The non-perturbative analyses of lower dimensional non-linear sigma models

低次元非線形シグマ模型の非摂動論的解析

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To my family and friends

Abstract

The Wilsonian renormalization group (WRG) method is suitable to formulate the quantum field theories nonperturbatively. The Wilsonian effective action, obtained by integrating out field variables with higher momenta, is a complicated functional of field variables and their derivatives. In order to study the low energy behavior, it is convenient to expand it into a power series of momenta. The lowest approximation of the WRG equation, called the local potential approximation, retains terms without derivatives and is well investigated. In this thesis, we study the next-to-leading order approximation of the WRG equation in the derivative expansion, namely, we discuss the Wilsonian effective action quadratic in momenta, which may be called the non-linear sigma model approximation. For simplicity, we impose the supersymmetry to forbid the appearance of the local potential terms. We focus on the supersymmetric nonlinear sigma model in two- or three-dimensions.

In non-linear sigma models, the coefficient of the kinetic term depends on field variables, and is interpreted as the metric tensor of the target manifold where field variables take values. The two-dimensional nonlinear sigma model is perturbatively renormalizable. The perturbative one-loop β function is proportional to the Ricci tensor of the target space. However, the nonperturbative β function obtained from the WRG method has the additional term which is proportional to the anomalous dimension of the scalar field. We construct the fixed point theory defined by the zero point of the nonperturbative β function. This theory has one free parameter, a, corresponding to the anomalous dimension of the scalar field. When the target space is of one complex dimension, the target space of the fixed point theory is a semi-infinite cigar, the two-dimensional black-hole proposed by Witten.

Three-dimensional nonlinear sigma models are not renormalizable within the perturbative method. We argue that some supersymmetric nonlinear sigma models are renormalizable using the nonperturbative WRG method. In the WRG approach, the renormalizability is equivalent to the existence of ultra-violet (UV) fixed point. We also reconfirm the existence of the UV fixed point by using the other nonperturbative analysis; the large-N expansion. Furthermore, we construct the novel conformal field theory defined as a fixed point theory of the nonperturbative WRG β function. This theory has one free parameter and interpolates the both UV and IR fixed points of the CP^N model.

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Contents

Chapter 1

Introduction

Quantum field theories have divergence because local fields defined at a point have infinite degrees of freedom. At first, such divergence appeared as the self-energy of the electron in the quantum electrodynamics (QED). Tomonaga, Feynman, Schwinger and Dyson invented the renormalization procedures for QED in the perturbative method. By expanding physical quantities in the power series of the coupling constant, they succeeded in removing all the divergences for perturbatively renormalizable theories, for example QED, φ^4 theory, yukawa theory and so on. This method is, however, applicable only to perturbatively renormalizable theories.

Wilson's idea, called the Wilsonian renormalization group (WRG) method is not restricted to the perturbation theory. In this formulation, all Green's functions are defined by the Euclidean path integral. When we are interested in the large distance behavior larger than a specified length scale, it is convenient to integrate over the fields with the wave length shorter than the length scale we are interested in. In this way, we can define the Wilsonian effective action which depends only on the fields with wave length larger than a specified length scale, the inverse of it is called the cutoff. The WRG equation describes the variation of the Wilsonian effective action when the cutoff scale is changed slightly. Since we integrate over the fileds with shorter wave length, the Wilsonian effective action is in general very complicated functional of fileds. The WRG equation for the most general effective action is simple and exact, but in practice we have to use some kind of approximation to deduce a tractable equation by throwing away irrelevant terms at the large distance. The simplest approximation, called the local potential approximation, was investigated in [1, 3, 4]. A nontrivial scalar theory has been found in three dimensions around Wilson-Fisher fixed point in the local potential approximation. This fixed point exists in any dimensions less than four, and the four dimensions is critical in scalar field theories. In this thesis, we consider the Wilsonian effective action quadratic in momenta in two- and three-dimensions. In general, such second derivative interaction of the scalar fields is described by the non-linear sigma model (NLσM) in which the coefficient of kinetic term depends on field variables. These field dependent coefficients of the kinetic terms can be interpreted as the metric tensors of the target manifolds where field variables take their values.

Furthermore, we consider the supersymmetric theory in this thesis to forbid the appearance of the local potential term for simplicity. Supersymmetry often plays an important role. For example, in supersymmetric theories, quantum corrections to the vacuum energies of scalar and spinor fields cancelle to each other, and the quadratic divergence of the Higgs mass disappears. In two- or three-dimensional field theories we are considering, the simplest supersymmetry is realized between a real scalar and a real (Majorana) fermion with two-components, and is called $\mathcal{N} = 1$ supersymmetry. An extended supersymmetry, called $\mathcal{N} = 2$ supersymmetry, is realized between a complex scalar and a complex (Dirac) fermion with two components. A marvelous property of certain class of NLσMs with $\mathcal{N} = 2$ supersymmetry is the absence of the local potential term. This class of NLσMs is characterized by the so-called Kähler potential whose second derivative is the metric tensor of the target manifold. Therefore, we concentrate on the renormalization of NLσMs with $\mathcal{N} = 2$ supersymmetry.

First, we consider 2-dimensional $\mathcal{N} = 2$ supersymmetric NLσM.

