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EQUIVARIANT K-RING OF G-MANIFOLD (U(n), ad_ρ) II

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Introduction

Let G be a compact Lie group. Throughout this paper K_G^* will denote the complex equivariant K-theory associated with the group G and $R(G)$ the ring of virtual complex representations of G.

Let V be a G-module over the field of the complex numbers and $U(V)$ the group of isometries of V with the action of G defined by conjugation. In [2], Hodgkin has announced the K_G-ring structure of $U(V)$ without proof. So we have proved a special case of Hodgkin's theorem in [4]. The purpose of this paper is to give a proof of the general case.

1. Statement of the theorems

Let G be a compact Lie group and ρ a unitary representation of G of dimension n. That is, ρ is a continuous homomorphism of G into a unitary group $U(n)$.

We consider $U(n)$ a differentiable G-manifold together with the adjoint operation ad_ρ: $G\times U(n)\to U(n)$, defined by

$$ad_\rho(g, u) = \rho(g)u\rho(g)^{-1} \qquad g\in G,\ u\in U(n)$$

and then we denote the G-manifold $U(n)$ by $(U(n), ad_\rho)$.

We denote by V the representation space of ρ over the field of the complex numbers $\boldsymbol{C}$, by $\underline{V}$ the product G-vector bundle with a fibre V over $U(n)$ and by $\lambda^k(\underline{V}) = \underline{\lambda^k(V)}$ the k-th exterior power of $\underline{V}$ for $1\leq k\leq n$. Then we can define an automorphism θ_k^G of $\lambda^k(\underline{V})$ by

$$\theta_k^G(u, z) = (u, \lambda^k(u)(z)) \qquad n\in U(n),\ z\in\lambda^k(V)\,.$$

Hence θ_k^G determines an element $[\lambda^k(\underline{V}), \theta_k^G]$ of $K_G^1(U(n), ad_\rho)$. Afterwards we shall use the same symbol θ_k^G in writing this induced element. Our main theorem is:

Theorem 1.1. *Let G be a compact Lie group and ρ a unitary representation of G of dimension n. Then*

$$K_G^*(U(n), ad_\rho) = \Lambda_{R(G)}(\theta_1^G, \cdots, \theta_n^G)$$

as an algebra over $R(G)$.

Theorem 1.1 has the following corollaries.

Corollary 1.2. *Let ρ be as in Theorem* 1.1 *and X a compact locally G-contractible G-space whose orbit space X/G has a finite covering dimensnion. Then the external tensor product homomorphism*

$$\mu: K_G^*(U(n), ad_\rho) \underset{R(G)}{\otimes} K_G^*(X) \to K_G^*((U(n), ad_\rho) \times X)$$

is an isomorphism.

Proof. Put $U=U(n)$ for the simplicity. $K_G^*(U, ad_\rho) \underset{R(G)}{\otimes} K_G^*(X)$ is an equivariant cohomology theory because $K_G^*(U, ad_\rho)$ is a free module over $R(G)$ and also $K_G^*((U, ad_\rho) \times X)$ is an equivariant cohomology theory. As easily checked, we can construct spectral sequences of Segal's type for these equivariant cohomology theories [5].

Let $\bar{X}$ denote the orbit space of X by G. There are two sheaves over $\bar{X}$, φ_* and τ_* whose stalks are respectively

$$\varphi_*(\bar{x}) = K_G^*(U, ad_\rho) \underset{R(G)}{\otimes} K_G^*(Gx)$$

and

$$\tau_*(\bar{x}) = K_G^*((U, ad_\rho) \times Gx)$$

where $\bar{x} \in \bar{X}$ and $Gx \subset X$ is the orbit of $x \in X$ lying over $\bar{x}$.

The external tensor product homomorphism μ induces a map of the spectral sequence

$$E_2^{p,q} = H^p(\bar{X}, \varphi_q) \Rightarrow K_G^*(U, ad_\rho) \underset{R(G)}{\otimes} K_G^*(X)$$

to the spectral sequence

$$E_2^{p,q} = H^p(\bar{X}, \tau_q) \Rightarrow K_G^*((U, ad_\rho) \times X) .$$

Let G_x denote the isotropy group at x. Since Gx is homeomorphic to G/G_x as a G-space we have

$$\begin{aligned} \varphi_*(\bar{x}) &\cong K_G^*(U, ad_\rho) \underset{R(G)}{\otimes} K_G^*(G/G_x) \\ &\cong K_G^*(U, ad_\rho) \underset{R(G)}{\otimes} R(G_x) \end{aligned}$$

