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MAXIMAL SOLUTIONS OF THE SCALAR CURVATURE EQUATION ON OPEN RIEMANNIAN MANIFOLDS

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1 Introduction

Let (M,g) be a Riemannian manifold with $\dim M = n \geq 3$, Δ_g the Laplacian of g, namely, $\Delta_g := \mathrm{trace}\nabla_g{}^2$, S_g the scalar curvature of g, and L_g the conformal Laplacian, that is, $L_g := -a_n\Delta_g + S_g$, where $a_n = 4(n-1)/(n-2)$. Define a new metric $\tilde{g}$ which is pointwise conformal to g by $\tilde{g} := u^{4/(n-2)}g$, where u is a positive smooth function on M. Then its scalar curvature $S_{\tilde{g}}$ is given by $L_g u = S_{\tilde{g}} u^{(n+2)/(n-2)}$. Hence, for any smooth function f on M, f can be realized as the scalar curvature of some metric which is pointwise conformal to g if and only if there is a smooth solution of the following partial differential equation:

$$\begin{cases} L_g u = f u^q \\ u > 0 \end{cases} \qquad \text{on } M,$$

where $q = (n+2)/(n-2) = 4/(n-2) + 1$. In this paper, we refer to this equation as "the equation (f,M)" for various functions f and manifolds M (e.g. $(-1,W)$, (f_0,Ω_i), etc.).

Now, we are interested in the problem of finding sufficient conditions for a function f to be realized by some complete conformal metric. This problem has been extensively investigated in various cases (see e.g. McOwen [10]). Recently, Cheng-Ni [6] proved the existence of a maximal solution U of the equation (f,M) in the case that (M,g) is the Euclidean space $(\boldsymbol{R}^n, g_0)$ or its open subset and that f is nonpositive and satisfies some conditions. In this paper, first we shall simplify their proof and generalize the result to the case that M is an open Riemannian manifold. This result is stated as follows.

Theorem I. *Let (M,g) be an open Riemannian manifold ($n = \dim M \geq 3$). Let f be a nonpositive smooth function on M. Suppose both of the following conditions hold*:
(a) *there exists a sequence $\{\Omega_i\}_{i\in N}$ of relatively compact domains of M with*

smooth boundaries which satisfies

$$\Omega_i \subset\subset \Omega_{i+1}, \ \bigcup_{i\in N} \Omega_i = M, \ \textit{and } f<0 \ \textit{on } \partial\Omega_i;$$

(b) *there exists a positive subsolution* u_- *of the equation* (f,M).
Then there is a smooth solution U *of the equation* (f,M) *which is maximal in the sense that* $U \geq u$ *for any solution* u *of the equation* (f,M).

Clearly, the condition (a) is satisfied if f is negative on the complement of some compact subset of M. As we shall see below (in Theorem V), when the condition (a) is not satisfied, the equation (f,M) does not necessarily possess a maximal solution. In Section 2, we give the proof of Theorem I.

Using this theorem, we want to find some geometric conditions for the above problem. Moreover, we investigate the asymptotic behavior of the maximal solution U. It should be mentioned here that Cheng-Ni [6] describe it in a special case by analyzing a certain ordinary differential equation. However, it seems difficult to apply the technique to our cases, and we employ different methods.

Throughout this paper, we denote the sectional (resp.Ricci) curvature of g by K_g (resp. Ric_g), the distance function to a point p (resp.a submanifold Σ) by r_p (resp.r_Σ) unless otherwise stated, and we use the notation "$f_0 \sim f$" to mean that the condition $C_1 f \leq f_0 \leq C_2 f$ holds for some positive constants C_1 and C_2.

Our main results are stated as follows.

Theorem II. *Let* (M,g) *be a complete, noncompact, simply connected Riemannian manifold* ($n=\dim M \geq 3$) *with nonpositive curvature. Let* f *be a nonpositive smooth function on* M *with the condition* (a) *in Theorem I. If* f *satisfies* $f \geq -C r_p{}^{-l}$ *near infinity for some point* $p \in M$, *a positive constant* C *and a number* $l>2$, *then there is a complete conformal metric* $\tilde{g}$ *on* M *with scalar curvature* f.

Moreover, suppose (M,g) *satisfies* $Ric_g/(n-1) \geq -A^2/(r_p{}^2+\varepsilon^2)$ *for positive numbers* A *and* ε *such that* $A^2 \leq (n-2)/n$. *Then the metric* $\tilde{g}$ *can be constructed so that its ratio to* g *is of order* $r_p{}^{l-2}$ *near infinity, if* f *satisfies* $f \sim -r_p{}^{-l}$ *near infinity.*

Theorem III. *Let* (M,g) *be a complete, noncompact Riemannian manifold* ($n=\dim M \geq 3$) *whose scalar curvature is bounded above by a negative number. Let* f *be a nonpositive smooth function on* M *with the condition* (a) *in Theorem I. If* f *is bounded, then there is a complete* *conformal metric $\tilde{g}$ on M with scalar curvature f.*

Moreover, suppose (M,g) is simply connected and satisfies $K_g \le -B^2$ and $Ric_g/(n-1) \ge -A^2$ for positive numbers A and B such that $(A/B)^2 \le (n-1)^2/n(n-2)$. Then the metric $\tilde{g}$ can be constructed so that its ratio to g is of order e^{lr_p} near infinity, if f satisfies $f \sim -e^{-lr_p}$ near infinity for some point $p \in M$ and a nonnegative number l.

Theorem IV. *Let $(\bar{M},\bar{g})$ be a compact Riemannian manifold $(n = \dim \bar{M} \ge 3)$, Σ a compact submanifold $(d = \dim \Sigma)$ of $\bar{M}$, and $(M,g) := (\bar{M}\setminus\Sigma, \bar{g}|_{\bar{M}\setminus\Sigma})$. Let f be a nonpositive smooth function on M with the condition* (a) *in Theorem I. If f satisfies $f \ge -Cr_\Sigma{}^l$ near Σ for a positive constant C and a nonnegative number $l > 2 - 4d/(n-2)$, then there is a complete conformal metric $\tilde{g}$ on M with scalar curvature f.*

Moreover, if f satisfies $f \sim -r_\Sigma{}^l$ near Σ, then the metric $\tilde{g}$ can be constructed so that its ratio to g is of order $r_\Sigma{}^{-(l+2)}$ near Σ.

