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EQUIVARIANT K-RING OF G-MANIFOLD (U(n), ad_ρ) II

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Introduction

Let G be a compact Lie group. Throughout this paper K_G^* will denote the complex equivariant K-theory associated with the group G and $R(G)$ the ring of virtual complex representations of G.

Let V be a G-module over the field of the complex numbers and $U(V)$ the group of isometries of V with the action of G defined by conjugation. In [2], Hodgkin has announced the K_G-ring structure of $U(V)$ without proof. So we have proved a special case of Hodgkin's theorem in [4]. The purpose of this paper is to give a proof of the general case.

1. Statement of the theorems

Let G be a compact Lie group and ρ a unitary representation of G of dimension n. That is, ρ is a continuous homomorphism of G into a unitary group $U(n)$.

We consider $U(n)$ a differentiable G-manifold together with the adjoint operation ad_ρ: $G\times U(n)\rightarrow U(n)$, defined by

$$ad_\rho(g, u) = \rho(g)u\rho(g)^{-1} \qquad g\in G,\ u\in U(n)$$

and then we denote the G-manifold $U(n)$ by $(U(n), ad_\rho)$.

We denote by V the representation space of ρ over the field of the complex numbers $\boldsymbol{C}$, by $\underline{V}$ the product G-vector bundle with a fibre V over $U(n)$ and by $\lambda^k(\underline{V})=\underline{\lambda^k(V)}$ the k-th exterior power of $\underline{V}$ for $1\leq k\leq n$. Then we can define an automorphism θ_k^G of $\lambda^k(\underline{V})$ by

$$\theta_k^G(u, z) = (u, \lambda^k(u)(z)) \qquad n\in U(n),\ z\in\lambda^k(V)\,.$$

Hence θ_k^G determines an element $[\lambda^k(\underline{V}), \theta_k^G]$ of $K_G^1(U(n), ad_\rho)$. Afterwards we shall use the same symbol θ_k^G in writing this induced element. Our main theorem is:

Theorem 1.1. *Let G be a compact Lie group and ρ a unitary representation of G of dimension n. Then*

$$K_G^*(U(n), ad_\rho) = \Lambda_{R(G)}(\theta_1^G, \cdots, \theta_n^G)$$

as an algebra over $R(G)$.

Theorem 1.1 has the following corollaries.

Corollary 1.2. *Let* ρ *be as in Theorem* 1.1 *and* X *a compact locally* G-*contractible* G-*space whose orbit space* X/G *has a finite covering dimensnion. Then the external tensor product homomorphism*

$$\mu\colon K_G^*(U(n), ad_\rho)\underset{R(G)}{\otimes} K_G^*(X)\to K_G^*((U(n), ad_\rho)\times X)$$

is an isomorphism.

Proof. Put $U=U(n)$ for the simplicity. $K_G^*(U, ad_\rho)\underset{R(G)}{\otimes} K_G^*(X)$ is an equivariant cohomology theory because $K_G^*(U, ad_\rho)$ is a free module over $R(G)$ and also $K_G^*((U, ad_\rho)\times X)$ is an equivariant cohomology theory. As easily checked, we can construct spectral sequences of Segal's type for these equivariant cohomology theories [5].

Let $\bar{X}$ denote the orbit space of X by G. There are two sheaves over $\bar{X}$, φ_* and τ_* whose stalks are respectively

$$\varphi_*(\bar{x}) = K_G^*(U, ad_\rho)\underset{R(G)}{\otimes} K_G^*(Gx)$$

and

$$\tau_*(\bar{x}) = K_G^*((U, ad_\rho)\times Gx)$$

where $\bar{x}\in\bar{X}$ and $Gx\subset X$ is the orbit of $x\in X$ lying over $\bar{x}$.

The external tensor product homomorphism μ induces a map of the spectral sequence

$$E_2^{p,q} = H^p(\bar{X}, \varphi_q)\Rightarrow K_G^*(U, ad_\rho)\underset{R(G)}{\otimes} K_G^*(X)$$

to the spectral sequence

$$E_2^{p,q} = H^p(\bar{X}, \tau_q)\Rightarrow K_G^*((U, ad_\rho)\times X)\,.$$

Let G_x denote the isotropy group at x. Since Gx is homeomorphic to G/G_x as a G-space we have

$$\begin{aligned}\varphi_*(\bar{x}) &\cong K_G^*(U, ad_\rho)\underset{R(G)}{\otimes} K_G^*(G/G_x)\\ &\cong K_G^*(U, ad_\rho)\underset{R(G)}{\otimes} R(G_x)\end{aligned}$$

