<table>
<thead>
<tr>
<th>Title</th>
<th>Maximal solutions of the scalar curvature equation on open Riemannian manifolds</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Kato, Shin</td>
</tr>
<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 31(2) P.355-P.371</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1994</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/5362">https://doi.org/10.18910/5362</a></td>
</tr>
<tr>
<td>DOI</td>
<td>10.18910/5362</td>
</tr>
<tr>
<td>rights</td>
<td></td>
</tr>
</tbody>
</table>
MAXIMAL SOLUTIONS OF THE SCALAR CURVATURE EQUATION ON OPEN RIEMANNIAN MANIFOLDS

SHIN KATO

(Received November 12, 1992)

1 Introduction

Let \((M, g)\) be a Riemannian manifold with \(\dim M = n \geq 3\), \(\Delta_g\) the Laplacian of \(g\), namely, \(\Delta_g := \text{trace}V_g^2\), \(S_g\) the scalar curvature of \(g\), and \(L_g\) the conformal Laplacian, that is, \(L_g := -a_n \Delta_g + S_g\), where \(a_n = 4(n-1)/(n-2)\). Define a new metric \(\tilde{g}\) which is pointwise conformal to \(g\) by \(\tilde{g} := u^4/(n-2)g\), where \(u\) is a positive smooth function on \(M\). Then its scalar curvature \(S_{\tilde{g}}\) is given by \(L_{\tilde{g}} u = S_{\tilde{g}} u^{(n+2)/(n-2)}\). Hence, for any smooth function \(f\) on \(M\), \(f\) can be realized as the scalar curvature of some metric which is pointwise conformal to \(g\) if and only if there is a smooth solution of the following partial differential equation:

\[
\begin{cases}
L_g u = fu^q \\
u > 0
\end{cases}
\]

on \(M\), where \(q = (n+2)/(n-2) = 4/(n-2) + 1\). In this paper, we refer to this equation as "the equation \((f,M)\)" for various functions \(f\) and manifolds \(M\) (e.g. \((-1, \mathbb{R}^-), \((f_0, \Omega_0)\), etc.).

Now, we are interested in the problem of finding sufficient conditions for a function \(f\) to be realized by some complete conformal metric. This problem has been extensively investigated in various cases (see e.g. McOwen [10]). Recently, Cheng-Ni [6] proved the existence of a maximal solution \(U\) of the equation \((f,M)\) in the case that \((M,g)\) is the Euclidean space \((\mathbb{R}^n, g_0)\) or its open subset and that \(f\) is nonpositive and satisfies some conditions. In this paper, first we shall simplify their proof and generalize the result to the case that \(M\) is an open Riemannian manifold. This result is stated as follows.

**Theorem I.** Let \((M,g)\) be an open Riemannian manifold \((n = \dim M \geq 3)\). Let \(f\) be a nonpositive smooth function on \(M\). Suppose both of the following conditions hold:

(a) there exists a sequence \(\{\Omega_i\}_{i \in \mathbb{N}}\) of relatively compact domains of \(M\) with
smooth boundaries which satisfies

\[ \Omega_i \subseteq \Omega_{i+1}, \quad \bigcup_{i \in \mathbb{N}} \Omega_i = M, \text{ and } f < 0 \text{ on } \partial \Omega_i; \]

(b) there exists a positive subsolution \( u_- \) of the equation \( (f, M) \).

Then there is a smooth solution \( U \) of the equation \( (f, M) \) which is maximal in the sense that \( U \geq u \) for any solution \( u \) of the equation \( (f, M) \).

Clearly, the condition (a) is satisfied if \( f \) is negative on the complement of some compact subset of \( M \). As we shall see below (in Theorem V), when the condition (a) is not satisfied, the equation \( (f, M) \) does not necessarily possess a maximal solution. In Section 2, we give the proof of Theorem I.

Using this theorem, we want to find some geometric conditions for the above problem. Moreover, we investigate the asymptotic behavior of the maximal solution \( U \). It should be mentioned here that Cheng-Ni [6] describe it in a special case by analyzing a certain ordinary differential equation. However, it seems difficult to apply the technique to our cases, and we employ different methods.

Throughout this paper, we denote the sectional (resp. Ricci) curvature of \( g \) by \( K_g \) (resp. \( \text{Ric}_g \)), the distance function to a point \( p \) (resp. a submanifold \( \Sigma \)) by \( r_p \) (resp. \( r_\Sigma \)) unless otherwise stated, and we use the notation "\( f_0 \sim f \)" to mean that the condition \( C_1 f \leq f_0 \leq C_2 f \) holds for some positive constants \( C_1 \) and \( C_2 \).

Our main results are stated as follows.

**Theorem II.** Let \( (M, g) \) be a complete, noncompact, simply connected Riemannian manifold \( (n = \dim M \geq 3) \) with nonpositive curvature. Let \( f \) be a nonpositive smooth function on \( M \) with the condition (a) in Theorem I. If \( f \) satisfies \( f \geq -Cr_p^{-1} \) near infinity for some point \( p \in M \), a positive constant \( C \) and a number \( l > 2 \), then there is a complete conformal metric \( \tilde{g} \) on \( M \) with scalar curvature \( f \).

Moreover, suppose \( (M, g) \) satisfies \( \text{Ric}_g/(n-1) \geq -A^2/(r_p^2 + \varepsilon^2) \) for positive numbers \( A \) and \( \varepsilon \) such that \( A^2 \leq (n-2)/n \). Then the metric \( \tilde{g} \) can be constructed so that its ratio to \( g \) is of order \( r_p^{l-2} \) near infinity, if \( f \) satisfies \( f \sim -r_p^{-1} \) near infinity.

**Theorem III.** Let \( (M, g) \) be a complete, noncompact Riemannian manifold \( (n = \dim M \geq 3) \) whose scalar curvature is bounded above by a negative number. Let \( f \) be a nonpositive smooth function on \( M \) with the condition (a) in Theorem I. If \( f \) is bounded, then there is a complete
conformal metric $\tilde{g}$ on $M$ with scalar curvature $f$. Moreover, suppose $(M, g)$ is simply connected and satisfies $K_g \leq -B^2$ and $\text{Ric}_g/(n-1) \geq -A^2$ for positive numbers $A$ and $B$ such that $(A/B)^2 \leq (n-1)^2/n(n-2)$. Then the metric $\tilde{g}$ can be constructed so that its ratio to $g$ is of order $e^{br}$ near infinity, if $f$ satisfies $f \sim -e^{-br}$ near infinity for some point $p \in M$ and a nonnegative number $l$.

