2.** *Let  $G$  be a compact Lie group and  $\rho$  a continuous homomorphism of  $G$  into  $U(n)$ . Then  $\Lambda_{R(G)}(\theta_1^G, \dots, \theta_n^G)$  is a subalgebra of  $K_G^*(U(n), ad_\rho)$ .*

Proof. We have a homomorphism  $\kappa_2$  of  $\Lambda_{R(G)}(\theta_1^G, \dots, \theta_n^G)$  into  $K_G^*(U(n), ad_\rho)$  as algebras defined by  $\kappa_2(\theta_k^G) = \theta_k^G, 1 \leq k \leq n$ . This homomorphism is well-defined by the same reason as  $\kappa_1$  in §2, Step 1 is so.

Let  $S$  be a Cartan subgroup of  $G$  and  $i_S: S \rightarrow G$  the inclusion of  $S$ . Then we have

$$K_S^*(U(n), ad_{\rho_S}) = \Lambda_{R(S)}(\theta_1^S, \dots, \theta_n^S)$$

from §2, Step 1 where  $\rho_S$  is the restriction of  $\rho$  onto  $S$ . Therefore if

$$\sum_{1 \leq i_1 < \dots < i_l \leq n} \alpha_{i_1 \dots i_l} \theta_{i_1}^G \dots \theta_{i_l}^G = 0$$

for  $\alpha_{i_1 \dots i_l} \in R(G)$  in  $K_G^*(U(n), ad_\rho)$ , then

$$i_S^*(\alpha_{i_1 \dots i_l}) = 0$$

for any Cartan subgroup  $S$  of  $G$ . So we get

$$\alpha_{i_1 \dots i_l} = 0, \quad 1 \leq i_1 < \dots < i_l \leq n$$

from Theorem 3.1. This shows that  $\kappa_2$  is injective. q.e.d.

Using the Segal's spectral sequence [5] we can easily check the following

**Lemma 3.3.** ([3], Proposition 2) *Let  $G$  be a compact Lie group. Let  $X$  and  $Y$  be compact locally  $G$ -contractible  $G$ -spaces such that the orbit spaces  $X/G$  and  $Y/G$  are of finite covering dimension. If  $K_G^*(X)$  or  $K_G^*(Y)$  is a free abelian group, then the external tensor product*

$$K_G^*(X) \otimes K_G^*(Y) \rightarrow K_{G \times G}^*(X \times Y)$$

*is an isomorphism.*

The following theorem is basic in proof of the general case.

**Theorem 3.4.** ([1], Proposition (4.9), [5], Proposition (3.8))

*Let  $G$  be a compact connected Lie group and  $i: T \rightarrow G$  the inclusion of a maximal torus. Then for each locally compact  $G$ -space  $X$  there is a natural homomorphism of  $K_G^*(X)$ -modules  $i_*: K_T^*(X) \rightarrow K_G^*(X)$  such that  $i_*(1) = 1$ , and hence  $i_* i^* = \text{identity}$ .*

**Theorem 3.5.** *Let  $G$  be a compact connected Lie group and  $\rho: G \rightarrow U(n)$  a*

unitary representation. Then, for each closed subgroup  $H$  of  $G$  we have

$$K_H^*(U(n), ad_{\rho_H}) = \Lambda_{R(H)}(\theta_1^H, \dots, \theta_n^H)$$

as an algebra over  $R(H)$  where  $\rho_H$  is the restriction of  $\rho$  onto  $H$ .

Proof. As in §2, we denote  $(U(n), ad_{\rho})$  by  $U(\rho)$ . Let  $\pi_1: U(\rho) \times G/H \rightarrow U(\rho)$  and  $\pi_2: U(\rho) \times G/H \rightarrow G/H$  be the projections. Let  $d: G \rightarrow G \times G$  be the diagonal map.

We consider the homomorphism

$$d^*: K_{G \times G}^*(U(\rho) \times G/H) \rightarrow K_G^*(U(\rho) \times G/H).$$

From Lemma 3.3 and §2, Step 2 we get

$$(1) \quad K_{G \times G}^*(U(\rho) \times G/H) \cong K_G^*(U(\rho)) \otimes K_G^*(G/H) \\ \cong \Lambda_{R(G)}(\theta_1^G, \dots, \theta_n^G) \otimes R(H).$$

From (1) we see that  $d^*$  induces a homomorphism

$$\mu_1: K_G^*(U(\rho)) \otimes K_G^*(G/H) \rightarrow K_G^*(U(\rho) \times G/H)$$

and then  $\mu_1$  is as follows:

$$\mu_1(x \otimes y) = \pi_1^*(x) \pi_2^*(y) \quad \text{for } x \in K_G^*(U(\rho)), y \in K_G^*(G/H).$$