and

$$\begin{aligned}\tau_*(\bar{x}) &\cong K_G^*((U, ad_\rho)\times G/G_x)\\ &\cong K_{G_x}^*(U, ad_{\rho'})\end{aligned}$$

where ρ' is the restriction of ρ onto G_x. Therefore, from Theorem 1.1 we see

$$\varphi_*(\bar{x}) \cong \Lambda_{R(G_x)}(\theta_1^G, \cdots, \theta_n^G),\ \tau_*(\bar{x}) \cong \Lambda_{R(G_x)}(\theta_1^{G_x}, \cdots, \theta_n^{G_x})$$

and so μ induces an isomorphism on the E_2-level. This permits the corollary.

Let X be a G-space as in Corollary 1.2 and E an n-dimensional complex G-vector bundle over X. Here we consider the unitary bundle π: $U(E) \to X$ of E (See [2], §3). For $1 \leq k \leq n$ we can define also an automoprhism θ_k^E of the G-vector bundle $\pi^*(\lambda^k(E)) = \lambda^k(\pi^*(E))$ over $U(E)$ by

$$\theta_k^E(u, z) = (u, \lambda^k(u)(z)) \qquad u \in U(E_x),\ z \in \lambda^k(E_x)$$

and we write θ_k^E for an element of $K_G^1(U(E))$ determined by θ_k^E. Then we have the following

Corollary 1.3.

$$K_G^*(U(E)) = \Lambda_{K_G^*(X)}(\theta_1^E, \cdots, \theta_n^E)$$

as an algebra over $K_G^*(X)$.

Proof. For the sake of simplicity, put $U = U(n)$ and $ad = ad_{1_{U(n)}}$, the adjoint operation of the identity representation of $U(n)$.

Let P be the associated principal bundle to E. Then P is a $G \times U$-space on which U acts freely: $P/U = X$ and

$$U(E) = P \underset{U}{\times} (U, ad).$$

We can regard (U, ad) as a $G \times U$-space where G acts on (U, ad) trivially. Then we have

$$K_{G \times U}^*(U, ad) \cong R(G) \otimes K_U^*(U, ad)$$

by a parallel proof to that of [5], Proposition (2.2).

From Corollary 1.2 we obtain

$$K_{G \times U}^*(P) \underset{R(G \times U)}{\otimes} K_{G \times U}^*(U, ad) \cong K_{G \times U}^*(P \times (U, ad)).$$

Hence we get

$$K_G^*(X) \underset{R(U)}{\otimes} K_U^*(U, ad) \cong K_G^*(U(E))$$

by [5], Proposition (2.1). This shows the corollary from Theorem 1.1.

In the following sections we shall give a proof of Theorem 1.1.

2. Proof when G is connected

The proof consists of two steps.

Step 1. *Proof when G is a compact abelian Lie group.*

For the sake of simplicity we write $U(\rho)$ for the G-manifold $(U(n), ad_\rho)$.

Since G is abelian, there exist 1-demensional representations of G, ρ_k: $G \to U(1)$ $1 \leq k \leq n$, such that ρ is equivalent to the sum $\rho_1 \oplus \cdots \oplus \rho_n$. Then

$$U(\rho) \cong U(\rho_1 \oplus \cdots \oplus \rho_n)$$

as a G-manifiold. So it suffices to show the theorem for $U(\rho)$, $\rho = \rho_1 \oplus \cdots \oplus \rho_n$.

Before beginning the proof of the theorem we prepare an elementary lemma. Let W be the representation space over $\boldsymbol{C}$ of the representation $1 \oplus \rho_1^{-1}\rho_2 \oplus \cdots \oplus \rho_1^{-1}\rho_n$. Then the unit sphere $S(W)$ in W is homeomorphic to the homogeneous space $U(\rho)/U(\rho_2 \oplus \cdots \oplus \rho_n)$ as a G-space where $U(\rho_2 \oplus \cdots \oplus \rho_n) = 1 \times U(\rho_2 \oplus \cdots \oplus \rho_n)$ and also $S(W)$ has a fixed point $p = (1, 0, \cdots, 0)$.

Lemma 2.1. *For each point $q = (z_1, \cdots, z_n)$ of $S(W)$ there exists a continuous map f: $[0, 1] \to U(n)$ such that $f(0)(p) = q$, $f(1) = 1$ and $\rho(g)f(t)\rho(g)^{-1} = f(t)$ for $g \in G_q$ and $t \in [0, 1]$ where G_q is the isotropy group at q.*

Proof. We shall prove Lemma 2.1 by induction on n. For the case of $n=1$ we have nothing to do. Assume that the assertion is true for $n<l$. In case of $n=l$ we consider two types of q as follows.