**Theorem IV.** Let $(\tilde{M}, \tilde{g})$ be a compact Riemannian manifold ($n = \text{dim } \tilde{M} \geq 3$), $\Sigma$ a compact submanifold ($d = \text{dim } \Sigma$) of $\tilde{M}$, and $(M, g) := (\tilde{M}, \Sigma, \tilde{g}|_{\tilde{M}\setminus \Sigma})$. Let $f$ be a nonpositive smooth function on $M$ with the condition (a) in Theorem I. If $f$ satisfies $f \geq -C r_{\Sigma}^{1}$ near $\Sigma$ for a positive constant $C$ and a nonnegative number $l > 2 - 4d/(n-2)$, then there is a complete conformal metric $\tilde{g}$ on $M$ with scalar curvature $f$.

Moreover, if $f$ satisfies $f \sim -r_{\Sigma}^{1}$ near $\Sigma$, then the metric $\tilde{g}$ can be constructed so that its ratio to $g$ is of order $r_{\Sigma}^{-\left(d+2\right)}$ near $\Sigma$.

In Theorems II and III, we have imposed the assumption on the upper bound for $A^2$ and $(A/B)^2$ respectively. However, these can be replaced with other conditions. See Remark 3.2 for details.

In the case that $f \sim -1$ i.e. $f$ is bounded between two negative constants near infinity (resp. $\Sigma$), the first assertion of Theorem III (resp. Theorem IV) was proved by Aviles-McOwen [3] (resp. [4]) essentially (see also [2]). However, we emphasize that $f$ need not to be bounded above by a negative constant near infinity (resp. $\Sigma$).

We shall now mention a result by Cheng-Ni [6] which states that, if a nonpositive function $f$ on $\mathbb{R}^n$ satisfies $f(x) \sim -|x|^{-1}$ near infinity for a number $l > 2$, then any solution $u$ of the equation $(f, \mathbb{R}^n)$ coincides with the maximal solution $U$ or a unique solution $u_h$ such that $u_h(x) \to h$ as $|x| \to +\infty$ for some positive number $h$. However, in our case, it would be difficult to describe all of the solutions of the equation $(f, M)$. In fact, we have the following

**Theorem V.** Let $(M, g)$ be as in Theorem IV with $\lambda_1(L_g) > 0$ and $d \leq (n-2)/2$, and $f$ a nonpositive smooth function on $M$. If $f$ satisfies $f \geq -C r_{\Sigma}^l$ near $\Sigma$ for a positive constant $C$ and a number $l > 2 - 4d/(n-2)$, then, for any positive $C^\alpha$-function $h$ on $\Sigma$ ($0 < \alpha < 1$), there is a unique solution $u_h$ of the equation $(f, M)$ such that $u_h(x)/G_{x}(x) \to h(x_\infty)$ as $x \to x_\infty \in \Sigma$ (in particular, the metric $\tilde{g} = u_h^{q-1}g$ is complete), where $G_x(x) := \int_{\Sigma} G(x, y) d\sigma_y$ and $G(x, y)$ is the Green function of $L_g$.

Moreover, if the support of $f$, $\text{supp } f$, is compact, then the equation $(f, M)$ does not possess a maximal solution.

The first assertion of this theorem was partially obtained in Delanoë
They proved that $u_h$ exists if $h$ is small enough. However, we claim that $u_h$ exists for an arbitrary $h$ when $f$ is nonpositive.

The proofs of Theorems II and III (resp. Theorems IV and V) are given in Section 3 (resp. Section 4).

The author would like to thank Professors A. Kasue, O. Kobayashi and S. Nayatani for valuable suggestion.

2 The existence of the maximal solution

In this section, we prove Theorem I. We first recall the following:

Method of Supersolutions and Subsolutions. Let $(M, g)$ be a Riemannian manifold $(\omega = \dim M \geq 3)$, and $f$ a smooth function on $M$. If there exist a supersolution $u_+$ and a subsolution $u_-$ of the equation $(f, M)$ such that $0 < u_- \leq u_+$, then the equation $(f, M)$ possesses a smooth solution $u$ satisfying $u_- \leq u \leq u_+$.

This is well-known and we omit the proof. We also recall the following significant result.

Theorem A (Aviles-McOwen [4]). Let $(W, g)$ be a compact Riemannian manifold with boundary $\partial W \neq \emptyset (n = \dim W \geq 3)$. The equation $(-1, W)$ possesses a solution $u$ such that $\varphi(x) \to +\infty$ as $x \to \partial W$.

To prove Theorem I, we prepare the following two lemmas.

Lemma 2.1. Let $(M, g)$ be an open Riemannian manifold, and $\Omega$ a relatively compact domain of $M$. Then there is a positive smooth function $\eta$ in $\Omega$ such that both $\eta$ and $L_g \eta$ are bounded below by a positive number in $\Omega$.