NLσMs in two dimensions are interesting for several reasons. First, they help us to understand various non-perturbative phenomena in four-dimensional gauge theories, such as confinement and dynamical mass generation [5, 6, 7]. Second, they provide a description of superstrings propagating in a curved space-time. In the latter case, the consistency of strings requires the $\mathcal{N} = 2$ superconformal symmetry of the NLσMs. Because $\mathcal{N} = 2$ supersymmetry and the scale invariance imply $\mathcal{N} = 2$ superconformal symmetry, these NLσMs have to be scale invariant. In quantum field theories, scale invariance, suffering from an anomaly due to divergent renormalization effects, is realized only at the fixed points of the renormalization group equation. Since field theories at the fixed points also describe phase transition, it is important to study these fixed-point theories of $\mathcal{N} = 2$ supersymmetric NLσMs. Then we construct the conformal fields theories realized at the fixed points of the nonperturbative β function.

Next, we consider 3-dimensional $\mathcal{N} = 2$ supersymmetric NLσM. The three dimensional NLσM is nonrenormalizable within the perturbative method. Therefore, we have to use nonperturbative methods to study the renormalizability of NLσMs. One of such nonperturbative methods is the large-N expansion, which has been applied to some examples of NLσMs [34]. For example, the $\mathbf{U}(N)$ invariant NLσM, in which field variables take values in the complex projective space CP^{N-1}, is renormalizable at the leading and the next-to-leading orders in the large-N expansion [35]. In fact, the next-to-leading order contribution to the β function in the $\mathrm{C}P^{N-1}$ model with $\mathcal{N} = 2$ supersymmetry vanishes in the $1/N$ expansion [35].

The Wilsonian renormalization group (WRG) offers another powerful tool suitable for nonperturbative studies. In the WRG approach, the renormalizability of NLσMs is equivalent to the existence of a nontrivial continuum limit, $\Lambda \to \infty$. When the ultraviolet (UV) cutoff Λ tends to infinity, we have to finetune the coupling constant to the critical value at the UV fixed point, so as to keep the observable quantities finite. Therefore, it is important to show the existence of the UV fixed point without using the perturbation theory.

This thesis is organized as follows: In Chapter 2, we give a brief review of the Wilsonian renormalization group and the approximation method. In Chapter 3, we derive the renormalization group equation for NLσM. In Chapter 4, we discuss the fixed points of the 2-dimensional $\mathcal{N} = 2$ supersymmetric NLσM. We compare the results with the perturbative ones. In Chapter 5, we investigate 3-dimensional supersymmetric NLσM using the non-perturbative Wilsonian renormalization group. In particular, we discuss the renormalizability of some 3-dimensional sigma models. We also construct the fixed point theory using nonperturbative β function. In Chapter 6, we investigate some 3-dimensional NLσM using the large-N expansion which is another nonperturbative method. We compare the results of both methods, to discuss the reliability of the approximation.

Chapter 2

The Wilsonian Renormalization Group

The WRG equation describes the variation of the Wilsonian effective action when the cutoff scale is changed [1, 2, 3, 4]. The renormalization group equation shows the relations among the infinite number of field theories and it is exact if we use the most general action. In practice, we have to introduce some kind of truncation of the functional differential equation describing the flow of an infinite number of coupling constants. The relevant truncation in the infrared region relies on the derivative expansion of the effective action [3].

In this Chapter, we give a brief review of the Wilsonian renormakization group for general Wilsonian effective action and the approaximation method.

2.1 Review of Wilsonian Renormalization Group (WRG)

Consider a general Euclidean quantum field theory with fields Ω_i in D-dimensions. The Euclidean path integral is

$$Z = \int [D\Omega_i] \exp\left[-S[\Omega]\right]. \tag{2.1.1}$$

Here $S[\Omega]$ is the most generic Euclidean action, which has the form

$$S[\Omega] \equiv \sum_n \frac{1}{n!} \int_{p_1} \cdots \int_{p_n} \hat{\delta}^{(D)}(p_1 + \cdots + p_n) g_{i_1,\cdots,i_n}(p_1, \cdots, p_n) \Omega_{i_1}(p_1) \cdots \Omega_{i_n}(p_n), \tag{2.1.2}$$

where g is a coupling constant, and

$$\hat{\delta}^{(D)} \equiv (2\pi)^D \delta^{(D)} \tag{2.1.3}$$

is the D-dimensional delta-function. The Fourier transformation of Ω is defined by

$$\Omega(x) \equiv \int_p \Omega(p) e^{-ipx}, \tag{2.1.4}$$

in which

$$\int_p \equiv \int \frac{d^D p}{(2\pi)^D}. \tag{2.1.5}$$

The basic idea in defining the effective action is as follows. We divide all fields Ω_i into two group, high frequency modes ($\Omega_{i>}$) and low frequency modes ($\Omega_{i<}$). After the high frequency modes are integrated out, the Wilsonian effective action (S_{eff}) is obtained as

$$\begin{aligned} Z &= \int [D\Omega_i] \exp\left[-S[\Omega_i]\right] \\ &= \int [D\Omega_{i>}][D\Omega_{i<}] \exp\left[-S[\Omega_{i<}, \Omega_{i>}]\right] \\ &\equiv \int [D\Omega_{i<}] \exp\left[-S_{eff}[\Omega_{i<}]\right]. \end{aligned} \tag{2.1.6}$$