(i) If $z_2 \cdots z_n \neq 0$, then

$$\rho_1(g) = \cdots = \rho_n(g) \qquad g \in G_q\,.$$

Namely $\rho(g)$ is a diagonal matrix for any $g \in G_q$. So it is sufficient to show the existence of a continuous map f: $[0, 1] \to U(n)$ such that $f(0)(p) = q$ and $f(1) = 1$. But this is clear because $U(n)$ acts on S^{2n-1} transitively and $U(n)$ is arcwise connected.

(ii) If there is an integer $k \geq 2$ such that $z_k = 0$, then we consider a subgroup, $U'(n-1)$, of $U(n)$ consisting of $(n-1)$-dimensional minors of which the (k, k)-component is 1, i.e.

$$\begin{array}{cc} & k \\ k & \begin{pmatrix} & 0 & \\ * & \vdots & * \\ & 0 & \\ 0 \cdots 0 & 1 & 0 \cdots 0 \\ & 0 & \\ * & \vdots & * \\ & 0 & \end{pmatrix} \end{array} \in U(n)\,.$$

Let ρ' be a continuous homomorphism of G into $U'(n-1)$ defined by

$$\rho' = \rho_1 \oplus \cdots \oplus \rho_{k-1} \oplus 1 \oplus \rho_{k+1} \oplus \cdots \oplus \rho_n\,.$$

In virtue of the inductive hypothesis there is a map f': $[0, 1] \to U'(n-1)$ satisfying the assertion mentioned in Lemma 2.1. Then we have

$$\rho'(g)f'(t)\rho'(g)^{-1} = \rho(g)f'(t)\rho(g)^{-1} \qquad g\in G_q,\ t\in[0, 1]\,.$$

Therefore when we put

$$f = if'$$

where i: $U'(n-1)\to U(n)$ is the inclusion of $U'(n-1)$, f: $G\to U(n)$ is a map which we require. q.e.d.

Now we proceed by induction on n to complete the step 1. In case of $n=1$, since G acts on $U(\rho_1)$ trivially we have

$$K_G^*(U(\rho_1))\cong R(G)\otimes K^*(U(1))$$

by [5], Proposition (2,2). $K^*(U(1))$ is an exterior algebra with one generator θ and by the above isomorphism θ_1^G corresponds to $\rho_1\otimes\theta$. Hence $K_G^*(U(\rho_1))=\Lambda_{R(G)}(\theta_1^G)$ is valid for any compact abelian Lie group G and any 1-demensional representation $\rho=\rho_1$ of G.

Let π: $U(\rho)\to S(W)(=U(\rho)/U(\rho_2\oplus\cdots\oplus\rho_n))$ be the projection. From [4], Lemma 1 we get

Lemma 2.2. *There exists an element g in $K_G^1(S(W))$ such that*

$$K_G^*(S(W)) = \Lambda_{R(G)}(g)$$

as an algebra over $R(G)$ and

$$\pi^*(g) = \textstyle\sum_{k=1}^n(-1)^k\rho_1^{-k}\theta_k^G \qquad \theta_k^G\in K_G^1(U(\rho))\,.$$

Proof. We observe the exact sequence of the pair $(D(W), S(W))$ where $D(W)$ is the unit disk in W. Then we see that

$$\tilde{K}_G^0(S(W)) = 0$$

and the coboundary homomorphism

$$\delta\colon K_G^1(S(W)) \to K_G^0(W)$$

is an isomorphism.

When we denote by λ_W the Thom class of the vector bundle $W\to P(=\text{a point})$, $K_G^0(W)$ is a free module over $R(G)$ generated by λ_W. So if we put $g=\delta^{-1}(\lambda_W)$, then

$$K_G^*(S(W)) = \Lambda_{R(G)}(g)\,.$$

Next we consider the following diagram

$$\begin{array}{ccc} K_T^*(U(n), ad_i) & \xrightarrow{\rho^*} & K_G^*(U(\rho)) \\ \uparrow{\scriptstyle \pi'^*} & & \uparrow{\scriptstyle \pi^*} \\ K_T^*(S(W')) & \xrightarrow{\rho^*} & K_G^*(S(W)) \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} \\ K_T^*(W') & \xrightarrow{\rho^*} & K_G^*(W) \end{array}$$

where i: $T\rightarrow U(n)$ is the inclusion map of the standard maximal torus T of $U(n)$, ρ^* the homomorphism induced by the continuous homomorphism $\rho=\rho_1\oplus\cdots\oplus\rho_n$: $G\rightarrow T$ and π': $U(n)\rightarrow S(W')$ represents the map π: $U(n)\rightarrow S(C\oplus W)$ in [4], §2. Then this diagram commutes and $\rho^*(\lambda_{W'})=\lambda_W$. Therefore we get

$$\pi^*(g)=\sum_{k=1}^{n}(-1)^k\rho_1^{-k}\theta_k^G$$

by [4], Lemma 1. q.e.d.