and

$$\begin{aligned} \tau_*(\bar{x}) &\cong K_G^*((U, ad_\rho) \times G/G_x) \\ &\cong K_{G_x}^*(U, ad_{\rho'}) \end{aligned}$$

where ρ' is the restriction of ρ onto G_x. Therefore, from Theorem 1.1 we see

$$\varphi_*(\tilde{x}) \cong \Lambda_{R(G_x)}(\theta_1^G, \cdots, \theta_n^G),\ \tau_*(\tilde{x}) \cong \Lambda_{R(G_x)}(\theta_1^{G_x}, \cdots, \theta_n^{G_x})$$

and so μ induces an isomorphism on the E_2-level. This permits the corollary.

Let X be a G-space as in Corollary 1.2 and E an n-dimensional complex G-vector bundle over X. Here we consider the unitary bundle $\pi: U(E) \to X$ of E (See [2], §3). For $1 \leq k \leq n$ we can define also an automoprhism θ_k^E of the G-vector bundle $\pi^*(\lambda^k(E)) = \lambda^k(\pi^*(E))$ over $U(E)$ by

$$\theta_k^E(u, z) = (u, \lambda^k(u)(z)) \qquad u \in U(E_x),\ z \in \lambda^k(E_x)$$

and we write θ_k^E for an element of $K_G^1(U(E))$ determined by θ_k^E. Then we have the following

Corollary 1.3.

$$K_G^*(U(E)) = \Lambda_{K_G^*(X)}(\theta_1^E, \cdots, \theta_n^E)$$

as an algebra over $K_G^*(X)$.

Proof. For the sake of simplicity, put $U = U(n)$ and $ad = ad_{U(n)}$, the adjoint operation of the identity representation of $U(n)$.

Let P be the associated principal bundle to E. Then P is a $G \times U$-space on which U acts freely: $P/U = X$ and

$$U(E) = P \underset{U}{\times} (U, ad)\,.$$

We can regard (U, ad) as a $G \times U$-space where G acts on (U, ad) trivially. Then we have

$$K_{G \times U}^*(U, ad) \cong R(G) \otimes K_U^*(U, ad)$$

by a parallel proof to that of [5], Proposition (2.2).

From Corollary 1.2 we obtain

$$K_{G \times U}^*(P) \underset{R(G \times U)}{\otimes} K_{G \times U}^*(U, ad) \cong K_{G \times U}^*(P \times (U, ad))\,.$$

Hence we get

$$K_G^*(X) \underset{R(U)}{\otimes} K_U^*(U, ad) \cong K_G^*(U(E))$$

by [5], Proposition (2.1). This shows the corollary from Theorem 1.1.

In the following sections we shall give a proof of Theorem 1.1.

2. Proof when G is connected

The proof consists of two steps.

Step 1. *Proof when G is a compact abelian Lie group.*

For the sake of simplicity we write $U(\rho)$ for the G-manifold $(U(n), ad_\rho)$.

Since G is abelian, there exist 1-demensional representations of G, ρ_k: $G\to U(1)$ $1\leq k\leq n$, such that ρ is equivalent to the sum $\rho_1\oplus\cdots\oplus\rho_n$. Then

$$U(\rho)\cong U(\rho_1\oplus\cdots\oplus\rho_n)$$

as a G-manifold. So it suffices to show the theorem for $U(\rho)$, $\rho=\rho_1\oplus\cdots\oplus\rho_n$.

Before beginning the proof of the theorem we prepare an elementary lemma. Let W be the representation space over $\boldsymbol{C}$ of the representation $1\oplus\rho_1^{-1}\rho_2\oplus\cdots\oplus\rho_1^{-1}\rho_n$. Then the unit sphere $S(W)$ in W is homeomorphic to the homogeneous space $U(\rho)/U(\rho_2\oplus\cdots\oplus\rho_n)$ as a G-space where $U(\rho_2\oplus\cdots\oplus\rho_n)=1\times U(\rho_2\oplus\cdots\oplus\rho_n)$ and also $S(W)$ has a fixed point $p=(1, 0, \cdots, 0)$.