In Theorems II and III, we have imposed the assumption on the upper bound for A^2 and $(A/B)^2$ respectively. However, these can be replaced with other conditions. See Remark 3.2 for details.

In the case that $f \sim -1$ i.e. f is bounded between two negative constants near infinity (resp.Σ), the first assertion of Theorem III (resp.Theorem IV) was proved by Aviles-McOwen [3] (resp. [4]) essentially (see also [2]). However, we emphasize that f need not to be bounded above by a negative constant near infinity (resp.Σ).

We shall now mention a result by Cheng-Ni [6] which states that, *if a nonpositive function f* on $\boldsymbol{R}^n$ *satisfies $f(x) \sim -|x|^{-l}$ near infinity for a number $l > 2$, then any solution u* of the equation $(f, \boldsymbol{R}^n)$ *coincides with the maximal solution U or a unique solution u_h such that $u_h(x) \to h$ as $|x| \to +\infty$ for some positive number h.* However, in our case, it would be difficult to describe all of the solutions of the equation (f,M). In fact, we have the following

Theorem V. *Let (M,g) be as in Theorem IV with $\lambda_1(L_{\bar{g}}) > 0$ and $d \le (n-2)/2$, and f a nonpositive smooth function on M. If f satisfies $f \ge -Cr_\Sigma{}^l$ near Σ for a positive constant C and a number $l > 2 - 4d/(n-2)$, then, for any positive C^α-function h on Σ $(0 < \alpha < 1)$, there is a unique solution u_h of the equation (f,M) such that $u_h(x)/G_\Sigma(x) \to h(x_\infty)$ as $x \to x_\infty \in \Sigma$ (in particular, the metric $\tilde{g} = u_h{}^{q-1}g$ is complete), where $G_\Sigma(x) := \int_\Sigma G(x,y)d\sigma_y$ and $G(x,y)$ is the Green function of $L_{\bar{g}}$.*

Moreover, if the support of f, supp f, *is compact, then the equation (f,M) does not possess a maximal solution.*

The first assertion of this theorem was partially obtained in Delanoë

[7] and Kato-Nayatani [9]. They proved that u_h exists if h is small enough. However, we claim that u_h exists for an arbitrary h when f is nonpositive.

The proofs of Theorems II and III (resp.Theorems IV and V) are given in Section 3 (resp.Section 4).

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2 The existence of the maximal solution

In this section, we prove Theorem I. We first recall the following:

Method of Supersolutions and Subsolutions. *Let (M,g) be a Riemannian manifold ($n=\dim M\geq 3$), and f a smooth function on M. If there exist a supersolution u_+ and a subsolution u_- of the equation (f,M)* such that $0<u_-\leq u_+$, *then the equation (f,M)* possesses a smooth solution u satisfying $u_-\leq u\leq u_+$.

This is well-known and we omit the proof. We also recall the following significant result.

Theorem A (Aviles-McOwen [4]). *Let $(\bar{W},g)$ be a compact Riemannian manifold with boundary $\partial W\neq\emptyset$($n=\dim W\geq 3$). The equation $(-1,W)$ possesses a solution u such that $u(x)\to+\infty$ as $x\to\partial W$.*

To prove Theorem I, we prepare the following two lemmas.

Lemma 2.1. *Let (M,g) be an open Riemannian manifold, and Ω a relatively compact domain of M. Then there is a positive smooth function η in Ω such that both η and $L_g\eta$ are bounded below by a positive number in Ω.*

Proof. Choose a relatively compact domain $\hat{\Omega}$ of M satisfying $\Omega\subset\subset\hat{\Omega}$. There exists a bounded positive smooth function ψ_0 defined on $\hat{\Omega}$ satisfying $L_g\psi_0\equiv 0$ (see e.g. Escobar [8, Proposition 1.3]). Set $\hat{g}:=\psi_0{}^{q-1}g|_{\hat{\Omega}}$, then we have $S_{\hat{g}}\equiv 0$. Let ξ be the first Dirichlet eigenfunction of $-\Delta_{\hat{g}}$ in $\hat{\Omega}$. Since ξ does not change sign, we can assume $\xi>0$. Set $\eta:=\psi_0\xi$. Then $\eta>0$ in $\hat{\Omega}$, and hence η is bounded below by a positive number in Ω. Moreover, since

$$L_g\eta=L_g(\psi_0\xi)=\psi_0{}^qL_{\hat{g}}\xi=a_n\psi_0{}^q\lambda_1(-\Delta_{\hat{g}})\xi>0 \text{ in } \hat{\Omega},$$

$L_g\eta$ is also bounded below by a positive number in Ω. q.e.d.

Lemma 2.2. *Let (M,g) and Ω be as in Lemma 2.1, f (resp. f_0) a smooth function in Ω, and u (resp. u_0) a solution of the equation (f,Ω) (resp. (f_0,Ω)). Suppose f_0 is nonpositive, $f_0 \leq f$ in Ω, and $\liminf_{x\to\partial\Omega}\{u(x) - u_0(x)\} \geq 0$. Then $u_0 \leq u$ in Ω.*

Proof. Let $\hat{g}$ be as in the proof of Lemma 2.1. Then $L_{\hat{g}}(\psi_0{}^{-1}u_0) = f_0 (\psi_0{}^{-1}u_0)^q$, and $L_{\hat{g}}(\psi_0{}^{-1}u) = f(\psi_0{}^{-1}u)^q \geq f_0(\psi_0{}^{-1}u)^q$ in Ω. Hence we get

$$\left\{a_n \Delta_{\hat{g}} + f_0 \psi_0{}^{1-q} \frac{u^q - u_0{}^q}{u - u_0}\right\} (\psi_0{}^{-1}u - \psi_0{}^{-1}u_0) \leq 0 \text{ in } \Omega.$$

Since

$$f_0 \psi_0{}^{1-q} \frac{u^q - u_0{}^q}{u - u_0} \leq 0 \text{ in } \Omega,$$

and $\liminf_{x\to\partial\Omega}\{u(x) - u_0(x)\} \geq 0$, by the maximal principle, we have $u - u_0 \geq 0$ in Ω. q.e.d.