Let $\mathfrak{M}$ be an exterior algebra over $R(G)$ generated by $\theta_1^G, \cdots, \theta_{n-1}^G$ where $\theta_k^G\in K_G^1(U(\rho))$ for $1\leq k\leq n-1$. Then we have a homomorphism

$$\kappa_1\colon \mathfrak{M}\rightarrow K_G^*(U(\rho))$$

of algebras, defined by $\kappa_1(\theta_k^G)=\theta_k^G$. Because, when we observe the homomorphism ρ^*: $K_T^*(U(n), ad_i)\rightarrow K_G^*(U(\rho))$ mentioned in the proof of Lemma 2.2 we get

$$(\theta_k^G)^2=\rho^*((\theta_k^T)^2)=0 \quad \text{for} \quad 1\leq k\leq n$$

since $(\theta_k^T)^2=0$ in $K_T^*(U(n), ad_i)$ by [4], Theorem 1 and also we get the relations $\theta_k^G\theta_l^G+\theta_l^G\theta_k^G=0$ for $1\leq k$, $l\leq n$ obviously since θ_k^G are the elements of $K_G^1(U(\rho))$. Morevoer, for each closed invariant subspace X of $S(W)$ we can define a homomorphism

$$\lambda\colon K_G^*(X)\underset{R(G)}{\otimes}\mathfrak{M}\rightarrow K_G^*(\pi^{-1}(X))$$

by

$$\lambda(x\otimes y)=\pi^*(x)j^*\kappa_1(y) \qquad x\in K_G^*(X),\ y\in\mathfrak{M}$$

where j: $\pi^{-1}(X)\rightarrow U(\rho)$ is the inclusion of $\pi^{-1}(X)$.

Under the assumption that the assertion of Theorem 1.1 in the step 1 is true for $n<l$ the following lemma is proved.

Lemma 2.3. *The homomorphism*

$$\lambda\colon K_G^*(S(W))\underset{R(G)}{\otimes}\mathfrak{M}\rightarrow K_G^*(U(\rho))$$

is an isomorphism.

Proof. Let $\overline{S(W)}$ denote the orbit space of $S(W)$ by G. We have two sheaves over $\overline{S(W)}$, φ_* and τ_* whose stalks are respectively

$$\varphi_*(\bar{q}) = K_G^*(Gq) \underset{R(G)}{\otimes} \mathfrak{M}$$

and

$$\tau_*(\bar{q}) = K_G^*(\pi^{-1}(Gq))$$

where $q \in S(W)$, $\bar{q} \in \overline{S(W)}$ and $Gq = \pi^{-1}(\bar{q})$.

Since $\mathfrak{M}$ is a free module over $R(G)$, $K_G^*(X) \underset{R(G)}{\otimes} \mathfrak{M}$ is an equivariant cohomology theory. Then λ induces a map of the spectral sequence [5]

$$E_2^{p,q} = H^p(\overline{S(W)}, \varphi_q) \Rightarrow K_G^*(S(W)) \underset{R(G)}{\otimes} \mathfrak{M}$$

to the spectral sequence

$$E_2^{p,q} = H^p(\overline{S(W)}, \tau_q) \Rightarrow K_G^*(U(\rho)) .$$

We shall prove that λ induces an isomorphism on the E_2-level. Clearly we have

$$\begin{aligned} \varphi_*(\bar{q}) &= K_G^*(Gq) \underset{R(G)}{\otimes} \mathfrak{M} \\ &\cong K_G^*(G/G_q) \underset{R(G)}{\otimes} \mathfrak{M} \\ &\cong \Lambda_{R(G_q)}(\theta_1^G, \cdots, \theta_{n-1}^G) . \end{aligned}$$