Lemma 2.1. *For each point $q=(z_1, \cdots, z_n)$ of $S(W)$ there exists a continuous map f: $[0, 1]\to U(n)$ such that $f(0)(p)=q$, $f(1)=1$ and $\rho(g)f(t)\rho(g)^{-1}=f(t)$ for $g\in G_q$ and $t\in[0, 1]$ where G_q is the isotropy group at q.*

Proof. We shall prove Lemma 2.1 by induction on n. For the case of $n=1$ we have nothing to do. Assume that the assertion is true for $n<l$. In case of $n=l$ we consider two types of q as follows.

(i) If $z_2\cdots z_n\neq 0$, then

$$\rho_1(g)=\cdots=\rho_n(g)\qquad g\in G_q\,.$$

Namely $\rho(g)$ is a diagonal matrix for any $g\in G_q$. So it is sufficient to show the existence of a continuous map f: $[0, 1]\to U(n)$ such that $f(0)(p)=q$ and $f(1)=1$. But this is clear because $U(n)$ acts on S^{2n-1} transitively and $U(n)$ is arcwise connected.

(ii) If there is an integer $k\geq 2$ such that $z_k=0$, then we consider a subgroup, $U'(n-1)$, of $U(n)$ consisting of $(n-1)$-dimensional minors of which the (k, k)-component is 1, i.e.

$$\begin{array}{c} \\ k \end{array}\begin{array}{c} \begin{array}{ccccc} & & k & & \end{array} \\ \begin{pmatrix} & 0 & & \\ * & \vdots & * & \\ & 0 & & \\ 0\cdots 0 & 1 & 0\cdots 0 & \\ & 0 & & \\ * & \vdots & * & \\ & 0 & & \end{pmatrix} \end{array}\in U(n)\,.$$

Let ρ' be a continuous homomorphism of G into $U'(n-1)$ defined by

$$\rho'=\rho_1\oplus\cdots\oplus\rho_{k-1}\oplus 1\oplus\rho_{k+1}\oplus\cdots\oplus\rho_n\,.$$

In virtue of the inductive hypothesis there is a map f': $[0, 1]\to U'(n-1)$ satisfying the assertion mentioned in Lemma 2.1. Then we have

$$\rho'(g)f'(t)\rho'(g)^{-1} = \rho(g)f'(t)\rho(g)^{-1} \qquad g\in G_q,\ t\in[0,1]\,.$$

Therefore when we put

$$f = if'$$

where i: $U'(n-1)\to U(n)$ is the inclusion of $U'(n-1)$, f: $G\to U(n)$ is a map which we require. q.e.d.

Now we proceed by induction on n to complete the step 1. In case of $n=1$, since G acts on $U(\rho_1)$ trivially we have

$$K_G^*(U(\rho_1))\cong R(G)\otimes K^*(U(1))$$

by [5], Proposition (2,2). $K^*(U(1))$ is an exterior algebra with one generator θ and by the above isomorphism θ_1^G corresponds to $\rho_1\otimes\theta$. Hence $K_G^*(U(\rho_1))=\Lambda_{R(G)}(\theta_1^G)$ is valid for any compact abelian Lie group G and any 1-demensional representation $\rho=\rho_1$ of G.

Let π: $U(\rho)\to S(W)(=U(\rho)/U(\rho_2\oplus\cdots\oplus\rho_n))$ be the projection. From [4], Lemma 1 we get

Lemma 2.2. *There exists an element g in $K_G^1(S(W))$ such that*

$$K_G^*(S(W)) = \Lambda_{R(G)}(g)$$

as an algebra over $R(G)$ and

$$\pi^*(g) = \textstyle\sum_{k=1}^{n}(-1)^k\rho_1^{-k}\theta_k^G \qquad \theta_k^G\in K_G^1(U(\rho))\,.$$

Proof. We observe the exact sequence of the pair $(D(W), S(W))$ where $D(W)$ is the unit disk in W. Then we see that

$$\tilde{K}_G^0(S(W)) = 0$$

and the coboundary homomorphism

$$\delta\colon K_G^1(S(W))\to K_G^0(W)$$

is an isomorphism.