Proof of Theorem I. By Theorem A, for any $i \in N$, the equation $(-1, \Omega_i)$ possesses a solution v_i such that $v_i(x) \to +\infty$ as $x \to \partial\Omega_i$. Set $u_{i-} := \gamma_i v_i$, where $\gamma_i = (\max_{\bar{\Omega}_i} |f|)^{-(n-2)/4}$. Then we have

$$L_g u_{i-} = -\gamma_i{}^{1-q} u_{i-}{}^q \leq f u_{i-}{}^q \text{ in } \Omega_i,$$

namely, u_i is a subsolution of the equation (f, Ω_i). On the other hand, by Lemma 2.1, there is a positive smooth function η_i in Ω_i such that both η_i and $L_g \eta_i$ are bounded below by a positive number in Ω_i. From the assumption (a), there exist an open subset Ω_i' of Ω_i such that $\Omega_i' \subset\subset \Omega_i$, and a positive number ε_i such that $f \leq -\varepsilon_i$ in $\Omega_i \setminus \bar{\Omega}_i'$. Set $u_{i+} := \delta_i(v_i + \beta_i \eta_i)$, where $\beta_i = \max_{\bar{\Omega}_i'}(v_i{}^q / L_g \eta_i)$ and $\delta_i = \varepsilon_i{}^{-(n-2)/4}$. Then we get

$$L_g u_{i+} = \delta_i(-v_i{}^q + \beta_i L_g \eta_i) \geq 0 \geq f u_{i+}{}^q \text{ on } \bar{\Omega}_i',$$

and

$$L_g u_{i+} \geq \delta_i L_g v_i = -\varepsilon_i(\delta_i v_i)^q \geq f u_{i+}{}^q \text{ in } \Omega_i \setminus \bar{\Omega}_i',$$

namely, u_{i+} is a supersolution of the equation (f, Ω_i). Since $u_{i+} \geq u_{i-} > 0$ in Ω_i, by the method of supersolutions and subsolutions, the equation (f, Ω_i) possesses a smooth solution u_i satisfying $u_{i+} \geq u_i \geq u_{i-}$.

Since $u_i(x) \to +\infty$ as $x \to \partial\Omega_i$, and both u_{i+1} and u_- are bounded above in Ω_i, by Lemma 2.2, we get $u_i \geq u_{i+1}$ and $u_i \geq u_-$ in Ω_i, that is,

$\{u_i\}_{i\in N}$ is monotonically decreasing and bounded below by u_-. Therefore, if we set $U:=\lim_{i\to+\infty} u_i$, then U is a smooth solution of the equation (f,M). By Lemma 2.2 again, it is clear that $u_i \geq u$ for any solution u of the equation (f,M). Hence we see that $U\geq u$ or U is the maximal solution. q.e.d.

Cheng-Ni [6, Propositions 2.11, 2.12 and 2.13] listed some properties of the maximal solution U in the case of $(\boldsymbol{R}^n,g_0)$, and their assertions are valid also in our case. Actually, we can prove the following proposition along the same lines as in Cheng-Ni [5, Theorem 2.8, Remark 2.9, and Proposition 2.10].

Proposition 2.3. *Let (M,g) be as in Theorem I, f (resp. f_0) a nonpositive function on M satisfying the condition* (a) *and* (b) *in Theorem I, and U (resp. U_0) the maximal solution of the equation (f,M) (resp. (f_0,M)).*
(1) *If $f_0\leq f$ on M, then $U_0\leq U$ on M.*
(2) *If u is a solution of the equation (f,M), then $u\equiv U$ or $u<U$ on M.*
(3) *If f is invariant under the action of some subgroup Γ of* $\mathrm{Isom}(M,g)$, *then U is also Γ-invariant.*

REMARK 2.4. In Theorem I, so far as the existence of a solution is concerned, we may assume f is positive on a compact subset of M. Actually, let f_0 be a nonpositive function on M with the conditions (a) and (b) in Theorem I, and U_0 the maximal solution of the equation (f_0,M). Let η_1 and Ω_1' be as in the proof of Theorem I, and χ a smooth nonnegative function on M such that $\chi\equiv 1$ on $\bar{\Omega}_1'$ and $\chi\equiv 0$ on $M\setminus\Omega_1$. Set $u_+:=\delta(U_0+\beta\chi\eta_1)$ and $u_-:=U_0$. Then, by choosing β and δ large enough, we get $L_g u_+\geq f_0 u_+{}^q$ on $M\setminus\Omega_1'$ and $L_g u_+\geq \varepsilon u_+{}^q$ in Ω_1' for some positive number ε depending only on f_0 and Ω_1'. Hence, *if a function f satisfies $f\equiv f_0$ on $M\setminus\Omega_1'$ and $f_0\leq f\leq\varepsilon$ in Ω_1', then there is a solution u of the equation (f,M) satisfying $u\sim U_0$ on M.* However, we do not know whether u is maximal or not.

3 The case of a complete simply connected manifold

In this section, we study the asymptotic behavior of the maximal solution U in the case that (M,g) is a complete simply connected manifold, and prove Theorems II and III. For this purpose, we prepare the following lemma.