Next we observe the stalks $\tau_*(\bar{q})$. Let f: $[0, 1] \to U(n)$ is a continuous map in Lemma 2.1. Then we have

$$\pi^{-1}(Gq) = \bigcup_{q \in G} (\rho(g) f(0) \rho(g)^{-1}) U(n-1)$$

and so we can define a G-map

$$\phi\colon G/G_q \times U(\rho_2 \oplus \cdots \oplus \rho_n) \to \pi^{-1}(Gq)$$

by

$$\phi(gG_q, u) = (\rho(g) f(0) \rho(g)^{-1}) u \qquad g \in G,\ u \in U(\rho_2 \oplus \cdots \oplus \rho_n)$$

because $\rho(g) f(0) \rho(g)^{-1} = f(0)$ for any $g \in G_q$. Further we can easily check that ϕ is an isomorphism. Therefore

$$\begin{aligned} \tau_*(\bar{q}) &= K_G^*(\pi^{-1}(Gq)) \\ &\cong K_G^*(G/G_q \times U(\rho_2 \oplus \cdots \oplus \rho_n)) \\ &\cong K_{G_q}^*(U(\rho_2 \oplus \cdots \oplus \rho_n)) . \end{aligned}$$

Thus we obtain

$$\tau_*(\bar{q}) \cong \Lambda_{R(G_q)}(\theta_1', \cdots, \theta_{n-1}')$$

by the inductive hypothesis where $\theta_k' = \theta_k^{G_q}$ for $1 \leq k \leq n-1$.

Here we consider the homomorphism

$$\lambda'\colon \Lambda_{R(G_q)}(\theta_1^G, \cdots, \theta_{n-1}^G) \to K_{G_q}^*(U(\rho_2 \oplus \cdots \oplus \rho_n))$$

induced by the homomorphism

$$\lambda\colon \varphi_*(\bar{q}) \to \tau_*(\bar{q})\,.$$

From the definition of θ_k^G we obtain easily

$$\lambda'(\theta_k^G) = [U(n-1)\times\lambda^k(V),\ \rho_k] \qquad (1\leq k\leq n-1)$$

where ξ_k is an automorphism of the product G_q-bundle $U(n-1)\times\lambda^k(V)$ given by

$$\xi_k(u, z) = (u, \lambda^k(f(0)u)(z)) \qquad u\in U(n-1),\ z\in\lambda^k(V)\,.$$

Since f is a homotopy from $f(0)$ to the identity element of $U(n)$ satisfying $\rho(g)f(t)=f(t)\rho(g)$ for any $g\in G_q$ and $t\in[0, 1]$, we get

$$\lambda'(\theta_k^G) = \begin{cases}\theta_1' & (k=1)\\ \theta_k'+\rho_1\theta_{k-1}' & (2\leq k\leq n-1)\,.\end{cases}$$

Hence we see that λ' is an isomorphism. This shows that λ induces an isomorphism on the E_2-level. Consequently we obtain Lemma 2.3. q.e.d.

Lemma 2.2 and lemma 2.3 show that the assertion in the case of $n=l$ is also true. This completes the step 1.

Step 2. Proof when G is connected.

Let T be a maximal torus of G and $i\colon T\to G$ the inclusion of T. Then from the step 1 we get

$$K_T^*(U(n), ad_{\rho_T}) = \Lambda_{R(T)}(\theta_1^T, \cdots, \theta_n^T)$$

where ρ_T is the restriction of ρ onto T and therefore, from [5], Proposition (3.8) and [4], Lemma 2 we get

$$K_G^*(U(n), ad_\rho)\cong K_T^*(U(n), ad_{\rho_T})^{W(G)}$$

where $W(G)$ is the Weyl group of G. This shows

$$K_G^*(U(n), ad_\rho) = \Lambda_{R(G)}(\theta_1^G, \cdots, \theta_n^G)\,.$$

3. Proof when G is not connected

We recall

Theorem 3.1. (Segal [6]) *Let G be a compact Lie group. Then the restric-*

tion $R(G)\to\sum_S R(S)$ is injective where S runs through the representatives of conjugacy classes of Cartan subgroups of G.

Then we have

Lemma 3.2. *Let G be a compact Lie group and ρ a continuous homomorphism of G into $U(n)$. Then $\Lambda_{R(G)}(\theta_1^G, \cdots, \theta_n^G)$ is a subalgebra of $K_G^*(U(n), ad_\rho)$.*

Proof. We have a homomorphism κ_2 of $\Lambda_{R(G)}(\theta_1^G, \cdots, \theta_n^G)$ into $K_G^*(U(n), ad_\rho)$ as algebras defined by $\kappa_2(\theta_k^G)=\theta_k^G$, $1\leq k\leq n$. This homomorphism is well-defined by the same reason as κ_1 in §2, Step 1 is so.