Since  $K_G^*(U(\rho) \times G/H) \cong K_H^*(U(\rho_H))$  and  $\Lambda_{R(H)}(\theta_1^H, \dots, \theta_n^H)$  is a subalgebra of  $K_H^*(U(\rho_H))$  by Lemma 3.2,  $\Lambda_{R(H)}(\pi_1^*(\theta_1^G), \dots, \pi_1^*(\theta_n^G))$  is a subalgebra of  $K_G^*(U(\rho) \times G/H)$  and also

$$(2) \quad \text{Im } \mu_1 = \Lambda_{R(H)}(\pi_1^*(\theta_1^G), \dots, \pi_1^*(\theta_n^G)).$$

Therefore if we prove that  $\mu_1$  is an epimorphism, then we obtain Theorem 3.5.

Let  $T$  be a maximal torus of  $G$ . First we consider the restriction  $\rho_T: T \rightarrow U(n)$  of  $\rho$  onto  $T$ . As the case of  $\rho: G \rightarrow U(n)$  we have

$$K_{T \times T}^*(U(\rho_T) \times G/H) \cong K_T^*(U(\rho_T)) \otimes K_T^*(G/H) \\ \cong \Lambda_{R(T)}(\theta_1^T, \dots, \theta_n^T) \otimes K_T^*(G/H)$$

and so the homomorphism

$$\mu_2: K_T^*(U(\rho_T)) \otimes K_T^*(G/H) \rightarrow K_T^*(U(\rho_T) \times G/H)$$

induced by  $d^*$ . Also we get

$$K_T^*(U(\rho_T) \times G/H) \cong K_T^*(U(\rho_T)) \otimes_{R(T)} K_T^*(G/H) \\ \cong \Lambda_{R(T)}(\theta_1^T, \dots, \theta_n^T) \otimes_{R(T)} K_T^*(G/H)$$

from §2, Step 1 and a parallel argument to Corollary 1.2.

Now we observe the following diagram

$$\begin{array}{ccc}
 \Lambda_{R(G)}(\theta_1^G, \dots, \theta_n^G) \otimes R(H) & & \Lambda_{R(T)}(\theta_1^T, \dots, \theta_n^T) \otimes K_T^*(G/H) \\
 \cong \downarrow & & \downarrow \cong \\
 K_G^*(U(\rho)) \otimes K_G^*(G/H) & \xrightleftharpoons[i_{1*} \otimes i_{2*}]{i_1^* \otimes i_2^*} & K_T^*(U(\rho_T)) \otimes K_T^*(G/H) \\
 \mu_1 \downarrow & & \downarrow \mu_2 \\
 K_G^*(U(\rho) \times G/H) & \xrightleftharpoons[j_*]{j^*} & K_T^*(U(\rho_T) \times G/H) \\
 \cong \uparrow & & \uparrow \cong \\
 K_H^*(U(\rho_H)) & & \Lambda_{R(T)}(\theta_1^T, \dots, \theta_n^T) \otimes_{R(T)} K_T^*(G/H)
 \end{array}$$

where  $i_1, i_2$  and  $j$  are the inclusion of  $T$ , and  $i_{1*}, i_{2*}$  and  $j_*$  denote the natural homomorphisms mentioned in Theorem 3.4.

For any  $x \in K_G^*(U(\rho) \times G/H)$  we can write

$$(3) \quad j^*(x) = \alpha \pi_2^*(y) + \sum_{1 \leq i_1 < \dots < i_s \leq n} \alpha_{i_1 \dots i_s} \pi_1^*(\theta_{i_1}^T \dots \theta_{i_s}^T) \pi_2^*(y_{i_1 \dots i_s})$$

for  $\alpha, \alpha_{i_1 \dots i_s} \in R(T)$  and  $y, y_{i_1 \dots i_s} \in K_T^*(G/H)$ .

Let put

$$z = 1 \otimes \alpha y + \sum_{1 \leq i_1 < \dots < i_s \leq n} \theta_{i_1}^T \dots \theta_{i_s}^T \otimes \alpha_{i_1 \dots i_s} y_{i_1 \dots i_s}$$

in  $K_T^*(U(\rho_T)) \otimes K_T^*(G/H)$ . Then from (3) we get

$$(4) \quad \mu_2(z) = j^*(x).$$

Moreover

$$(i_{1*} \otimes i_{2*})(z) = 1 \otimes i_{2*}(\alpha y) + \sum_{1 \leq i_1 < \dots < i_s \leq n} \theta_{i_1}^G \dots \theta_{i_s}^G \otimes i_{2*}(\alpha_{i_1 \dots i_s} y_{i_1 \dots i_s})$$

since  $i_1^* \theta_k^G = \theta_k^T$   $1 \leq k \leq n$  and  $i_{1*} i_1^* = 1$ , and