Lemma 3.1. *Let (M,g) be as in Theorem I with the condition* (P) *there exists a positive smooth function ψ on M satisfying $L_g\psi>0$.*

Let f (resp. f_0) be a nonpositive function on M satisfying the condition (a) *and* (b) *in Theorem I, and U (resp. U_0) the maximal solution of the equation (f,M) (resp. (f_0,M)). Suppose $f_0 \leq f$ on M, $f_0 \sim f$ on the complement of some compact subset of M, and ψ/U_0 is bounded on M. Then $U_0 \sim U$ on M.*

Proof. We may assume $f_0 \sim f$ on $M \setminus \bar{\Omega}_1$, that is, $f \leq Cf_0$ on $M \setminus \bar{\Omega}_1$ for a positive constant $C<1$, without loss of generality. By the proof of Theorem I, for any $i \in N$, the equation (f_0,Ω_i) possesses a solution $u_{0,i}$ satisfying $u_{0,i}(x) \to +\infty$ as $x \to \partial\Omega_i$. Obviously

$$L_g u_{0,i} = f_0 u_{0,i}{}^q \leq f u_{0,i}{}^q \text{ in } \Omega_i,$$

namely, $u_{0,i}$ is a subsolution of the equation (f,Ω_i). On the other hand, if we set $u_{i+} := \delta(u_{0,i} + \beta\psi)$, where $\beta = \max_{\bar{\Omega}_1}(|f_0| u_{0,2}{}^q / L_g\psi)$ and $\delta = C^{-(n-2)/4}$, then we get, for any $i \geq 2$,

$$\begin{aligned} L_g u_{i+} &= \delta(f_0 u_{0,i}{}^q + \beta L_g \psi) \geq \delta(f_0 u_{0,i}{}^q + |f_0| u_{0,2}{}^q) \\ &= \delta|f_0|(u_{0,2}{}^q - u_{0,i}{}^q) \geq 0 \geq f u_{i+}{}^q \text{ on } \bar{\Omega}_1, \end{aligned}$$

and

$$L_g u_{i+} \geq \delta f_0 u_{0,i}{}^q = C f_0 (\delta u_{0,i})^q \geq f u_{i+}{}^q \quad \text{on} \quad \Omega_i \setminus \bar{\Omega}_1,$$

namely, u_{i+} is a supersolution of the equation (f,Ω_i). Since $u_{i+} \geq u_{0,i} > 0$ in Ω_i, by the method of supersolutions and subsolutions, the equation (f,Ω_i) possesses a solution u_i of the equation (f,Ω_i) satisfying $u_{i+} \geq u_i \geq u_{0,i}$.

Now, it is clear that $\lim_{i \to +\infty} u_i = U$. Therefore we have $U_0 \leq U \leq \delta (U_0 + \beta\psi)$, or $1 \leq U/U_0 \leq \delta(1 + \beta\psi/U_0)$ on M. Since we assume ψ/U_0 is bounded, we conclude that $U_0 \sim U$ on M. q.e.d.

Remark that the assumption "$f_0 \leq f$ on M" in this lemma can be replaced by the condition "$C_1 f_0 \leq f$ on M for a positive constant C_1".

Proof of Theorem II. Write $r := r_p$ for convenience. Since $K_g \leq 0$, we see $\Delta_g r \geq (n-1)/r$ or $r\Delta_g r \geq n-1$ on M. Set $u_0 := (r^2+1)^{\alpha-(n-2)/2}$, where $\alpha = (l+2)(n-2)/8$, and put $f_0 := u_0{}^{-q} L_g u_0$. Then we get

$$\begin{aligned} f_0 = -2a_n(r^2+1)^{-l/2-1}\Bigg[& 2\left(\alpha - \frac{n-2}{2}\right)^2 r^2 \\ & + \left(\alpha - \frac{n-2}{2}\right)\{(r^2+1)(-1+r\Delta_g r)+2\} + \frac{|S_g|(r^2+1)^2}{2a_n}\Bigg] \end{aligned}$$

$$\leq -2a_n\left(\alpha-\frac{n-2}{2}\right)(n-2)(r^2+1)^{-l/2}<0$$

since $\alpha>(n-2)/2$. Hence, by the assumption, there is a positive constant C_1 such that $C_1f_0\leq f$ on M. Set $u_-:=C_1{}^{-(n-2)/4}u_0$. Then we have

$$L_gu_-=C_1f_0u_-{}^q\leq fu_-{}^q \text{ on } M,$$

namely, u_- is a positive subsolution of the equation (f,M). Therefore, by Theorem I, the equation (f,M) possesses a maximal solution U which satisfies

$$U\geq u_-=C_1{}^{-(n-2)/4}(r^2+1)^{\alpha-(n-2)/2} \text{ on } M.$$

Since $u_-{}^{q-1}\sim r^{l-2}$ near infinity and $l-2>0$, it is clear that $\tilde{g}=U^{q-1}g$ is complete.

To prove the second assertion, set

$$v_\gamma:=\left(\frac{\gamma}{\gamma-r^2}\right)^\alpha, \text{ and } u_\gamma:=v_\gamma u_0$$

for any $\gamma\geq 1$. If $r<\sqrt{\gamma}$, then clearly

$$v_\gamma{}^{1-q}=\left(\frac{\gamma-r^2}{\gamma}\right)^{\alpha(q-1)}<1,$$

$$\left(\frac{r^2+1}{\gamma-r^2}\right)v_\gamma{}^{1-q}\leq\frac{(r^2+1)(\gamma-r^2)}{\gamma^2}\leq 1,$$

$$\left(\frac{r^2+1}{\gamma-r^2}\right)^2v_\gamma{}^{1-q}\leq\frac{(r^2+1)^2}{\gamma^2}<4.$$