Let S be a Cartan subgroup of G and i_S: $S\to G$ the inclusion of S. Then we have

$$K_S^*(U(n), ad_{\rho_s}) = \Lambda_{R(S)}(\theta_1^S, \cdots, \theta_n^S)$$

from §2, Step 1 where ρ_S is thr restriction of ρ onto S. Therefore if

$$\sum_{1\leq i_1<\cdots<i_l\leq n}\alpha_{i_1\cdots i_l}\theta_{i_1}^G\cdots\theta_{i_l}^G = 0$$

for $\alpha_{i_1\cdots i_l}\in R(G)$ in $K_G^*(U(n), ad_\rho)$, then

$$i_S^*(\alpha_{i_1\cdots i_l}) = 0$$

for any Cartan subgroup S of G. So we get

$$\alpha_{i_1\cdots i_l} = 0\,, \quad 1\leq i_1<\cdots<i_l\leq n$$

from Theorem 3.1. This shows that κ_2 is injective. q.e.d.

Using the Segal's spectral sequence [5] we can easily check the following

Lemma 3.3. ([3], Proposition 2) *Let G be a compact Lie group. Let X and Y be compact locally G-contractible G-spaces such that the orbit spaces X/G and Y/G are of finite covering dimension. If $K_G^*(X)$ or $K_G^*(Y)$ is a free abelian group, then the external tensor product*

$$K_G^*(X)\otimes K_G^*(Y) \to K_{G\times G}^*(X\times Y)$$

is an isomorphism.

The following theorem is basic in proof of the general case.

Theorem 3.4. ([1], Proposition (4.9), [5], Proposition (3.8))
Let G be a compact connected Lie group and i: $T\to G$ the inclusion of a maximal torus. Then for each locally compact G-space X there is a natural homomorphism of $K_G^*(X)$-modules i_*: $K_T^*(X)\to K_G^*(X)$ such that $i_*(1)=1$, and hence $i_*i^*=identity$.

Theorem 3.5. *Let G be a compact connected Lie group and $\rho: G\to U(n)$ a* *unitary representation. Then, for each closed subgroup H of G we have*

$$K_H^*(U(n), ad_{\rho_H}) = \Lambda_{R(H)}(\theta_1^H, \cdots, \theta_n^H)$$

as an algebra over $R(H)$ where ρ_H is the restriction of ρ onto H.

Proof. As in §2, we denote $(U(n), ad_\rho)$ by $U(\rho)$. Let π_1: $U(\rho)\times G/H \to U(\rho)$ and π_2: $U(\rho)\times G/H \to G/H$ be the projections. Let d: $G \to G\times G$ be the diagonal map.

We consider the homomorphism

$$d^*: K_{G\times G}^*(U(\rho)\times G/H) \to K_G^*(U(\rho)\times G/H).$$

From Lemma 3.3 and §2, Step 2 we get

$$(1)\qquad \begin{aligned} K_{G\times G}^*(U(\rho)\times G/H) &\cong K_G^*(U(\rho))\otimes K_G^*(G/H) \\ &\cong \Lambda_{R(G)}(\theta_1^G, \cdots, \theta_n^G)\otimes R(H). \end{aligned}$$

From (1) we see that d^* induces a homomorphism

$$\mu_1: K_G^*(U(\rho))\otimes K_G^*(G/H) \to K_G^*(U(\rho)\times G/H)$$

and then μ_1 is as follows:

$$\mu_1(x\otimes y) = \pi_1^*(x)\pi_2^*(y) \qquad \text{for } x\in K_G^*(U(\rho)), y\in K_G^*(G/H).$$

Since $K_G^*(U(\rho)\times G/H) \cong K_H^*(U(\rho_H))$ and $\Lambda_{R(H)}(\theta_1^H, \cdots, \theta_n^H)$ is a subalgebra of $K_H^*(U(\rho_H))$ by Lemma 3.2, $\Lambda_{R(H)}(\pi_1^*(\theta_1^G), \cdots, \pi_1^*(\theta_n^G))$ is a subalgebra of $K_G^*(U(\rho)\times G/H)$ and also

$$(2)\qquad \operatorname{Im}\mu_1 = \Lambda_{R(H)}(\pi_1^*(\theta_1^G), \cdots, \pi_1^*(\theta_n^G)).$$

Therefore if we prove that μ_1 is an epimorphism, then we obtain Theorem 3.5.