When we denote by λ_W the Thom class of the vector bundle $W\to P(=$a point$)$, $K_G^0(W)$ is a free module over $R(G)$ generated by λ_W. So if we put $g=\delta^{-1}(\lambda_W)$, then

$$K_G^*(S(W)) = \Lambda_{R(G)}(g)\,.$$

Next we consider the following diagram

$$\begin{array}{ccc} K_T^*(U(n), ad_i) & \xrightarrow{\rho^*} & K_G^*(U(\rho)) \\ \pi'^* \uparrow & & \uparrow \pi^* \\ K_T^*(S(W')) & \xrightarrow{\rho^*} & K_G^*(S(W)) \\ \delta \downarrow & & \downarrow \delta \\ K_T^*(W') & \xrightarrow{\rho^*} & K_G^*(W) \end{array}$$

where i: $T \to U(n)$ is the inclusion map of the standard maximal torus T of $U(n)$, ρ^* the homomorphism induced by the continuous homomorphism $\rho = \rho_1 \oplus \cdots \oplus \rho_n$: $G \to T$ and π': $U(n) \to S(W')$ represents the map π: $U(n) \to S(\boldsymbol{C} \oplus W)$ in [4], §2. Then this diagram commutes and $\rho^*(\lambda_{W'}) = \lambda_W$. Therefore we get

$$\pi^*(g) = \sum_{k=1}^{n} (-1)^k \rho_1^{-k} \theta_k^G$$

by [4], Lemma 1. q.e.d.

Let $\mathfrak{M}$ be an exterior algebra over $R(G)$ generated by $\theta_1^G, \cdots, \theta_{n-1}^G$ where $\theta_k^G \in K_G^1(U(\rho))$ for $1 \leq k \leq n-1$. Then we have a homomorphism

$$\kappa_1 : \mathfrak{M} \to K_G^*(U(\rho))$$

of algebras, defined by $\kappa_1(\theta_k^G) = \theta_k^G$. Because, when we observe the homomorphism ρ^*: $K_T^*(U(n), ad_i) \to K_G^*(U(\rho))$ mentioned in the proof of Lemma 2.2 we get

$$(\theta_k^G)^2 = \rho^*((\theta_k^T)^2) = 0 \quad \text{for} \quad 1 \leq k \leq n$$

since $(\theta_k^T)^2 = 0$ in $K_T^*(U(n), ad_i)$ by [4], Theorem 1 and also we get the relations $\theta_k^G \theta_l^G + \theta_l^G \theta_k^G = 0$ for $1 \leq k, l \leq n$ obviously since θ_k^G are the elements of $K_G^1(U(\rho))$. Morevoer, for each closed invariant subspace X of $S(W)$ we can define a homomorphism

$$\lambda : K_G^*(X) \underset{R(G)}{\otimes} \mathfrak{M} \to K_G^*(\pi^{-1}(X))$$

by

$$\lambda(x \otimes y) = \pi^*(x) j^* \kappa_1(y) \qquad x \in K_G^*(X),\ y \in \mathfrak{M}$$

where j: $\pi^{-1}(X) \to U(\rho)$ is the inclusion of $\pi^{-1}(X)$.

Under the assumption that the assertion of Theorem 1.1 in the step 1 is true for $n < l$ the following lemma is proved.

Lemma 2.3. *The homomorphism*

$$\lambda : K_G^*(S(W)) \underset{R(G)}{\otimes} \mathfrak{M} \to K_G^*(U(\rho))$$

is an isomorphism.

Proof. Let $\overline{S(W)}$ denote the orbit space of $S(W)$ by G. We have two

sheaves over $\overline{S(W)}$, φ_* and τ_* whose stalks are respectively

$$\varphi_*(\bar{q}) = K_G^*(Gq) \underset{R(G)}{\otimes} \mathfrak{M}$$

and

$$\tau_*(\bar{q}) = K_G^*(\pi^{-1}(Gq))$$

where $q \in S(W)$, $\bar{q} \in \overline{S(W)}$ and $Gq = \pi^{-1}(\bar{q})$.

Since $\mathfrak{M}$ is a free module over $R(G)$, $K_G^*(X) \underset{R(G)}{\otimes} \mathfrak{M}$ is an equivariant cohomology theory. Then λ induces a map of the spectral sequence [5]

$$E_2^{p,q} = H^p(\overline{S(W)}, \varphi_q) \Rightarrow K_G^*(S(W)) \underset{R(G)}{\otimes} \mathfrak{M}$$

to the spectral sequence

$$E_2^{p,q} = H^p(\overline{S(W)}, \tau_q) \Rightarrow K_G^*(U(\rho)) .$$

We shall prove that λ induces an isomorphism on the E_2-level. Clearly we have

$$\begin{aligned} \varphi_*(\bar{q}) &= K_G^*(Gq) \underset{R(G)}{\otimes} \mathfrak{M} \\ &\cong K_G^*(G/G_q) \underset{R(G)}{\otimes} \mathfrak{M} \\ &\cong \Lambda_{R(G_q)}(\theta_1^G, \cdots, \theta_{n-1}^G) . \end{aligned}$$