On the other hand, since $Ric_g/(n-1)\geq -6/(r^2+\varepsilon^2)=-(r+\varepsilon^{-2}r^3)''/(r+\varepsilon^{-2}r^3)$, by the comparison theorem, we can estimate

$$\Delta_gr\leq(n-1)\frac{(r+\varepsilon^{-2}r^3)'}{r+\varepsilon^{-2}r^3}\leq\frac{3(n-1)}{r},$$

or $r\Delta_gr\leq 3(n-1)$ on M. Moreover, it follows that $f_0\sim -r^{-l}$ near infinity. Hence, by direct computation, we obtain

$$
\begin{aligned}
-u_\gamma{}^{-q}L_g u_\gamma &= v_\gamma{}^{1-q}(-u_0{}^{-q}L_g u_0) + a_n v_\gamma{}^{-q} u_0{}^{-q}(2\nabla_g v_\gamma \cdot \nabla_g u_0 + u_0 \Delta_g v_\gamma) \\
&= v_\gamma{}^{1-q}|f_0| + 2a_n\alpha(r^2+1)^{-l/2-1}\left\{4\left(\alpha - \frac{n-2}{2}\right)r^2\left(\frac{r^2+1}{\gamma-r^2}\right)\right. \\
&\quad \left. + 2(\alpha+1)r^2\left(\frac{r^2+1}{\gamma-r^2}\right)^2 + (r^2+1)(1+r\Delta_g r)\left(\frac{r^2+1}{\gamma-r^2}\right)\right\} v_\gamma{}^{1-q} \\
&\le |f_0| + 2a_n\alpha\left\{4\left(\alpha-\frac{n-2}{2}\right) + 8(\alpha+1) + (3n-2)\right\}(r^2+1)^{-l/2} \\
&\le C_2|f_0| = -C_2 f_0
\end{aligned}
$$

for some positive constant C_2 independent of γ. Therefore, if we set $u_{\gamma+} := C_2{}^{(n-2)/4}u_\gamma$, then it follows that

$$
L_g u_{\gamma+} \ge f_0 u_{\gamma+}{}^q \quad \text{when} \quad r < \sqrt{\gamma},
$$

namely, $u_{\gamma+}$ is a supersolution of the equation (f_0, $B_{\sqrt{\gamma}}(p)$). Since $u_{\gamma+} \to +\infty$ as $r \to \sqrt{\gamma}$, by Lemma 2.2, the maximal solution U_0 of the equation (f_0,M) satisfies

$$
U_0 \le u_{\gamma+} = C_2{}^{(n-2)/4}u_\gamma \quad \text{in} \quad B_{\sqrt{\gamma}}(p).
$$

Hence we have

$$
U_0 \le \lim_{\gamma\to+\infty} u_{\gamma+} = C_2{}^{(n-2)/4}u_0 \quad \text{on} \quad M.
$$

Combining this inequality and $U_0 \ge u_0$, we obtain $U_0 \sim u_0 \sim r^{(l-2)(n-2)/4}$ near infinity.

On the other hand, the function $\psi := (r^2+\varepsilon^2)^{-(n-2)/4}$ satisfies

$$
\begin{aligned}
L_g\psi &= (n-1)(r^2+\varepsilon^2)^{-(n+6)/4} \\
&\quad \times \left\{-(n+2)r^2 + 2(r^2+\varepsilon^2)(1+r\Delta_g r) + \frac{S_g(r^2+\varepsilon^2)^2}{n-1}\right\} \\
&\ge (n-1)(r^2+\varepsilon^2)^{-(n+6)/4}\{-(n+2)r^2 + 2n(r^2+\varepsilon^2) - (n-2)(r^2+\varepsilon^2)\} \\
&= (n-1)(n+2)\varepsilon^2(r^2+\varepsilon^2)^{-(n+6)/4} > 0 \quad \text{on} \quad M,
\end{aligned}
$$

that is, the condition (P) holds. Since ψ/U_0 is bounded above, by Lemma 3.1, we see that the maximal solution U of the equation (f,M) satisfies the estimate $U \sim r^{(l-2)(n-2)/4}$ near infinity. q.e.d.

Proof of Theorem III. Set $u_-:=\{\min_M|S_g|/\max_M|f|\}^{(n-2)/4}$. Then we have

$$L_g u_- = S_g u_- \leq -\min_M |S_g| u_- = -\max_M |f| u_-{}^q \leq f u_-{}^q \text{ on } M,$$

namely, u_- is a positive subsolution of the equation (f,M). Therefore, by Theorem I, the equation (f,M) possesses a maximal solution U which satisfies

$$U \geq u_- = \left(\frac{\min_M |S_g|}{\max_M |f|}\right)^{(n-2)/4} > 0 \quad \text{on} \quad M.$$

It is clear that $\tilde{g}=U^{q-1}g$ is complete.

To prove the second assertion, write $r:=r_p$, $c:=\cosh Br_p$, and $s:=\sinh Br_p$ for convenience. Since $K_g \leq -B^2$ and $Ric_g/(n-1) \geq -A^2$, by the comparison theorem, we can estimate

$$\begin{aligned}\frac{(n-1)Bc}{s} &= (n-1)B \coth Br \\ &\leq \Delta_g r \leq (n-1)A \coth Ar \leq (n-1)A \coth Br = \frac{(n-1)Ac}{s},\end{aligned}$$

or

$$n-1 \leq \frac{s}{Bc}\Delta_g r \leq (n-1)\frac{A}{B} \quad \text{on} \quad M.$$