Our aim is to determine the flow of S_{eff} by continuously changing the cutoff Λ, which represents the boundary between higher and lower momentum modes. To carry this out, we change the cutoff Λ infinitesimally to $\Lambda(\delta t) = \Lambda e^{-\delta t}$. Then, in the effective action at Λ, we again divide all fields into higher modes$[\Omega_s]$ and lower modes. The fields $[\Omega_s]$ that are nonzero value only in the momentum shell between $\Lambda(\delta t)$ and Λ are integrated out, and we thereby obtain the effective action for the new cutoff $\Lambda(\delta t)$. Hereafter, we write S_{eff} simply as S, and we assume that Z is cutoff independent:

$$\begin{aligned} Z &= \int [D\Omega]_{\Lambda(\delta t)}[D\Omega_s] \exp\left[-S[\Omega+\Omega_s;\Lambda]\right] \\ &= \int [D\Omega]_{\Lambda(\delta t)}[D\Omega_s] \exp\left[-\left(S[\Omega;\Lambda]| + \frac{\delta S}{\delta \Omega_i}\Omega_s^i + \frac{1}{2}\Omega_s^i \frac{\delta^2 S}{\delta\Omega^i\delta\Omega^j}\Omega_s^j + O(\Omega_s^3)\right)\right] \\ &= \int [D\Omega]_{\Lambda(\delta t)} \exp\left[-\left(S[\Omega;\Lambda]| + \frac{1}{2}\int_{p'} tr\ln\left(\frac{\delta^2 S}{\delta\Omega^i\delta\Omega^j}\right)\right.\right. \\ &\qquad \left.\left. -\frac{1}{2}\int_{p'}\int_{q'} \frac{\delta S}{\delta\Omega^i}\left(\frac{\delta^2 S}{\delta\Omega^i\delta\Omega^j}\right)^{-1}\frac{\delta S}{\delta\Omega^j} + O((\delta t)^2)\right)\right] \end{aligned} \tag{2.1.7}$$

$$\equiv \int [D\Omega]_{\Lambda(\delta t)} \exp\left[-S[\Omega;\Lambda(\delta t)]\right]. \tag{2.1.8}$$

Here,

$$\int_{p'} = \int \frac{d\Omega_D}{(2\pi)^D}\int_{\Lambda(\delta t)}^{\Lambda} dp' \sim O(\delta t), \tag{2.1.9}$$

where $\int d\Omega_D$ is the surface integral of the D-dimensional unit sphere. We can drop terms of order $O(\Omega_s^3)$, because such terms vanish as $\delta t \to 0$. In Eq.(2.1.7), the action $S[\Omega;\Lambda]|$ is obtained from $S[\Omega;\Lambda]$ by dropping all fields with momentum above $\Lambda(\delta t)$ giving

$$S[\Omega;\Lambda]| = \sum_n \frac{1}{n!}\int_{p_1}\cdots\int_{p_n}\hat{\delta}(p_1+\cdots+p_n)g(\Lambda)\Omega_{i_1}\cdots\Omega_{i_n}. \tag{2.1.10}$$

Then, the action $S[\Omega;\Lambda(\delta t)]$ in Eq.(2.1.8) depends on the fields with the momenta below the lower cutoff, and the coupling constants that are defined at the lower cutoff $\Lambda(\delta t)$:

$$S[\Omega;\Lambda(\delta t)] = \sum_n \frac{1}{n!}\int_{p_1}\cdots\int_{p_n}\hat{\delta}(p_1+\cdots+p_n)g(\Lambda(\delta t))\Omega_{i_1}\cdots\Omega_{i_n}. \tag{2.1.11}$$

The difference between Eq.(2.1.10) and Eq.(2.1.11) is represented by the difference in the coupling constants when the cutoff is changed:

$$S[\Omega;\Lambda(\delta t)] - S[\Omega;\Lambda]|$$

$$
\begin{aligned}
&= \sum_n \frac{1}{n!} \int_{p_1} \cdots \int_{p_n} \hat{\delta}(p_1 + \cdots + p_n) \left[g(\Lambda(\delta t)) - g(\Lambda) \right] \Omega_{i_1} \cdots \Omega_{i_n} \\
&= -\delta t \sum_n \frac{1}{n!} \int_{p_1} \cdots \int_{p_n} \hat{\delta}(p_1 + \cdots + p_n) \left(\Lambda \frac{\partial}{\partial \Lambda} g(\Lambda) \right) \Omega_{i_1}(p_1) \cdots \Omega_{i_n}(p_n) \\
&= \frac{1}{2} \int_{p'} tr \ln \left(\frac{\delta^2 S}{\delta \Omega^i \delta \Omega^j} \right) - \frac{1}{2} \int_{p'} \int_{q'} \frac{\delta S}{\delta \Omega^i} \left(\frac{\delta^2 S}{\delta \Omega^i \delta \Omega^j} \right)^{-1} \frac{\delta S}{\delta \Omega^j}. \qquad (2.1.12)
\end{aligned}
$$

If the coupling constants depend on the momenta, this cutoff dependence of the coupling constants is given by

$$
\Lambda \frac{\partial}{\partial \Lambda} g(\Lambda) = \Lambda \frac{d}{d\Lambda} g(\Lambda) - \sum_i p_i^\mu \frac{\partial}{\partial p_i^\mu} g(\Lambda). \qquad (2.1.13)
$$