Proof. Choose a relatively compact domain $\hat{\Omega}$ of $M$ satisfying $\Omega \subset \subset \hat{\Omega}$. There exists a bounded positive smooth function $\psi_0$ defined on $\hat{\Omega}$ satisfying $L_g \psi_0 \equiv 0$ (see e.g. Escobar [8, Proposition 1.3]). Set $\hat{g} := \psi_0^{-1} g|_{\hat{\Omega}}$, then we have $S_{\hat{g}} \equiv 0$. Let $\xi$ be the first Dirichlet eigenfunction of $-\Delta_{\hat{g}}$ in $\hat{\Omega}$. Since $\xi$ does not change sign, we can assume $\xi > 0$. Set $\eta := \psi_0 \xi$. Then $\eta > 0$ in $\hat{\Omega}$, and hence $\eta$ is bounded below by a positive number in $\Omega$. Moreover, since

$$L_g \eta = L_g (\psi_0 \xi) = \psi_0 \xi L_g \xi \equiv a_0 \psi_0 \xi (-\Delta_{\hat{g}}) \xi > 0 \text{ in } \hat{\Omega},$$

$L_g \eta$ is also bounded below by a positive number in $\Omega$. q.e.d.
Lemma 2.2. Let \((M,g)\) and \(\Omega\) be as in Lemma 2.1, \(f\) (resp. \(f_0\)) a smooth function in \(\Omega\), and \(u\) (resp. \(u_0\)) a solution of the equation \((f,\Omega)\) (resp. \((f_0,\Omega))\). Suppose \(f_0\) is nonpositive, \(f_0 \leq f\) in \(\Omega\), and \(\lim\inf_{x \to \partial \Omega} \{u(x) - u_0(x)\} \geq 0\). Then \(u_0 \leq u\) in \(\Omega\).

Proof. Let \(g\) be as in the proof of Lemma 2.1. Then \(L_g(\psi_0^{-1}u_0) = f_0(\psi_0^{-1}u_0)^q\), and \(L_g(\psi_0^{-1}u) = f(\psi_0^{-1}u)^q \geq f_0(\psi_0^{-1}u)^q\) in \(\Omega\). Hence we get

\[
\begin{cases}
\alpha u + f_0 \psi_0^{1-q} \frac{u^q - u_0^q}{u - u_0} (\psi_0^{-1}u - \psi_0^{-1}u_0) \leq 0 \text{ in } \Omega.
\end{cases}
\]

Since

\[
\frac{f_0 \psi_0^{1-q} \frac{u^q - u_0^q}{u - u_0}}{u - u_0} \leq 0 \text{ in } \Omega,
\]

and \(\lim\inf_{x \to \partial \Omega} \{u(x) - u_0(x)\} \geq 0\), by the maximal principle, we have \(u - u_0 \geq 0\) in \(\Omega\). q.e.d.

Proof of Theorem I. By Theorem A, for any \(i \in \mathbb{N}\), the equation \((-1,\Omega_i)\) possesses a solution \(v_i\) such that \(v_i(x) \to +\infty\) as \(x \to \partial \Omega_i\). Set \(u_{i-} := \gamma_i v_i\), where \(\gamma_i := (\max_{\Omega_i} |f|)^{-(n-2)/4}\). Then we have

\[
L_g u_{i-} = -\gamma_i^{1-q} u_{i-}^q \leq f u_{i-}^q \text{ in } \Omega_i,
\]

namely, \(u_i\) is a subsolution of the equation \((f,\Omega_i)\). On the other hand, by Lemma 2.1, there is a positive smooth function \(\eta_i\) in \(\Omega_i\) such that both \(\eta_i\) and \(L_g \eta_i\) are bounded below by a positive number in \(\Omega_i\). From the assumption (a), there exist an open subset \(\Omega'_i\) of \(\Omega_i\) such that \(\Omega'_i \subset \subset \Omega_i\), and a positive number \(\delta_i\) such that \(f \leq -\delta_i \text{ in } \Omega_i \setminus \Omega'_i\). Set \(u_{i+} := \delta_i (v_i + \beta_i \eta_i)\), where \(\beta_i = \max_{\Omega_i} |v_i|^q / L_g \eta_i\) and \(\delta_i = \epsilon_i^{-(n-2)/4}\). Then we get

\[
L_g u_{i+} = \delta_i (-v_i^q + \beta_i L_g \eta_i) \geq 0 \geq f u_{i+}^q \text{ on } \partial \Omega_i,
\]

and

\[
L_g u_{i+} \geq \delta_i L_g v_i = -\epsilon_i (\delta_i v_i)^q \geq f u_{i+}^q \text{ in } \Omega_i \setminus \partial \Omega_i,
\]

namely, \(u_{i+}\) is a supersolution of the equation \((f, \Omega_i)\). Since \(u_{i+} \geq u_{i-} > 0\) in \(\Omega_i\), by the method of supersolutions and subsolutions, the equation \((f, \Omega_i)\) possesses a smooth solution \(u_i\) satisfying \(u_{i+} \geq u_i \geq u_{i-}\).

Since \(u_i(x) \to +\infty\) as \(x \to \partial \Omega_i\), and both \(u_{i+}\) and \(u_{-}\) are bounded above in \(\Omega_i\), by Lemma 2.2, we get \(u_i \geq u_{i+1}\) and \(u_i \geq u_{-}\) in \(\Omega_i\), that is,
$\{u_i\}_{i \in \mathbb{N}}$ is monotonically decreasing and bounded below by $u_-$. Therefore, if we set $U := \lim_{i \to +\infty} u_i$, then $U$ is a smooth solution of the equation $(f, M)$. By Lemma 2.2 again, it is clear that $u_i \geq u$ for any solution $u$ of the equation $(f, M)$. Hence we see that $U \geq u$ or $U$ is the maximal solution.

Cheng-Ni [6, Propositions 2.11, 2.12 and 2.13] listed some properties of the maximal solution $U$ in the case of $(\mathbb{R}^n, g_0)$, and their assertions are valid also in our case. Actually, we can prove the following proposition along the same lines as in Cheng-Ni [5, Theorem 2.8, Remark 2.9, and Proposition 2.10].

**Proposition 2.3.** Let $(M, g)$ be as in Theorem I, $f$ (resp. $f_0$) a nonpositive function on $M$ satisfying the condition (a) and (b) in Theorem I, and $U$ (resp. $U_0$) the maximal solution of the equation $(f, M)$ (resp. $(f_0, M)$).

1. If $f_0 \leq f$ on $M$, then $U_0 \leq U$ on $M$.
2. If $f$ is a solution of the equation $(f, M)$, then $u \equiv U$ or $u \leq U$ on $M$.
3. If $f$ is invariant under the action of some subgroup $\Gamma$ of Isom$(M, g)$, then $U$ is also $\Gamma$-invariant.