Set

$$u_0 := c^{\alpha-(n-2)/2}, \quad v_\gamma := \left(\frac{\gamma}{\gamma-c}\right)^\alpha, \quad \text{and} \quad u_\gamma := v_\gamma u_0$$

for any $\gamma \geq 1$, where $\alpha=(1+l/2B)(n-2)/2$, and put $f_0:=u_0{}^{-q}L_g u_0$. Then we get

$$\begin{aligned}f_0 &= -a_n B^2 c^{-l/B-2}\left\{\left(\alpha-\frac{n-2}{2}\right)^2 s^2 + \left(\alpha-\frac{n-2}{2}\right)\left(\frac{cs}{B}\Delta_g r+1\right)+\frac{|S_g|c^2}{a_n B^2}\right\} \\ &\leq -n(n-1)B^2 c^{-l/B} < 0\end{aligned}$$

since $\alpha \geq (n-2)/2$. Hence, by the assumption, there is a positive constant C_1 such that $C_1 f_0 \leq f$ on M. Moreover, it follows that $f_0 \sim -e^{-lr}$ near infinity. On the other hand, if $r<B^{-1}\cosh^{-1}\gamma$, then clearly

$$v_\gamma^{1-q}=\left(\frac{\gamma-c}{\gamma}\right)^{\alpha(q-1)}<1,$$

$$\left(\frac{c}{\gamma-c}\right)v_\gamma^{1-q}\leq\frac{c(\gamma-c)}{\gamma^2}\leq\frac{1}{4},$$

$$\left(\frac{c}{\gamma-c}\right)^2 v_\gamma^{1-q}\leq\frac{c^2}{\gamma^2}<1.$$

Hence, by direct computation, we obtain

$$\begin{aligned}
-u_\gamma^{-q}L_g u_\gamma &= v_\gamma^{1-q}(-u_0^{-q}L_g u_0)+a_n v_\gamma^{-q}u_0^{-q}(2\nabla_g v_\gamma\cdot\nabla_g u_0+u_0\Delta_g v_\gamma)\\
&= v_\gamma^{1-q}|f_0|+a_n\alpha B^2 c^{-l/B-1}\left\{2\left(\alpha-\frac{n-2}{2}\right)s^2\left(\frac{c}{\gamma-c}\right)\right.\\
&\quad \left.+(\alpha+1)s^2\left(\frac{c}{\gamma-c}\right)^2+c^2\left(1+\frac{s}{Bc}\Delta_g r\right)\left(\frac{c}{\gamma-c}\right)\right\}v_\gamma^{1-q}\\
&\leq |f_0|+a_n\alpha B^2\left[\frac{1}{2}\left(\alpha-\frac{n-2}{2}\right)+(\alpha+1)+\frac{1}{4}\left\{1+(n-1)\frac{A}{B}\right\}\right]c^{-l/B}\\
&\leq C_2|f_0|=-C_2 f_0
\end{aligned}$$

for some positive constant C_2 independent of γ. Therefore, if we set $u_{\gamma+}:=C_2^{(n-2)/4}u_\gamma$, then it follows that

$$L_g u_{\gamma+}\geq f_0 u_{\gamma+}^{\,q} \text{ when } r<\frac{1}{B}\cosh^{-1}\gamma,$$

namely, $u_{\gamma+}$ is a supersolution of the equation $(f_0, B_{B^{-1}\cosh^{-1}\gamma}(p))$. Since $u_{\gamma+}\to+\infty$ as $r\to B^{-1}\cosh^{-1}\gamma$, by Lemma 2.2, the maximal solution U_0 of the equation (f_0,M) satisfies

$$U_0\leq u_{\gamma+}=C_2^{(n-2)/4}u_\gamma \text{ in } B_{B^{-1}\cosh^{-1}\gamma}(p).$$

Hence we have

$$U_0\leq\lim_{\gamma\to+\infty}u_{\gamma+}=C_2^{(n-2)/4}u_0 \text{ on } M.$$

Combining this inequality and $U_0\geq u_0$, we obtain $U_0\sim u_0\sim e^{(n-2)lr/4}$ near infinity.

On the other hand, the function $\psi := c^{-(n-1)/2}$ satisfies

$$\begin{aligned} L_g\psi &= \frac{(n-1)^2}{n-2}B^2c^{-(n+3)/2}\left\{-(n+1)s^2+2c^2\left(1+\frac{s}{Bc}\Delta_g r\right)+\frac{(n-2)c^2S_g}{(n-1)^2B^2}\right\} \\ &\geq \frac{(n-1)^2}{n-2}B^2c^{-(n+3)/2}\{-(n+1)s^2+2nc^2-(n-1)c^2\} \\ &= \frac{(n-1)^2(n+1)}{n-2}\mathrm{B}^2c^{-(n+3)/2}>0 \text{ on } M, \end{aligned}$$

that is, the condition (P) holds. Since ψ/U_0 is bounded above, by Lemma 3.1, we see that the maximal solution U of the equation (f,M) satisfies the estimate $U\sim e^{(n-2)lr/4}$ near infinity. q.e.d.

REMARK 3.2. In Theorem II, even if we do not assume $A^2\leq(n-2)/n$, we can estimate, by the comparison theorem, that $r\Delta_g r\leq\beta(n-1)$ on M, where β is a positive number satisfying the inequality $-(r+\varepsilon_1r^3+\varepsilon_2r^\beta)''/(r+\varepsilon_1r^3+\varepsilon_2r^\beta)\leq-A^2/(r^2+\varepsilon^2)$ for some positive numbers ε_1 and ε_2. Therefore, as we mentioned in Section 1, we can prove the assertion of Theorem II under the condition (P) with $\psi\leq Cr^{(l-2)(n-2)/4}$ in place of the assumption $A^2\leq(n-2)/n$. Similarly, we can prove the assertion of Theorem III under the condition (P) with $\psi\leq Ce^{(n-2)lr/4}$ in place of the assumption $(A/B)^2\leq(n-1)^2/n(n-2)$.

4 The case of a domain of a compact manifold

In this section, we first study the asymptotic behavior of the maximal solution U in the case that (M,g) is a subdomain of a compact manifold, and prove Theorem IV. Secondaly, we prove Theorem V which involves a nonexistence result of the maximal solution.

Proof of Theorem IV. Write $r:=r_\Sigma$ for convenience. In the case $d<n-2$, choose a negative function f_0 on M such that $f_0\leq f$ on M, and $f_0\sim -r^l$ near Σ. From Kato-Nayatani [9, Theorems 2(b) and 3], if $\lambda_1(L_{\bar g})>0$, then the equation (f_0,M) posseses a solution u_0 satisfying $u_0\sim r^{-(l+2)(n-2)/4}$ near Σ. Also when $\lambda_1(L_{\bar g})\leq 0$, we can prove the same assertion by the same method after replacing the Green function of $L_{\bar g}$ by that of $-\Delta_{\bar g}$.