Next we observe the stalks $\tau_*(\bar{q})$. Let f: $[0, 1] \to U(n)$ is a continuous map in Lemma 2.1. Then we have

$$\pi^{-1}(Gq) = \bigcup_{g \in G} (\rho(g) f(0) \rho(g)^{-1}) U(n-1)$$

and so we can define a G-map

$$\phi: G/G_q \times U(\rho_2 \oplus \cdots \oplus \rho_n) \to \pi^{-1}(Gq)$$

by

$$\phi(gG_q, u) = (\rho(g) f(0) \rho(g)^{-1}) u \qquad g \in G,\ u \in U(\rho_2 \oplus \cdots \oplus \rho_n)$$

because $\rho(g) f(0) \rho(g)^{-1} = f(0)$ for any $g \in G_q$. Further we can easily check that ϕ is an isomorphism. Therefore

$$\begin{aligned} \tau_*(\bar{q}) &= K_G^*(\pi^{-1}(Gq)) \\ &\cong K_G^*(G/G_q \times U(\rho_2 \oplus \cdots \oplus \rho_n)) \\ &\cong K_{G_q}^*(U(\rho_2 \oplus \cdots \oplus \rho_n)) . \end{aligned}$$

Thus we obtain

$$\tau_*(\bar{q}) \cong \Lambda_{R(G_q)}(\theta_1', \cdots, \theta_{n-1}')$$

by the inductive hypothesis where $\theta_k' = \theta_k^{G_q}$ for $1 \leq k \leq n-1$.

Here we consider the homomorphism

$$\lambda' : \Lambda_{R(G_q)}(\theta_1^G, \cdots, \theta_{n-1}^G) \to K_{G_q}^*(U(\rho_2 \oplus \cdots \oplus \rho_n))$$

induced by the homomorphism

$$\lambda : \varphi_*(\bar{q}) \to \tau_*(\bar{q}) .$$

From the definition of θ_k^G we obtain easily

$$\lambda'(\theta_k^G) = [U(n-1) \times \lambda^k(V), \rho_k] \qquad (1 \leq k \leq n-1)$$

where ξ_k is an automorphism of the product G_q-bundle $U(n-1) \times \lambda^k(V)$ given by

$$\xi_k(u, z) = (u, \lambda^k(f(0)u)(z)) \qquad u \in U(n-1),\ z \in \lambda^k(V) .$$

Since f is a homotopy from $f(0)$ to the identity element of $U(n)$ satisfying $\rho(g)f(t) = f(t)\rho(g)$ for any $g \in G_q$ and $t \in [0, 1]$, we get

$$\lambda'(\theta_k^G) = \begin{cases} \theta_1' & (k = 1) \\ \theta_k' + \rho_1 \theta_{k-1}' & (2 \leq k \leq n-1) . \end{cases}$$

Hence we see that λ' is an isomorphism. This shows that λ induces an isomorphism on the E_2-level. Consequently we obtain Lemma 2.3. q.e.d.

Lemma 2.2 and lemma 2.3 show that the assertion in the case of $n = l$ is also true. This completes the step 1.

Step 2. *Proof when G is connected.*

Let T be a maximal torus of G and i: $T \to G$ the inclusion of T. Then from the step 1 we get

$$K_T^*(U(n), ad_{\rho_T}) = \Lambda_{R(T)}(\theta_1^T, \cdots, \theta_n^T)$$

where ρ_T is the restriction of ρ onto T and therefore, from [5], Proposition (3.8) and [4], Lemma 2 we get

$$K_G^*(U(n), ad_\rho) \cong K_T^*(U(n), ad_{\rho_T})^{W(G)}$$

where $W(G)$ is the Weyl group of G. This shows

$$K_G^*(U(n), ad_\rho) = \Lambda_{R(G)}(\theta_1^G, \cdots, \theta_n^G) .$$

3. Proof when G is not connected

We recall

Theorem 3.1. (Segal [6]) *Let G be a compact Lie group. Then the restric-*

tion $R(G)\to\sum_S R(S)$ is injective where S runs through the representatives of conjugacy classes of Cartan subgroups of G.