In order to derive a differential equation for S, we transform all fields and coupling constants into dimensionless quantities. The mass dimension of a coupling constant g is given by

$$
\dim[g] \quad = D - \textstyle\sum_{\Omega_i} (d_{\Omega_i} + \gamma_{\Omega_i}), \qquad (2.1.14)
$$

where D denotes the space-time dimension, and d_{Ω_i} and γ_{Ω_i} are the canonical and anomalous dimensions of Ω_i. Because the cutoff dependence of the coupling constant can be written

$$
\Lambda \frac{d}{d\Lambda} g = \Lambda \frac{\partial}{\partial \Lambda} g - \dim[g] \cdot g + \sum_i p_i^\mu \frac{\partial}{\partial p_i^\mu} g(\Lambda), \qquad (2.1.15)
$$

the WRG equation for the effective action S is

$$
\begin{aligned}
\Lambda \frac{d}{d\Lambda} S &= -\frac{1}{\delta t} \left[S[\Omega; \Lambda(\delta t)] - S[\Omega; \Lambda] \right]] \\
&\quad - \left[D - \sum_{\Omega_i} \int_p \hat{\Omega}_i(p) \left(d_{\Omega_i} + \gamma_{\Omega_i} + \hat{p}^\mu \frac{\partial}{\partial \hat{p}^\mu} \right) \frac{\delta}{\delta \hat{\Omega}_i(p)} \right] \hat{S} \\
&\equiv -\frac{d}{dt} \hat{S}, \qquad (2.1.16)
\end{aligned}
$$

where the caret indicates dimensionless quantities.

Using Eq.(2.1.12), we obtain the WRG equation for the dimensionless action:

$$
\begin{aligned}
\frac{d}{dt} S[\Omega; t] &= \frac{1}{2\delta t} \int_{p'} tr \ln \left(\frac{\delta^2 S}{\delta \Omega^i \delta \Omega^j} \right) \\
&\quad - \frac{1}{2\delta t} \int_{p'} \int_{q'} \frac{\delta S}{\delta \Omega^i(p')} \left(\frac{\delta^2 S}{\delta \Omega^i(p') \delta \Omega^j(q')} \right)^{-1} \frac{\delta S}{\delta \Omega^j(q')} \\
&\quad + \left[D - \sum_{\Omega_i} \int_p \hat{\Omega}_i(p) \left(d_{\Omega_i} + \gamma_{\Omega_i} + \hat{p}^\mu \frac{\partial}{\partial \hat{p}^\mu} \right) \frac{\delta}{\delta \hat{\Omega}_i(p)} \right] \hat{S}.
\end{aligned}
\qquad (2.1.17)
$$

Here, the terms

$$
\int_{p'} tr \ln \left(\frac{\delta^2 S}{\delta \Omega^i \delta \Omega^j} \right) \qquad (2.1.18)
$$

and

$$
\int_{p'} \int_{q'} \frac{\delta S}{\delta \Omega^i(p')} \left(\frac{\delta^2 S}{\delta \Omega^i(p') \delta \Omega^j(q')} \right)^{-1} \frac{\delta S}{\delta \Omega^j(q')} \qquad (2.1.19)
$$

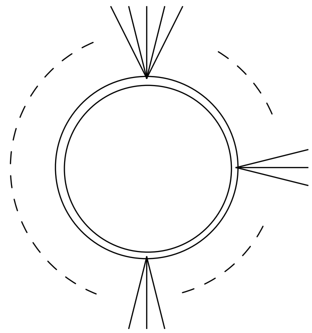

Figure 2.1: One-loop diagram.

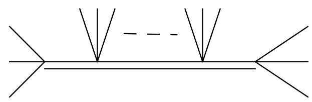

Figure 2.2: Dumbbell diagram.

correspond to the one-loop and dumbbell diagrams, respectively, displayed in Fig. 1 and 2.

In these figures, the single and double lines represent the lower and higher modes, respectively, and the latter are integrated out in Eq.(2.1.17). In the limit $\delta t \to 0$, the contribution of the higher-order diagrams disappears.

If there are fermionic fields in addition to bosonic fields, the WRG equation (2.1.17) can be rewritten

$$\begin{aligned} \frac{d}{dt}S[\Omega;t] &= \frac{1}{2\delta t}\int_{p'} str\ln\left(\frac{\delta^2 S}{\delta\Omega^i\delta\Omega^j}\right) \\ &\quad -\frac{1}{2\delta t}\int_{p'}\int_{q'}(-1)^F\frac{\delta S}{\delta\Omega^i(p')}\left(\frac{\delta^2 S}{\delta\Omega^i(p')\delta\Omega^j(q')}\right)^{-1}\frac{\delta S}{\delta\Omega^j(q')} \\ &\quad +\left[D-\sum_{\Omega^i}\int_p \hat{\Omega}^i(p)(d_{\Omega^i}+\gamma_{\Omega^i}+\hat{p}^\mu\frac{\partial}{\partial\hat{p}^\mu})\frac{\delta}{\delta\hat{\Omega}^i(p)}\right]\hat{S}, \end{aligned} \tag{2.1.20}$$

where

$$\begin{aligned} str\ln\left(\frac{\delta^2 S}{\delta\Omega^i\delta\Omega^j}\right) &= str\ln M_{i\bar{j}} \\ &= str\ln\begin{pmatrix} M_{BB} & M_{BF} \\ M_{FB} & M_{FF}\end{pmatrix} \\ &= tr\ln M_{BB} - tr\ln N_{FF}, \end{aligned} \tag{2.1.21}$$

with

$$N_{FF} = M_{FF} - M_{FB}M_{BB}^{-1}M_{BF}. \tag{2.1.22}$$

2.2 The approximation method

The Wilsonian effective action $S[\Omega;t]$ in Eq.(2.1.17) includes an infinite number of coupling constants $g_{i_1,\cdots,i_n}$, and Eq.(2.1.17) gives as infinite number of differential equations involving them. To make these equations more tractable, we usually expand the effective action in powers of derivatives and retain the first few terms. We often introduce a symmetry (supersymmetry, gauge symmetry, Z_2 symmetry, etc) to further decrease the number of independent coupling constants.