On the other hand, in the case $d\geq n-2$, from Aviles-McOwen [4], the equation $(-1,M)$ possesses a solution v satisfying $v\sim r^{-(n-2)/2}$ near

Σ. Set $u_0 := \gamma(v^{(l+2)/2} + \beta v)$ and $f_0 := u_0{}^{-q} L_g u_0$, where $\beta = \max\{-(l/2)\min_M \{v^{l/2}(1 + S_g v^{1-q})\}, 0\}$. By direct computation, we get

$$\begin{aligned} f_0 &= -\gamma^{1-q} v^{-q(l+2)/2}(1+\beta v^{-l/2})^{-q} \\ &\quad \times \left\{ \frac{l(l+2)(n-1)}{n-2} v^{(l-2)/2} |\nabla_g v|^2 + \frac{l+2}{2} v^{l/2+q} + \frac{l}{2} S_g v^{(l+2)/2} + \beta v^q \right\} \\ &\le -\gamma^{1-q}\{1+\beta(\min_M v)^{-l/2}\}^{-q} v^{-2l/(n-2)} < 0. \end{aligned}$$

Since $v^{-2l/(n-2)} \sim r^l$ near Σ, if we choose a small γ, then we have $f_0 \le f$.

Thus, in both cases, u_0 is a positive subsolution of the equation (f,M). Therefore, by Theorem I, the equation (f,M) possesses a maximal solution U which satisfies

$$U \ge u_0 \ge C_1 r^{-(l+2)(n-2)/4} \text{ near } \Sigma$$

for a positive constant C_1. Since l is nonnegative, we have

$$U^{q-1} \ge C_1{}^{q-1} r^{-(l+2)} \ge C_1{}^{q-1} r^{-2} \text{ near } \Sigma$$

from which it follows that $\tilde{g} = U^{q-1} g$ is complete.

To get the upper estimate of U, we use the lower estimate of U as above and the method of Aviles [1]. Choose a constant C_2 such that $f - S_g U^{1-q} \le -a_n C_2 r^l$ near Σ. Then there is a positive number δ such that

$$\Delta_g U \ge C_2 r^l U^q \text{ for } r_\Sigma < \delta.$$

For any $y \in B_{\delta/2}(\Sigma)$, let H_y be the Green function of $-\Delta_g$ in $B_{\delta/2}(y)$ centered at y. It is well-known that there are positive constants C_3, C_4 and C_5 independent of y such that

$$C_3 r_y{}^{2-n} \le H_y \le C_4 r_y{}^{2-n} \text{ and } |\nabla_g H_y| \le C_5 r_y{}^{1-n} \text{ for } r_y < \frac{\delta}{4}.$$

Let B_y be the connected component of the set $\{x \in M \mid h_y(x) > 2^{n-2} C_4 r(y)^{2-n}\}$ which contains the point y. Clearly $B_y \subset B_{r(y)/2}(y)$. Set $w_y := C_y\{r(y)^2/4 - (C_4/H_y)^{2/(n-2)}\}^{-(n-2)/2}$. Then it follows that

$$\Delta_g w_y \le C_6 r(y)^2 C_y{}^{1-q} w_y{}^q \text{ in } B_y,$$

where $C_6 = n C_4{}^{2/(n-2)} C_5{}^2 / 4(n-2) C_3{}^{2(n-1)/(n-2)}$. If we choose $C_y := \{2^l$

$C_6/C_2 r(y)^{l-2}\}^{(n-2)/4}$, then we have

$$\Delta_g w_y \le \frac{C_2}{2^l} r(y)^l w_y{}^q \text{ in } B_y.$$

On the other hand, it follows easily that

$$\Delta_g U \ge \frac{C_2}{2^l} r(y)^l U^q \text{ in } B_y.$$

Since $w_y(x) \to +\infty$ as $x \to \partial B_y$, and U is bounded above in B_y, by the maximal principle, we get $U \le w_y$ in B_y. Hence we conclude that

$$U(y) \le w_y(y) = 2^{n-2} C_y r(y)^{2-n} \le C_7 r(y)^{-(l+2)(n-2)/4} \text{ for } y \in B_{\delta/2}(\Sigma)$$

for a positive constant C_7. This completes the proof. q.e.d.

REMARK 4.1. In the case that $\lambda_1(L_{\bar{g}}) > 0$ and $d \le (n-2)/2$, we can improve the assumption "$f \ge -Cr_\Sigma{}^l$" as follows:

$$\int_M G(x,y)|f(y)| r_\Sigma(y)^{(d-n+2)(n+2)/(n-2)} dv_y \le C r_\Sigma(x)^{d-n+2} \text{ on } M$$

for some positive constant C. Actually, from Kato-Nayatani [9, Theorem 2′], we can construct a positive subsolution u_- of the equation (f,M) under this condition.

Proof of Theorem V. For any positive C^α-function h on Σ, define the function $G_{\Sigma,h}$ by $G_{\Sigma,h}(x) := \int_\Sigma G(x,y)h(y)d\sigma_y$. From Delanoë [7, Theorem 5] or Kato-Nayatani [9, Theorem 2(a) and Remark after Proposition 2], there is a positive number γ_0 such that, for any positive number $\mu \le \gamma_0$, the equation (f,M) possesses a solution v_μ satisfying $v_\mu / G_{\Sigma,h} \le \mu$ and $\lim_{x\to\Sigma}\{v_\mu(x)/G_{\Sigma,h}(x)\} = \mu$. In particular, $v_\mu{}^{q-1} g$ is complete for any such v_μ.

Let γ be the supremum of such γ_0's. Suppose $\gamma < +\infty$. Then, for any $\mu < \gamma$, the equation (f,M) possesses a solution v_μ as above. Since f is nonpositive, it follows from Lemma 2.2 that $\{v_\mu\}_{\mu<\gamma}$ is monotonically increasing and bounded above by $\gamma G_{\Sigma,h}$. Therefore, if we set $v_\gamma := \lim_{\mu\to\gamma} v_\mu$, then v_γ is a solution of the equation (f,M) with the same properties as above.