Then we have

Lemma 3.2. *Let G be a compact Lie group and ρ a continuous homomorphism of G into $U(n)$. Then $\Lambda_{R(G)}(\theta_1^G, \cdots, \theta_n^G)$ is a subalgebra of $K_G^*(U(n), ad_\rho)$.*

Proof. We have a homomorphism κ_2 of $\Lambda_{R(G)}(\theta_1^G, \cdots, \theta_n^G)$ into $K_G^*(U(n), ad_\rho)$ as algebras defined by $\kappa_2(\theta_k^G)=\theta_k^G$, $1\leq k\leq n$. This homomorphism is well-defined by the same reason as κ_1 in §2, Step 1 is so.

Let S be a Cartan subgroup of G and $i_S: S\to G$ the inclusion of S. Then we have

$$K_S^*(U(n), ad_{\rho_S}) = \Lambda_{R(S)}(\theta_1^S, \cdots, \theta_n^S)$$

from §2, Step 1 where ρ_S is thr restriction of ρ onto S. Therefore if

$$\sum_{1\leq i_1<\cdots<i_l\leq n}\alpha_{i_1\cdots i_l}\theta_{i_1}^G\cdots\theta_{i_l}^G = 0$$

for $\alpha_{i_1\cdots i_l}\in R(G)$ in $K_G^*(U(n), ad_\rho)$, then

$$i_S^*(\alpha_{i_1\cdots i_l}) = 0$$

for any Cartan subgroup S of G. So we get

$$\alpha_{i_1\cdots i_l} = 0\,, \quad 1\leq i_1<\cdots<i_l\leq n$$

from Theorem 3.1. This shows that κ_2 is injective. q.e.d.

Using the Segal's spectral sequence [5] we can easily check the following

Lemma 3.3. ([3], Proposition 2) *Let G be a compact Lie group. Let X and Y be compact locally G-contractible G-spaces such that the orbit spaces X/G and Y/G are of finite covering dimension. If $K_G^*(X)$ or $K_G^*(Y)$ is a free abelian group, then the external tensor product*

$$K_G^*(X)\otimes K_G^*(Y)\to K_{G\times G}^*(X\times Y)$$

is an isomorphism.

The following theorem is basic in proof of the general case.

Theorem 3.4. ([1], Proposition (4.9), [5], Proposition (3.8))
Let G be a compact connected Lie group and $i: T\to G$ the inclusion of a maximal torus. Then for each locally compact G-space X there is a natural homomorphism of $K_G^*(X)$-modules $i_*: K_T^*(X)\to K_G^*(X)$ such that $i_*(1)=1$, and hence $i_*i^*=$identity.

Theorem 3.5. *Let G be a compact connected Lie group and $\rho: G\to U(n)$ a*

unitary representation. Then, for each closed subgroup H of G we have

$$K_H^*(U(n), ad_{\rho_H}) = \Lambda_{R(H)}(\theta_1^H, \cdots, \theta_n^H)$$

as an algebra over $R(H)$ where ρ_H is the restriction of ρ onto H.

Proof. As in §2, we denote $(U(n), ad_\rho)$ by $U(\rho)$. Let $\pi_1\colon U(\rho)\times G/H \to U(\rho)$ and $\pi_2\colon U(\rho)\times G/H \to G/H$ be the projections. Let $d\colon G\to G\times G$ be the diagonal map.

We consider the homomorphism

$$d^*\colon K_{G\times G}^*(U(\rho)\times G/H) \to K_G^*(U(\rho)\times G/H)\,.$$

From Lemma 3.3 and §2, Step 2 we get

$$\begin{aligned}(1)\qquad K_{G\times G}^*(U(\rho)\times G/H) &\cong K_G^*(U(\rho))\otimes K_G^*(G/H)\\ &\cong \Lambda_{R(G)}(\theta_1^G, \cdots, \theta_n^G)\otimes R(H)\,.\end{aligned}$$

From (1) we see that d^* induces a homomorphism

$$\mu_1\colon K_G^*(U(\rho))\otimes K_G^*(G/H) \to K_G^*(U(\rho)\times G/H)$$

and then μ_1 is as follows:

$$\mu_1(x\otimes y) = \pi_1^*(x)\pi_2^*(y) \qquad \text{for } x\in K_G^*(U(\rho)),\ y\in K_G^*(G/H)\,.$$

Since $K_G^*(U(\rho)\times G/H)\cong K_H^*(U(\rho_H))$ and $\Lambda_{R(H)}(\theta_1^H, \cdots, \theta_n^H)$ is a subalgebra of $K_H^*(U(\rho_H))$ by Lemma 3.2, $\Lambda_{R(H)}(\pi_1^*(\theta_1^G), \cdots, \pi_1^*(\theta_n^G))$ is a subalgebra of $K_G^*(U(\rho)\times G/H)$ and also

$$(2)\qquad \operatorname{Im}\mu_1 = \Lambda_{R(H)}(\pi_1^*(\theta_1^G), \cdots, \pi_1^*(\theta_n^G))\,.$$

Therefore if we prove that μ_1 is an epimorphism, then we obtain Theorem 3.5.