**Remark 2.4.** In Theorem I, so far as the existence of a solution is concerned, we may assume $f$ is positive on a compact subset of $M$. Actually, let $f_0$ be a nonpositive function on $M$ with the conditions (a) and (b) in Theorem I, and $U_0$ the maximal solution of the equation $(f_0, M)$. Let $\eta_1$ and $\Omega_1$ be as in the proof of Theorem I, and $\chi$ a smooth nonnegative function on $M$ such that $\chi \equiv 1$ on $\tilde{\Omega}_1$ and $\chi \equiv 0$ on $M \setminus \Omega_1$. Set $u_+ := \delta(U_0 + \beta \chi \eta_1)$ and $u_- := U_0$. Then, by choosing $\beta$ and $\delta$ large enough, we get $L_g u_+ \geq f_0 u_+^q$ on $M \setminus \Omega_1$ and $L_g u_+ \geq \epsilon u_+^q$ in $\Omega_1$ for some positive number $\epsilon$ depending only on $f_0$ and $\Omega_1$. Hence, if a function $f$ satisfies $f \equiv f_0$ on $M \setminus \Omega_1$ and $f_0 \leq f \leq \epsilon$ in $\Omega_1$, then there is a solution $u$ of the equation $(f, M)$ satisfying $u \sim U_0$ on $M$. However, we do not know whether $u$ is maximal or not.

**3 The case of a complete simply connected manifold**

In this section, we study the asymptotic behavior of the maximal solution $U$ in the case that $(M, g)$ is a complete simply connected manifold, and prove Theorems II and III. For this purpose, we prepare the following lemma.

**Lemma 3.1.** Let $(M, g)$ be as in Theorem I with the condition (P) there exists a positive smooth function $\psi$ on $M$ satisfying $L_g \psi > 0$. 
Let $f$ (resp. $f_0$) be a nonpositive function on $M$ satisfying the condition (a) and (b) in Theorem I, and $U$ (resp. $U_0$) the maximal solution of the equation $(f,M)$ (resp. $(f_0,M)$). Suppose $f_0 \leq f$ on $M$, $f_0 \sim f$ on the complement of some compact subset of $M$, and $\psi/U_0$ is bounded on $M$. Then $U_0 \sim U$ on $M$.

Proof. We may assume $f_0 \sim f$ on $M \setminus \Omega_1$, that is, $f \leq Cf_0$ on $M \setminus \Omega_1$ for a positive constant $C < 1$, without loss of generality. By the proof of Theorem I, for any $i \in \mathbb{N}$, the equation $(f_0,\Omega_i)$ possesses a solution $u_{0,i}$ satisfying $u_{0,i}(x) \to +\infty$ as $x \to \partial \Omega_i$. Obviously

$$L_g u_{0,i} = f_0 u_{0,i} \leq f u_{0,i}$$

in $\Omega_i$, namely, $u_{0,i}$ is a subsolution of the equation $(f,\Omega_i)$. On the other hand, if we set $u_{i+} := \delta(u_{0,i} + \beta \psi)$, where $\beta = \max_{\Omega_i}|f_0| u_{0,2}^q/L_g \psi$ and $\delta = C^{-(n-2)/q}$, then we get, for any $i \geq 2$,

$$L_g u_{i+} = \delta (f_0 u_{0,i}^q + \beta L_g \psi) \geq \delta (f_0 u_{0,i}^q + |f_0| u_{0,2}^g)$$

$$= \delta |f_0|(u_{0,2}^g - u_{0,i}^q) \geq 0 \geq f u_{i+}^q$$

on $\Omega_i$, and

$$L_g u_{i+} \geq C f_0 (\delta u_{0,i})^q \geq f u_{i+}^q$$

on $\Omega_i \setminus \Omega_1$, namely, $u_{i+}$ is a supersolution of the equation $(f,\Omega_i)$. Since $u_{i+} \geq u_{0,i} > 0$ in $\Omega_i$, by the method of supersolutions and subsolutions, the equation $(f,\Omega_i)$ possesses a solution $u_i$ of the equation $(f,\Omega_i)$ satisfying $u_{i+} \geq u_i \geq u_{0,i}$.

Now, it is clear that $\lim_{i \to +\infty} u_i = U$. Therefore we have $U_0 \leq U \leq \delta (U_0 + \beta \psi)$, or $1 \leq U/U_0 \leq \delta (1 + \beta \psi/U_0)$ on $M$. Since we assume $\psi/U_0$ is bounded, we conclude that $U_0 \sim U$ on $M$.

q.e.d.

Remark that the assumption "$f_0 \leq f$ on $M$" in this lemma can be replaced by the condition "$C f_0 \leq f$ on $M$ for a positive constant $C_1$".

Proof of Theorem II. Write $r := r_p$ for convenience. Since $K_g \leq 0$, we see $\Delta_g r \geq (n-1)/r$ or $r \Delta_g r \geq n-1$ on $M$. Set $u_0 := (r^2 + 1)^{\alpha-(n-2)/2}$, where $\alpha = (l + 2)(n-2)/8$, and put $f_0' := u_0^{-q} L_g u_0$. Then we get

$$f_0 = -2 a_n (r^2 + 1)^{-\alpha/2 - 1} \left[ 2 \left( \alpha - \frac{n-2}{2} \right)^2 r^2 \right. + \left( \alpha - \frac{n-2}{2} \right) \left( (r^2 + 1)(-1 + r \Delta_g r) + 2 \right) + \frac{|S_g| (r^2 + 1)^2}{2 a_n} \right]$$
\[ \leq -2a_n \left( \alpha - \frac{n-2}{2} \right) (n-2)(r^2 + 1)^{-l/2} < 0 \]

since \( \alpha > (n-2)/2 \). Hence, by the assumption, there is a positive constant \( C_1 \) such that \( C_1 f_0 \leq f \) on \( M \). Set \( u_- := C_1^{-(-n-2)/4} u_0 \). Then we have

\[ L_g u_- = C_1 f_0 u_-^q \leq f u_-^q \text{ on } M, \]

namely, \( u_- \) is a positive subsolution of the equation \( (f,M) \). Therefore, by Theorem I, the equation \( (f,M) \) possesses a maximal solution \( U \) which satisfies

\[ U \geq u_- = C_1^{-(n-2)/4} (r^2 + 1)^{-(n-2)/2} \text{ on } M. \]

Since \( u_-^q \sim r^{l-2} \) near infinity and \( l-2 > 0 \), it is clear that \( \tilde{g} = U^{q-1} g \) is complete.