Put $w_\gamma := \gamma - v_\gamma / G_{\Sigma,h}$. Then it is clear that w_γ is nonnegative, $\max_M$

$w_\gamma<\gamma$, and $\lim_{x\to\Sigma}w_\gamma(x)=0$. Choose a positive number δ satisfying $1<\delta<\gamma/\max_M w_\gamma$, and set $v_{\varepsilon\gamma-}:=\varepsilon(\gamma-\delta w_\gamma)G_{\Sigma,h}$. Then we get, for any $\varepsilon\leq\delta^{(n-2)/4}$,

$$L_g v_{\varepsilon\gamma-}=\varepsilon\delta L_g v_\gamma=\varepsilon\delta f v_\gamma{}^q=f v_{\varepsilon\gamma-}{}^q\varepsilon^{1-q}\delta\left(\frac{\gamma-w_\gamma}{\gamma-\delta w_\gamma}\right)^q\leq f v_{\varepsilon\gamma-}{}^q \quad \text{on} \quad M,$$

namely, $v_{\varepsilon\gamma-}$ is a subsolution of the equation (f,M). On the other hand, if we set $v_{\varepsilon\gamma+}:=\varepsilon\gamma G_{\Sigma,h}$, then we have

$$L_g v_{\varepsilon\gamma+}=0\geq f v_{\varepsilon\gamma+}{}^q \quad \text{on} \quad M,$$

namely, $v_{\varepsilon\gamma+}$ is a supersolution of the equation (f,M). Since $v_{\varepsilon\gamma+}\geq v_{\varepsilon\gamma-}>0$, by the method of supersolutions and subsolutions, the equation (f,M) possesses a solution $v_{\varepsilon\gamma}$ satisfying $v_{\varepsilon\gamma+}\geq v_{\varepsilon\gamma}\geq v_{\varepsilon\gamma-}$. It is clear that $v_{\varepsilon\gamma}/G_{\Sigma,h}\leq\varepsilon\gamma$ and $\lim_{x\to\Sigma}\{v_{\varepsilon\gamma}/G_{\Sigma,h}(x)\}=\varepsilon\gamma$. This contradicts the definition of γ since $\delta^{(n-2)/4}\gamma>\gamma$.

Hence we conclude that $\gamma=+\infty$. In particular, the equation (f,M) possesses a solution $u_h:=v_1$ satisfying $u_h(x)/G_{\Sigma,h}(x)\to 1$ as $x\to\Sigma$. On the other hand, since h is a C^α-function, it follows easily that $G_{\Sigma,h}(x)/G_\Sigma(x)\to h(x_\infty)$ as $x\to x_\infty\in\Sigma$. Therefore we get

$$\frac{u_h(x)}{G_\Sigma(x)}=\frac{u_h(x)}{G_{\Sigma,h}(x)}\times\frac{G_{\Sigma,h}(x)}{G_\Sigma(x)}\to h(x_\infty) \text{ as } x\to x_\infty\in\Sigma.$$

The uniqueness is established by the same method as in Cheng-Ni [6, Theorem 3.1].

Now, if the equation (f,M) possesses a maximal solution U, then it holds that $U\geq u_h$ for any h. Hence we have $\lim_{x\to\Sigma}U(x)/G_\Sigma(x)=+\infty$. On the other hand, if $\operatorname{supp} f$ is compact, then it is clear that

$$\Phi(x):=\int_{\bar{M}}G(x,y)f(y)U(y)^q dy$$

is a bounded smooth function on $\bar{M}$ satisfying $L_{\bar{g}}\Phi=fU^q$. Set $\Psi:=(U-\Phi)/G_\Sigma$, and $\hat{g}:=G_\Sigma{}^{q-1}g$. Then we see that

$$\Delta_{\hat{g}}\Psi=-\frac{1}{a_n}L_{\hat{g}}\Psi=-\frac{1}{a_n}G_\Sigma{}^{-q}L_g(U-\Phi)=0 \text{ and } \lim_{x\to\Sigma}\Psi(x)=+\infty.$$

This contradicts the maximal principle. Therefore the equation (f,M) does not possess a maximal solution when $\operatorname{supp} f$ is compact. q.e.d.

More generally, it follows from the similar argument as in the final part of the proof of Theorem V that, if $\operatorname{supp} f$ is compact, then any solution u of the equation (f,M) satisfies $u(x)/G_\Sigma(x)-\Psi(x)\to 0$ as $x\to\Sigma$ for some positive function Ψ such that $\Delta_{\hat{g}}\Psi=0$. On the other hand, if Σ is a point i.e. $\Sigma=\{p\}$, then the minimal positive Green function of $-\Delta_{\hat{g}}$ is $G(x,y)/G(x,p)G(y,p)$. Since

$$\frac{G(x,y)/G(x,p)G(y,p)}{G(x',y)/G(x',p)G(y,p)}\to 1 \quad \text{as} \quad y\to p$$

for any x and $x'\in M$, the Martin boundary of $(M,\hat{g})$ is a point, and hence any positive harmonic function on $(M,\hat{g})$ must be a constant. Thus we have the following result which involves a higher dimensional version of Cheng-Ni [5, Theorem III].

Proposition 4.2. *Let $(\bar{M},\bar{g})$ be a compact Riemannian manifold ($n=\dim M\geq 3$) with $\lambda_1(L_{\bar{g}})>0$, p a point in $\bar{M}$, and $(M,g):=(\bar{M}\setminus\{p\}, \bar{g}|_{\bar{M}\setminus\{p\}})$. Let f be a nonpositive smooth function on M. If $\operatorname{supp} f$ is compact, then an arbitrary solution u of the equation (f,M) coincides with u_h (as in Theorem V) for some positive number h.*

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