$$(5) \quad \begin{aligned} \mu_1((i_{1*} \otimes i_{2*})(z)) &= \pi_2^* i_{2*}(\alpha y) + \sum_{1 \leq i_1 < \dots < i_s \leq n} \pi_1^*(\theta_{i_1}^G \dots \theta_{i_s}^G) \pi_2^* i_{2*}(\alpha_{i_1 \dots i_s} y_{i_1 \dots i_s}) \\ &= j_* \pi_2^*(\alpha y) + \sum_{1 \leq i_1 < \dots < i_s \leq n} \pi_1^*(\theta_{i_1}^G \dots \theta_{i_s}^G) j_* \pi_2^*(\alpha_{i_1 \dots i_s} y_{i_1 \dots i_s}) \end{aligned}$$

because of  $j_* \pi_2^* = \pi_2^* i_{2*}$ . By Theorem 3.4,  $j_*$  is the homomorphism of  $K_G^*(U(\rho) \times G/H)$ -modules. Therefore (5) shows

$$(6) \quad \mu_1((i_{1*} \otimes i_{2*})(z)) = j_*(\alpha \pi_2^*(y) + \sum_{1 \leq i_1 < \dots < i_s \leq n} \alpha_{i_1 \dots i_s} \pi_1^*(\theta_{i_1}^T \dots \theta_{i_s}^T) \pi_2^*(y_{i_1 \dots i_s}))$$

because of  $\pi_1^* i_1^* = j^* \pi_1^*$ .

From (3) and (6) we obtain

$$(7) \quad \mu_1((i_{1*} \otimes i_{2*})(z)) = j_* j^*(x) = x$$

and so we see that  $\mu_1$  is an epimorphism. Hence (2) and (7) conclude

$$K_H^*(U(\rho_H)) = \Lambda_{RCHD}(\theta_1^H, \dots, \theta_n^H).$$

q.e.d.

*Proof of the general case.* Let  $G$  be a compact Lie group and  $\rho: G \rightarrow U(n)$  a unitary representation of  $G$ .

Embed  $G$  in a unitary group  $U(m)$  and consider an embedding

$$f: G \rightarrow U(n) \times U(m)$$

defined by

$$f(g) = (\rho(g), g) \quad g \in G.$$

Let  $\pi: U(n) \times U(m) \rightarrow U(n)$  be the projection. If we regard  $G$  as a closed subgroup of  $U(n) \times U(m)$  by  $f$ , then  $\rho$  is the restriction of  $\pi$  onto  $G$ . Therefore, from Theorem 3.5 we get

$$K_G^*(U(n), ad_\rho) = \Lambda_{RCG}(\theta_1^G, \dots, \theta_n^G).$$

This completes the proof of Theorem 1.1.

#### 4. The special unitary group $SU(n)$

Let  $G$  be a compact Lie group and  $\rho: G \rightarrow U(n)$  a unitary representation of  $G$ . Then  $SU(n)$  becomes a  $G$ -submanifold of  $(U(n), ad_\rho)$  which we denote by  $(SU(n), ad_\rho)$ .

Let  $j: SU(n) \rightarrow U(n)$  be the inclusion of  $SU(n)$ . We use the same symbol  $\theta_k^G$  for the image of  $\theta_k^G \in K_G^*(U(n), ad_\rho)$  by  $j^*$  for  $1 \leq k \leq n-1$ . In particular,  $j^*(\theta_n^G) = 0$ .

Let  $T$  be the standard maximal torus of  $U(n)$  and  $i: T \rightarrow U(n)$  the inclusion of  $T$ . Then, by a parallel proof to that in [4] we obtain

**Proposition 4.1.** *Using the notation of [4], Lemma 1 we have*

(i)  $K_{\mathbb{Z}}^*(S(C \oplus W)) = K_{\mathbb{Z}}^*(SU(n)/SU(n-1))$  is an exterior algebra over  $R(T)$  with one generator  $g$  satisfying

$$\pi^*(g) = \sum_{k=1}^{n-1} (-1)^k \rho_1^{-k} \theta_k^T$$

where  $\pi: SU(n) \rightarrow S(C \oplus W) (= SU(n)/SU(n-1))$  is the projection, and therefore

(ii)  $K_{\mathbb{Z}}^*(SU(n), ad_i) = \Lambda_{RC(T)}(\theta_1^T, \dots, \theta_{n-1}^T)$  as an algebra over  $R(T)$ .

From Proposition 4.1 an analogous statement can be made as follows.

**Proposition 4.2.** *Let  $G$  be a compact Lie group and  $\rho: G \rightarrow U(n)$  a unitary representation of  $G$ . Then*

$$K_G^*(SU(n), ad_p) = \Lambda_{R(G)}(\theta_1^G, \dots, \theta_{n-1}^G)$$

as an algebra over  $R(G)$ .

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