To prove the second assertion, set

\[ v_\gamma := \left( \frac{\gamma}{\gamma - r^2} \right)^\alpha, \quad u_\gamma := v_\gamma u_0 \]

for any \( \gamma \geq 1 \). If \( r < \sqrt{\gamma} \), then clearly

\[ v_\gamma^{1-q} = \left( \frac{\gamma - r^2}{\gamma} \right)^{\alpha(q-1)} < 1, \]

\[ \left( \frac{r^2 + 1}{\gamma - r^2} \right) v_\gamma^{1-q} \leq \frac{(r^2 + 1)(\gamma - r^2)}{\gamma^2} \leq 1, \]

\[ \left( \frac{r^2 + 1}{\gamma - r^2} \right)^2 v_\gamma^{1-q} \leq \frac{(r^2 + 1)^2}{\gamma^2} < 4. \]

On the other hand, since \( Ric_g/(n-1) \geq -6/(r^2 + \varepsilon^2) = -(r + \varepsilon^{-2} r^3)^\gamma / (r + \varepsilon^{-2} r^3) \), by the comparison theorem, we can estimate

\[ \Delta_g r \leq (n-1) \frac{(r + \varepsilon^{-2} r^3)^\gamma}{r + \varepsilon^{-2} r^3} \leq \frac{3(n-1)}{r}, \]

or \( r \Delta_g r \leq 3(n-1) \) on \( M \). Moreover, it follows that \( f_0 \sim -r^{-l} \) near infinity. Hence, by direct computation, we obtain
for some positive constant $C_2$ independent of $\gamma$. Therefore, if we set $u_{\gamma+} := C_2^{(n-2)/4}u_\gamma$, then it follows that

$$L_gu_{\gamma+} \geq f_0u_{\gamma+}q$$

when $r < \sqrt{\gamma}$,

namely, $u_{\gamma+}$ is a supersolution of the equation $(f_0, B_\sqrt{\gamma}(p))$. Since $u_{\gamma+} \to +\infty$ as $r \to \sqrt{\gamma}$, by Lemma 2.2, the maximal solution $U_0$ of the equation $(f_0,M)$ satisfies

$$U_0 \leq u_{\gamma+} = C_2^{(n-2)/4}u_\gamma \quad \text{in} \quad B_\sqrt{\gamma}(p).$$

Hence we have

$$U_0 \leq \lim_{\gamma \to +\infty} u_{\gamma+} = C_2^{(n-2)/4}u_0 \quad \text{on} \quad M.$$ 

Combining this inequality and $U_0 \geq u_0$, we obtain $U_0 \sim u_0 \sim r^{(l-2)(n-2)/4}$ near infinity.

On the other hand, the function $\psi := (r^2 + \varepsilon^2)^{-(n-2)/4}$ satisfies

$$L_g\psi = (n-1)(r^2 + \varepsilon^2)^{-(n+6)/4}$$

$$\times \left\{ - (n+2)r^2 + 2(r^2 + \varepsilon^2)(1 + r\Delta_g r) + \frac{S_g(r^2 + \varepsilon^2)^2}{n-1} \right\}$$

$$\geq (n-1)(r^2 + \varepsilon^2)^{-(n+6)/4} \left\{ -(n+2)r^2 + 2n(r^2 + \varepsilon^2) - (n-2)(r^2 + \varepsilon^2) \right\}$$

$$= (n-1)(n+2)\varepsilon^2(r^2 + \varepsilon^2)^{-(n+6)/4} > 0 \quad \text{on} \quad M,$$

that is, the condition (P) holds. Since $\psi/U_0$ is bounded above, by Lemma 3.1, we see that the maximal solution $U$ of the equation $(f,M)$ satisfies the estimate $U \sim r^{(l-2)(n-2)/4}$ near infinity. q.e.d.
Proof of Theorem III. Set $u_- := \{\min_M |S_g|/\max_M |f|\}^{(n-2)/4}$. Then we have

$$L_g u_- = S_g u_- \leq -\min_M |S_g| u_- = -\max_M |f| u_- q \leq f u_- q$$ on $M$,

namely, $u_-$ is a positive subsolution of the equation $(f,M)$. Therefore, by Theorem I, the equation $(f,M)$ possesses a maximal solution $U$ which satisfies

$$U \geq u_- = \left(\frac{\min_M |S_g|}{\max_M |f|}\right)^{(n-2)/4} > 0 \quad \text{on} \quad M.$$

It is clear that $\bar{g} = U^{n-1} g$ is complete.

To prove the second assertion, write $r = r_p$, $c = \cosh Br_p$, and $s = \sinh Br_p$ for convenience. Since $K_g \leq -B^2$ and $\text{Ric}_g/(n-1) \geq -A^2$, by the comparison theorem, we can estimate

$$\frac{(n-1)Bc}{s} = (n-1)B \coth Br$$

$$\leq \Delta_g r \leq (n-1)A \coth Ar \leq (n-1)A \coth Br = \frac{(n-1)Ac}{s},$$

or

$$n-1 \leq \frac{s}{Bc} \Delta_g r \leq (n-1) \frac{A}{B} \quad \text{on} \quad M.$$