Consider, for example, a single real scalar field theory that is invariant under $\varphi \to -\varphi$ (Z_2 symmetry). In this case, we can expand the most generic effective action as

$$S[\varphi] = \int d^D x \left[V[\varphi] + \frac{1}{2} K[\varphi](\partial_\mu \varphi)^2 + H_1[\varphi](\partial_\mu \varphi)^4 + H_2[\varphi](\partial_\mu \partial^\mu \varphi)^2 + \cdots \right],$$

where $V[\varphi], K[\varphi], H_1[\varphi], \cdots$ are functions of φ. To second order in derivatives, this action becomes

$$S[\varphi] = \int d^D x \left[V[\varphi] + \frac{1}{2} K[\varphi](\partial_\mu \varphi)^2 \right]. \tag{2.2.1}$$

Substituting this $S[\varphi]$ into Eq.(2.1.17) and expanding the right-hand side of Eq.(2.1.17) up to $O(\partial^2)$, we obtain two differential equations for $V[\varphi]$ and $K[\varphi]$.

In scalar field theories, often only the local potential term $V[\varphi]$ is retained. In this thesis, we derive the Wilsonian renormalization group equation to the first nontrivial order of the derivative expansion in the $\mathcal{N} = 2$ SNLσM. Because of the reparametrization invariance of the target manifold, the Lagrangian of the nonlinear sigma model is proportional to the metric of this manifold, and the potential term is absent. Therefore, the first nontrivial order of the derivative expansion is already second order in the derivatives [8] [9].

Chapter 3

The Flow Equation for Supersymmetric Non-linear Sigma Models

In this chapter, we derive the supersymmetric Wilsonian renormalization group equation for 2- and 3-dimensional $\mathcal{N}=2$ supersymmetric non-linear sigma models. These supersymmetric non-linear sigma models are obtained by dimensional reduction of 4-dimennsional $\mathcal{N}=1$ supersymmetric theory. To derive the supersymmetric flow equation, we use the property of the Kähler manifold which is the target spaces of these non-linear sigma models.

3.1 Supersymmetric nonlinear sigma model

For the $\mathcal{N}=2$ supersymmetric nonlinear sigma model (SNLσM) in two dimensions ($D=2$), the action is determined by the Kähler potential $K[\Phi,\Phi^\dagger]$:

$$S = \int dV K[\Phi,\Phi^\dagger], \tag{3.1.1}$$

where

$$\int dV \equiv \int d^2x d^2\theta d^2\bar{\theta}.$$

This is also true for the $\mathcal{N}=2$ SNLσM in $D=3$ and the $\mathcal{N}=1$ SNLσM in $D=4$. For this reason, we replace the number of space-time dimensions, 2, by D. Here, Φ is a chiral superfield and can be written in terms of component fields as

$$\begin{aligned}
\Phi^i(y) &= \varphi^i(y) + \sqrt{2}\theta\psi^i(y) + \theta\theta F^i(y) \\
&= \varphi^i(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\varphi^i(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^\mu\partial_\mu\varphi^i(x) \\
&\quad + \sqrt{2}\theta\psi^i(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi^i(x)\sigma^\mu\bar{\theta} + \theta\theta F^i(x) &(3.1.2)\\
&\equiv \varphi^i(x) + \delta\Phi^i(x), &(3.1.3)
\end{aligned}$$

where

$$y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}. \tag{3.1.4}$$

Using (3.1.3), we expand the action (3.1.1) around the scalar fields (φ, φ^*). It is sufficient to expand this action to $O((\delta\Phi)^4)$, because $\delta\Phi$ (and $\delta\Phi^\dagger$) contains at least one factor of θ. Integrating over the Grassmann numbers (θ and $\bar{\theta}$), we obtain the SNLσM action written in terms of the component fields:

$$\begin{aligned} S &= \int d^D x \Bigg[g_{n\bar{m}} \left(\partial^\mu \varphi^n \partial_\mu \varphi^{*\bar{m}} + \frac{i}{2} \bar{\psi}^{\bar{m}} \sigma^\mu (D_\mu \psi)^n + \frac{i}{2} \psi^n \bar{\sigma}^\mu (D_\mu \bar{\psi})^{\bar{m}} + \bar{F}^{\bar{m}} F^n \right) \\ &\quad - \frac{1}{2} K_{,nm\bar{l}} \bar{F}^{\bar{l}} \psi^n \psi^m - \frac{1}{2} K_{,n\bar{m}\bar{l}} F^n \bar{\psi}^{\bar{m}} \bar{\psi}^{\bar{l}} + \frac{1}{4} K_{,nm\bar{k}\bar{l}} (\bar{\psi}^{\bar{k}} \bar{\psi}^{\bar{l}})(\psi^n \psi^m) \Bigg]. \end{aligned} \tag{3.1.5}$$

First, we consider the bosonic part of the action (3.1.5). The loop correction term in the WRG Eq.(2.1.17) cannot be written in covariant form in general. We use the Kähler normal coordinates (KNC) expansion of the action to obtain a covariant expression for the loop correction [10] [11].

The Kähler normal coordinates (ω, ω^*) are defined by the following condition:

$$K_{,\bar{j}i_1\cdots i_n}(\omega, \omega^*)|_0 = 0. \tag{3.1.6}$$

Here, the index "0" indicates that the left-hand side is evaluated at the origin of the KNC, which is the expansion point.

The Riemann normal coordinates (RNC) are well known in the context of perturbative calculations [24] [25]. All geodesics in the RNC become straight lines. The RNC in Kähler manifolds, however, are not chiral, and the coordinate transformation from the holomorphic coordinates to the RNC is not holomorphic. In contrast, in the KNC, geodesics cannot become straight lines, but the KNC preserves the holomorphy.