Let T be a maximal torus of G. First we consider the restriction ρ_T: $T \to U(n)$ of ρ onto T. As the case of ρ: $G \to U(n)$ we have

$$\begin{aligned} K_{T\times T}^*(U(\rho_T)\times G/H) &\cong K_T^*(U(\rho_T))\otimes K_T^*(G/H) \\ &\cong \Lambda_{R(T)}(\theta_1^T, \cdots, \theta_n^T)\otimes K_T^*(G/H) \end{aligned}$$

and so the homomorphism

$$\mu_2: K_T^*(U(\rho_T))\otimes K_T^*(G/H) \to K_T^*(U(\rho_T)\times G/H)$$

induced by d^*. Also we get

$$\begin{aligned} K_T^*(U(\rho_T)\times G/H) &\cong K_T^*(U(\rho_T)) \underset{R(T)}{\otimes} K_T^*(G/H) \\ &\cong \Lambda_{R(T)}(\theta_1^T, \cdots, \theta_n^T) \underset{R(T)}{\otimes} K_T^*(G/H) \end{aligned}$$

from §2, Step 1 and a parallel argument to Corollary 1.2.

Now we observe the following diagram

$$\begin{tikzcd}
\Lambda_{R(G)}(\theta_1^G, \cdots, \theta_n^G)\otimes R(H) \arrow[d, "\cong"'] & & \Lambda_{R(T)}(\theta_1^T, \cdots, \theta_n^T)\otimes K_T^*(G/H) \arrow[d, "\cong"] \\
K_G^*(U(\rho))\otimes K_G^*(G/H) \arrow[d, "\mu_1"'] \arrow[rr, shift left, "i_1^*\otimes i_2^*"] & & K_T^*(U(\rho_T))\otimes K_T^*(G/H) \arrow[ll, shift left, "i_{1*}\otimes i_{2*}"] \arrow[d, "\mu_2"] \\
K_G^*(U(\rho)\times G/H) \arrow[rr, shift left, "j^*"] & & K_T^*(U(\rho_T)\times G/H) \arrow[ll, shift left, "j_*"] \\
K_H^*(U(\rho_H)) \arrow[u, "\cong"] & & \Lambda_{R(T)}(\theta_1^T, \cdots, \theta_n^T)\underset{R(T)}{\otimes} K_T^*(G/H) \arrow[u, "\cong"']
\end{tikzcd}$$

where i_1, i_2 and j are the inclusion of T, and i_{1*}, i_{2*} and j_* denote the natural homomorphisms mentioned in Theorem 3.4.

For any $x\in K_G^*(U(\rho)\times G/H)$ we can write

$$(3)\qquad j^*(x) = \alpha\pi_2^*(y)+\textstyle\sum_{1\leq i_1<\cdots<i_s\leq n}\alpha_{i_1\cdots i_s}\pi_1^*(\theta_{i_1}^T\cdots\theta_{i_s}^T)\pi_2^*(y_{i_1\cdots i_s})$$

for α, $\alpha_{i_1\cdots i_s}\in R(T)$ and y, $y_{i_1\cdots i_s}\in K_T^*(G/H)$.

Let put

$$z = 1\otimes\alpha y+\textstyle\sum_{1\leq i_1<\cdots<i_s\leq n}\theta_{i_1}^T\cdots\theta_{i_s}^T\otimes\alpha_{i_1\cdots i_s}y_{i_1\cdots i_s}$$

in $K_T^*(U(\rho_T))\otimes K_T^*(G/H)$. Then from (3) we get

$$(4)\qquad \mu_2(z) = j^*(x)\,.$$

Moreover

$$(i_{1*}\otimes i_{2*})(z) = 1\otimes i_{2*}(\alpha y)+\textstyle\sum_{1\leq i_1<\cdots<i_s\leq n}\theta_{i_1}^G\cdots\theta_{i_s}^G\otimes i_{2*}(\alpha_{i_1\cdots i_s}y_{i_1\cdots i_s})$$

since $i_1^*\theta_k^G=\theta_k^T$ $1\leq k\leq n$ and $i_{1*}i_1{}^*=1$, and

$$\begin{aligned}(5)\qquad \mu_1((i_{1*}\otimes i_{2*})(z)) &= \pi_2^*i_{2*}(\alpha y)+\textstyle\sum_{1\leq i_1<\cdots<i_s\leq n}\pi_1^*(\theta_{i_1}^G\cdots\theta_{i_s}^G)\pi_2^*i_{2*}(\alpha_{i_1\cdots i_s}y_{i_1\cdots i_s})\\ &= j_*\pi_2^*(\alpha y)+\textstyle\sum_{1\leq i_1<\cdots<i_s\leq n}\pi_1^*(\theta_{i_1}^G\cdots\theta_{i_s}^G)j_*\pi_2^*(\alpha_{i_1\cdots i_s}y_{i_1\cdots i_s})\end{aligned}$$