Set

$$u_0 := c^{\alpha-(n-2)/2}, \quad v_\gamma := \left(\frac{\gamma}{\gamma-c}\right)^\alpha, \quad \text{and} \quad u_\gamma := v_\gamma u_0$$

for any $\gamma \geq 1$, where $\alpha = (1+1/2B)(n-2)/2$, and put $f_0 := u_0^{-q} L_g u_0$. Then we get

$$f_0 = -a_n B^2 c^{-1/B-2} \left\{ \left(\alpha - \frac{n-2}{2}\right)^2 s^2 + \left(\alpha - \frac{n-2}{2}\right) \left(\frac{cs}{B} \Delta_g r + 1\right) + \frac{|S_g|^2 c^2}{a_n B^2} \right\}$$

$$\leq -n(n-1)B^2 c^{-1/B} < 0$$

since $\alpha \geq (n-2)/2$. Hence, by the assumption, there is a positive constant $C_1$ such that $C_1 f_0 \leq f$ on $M$. Moreover, it follows that $f_0 \sim -e^{-lr}$ near
infinity. On the other hand, if $r < B^{-1} \cosh^{-1} \gamma$, then clearly

$$v_\gamma^{1-q} = \left(\frac{\gamma - c}{\gamma}\right)^{2(q-1)} < 1,$$

$$\left(\frac{c}{\gamma - c}\right) v_\gamma^{1-q} \leq \frac{c(\gamma - c)}{\gamma^2} \leq \frac{1}{4},$$

$$\left(\frac{c}{\gamma - c}\right)^2 v_\gamma^{1-q} \leq \frac{c^2}{\gamma^2} < 1.$$

Hence, by direct computation, we obtain

$$-u_\gamma^{-q} L_g u_\gamma = v_\gamma^{1-q}(-u_0^{-q} L_g u_0) + a_n v_\gamma^{-q} u_0^{-q}(2\nabla_g v_\gamma \cdot \nabla_g u_0 + u_0 \Delta_g v_\gamma)$$

$$= v_\gamma^{1-q} |f_0| + a_n B^2 c^{-1/\beta} \left\{ 2 \left(\alpha - \frac{n-2}{2}\right) s^2 \left(\frac{c}{\gamma - c}\right) + (\alpha + 1) s^2 \left(1 + \frac{s}{B c} \Delta_g r\right) \left(\frac{c}{\gamma - c}\right) v_\gamma^{1-q} \right\}$$

$$\leq |f_0| + a_n B^2 \left[ \frac{1}{2} \left(\alpha - \frac{n-2}{2}\right) + (\alpha + 1) + \frac{1}{4} \left(1 + (n-1) A \right) \right] c^{-1/\beta}$$

$$\leq C_2 |f_0| = -C_2 f_0$$

for some positive constant $C_2$ independent of $\gamma$. Therefore, if we set $u_{\gamma+} = C_2^{(n-2)/4} u_\gamma$, then it follows that

$$L_g u_{\gamma+} \geq f_0 u_{\gamma+}$$

when $r < \frac{1}{B} \cosh^{-1} \gamma$, namely, $u_{\gamma+}$ is a supersolution of the equation $(f_0, B_{-1} \cosh^{-1} \gamma \langle p \rangle)$. Since $u_{\gamma+} \to +\infty$ as $r \to B^{-1} \cosh^{-1} \gamma$, by Lemma 2.2, the maximal solution $U_0$ of the equation $(f_0, M)$ satisfies

$$U_0 \leq u_{\gamma+} = C_2^{(n-2)/4} u_\gamma$$

in $B_{-1} \cosh^{-1} \gamma \langle p \rangle$. Hence we have

$$U_0 \leq \lim_{\gamma \to +\infty} u_{\gamma+} = C_2^{(n-2)/4} u_0$$

on $M$. Combining this inequality and $U_0 \geq u_0$, we obtain $U_0 \sim u_0 \sim e^{(n-2)l r/4}$ near
infinity.

On the other hand, the function \( \psi := c^{-(n-1)/2} \) satisfies

\[
L_g \psi = \frac{(n-1)^2}{n-2} B^2 c^{-(n+3)/2} \left\{ -(n+1)s^2 + 2c^2 \left( 1 + \frac{s}{Bc} - \frac{(n-2)c^2S_b}{(n-1)^2B^2} \right) \right\}
\]

\[
\geq \frac{(n-1)^2}{n-2} B^2 c^{-(n+3)/2} \left\{ -(n+1)s^2 + 2nc^2 - (n-1)c^2 \right\}
\]

\[
= \frac{(n-1)^2(n+1)}{n-2} B^2 c^{-(n+3)/2} > 0 \text{ on } M,
\]

that is, the condition (P) holds. Since \( \psi/U_0 \) is bounded above, by Lemma 3.1, we see that the maximal solution \( U \) of the equation \((f,M)\) satisfies the estimate \( U \sim e^{(n-2)lr/4} \) near infinity. q.e.d.

** Remark 3.2.** In Theorem II, even if we do not assume \( A^2 \leq (n-2)/n \), we can estimate, by the comparison theorem, that \( r\Delta_g r \leq \beta(n-1) \) on \( M \), where \( \beta \) is a positive number satisfying the inequality \( -(r + \varepsilon_1 r^3 + \varepsilon_2 r^2)^2 \leq -A^2/(r^2 + \varepsilon_2) \) for some positive numbers \( \varepsilon_1 \) and \( \varepsilon_2 \). Therefore, as we mentioned in Section 1, we can prove the assertion of Theorem II under the condition (P) with \( \psi \leq C r^{(l-2)(n-2)/4} \) in place of the assumption \( A^2 \leq (n-2)/n \). Similarly, we can prove the assertion of Theorem III under the condition (P) with \( \psi \leq C e^{(n-2)lr/4} \) in place of the assumption \( (A/B)^2 \leq (n-1)^2/n(n-2) \).

4 The case of a domain of a compact manifold

In this section, we first study the asymptotic behavior of the maximal solution \( U \) in the case that \((M,g)\) is a subdomain of a compact manifold, and prove Theorem IV. Secondaly, we prove Theorem V which involves a nonexistence result of the maximal solution.

** Proof of Theorem IV.** Write \( r = r_\Sigma \) for convenience. In the case \( d < n-2 \), choose a negative function \( f_0 \) on \( M \) such that \( f_0 \leq f \) on \( M \), and \( f_0 \sim -r^l \) near \( \Sigma \). From Kato-Nayatani [9, Theorems 2(b) and 3], if \( \lambda_1(L_g) > 0 \), then the equation \((f_0,M)\) possesses a solution \( u_0 \) satisfying \( u_0 \sim r^{-l(n-2)/4} \) near \( \Sigma \). Also when \( \lambda_1(L_g) \leq 0 \), we can prove the same assertion by the same method after replacing the Green function of \( L_g \) by that of \( -\Delta_g \).