Let T be a maximal torus of G. First we consider the restriction $\rho_T\colon T\to U(n)$ of ρ onto T. As the case of $\rho\colon G\to U(n)$ we have

$$\begin{aligned}K_{T\times T}^*(U(\rho_T)\times G/H) &\cong K_T^*(U(\rho_T))\otimes K_T^*(G/H)\\ &\cong \Lambda_{R(T)}(\theta_1^T, \cdots, \theta_n^T)\otimes K_T^*(G/H)\end{aligned}$$

and so the homomorphism

$$\mu_2\colon K_T^*(U(\rho_T))\otimes K_T^*(G/H) \to K_T^*(U(\rho_T)\times G/H)$$

induced by d^*. Also we get

$$\begin{aligned}K_T^*(U(\rho_T)\times G/H) &\cong K_T^*(U(\rho_T))\underset{R(T)}{\otimes} K_T^*(G/H)\\ &\cong \Lambda_{R(T)}(\theta_1^T, \cdots, \theta_n^T)\underset{R(T)}{\otimes} K_T^*(G/H)\end{aligned}$$

from §2, Step 1 and a parallel argument to Corollary 1.2.

Now we observe the following diagram

$$\begin{array}{ccc}
\Lambda_{R(G)}(\theta_1^G, \cdots, \theta_n^G)\otimes R(H) & & \Lambda_{R(T)}(\theta_1^T, \cdots, \theta_n^T)\otimes K_T^*(G/H) \\
\cong\downarrow & & \downarrow\cong \\
K_G^*(U(\rho))\otimes K_G^*(G/H) & \underset{i_{1*}\otimes i_{2*}}{\overset{i_1^*\otimes i_2^*}{\rightleftarrows}} & K_T^*(U(\rho_T))\otimes K_T^*(G/H) \\
\mu_1\downarrow & & \downarrow\mu_2 \\
K_G^*(U(\rho)\times G/H) & \underset{j_*}{\overset{j^*}{\rightleftarrows}} & K_T^*(U(\rho_T)\times G/H) \\
\cong\uparrow & & \uparrow\cong \\
K_H^*(U(\rho_H)) & & \Lambda_{R(T)}(\theta_1^T, \cdots, \theta_n^T)\underset{R(T)}{\otimes} K_T^*(G/H)
\end{array}$$

where i_1, i_2 and j are the inclusion of T, and i_{1*}, i_{2*} and j_* denote the natural homomorphisms mentioned in Theorem 3.4.

For any $x\in K_G^*(U(\rho)\times G/H)$ we can write

$$(3)\qquad j^*(x) = \alpha\pi_2^*(y)+\sum_{1\leq i_1<\cdots<i_s\leq n}\alpha_{i_1\cdots i_s}\pi_1^*(\theta_{i_1}^T\cdots\theta_{i_s}^T)\pi_2^*(y_{i_1\cdots i_s})$$

for α, $\alpha_{i_1\cdots i_s}\in R(T)$ and y, $y_{i_1\cdots i_s}\in K_T^*(G/H)$.

Let put

$$z = 1\otimes\alpha y+\sum_{1\leq i_1<\cdots<i_s\leq n}\theta_{i_1}^T\cdots\theta_{i_s}^T\otimes\alpha_{i_1\cdots i_s}y_{i_1\cdots i_s}$$

in $K_T^*(U(\rho_T))\otimes K_T^*(G/H)$. Then from (3) we get

$$(4)\qquad \mu_2(z) = j^*(x)\,.$$

Moreover

$$(i_{1*}\otimes i_{2*})(z) = 1\otimes i_{2*}(\alpha y)+\sum_{1\leq i_1<\cdots<i_s\leq n}\theta_{i_1}^G\cdots\theta_{i_s}^G\otimes i_{2*}(\alpha_{i_1\cdots i_s}y_{i_1\cdots i_s})$$