Let us decompose some arbitrary holomorphic coordinates (z^i) into a background field φ^i and small fluctuations π^i around it: $z^i = \varphi^i + \pi^i$. The coordinate transformation from the coordinates z^i to the KNC ω^i is given by

$$\omega^i \;=\; \pi^i + \sum_{n=2}^{\infty} \frac{1}{n!} [g^{i\bar{j}} K_{,\bar{j}i_1\cdots i_n}(z^i, z^{*\bar{i}})]_\varphi \pi^{i_1} \cdots \pi^{i_n}. \tag{3.1.7}$$

These KNC ω^i are known to transforme as holomorphic tangent vectors. Because they have well-defined transformation properties, we can use a convenient coordinate system for our calculation.

In the D-dimensional sigma model action,

$$S_{scalar} = \int d^D x g_{i\bar{j}}(\varphi, \varphi^*) \partial^\mu \varphi^i \partial_\mu \varphi^{*\bar{j}}, \tag{3.1.8}$$

the background fields φ^i are regarded as the lower frequency modes and the fluctuations π^i as the higher frequency modes. The Euclidean path integral Z in Eq.(2.1.7) is written

$$Z = \int [D\varphi][D\varphi^*][D\pi][D\pi^*] \exp\Big[- S[\varphi + \pi, \varphi^* + \pi^*] \Big]. \tag{3.1.9}$$

To calculate this path integral, we use the KNC system with the ω^i defined by Eq.(3.1.7). We regard the background field φ^i as the origin of the KNC and transform the fluctuations π^i to the KNC fields $\bar{\pi}^i$ according to

$$Z \xrightarrow{KNC} \int [D\varphi][D\varphi^*][D\bar{\pi}][D\bar{\pi}^*] \exp\Big[- \bar{S}[\varphi + \bar{\pi}, \varphi^* + \bar{\pi}^*] \Big], \tag{3.1.10}$$

where the bars indicate quantities in the KNC system, and $\bar{S}$ is given by

$$\bar{S}[\varphi + \bar{\pi}, \varphi^* + \bar{\pi}^*] = \int d^D x \bar{g}_{i\bar{j}}(\varphi + \bar{\pi}, \varphi^* + \bar{\pi}^*) \partial_\mu (\varphi + \bar{\pi})^i \partial^\mu (\varphi^* + \bar{\pi}^*)^{\bar{j}}. \tag{3.1.11}$$

The KNC fields $\bar{\pi}^i(x)$ can be expanded in the tangent vectors $\bar{\omega}^i(x)$ by solving Eq.(3.1.7) for π^i and using the KNC system

$$\bar{\pi}^i(x) = \bar{\omega}^i(x) - \frac{1}{2}\bar{\Gamma}^i{}_{k_1k_2}|_\varphi \bar{\omega}^{k_1}(x)\bar{\omega}^{k_2}(x) + O(\bar{\omega}^3). \tag{3.1.12}$$

When no space-time derivatives act on $\bar{\pi}^i$, the KNC fields $\bar{\pi}^i$ coincide with the tangent vector fields: $\bar{\pi}^i(x) = \bar{\omega}^i(x)$. Substituting Eq.(3.1.12) into Eq.(3.1.11) and using the property of the KNC (3.1.6), we obtain the expansion of the sigma model action to second order in the fluctuations of the KNC:

$$\begin{aligned}\bar{S}[\varphi+\bar{\pi}, \varphi^*+\bar{\pi}^*] = \int d^Dx \bar{g}_{i\bar{j}}|_\varphi &\left(\partial_\mu\varphi^i\partial^\mu\varphi^{*\bar{j}} + \partial_\mu\varphi^i\partial^\mu\bar{\omega}^{*\bar{j}} + \partial_\mu\bar{\omega}^i\partial^\mu\varphi^{*\bar{j}} + \partial_\mu\bar{\omega}^i\partial^\mu\bar{\omega}^{*\bar{j}}\right)\\ &+\bar{g}_{i\bar{j},k\bar{l}}|_\varphi \left(\partial_\mu\varphi^i\partial^\mu\varphi^{*\bar{j}}\bar{\omega}^k\bar{\omega}^{*\bar{l}} + \frac{1}{2}\partial_\mu\varphi^i\partial^\mu\varphi^k\bar{\omega}^{*\bar{j}}\bar{\omega}^{*\bar{l}} + \frac{1}{2}\partial_\mu\varphi^{*\bar{j}}\partial^\mu\varphi^{*\bar{l}}\bar{\omega}^i\bar{\omega}^k\right)\\ &+O(\bar{\omega}^3).\end{aligned} \tag{3.1.13}$$