On the other hand, in the case \( d \geq n-2 \), from Aviles-McOwen [4], the equation \((-1,M)\) possesses a solution \( v \) satisfying \( v \sim r^{-(n-2)/2} \) near
Σ. Set \( u_0 = γ(v^{l+2}/2 + βv) \) and \( f_0 = u_0^{-q}Lq u_0 \), where \( β = \max\{-(l/2)\min_M \{v^{l/2}(1+ S_g v^{1-q}\}, 0\} \). By direct computation, we get

\[
f_0 = -γ^{-1-q}v^{q(l+2)/2}(1 + βv^{-l/2})^{-q}
\]

\[
- \frac{l(l+2)(n-1)}{n-2} v^{l/2} + \frac{l+2}{2} v^{l/2} + q \frac{l}{2} S_g v^{(l+2)/2} + βv^q
\]

\[
\leq -γ^{-1-q} \{1 + β(\min_M v)^{-l/2}\}^{-q} v^{-2l/(n-2)} < 0.
\]

Since \( v^{-2l/(n-2)} \sim r \) near \( Σ \), if we choose a small \( γ \), then we have \( f_0 ≤ f \).

Thus, in both cases, \( u_0 \) is a positive subsolution of the equation \((f,M)\). Therefore, by Theorem I, the equation \((f,M)\) possesses a maximal solution \( U \) which satisfies

\[
U ≥ u_0 ≥ C_1 r^{-(l+2)(n-2)/4} \quad \text{near} \, Σ
\]

for a positive constant \( C_1 \). Since \( l \) is nonnegative, we have

\[
U^{q-1} ≥ C_1 q^{-1} r^{-(l+2)} ≥ C_1 q^{-1} r^{-2} \quad \text{near} \, Σ
\]

from which it follows that \( \tilde{g} = U^{q-1}g \) is complete.

To get the upper estimate of \( U \), we use the lower estimate of \( U \) as above and the method of Aviles [1]. Choose a constant \( C_2 \) such that \( f - S_g U^{1-q} \leq -a_k C_2 r \) near \( Σ \). Then there is a positive number \( δ \) such that

\[
\Delta g U ≥ C_2 r U^q \quad \text{for} \, r_0 < δ.
\]

For any \( y \in B_{\delta/2}(Σ) \), let \( H_y \) be the Green function of \( -\Delta g \) in \( B_{\delta/2}(y) \) centered at \( y \). It is well-known that there are positive constants \( C_3, C_4 \) and \( C_5 \) independent of \( y \) such that

\[
C_3 r_y^{2-n} ≤ H_y ≤ C_4 r_y^{2-n} \quad \text{and} \quad |\nabla g H_y| ≤ C_5 r_y^{1-n} \quad \text{for} \, r_y < \frac{δ}{4}
\]

Let \( B_y \) be the connected component of the set \( \{x \in M | h_y(x) > 2^{n-2} C_4 r(y)^{2-n}\} \) which contains the point \( y \). Clearly \( B_y ⊂ B_{\delta y/2}(y) \). Set \( \nu_y = C_y \{r(y)^{2/4 - (C_4/H_y)^{2/(n-2)}}\}^{-(n-2)/2} \). Then it follows that

\[
\Delta g \nu_y ≤ C_6 r(y)^{2} C_y \nu_y^{1-q} \nu_y^q \quad \text{in} \, B_y,
\]

where \( C_6 = n C_4^{2/(n-2)} C_5^{2/4(n-2)} C_3^{2(n-1)/(n-2)} \). If we choose \( C_y = \{2^l \)
$C_6/C_2r(y)^{1-2}((n-2)/4$, then we have
\[
\Delta_g w = \frac{C_2}{2} r(y)^2 w_y \text{ in } B_y.
\]

On the other hand, it follows easily that
\[
\Delta_g U \geq \frac{C_2}{2} r(y)^2 U^q \text{ in } B_y.
\]

Since $w_y(x) \to +\infty$ as $x \to \partial B_y$, and $U$ is bounded above in $B_y$, by the maximal principle, we get $U \leq w_y$ in $B_y$. Hence we conclude that
\[
U(y) \leq w_y(y) = 2^{n-2} C_6 r(y)^{2-n} \leq C_7 r(y)^{(l+2)(n-2)/4} \text{ for } y \in B_{\bar{y}}(\Sigma)
\]
for a positive constant $C_7$. This completes the proof. \text{q.e.d.}

\textbf{Remark 4.1.} In the case that $\lambda_1(L_g) > 0$ and $d \leq (n-2)/2$, we can improve the assumption "$f \geq -C_r^{1-h}" as follows:
\[
\int_M G(x,y)|f(y)|r^2 G(x,y)h(y)d\sigma_x \\
\leq C G^2 r^{-1} \text{ on } M
\]
for some positive constant $C$. Actually, from Kato-Nayatani [9, Theorem 2'], we can construct a positive subsolution $u_-$ of the equation $(f,M)$ under this condition.

\textbf{Proof of Theorem V.} For any positive $C^2$-function $h$ on $\Sigma$, define the function $G_{\Sigma,h}$ by $G_{\Sigma,h}(x) := \int_{\Sigma} G(x,y)h(y)d\sigma_y$. From Delanoë [7, Theorem 5] or Kato-Nayatani [9, Theorem 2(a) and Remark after Proposition 2], there is a positive number $\gamma_0$ such that, for any positive number $\mu \leq \gamma_0$, the equation $(f,M)$ possesses a solution $v_\mu$ satisfying $v_\mu/G_{\Sigma,h} \leq \mu$ and $\lim_{x \to \Sigma} [v_\mu(x)/G_{\Sigma,h}(x)] = \mu$. In particular, $v_\mu^{-1} g$ is complete for any such $v_\mu$.

Let $\gamma$ be the supremum of such $\gamma_0$'s. Suppose $\gamma < +\infty$. Then, for any $\mu < \gamma$, the equation $(f,M)$ possesses a solution $v_\mu$ as above. Since $f$ is nonpositive, it follows from Lemma 2.2 that $\{v_\mu\} \mu < \gamma$ is monotonically increasing and bounded above by $\gamma G_{\Sigma,h}$. Therefore, if we set $v_\gamma := \lim_{\mu \to \gamma} v_\mu$, then $v_\gamma$ is a solution of the equation $(f,M)$ with the same properties as above.