since $i_1^*\theta_k^G=\theta_k^T$ $1\leq k\leq n$ and $i_{1*}i_1^*=1$, and

$$\begin{aligned}(5)\qquad \mu_1((i_{1*}\otimes i_{2*})(z)) &= \pi_2^*i_{2*}(\alpha y)+\sum_{1\leq i_1<\cdots<i_s\leq n}\pi_1^*(\theta_{i_1}^G\cdots\theta_{i_s}^G)\pi_2^*i_{2*}(\alpha_{i_1\cdots i_s}y_{i_1\cdots i_s}) \\ &= j_*\pi_2^*(\alpha y)+\sum_{1\leq i_1<\cdots<i_s\leq n}\pi_1^*(\theta_{i_1}^G\cdots\theta_{i_s}^G)j_*\pi_2^*(\alpha_{i_1\cdots i_s}y_{i_1\cdots i_s})\end{aligned}$$

because of $j_*\pi_2^*=\pi_2^*i_{2*}$. By Theorem 3.4, j_* is the homomorphism of $K_G^*(U(\rho)\times G/H)$-modules. Therefore (5) shows

$$(6)\qquad \mu_1((i_{1*}\otimes i_{2*})(z)) = j_*(\alpha\pi_2^*(y)+\sum_{1\leq i_1<\cdots<i_s\leq n}\alpha_{i_1\cdots i_s}\pi_1^*(\theta_{i_1}^T\cdots\theta_{i_s}^T)\pi_2^*(y_{i_1\cdots i_s}))$$

because of $\pi_1^*i_1^*=j^*\pi_1^*$.

From (3) and (6) we obtain

$$(7)\qquad \mu_1((i_{1*}\otimes i_{2*})(z)) = j_*j^*(x) = x$$

and so we see that μ_1 is an epimorphism. Hence (2) and (7) conclude

$$K_H^*(U(\rho_H)) = \Lambda_{R(H)}(\theta_1^H, \cdots, \theta_n^H) .$$

q.e.d.

Proof of the general case. Let G be a compact Lie group and $\rho: G \to U(n)$ a unitary representation of G.

Embed G in a unitary group $U(m)$ and consider an embedding

$$f: G \to U(n) \times U(m)$$

defined by

$$f(g) = (\rho(g), g) \qquad g \in G .$$

Let $\pi: U(n) \times U(m) \to U(n)$ be the projection. If we regard G as a closed subgroup of $U(n) \times U(m)$ by f, then ρ is the restriction of π onto G. Therefore, from Theorem 3.5 we get

$$K_G^*(U(n), ad_\rho) = \Lambda_{R(G)}(\theta_1^G, \cdots, \theta_n^G) .$$

This completes the proof of Theorem 1.1.

4. The special unitary group $SU(n)$

Let G be a compact Lie group and $\rho: G \to U(n)$ a unitary representation of G. Then $SU(n)$ becomes a G-submanifold of $(U(n), ad_\rho)$ which we denote by $(SU(n), ad_\rho)$.

Let $j: SU(n) \to U(n)$ be the inclusion of $SU(n)$. We use the same symbol θ_k^G for the image of $\theta_k^G \in K_G^1(U(n), ad_\rho)$ by j^* for $1 \leq k \leq n-1$. In particular, $j^*(\theta_n^G) = 0$.

Let T be the standard maximal torus of $U(n)$ and $i: T \to U(n)$ the inclusion of T. Then, by a parallel proof to that in [4] we obtain

Proposition 4.1. *Using the notation of* [4], *Lemma* 1 *we have*

(i) $K_T^*(S(\boldsymbol{C} \oplus W)) = K_T^*(SU(n)/SU(n-1))$ *is an exterior algebra over* $R(T)$ *with one generator* g *satisfying*

$$\pi^*(g) = \sum_{k=1}^{n-1} (-1)^k \rho_1^{-k} \theta_k^T$$

where $\pi: SU(n) \to S(\boldsymbol{C} \oplus W) (= SU(n)/SU(n-1))$ *is the projection, and therefore*

(ii) $K_T^*(SU(n), ad_i) = \Lambda_{R(T)}(\theta_1^T, \cdots, \theta_{n-1}^T)$

as an algebra over $R(T)$.

From Proposition 4.1 an analogous statement can be made as follows.

Proposition 4.2. *Let* G *be a compact Lie group and* $\rho: G \to U(n)$ *a unitary representation of* G. *Then*

$$K_G^*(SU(n), ad_\rho) = \Lambda_{R(G)}(\theta_1^G, \cdots, \theta_{n-1}^G)$$

as an algenra over $R(G)$.

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