From the above expansion, for the matrix in Eq.(2.1.17) we can read off

$$\left(\frac{\delta^2 S}{\delta\Omega^i(p')\delta\Omega^j(q')}\right) = M_{ij}(q', p'), \tag{3.1.14}$$

whose trace corresponds to the one loop correction term. These matrix elements are in momentum space

$$M(q', p') = \begin{pmatrix} M_{k\bar{l}} & M_{\bar{k}\bar{l}} \\ M_{kl} & M_{\bar{k}l} \end{pmatrix}, \tag{3.1.15}$$

where

$$\begin{aligned} M_{k\bar{l}} &\equiv \frac{\delta^2\bar{S}}{\delta\varphi^{*\bar{l}}(q')\delta\varphi^k(p')}\\ &= \int_{p',q',P}(-p'\cdot q')\bar{g}_{k\bar{l}}(P)\hat{\delta}(p'+q'+P)\\ &\quad+\int_{p',q',p,q,P''}(-p\cdot q)\bar{g}_{i\bar{j},k\bar{l}}(P''+p'+q')\varphi^i(p)\varphi^{*\bar{j}}\hat{\delta}(p'+q'+p+q+P''), \end{aligned} \tag{3.1.16}$$

$$M_{\bar{k}\bar{l}} = \int_{p',q',p,q,P''}(-p\cdot q)\bar{g}_{i\bar{k},j\bar{l}}(P''+p'+q')\varphi^i(p)\varphi^j(q)\hat{\delta}(p'+q'+p+q+P''), \tag{3.1.17}$$

$$M_{kl} = \int_{p',q',p,q,P''}(-p\cdot q)\bar{g}_{k\bar{i},l\bar{j}}(P''+p'+q')\varphi^{*\bar{i}}(p)\varphi^{*\bar{j}}(q)\hat{\delta}(p'+q'+p+q+P''), \tag{3.1.18}$$

$$\begin{aligned} M_{\bar{k}l} &= \int_{p',q',P}(-p'\cdot q')\bar{g}_{l\bar{k}}(P)\hat{\delta}(p'+q'+P)\\ &\quad+\int_{p',q',p,q,P''}(-p\cdot q)\bar{g}_{i\bar{j},l\bar{k}}(P''+p'+q')\varphi^i(p)\varphi^{*\bar{j}}\hat{\delta}(p'+q'+p+q+P''). \end{aligned} \tag{3.1.19}$$

We write the (1,1) element (3.1.16) as

$$M_{k\bar{l}} \equiv \langle\bar{l}; -q'|\hat{M}_1|k; p'\rangle, \tag{3.1.20}$$

to define the operator $\hat{M}_1$. Here, the states $|k;p\rangle$ are defined by

$$\langle k;p|\bar{l};q\rangle = \langle\bar{l};p|k;q\rangle = \hat{\delta}(q-p)\delta_{k\bar{l}}, \tag{3.1.21}$$

$$\int_p \sum_{k\bar{l}} \Big[|k;p\rangle\delta^{\bar{l}k}\langle\bar{l};p| + |\bar{l};p\rangle\delta^{\bar{l}k}\langle k;p| \Big] = 1. \tag{3.1.22}$$

From Eq.(3.1.20), the explicit form of the operator $\hat{M}_1$ is

$$(\hat{M}_1)_{k\bar{l}} = \hat{p}\bar{g}_{k\bar{l}}\hat{p} + \bar{g}_{i\bar{j},k\bar{l}}\partial_\mu\varphi^i\partial^\mu\varphi^{*\bar{j}}. \tag{3.1.23}$$

Similarly, from the other elements of the matrix, (3.1.17), (3.1.18) and (3.1.19), we can define the following operators:

$$\begin{aligned}
(\hat{M}_2)_{\bar{k}\bar{l}} &= \bar{g}_{i\bar{k},j\bar{l}}\partial_\mu\varphi^i\partial^\mu\varphi^j, & (3.1.24)\\
(\hat{M}_3)_{kl} &= \bar{g}_{k\bar{i},l\bar{j}}\partial_\mu\varphi^{*\bar{i}}\partial^\mu\varphi^{*\bar{j}}, & (3.1.25)\\
(\hat{M}_4)_{\bar{k}l} &= \hat{p}\bar{g}_{\bar{k}l}\hat{p} + \bar{g}_{i\bar{j},\bar{k}l}\partial_\mu\varphi^i\partial^\mu\varphi^{*\bar{j}}. & (3.1.26)
\end{aligned}$$

The one loop correction term to $O(\partial^2)$ can be decomposed as

$$\int_{p'} tr\ln M_{ij}(-p',p') = \int_{p'} tr\ln\hat{M}_{1i\bar{j}} + \int_{p'} tr\ln\hat{M}_{4i\bar{j}}, \tag{3.1.27}$$

because $\hat{M}_2$ and $\hat{M}_3$ contribute only to higher-order derivative terms. Noting that $\bar{g}_{i\bar{j},k}$ and $\bar{g}_{i\bar{j},\bar{k}}$ are zero in the KNC, we can calculate this trace:

$$\begin{aligned}
\int_{p'} tr\ln M_{ij}(-p',p') &= \frac{2}{(2\pi)^D}(\delta t)\int d\Omega_D\int d^Dx tr\ln\bar{g}_{i\bar{j}}\\
&\quad+\frac{2}{(2\pi)^D}(\delta t)\int d\Omega_D\int d^Dx\bar{g}_{i\bar{j},k\bar{l}}\bar{g}^{k\bar{l}}\partial_\mu\varphi^i\partial^\mu\varphi^{*\bar{j}}.
\end{aligned} \tag{3.1.28}$$

Now we consider the contribution of the fermion part in the supersymmetric action (3.1.5). It turns out that there is no fermion contribution to the bosonic action, except for one equal to the first term in (3.1.28) with opposite sign. Therefore, the first term on the right-hand side of Eq.(3.1.28) cancels with this contribution from the fermionic part. In the KNC, the second term can be written

$$-\frac{2}{(2\pi)^D}(\delta t)\int d\Omega_D\int d^Dx\bar{R}_{i\bar{j}}\partial_\mu\varphi^i\partial^\mu\varphi^{*\bar{j}}. \tag{3.1.29}$$

Finally, we transform this result from the KNC to the original coordinates. Then, the first term of the WRG Eq.(2.1.17) is

$$\frac{1}{2\delta t}\int_{p'} tr\ln M_{ij} = -\frac{1}{(2\pi)^D}\int d\Omega_D\int d^DxR_{i\bar{j}}\partial_\mu\varphi^i\partial^\mu\varphi^{*\bar{j}}. \tag{3.1.30}$$

The second term of Eq.(2.17), the contribution of the dumbell diagram, vanishes to order $O(\partial^2)$ in the derivative expansion, because external lines carrying soft momenta in the derivative expansion cannot satisfy the energy-momentum conservation law with the hard shell-momenta carried by the internal double lines in the dumbell diagram.