because of $j_*\pi_2^*=\pi_2^*i_{2*}$. By Theorem 3.4, j_* is the homomorphism of $K_G^*(U(\rho)\times G/H)$-modules. Therefore (5) shows

$$(6)\qquad \mu_1((i_{1*}\otimes i_{2*})(z)) = j_*(\alpha\pi_2^*(y)+\textstyle\sum_{1\leq i_1<\cdots<i_s\leq n}\alpha_{i_1\cdots i_s}\pi_1^*(\theta_{i_1}^T\cdots\theta_{i_s}^T)\pi_2^*(y_{i_1\cdots i_s}))$$

because of $\pi_1^*i_1^*=j^*\pi_1^*$.

From (3) and (6) we obtain

$$(7)\qquad \mu_1((i_{1*}\otimes i_{2*})(z)) = j_*j^*(x) = x$$

and so we see that μ_1 is an epimorphism. Hence (2) and (7) conclude

$$K_H^*(U(\rho_H)) = \Lambda_{R(H)}(\theta_1^H, \cdots, \theta_n^H).$$

q.e.d.

Proof of the general case. Let G be a compact Lie group and $\rho: G \to U(n)$ a unitary representation of G.

Embed G in a unitary group $U(m)$ and consider an embedding

$$f: G \to U(n) \times U(m)$$

defined by

$$f(g) = (\rho(g), g) \qquad g \in G.$$

Let $\pi: U(n) \times U(m) \to U(n)$ be the projection. If we regard G as a closed subgroup of $U(n) \times U(m)$ by f, then ρ is the restriction of π onto G. Therefore, from Theorem 3.5 we get

$$K_G^*(U(n), ad_\rho) = \Lambda_{R(G)}(\theta_1^G, \cdots, \theta_n^G).$$

This completes the proof of Theorem 1.1.

4. The special unitary group $SU(n)$

Let G be a compact Lie group and $\rho: G \to U(n)$ a unitary representation of G. Then $SU(n)$ becomes a G-submanifold of $(U(n), ad_\rho)$ which we denote by $(SU(n), ad_\rho)$.

Let $j: SU(n) \to U(n)$ be the inclusion of $SU(n)$. We use the same symbol θ_k^G for the image of $\theta_k^G \in K_G^1(U(n), ad_\rho)$ by j^* for $1 \leq k \leq n-1$. In particular, $j^*(\theta_n^G) = 0$.

Let T be the standard maximal torus of $U(n)$ and $i: T \to U(n)$ the inclusion of T. Then, by a parallel proof to that in [4] we obtain

Proposition 4.1. *Using the notation of* [4], *Lemma* 1 *we have*

(i) $K_T^*(S(C \oplus W)) = K_T^*(SU(n)/SU(n-1))$ *is an exterior algebra over* $R(T)$ *with one generator* g *satisfying*

$$\pi^*(g) = \textstyle\sum_{k=1}^{n-1}(-1)^k \rho_1^{-k} \theta_k^T$$

where $\pi: SU(n) \to S(C \oplus W) (= SU(n)/SU(n-1))$ *is the projection, and therefore*

(ii) $K_T^*(SU(n), ad_i) = \Lambda_{R(T)}(\theta_1^T, \cdots, \theta_{n-1}^T)$
as an algebra over $R(T)$.

From Proposition 4.1 an analogous statement can be made as follows.

Proposition 4.2. *Let* G *be a compact Lie group and* $\rho: G \to U(n)$ *a unitary representation of* G. *Then*

$$K_G^*(SU(n),\ ad_\rho) = \Lambda_{R(G)}(\theta_1^G,\ \cdots,\ \theta_{n-1}^G)$$

as an algenra over $R(G)$.

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References

[1] M.F. Atiyah: *Bott periodicity and the index of elliptic operators*, Quart. J. Math. Oxford Ser. **19** (1968), 113–140.

[2] L. Hodgkin: *An equivariant Künneth formula in K-theory*, 1968, University of Warwick, preprint.

[3] H. Minami: *A Künneth formula for equivariant K-theory*, Osaka J. Math. **6** (1969), 143–146.

[4] ———: *Equivariant K-ring of G-manifold* $(U(n), ad_G)$ I, Osaka J. Math. **9** (1972), 367–377.

[5] G.B. Segal: *Equivariant K-theory*, Inst. Hautes Etudes Sci. Publ. Math. **34** (1968), 129–151.

[6] ———: *The representation ring of a compact Lie group*, Inst. Hautes Etudes Sci. Publ. Math. **34** (1968), 113–128.