Put $w_\gamma := \gamma - v_\gamma/G_{\Sigma,h}$. Then it is clear that $w_\gamma$ is nonnegative, $\max_M$
MAXIMAL SOLUTIONS

369

\( w_y < \gamma \), and \( \lim_{x \to \Sigma} w_y(x) = 0 \). Choose a positive number \( \delta \) satisfying

\[ 1 < \delta < \gamma / \max_M w_y \], and set \( v_{e \gamma -} := \varepsilon (\gamma - \delta w_y) G_{\Sigma, h} \). Then we get, for any \( \varepsilon \leq \delta^{(m-2)/4} \),

\[ L_g v_{e \gamma -} = \varepsilon \delta L_g v_y = \varepsilon \delta f v_y^q = f v_{e \gamma -}^q \leq f v_{e \gamma -}^q \quad \text{on } M, \]

namely, \( v_{e \gamma -} \) is a subsolution of the equation \((f, M)\). On the other hand, if we set \( v_{e \gamma +} := \varepsilon \gamma G_{\Sigma, h} \), then we have

\[ L_g v_{e \gamma +} = 0 \geq f v_{e \gamma +}^q \quad \text{on } M, \]

namely, \( v_{e \gamma +} \) is a supersolution of the equation \((f, M)\). Since \( v_{e \gamma +} \geq v_{e \gamma -} > 0 \), by the method of supersolutions and subsolutions, the equation \((f, M)\) possesses a solution \( v_{e \gamma} \) satisfying \( v_{e \gamma -} \geq v_{e \gamma} \geq v_{e \gamma +} \). It is clear that

\[ v_{e \gamma} / G_{\Sigma, h} \leq \varepsilon \gamma \] and \( \lim_{x \to \Sigma} \{ v_{e \gamma} / G_{\Sigma, h}(x) \} = \varepsilon \gamma \). This contradicts the definition of \( \gamma \) since \( \delta^{(m-2)/4} \gamma > \gamma \).

Hence we conclude that \( \gamma = + \infty \). In particular, the equation \((f, M)\) possesses a solution \( u_h := v_1 \) satisfying \( u_h(x) / G_{\Sigma, h}(x) \to 1 \) as \( x \to \Sigma \). On the other hand, since \( h \) is a \( C^\alpha \)-function, it follows easily that \( G_{\Sigma, h}(x) / G_{\Sigma}(x) \to h(x_\infty) \) as \( x \to x_\infty \in \Sigma \). Therefore we get

\[
\frac{u_h(x)}{G_{\Sigma}(x)} = \frac{u_h(x)}{G_{\Sigma, h}(x)} \times \frac{G_{\Sigma, h}(x)}{G_{\Sigma}(x)} \to h(x_\infty) \quad \text{as } x \to x_\infty \in \Sigma.
\]

The uniqueness is established by the same method as in Cheng-Ni [6, Theorem 3.1].

Now, if the equation \((f, M)\) possesses a maximal solution \( U \), then it holds that \( U \geq u_h \) for any \( h \). Hence we have \( \lim_{x \to \Sigma} U(x) / G_{\Sigma}(x) = + \infty \). On the other hand, if \( \text{supp} f \) is compact, then it is clear that

\[
\Phi(x) := \int_{\bar{\Sigma}} G(x, y)f(y)U(y)dy
\]

is a bounded smooth function on \( \bar{\Sigma} \) satisfying \( L_g \Phi = fU^q \). Set \( \Psi := (U - \Phi) / G_{\Sigma} \), and \( \check{g} := G_{\Sigma}^{-1}g \). Then we see that

\[
\Delta g \Psi = - \frac{1}{a_n} L_g \Psi = - \frac{1}{a_n} G_{\Sigma}^{-q} L_g (U - \Phi) = 0 \quad \text{and} \quad \lim_{x \to \Sigma} \Psi(x) = + \infty.
\]

This contradicts the maximal principle. Therefore the equation \((f, M)\) does not possess a maximal solution when \( \text{supp} f \) is compact. q.e.d.
More generally, it follows from the similar argument as in the final part of the proof of Theorem V that, if \( \text{supp } f \) is compact, then any solution \( u \) of the equation \((f, M)\) satisfies \( u(x)/G(x) - \Psi(x) \to 0 \) as \( x \to \Sigma \) for some positive function \( \Psi \) such that \( \Delta ^2 \Psi = 0 \). On the other hand, if \( \Sigma \) is a point i.e. \( \Sigma = \{ p \} \), then the minimal positive Green function of \( -\Delta _g \) is \( G(x, y)/G(x, p)G(y, p) \). Since
\[
\frac{G(x, y)/G(x, p)G(y, p)}{G(x', y)/G(x', p)G(y, p)} \to 1 \quad \text{as} \quad y \to p
\]
for any \( x \) and \( x' \in M \), the Martin boundary of \((M, g)\) is a point, and hence any positive harmonic function on \((M, g)\) must be a constant. Thus we have the following result which involves a higher dimensional version of Cheng-Ni [5, Theorem III].

**Proposition 4.2.** Let \((\bar{M}, \bar{g})\) be a compact Riemannian manifold \((n = \dim M \geq 3)\) with \( \lambda _1 (L_g) > 0 \), \( p \) a point in \( \bar{M} \), and \((M, g) := (\bar{M} \setminus \{ p \}, \bar{g}|_{\bar{M} \setminus \{ p \}})\). Let \( f \) be a nonpositive smooth function on \( M \). If \( \text{supp } f \) is compact, then an arbitrary solution \( u \) of the equation \((f, M)\) coincides with \( u_h \) (as in Theorem V) for some positive number \( h \).

---

**References**


Department of Mathematics  
Nara Women’s University  
Nara 630  
Japan

*Present Address:*  
Department of Mathematics  
Osaka University  
Toyonaka, Osaka 560  
Japan