A similar derivation can be applied to the $D=2$ $\mathcal{N}=2$, the $D=3$ $\mathcal{N}=2$ and the $D=4$ $\mathcal{N}=1$ SNLσM. We obtain the WRG equations of the scalar part action for these SNLσM as follows:

- In the case of $D=2$ $\mathcal{N}=2$ SNLσM,

$$\begin{aligned}&\frac{d}{dt}\int d^2x g_{i\bar{j}}(\partial_\mu\varphi)^i(\partial^\mu\varphi^*)^{\bar{j}}\\ =\ &\int d^2x\Bigg[-\frac{1}{2\pi}R_{i\bar{j}}-\gamma\Big[\varphi^k g_{i\bar{j},k}+\varphi^{*\bar{k}}g_{i\bar{j},\bar{k}}+2g_{i\bar{j}}\Big]\Bigg](\partial_\mu\varphi)^i(\partial^\mu\varphi^*)^{\bar{j}}.\end{aligned} \tag{3.1.31}$$

- In the case of $D=3$ $\mathcal{N}=2$ SNLσM,

$$\begin{aligned}&\frac{d}{dt}\int d^3x g_{i\bar{j}}(\partial_\mu\varphi)^i(\partial^\mu\varphi^*)^{\bar{j}}\\ =\ &\int d^3x\Bigg[-\frac{1}{2\pi^2}R_{i\bar{j}}-\gamma\Big[\varphi^k g_{i\bar{j},k}+\varphi^{*\bar{k}}g_{i\bar{j},\bar{k}}+2g_{i\bar{j}}\Big]\\ &\qquad -\frac{1}{2}\Big[\varphi^k g_{i\bar{j},k}+\varphi^{*\bar{k}}g_{i\bar{j},\bar{k}}\Big]\Bigg](\partial_\mu\varphi)^i(\partial^\mu\varphi^*)^{\bar{j}}.\end{aligned} \tag{3.1.32}$$

- In the case of $D=4$ $\mathcal{N}=1$ SNLσM,

$$\begin{aligned}&\frac{d}{dt}\int d^4x g_{i\bar{j}}(\partial_\mu\varphi)^i(\partial^\mu\varphi^*)^{\bar{j}}\\ =\ &\int d^4x\Bigg[-\frac{1}{8\pi^2}R_{i\bar{j}}-\gamma\Big[\varphi^k g_{i\bar{j},k}+\varphi^{*\bar{k}}g_{i\bar{j},\bar{k}}+2g_{i\bar{j}}\Big]\\ &\qquad -\Big[\varphi^k g_{i\bar{j},k}+\varphi^{*\bar{k}}g_{i\bar{j},\bar{k}}\Big]\Bigg](\partial_\mu\varphi)^i(\partial^\mu\varphi^*)^{\bar{j}}.\end{aligned} \tag{3.1.33}$$

In two dimensions, for example, the β function of the Kähler metric is

$$\begin{aligned}\frac{d}{dt}g_{i\bar{j}} &= -\frac{1}{2\pi}R_{i\bar{j}}-\gamma\Big[\varphi^k g_{i\bar{j},k}+\varphi^{*\bar{k}}g_{i\bar{j},\bar{k}}+2g_{i\bar{j}}\Big]\\ &\equiv -\beta(g_{i\bar{j}}).\end{aligned} \tag{3.1.34}$$

We have concentrated on the discussion of the WRG equation for the bosonic part of the action. In supersymmetric theories, Eq.(3.1.34) has to be the scalar part of the relation among superfields. Other parts will have contributions both from bosonic and fermionic parts. Here, we simply assume supersymmetry to derive a supersymmetric relation. In this paper, we have used a straight cut-off in momentum space to obtain the WRG equation. We need some modification or counterterms to maintain supersymmetry. If we assume supersymmetry, the WRG equation for the Kähler potential should be

$$\begin{aligned}\frac{d}{dt}\int dV K[\Phi,\Phi^\dagger] &= \int dV\triangle K_1[\Phi,\Phi^\dagger]\\ &+\left[2-\sum_{\Omega^i}\int_p\hat{\Omega}^i(p)(d_{\Omega^i}+\gamma_{\Omega^i}+\hat{p}^\mu\frac{\partial}{\partial\hat{p}^\mu})\frac{\delta}{\delta\hat{\Omega}^i(p)}\right]\hat{S},\end{aligned} \tag{3.1.35}$$

where Ω stands for $\varphi, \varphi^*, \psi, \bar{\psi}, F$ and $\bar{F}$, and $\triangle K_1$ is the one-loop correction:

$$\triangle K_1 = \frac{1}{2\pi} \ln \det g_{k\bar{l}}[\Phi, \Phi^\dagger]. \tag{3.1.36}$$

Expanding Eq.(3.1.35) around scalar fields, as in Eq.(3.1.5), we can obtain the WRG equations for various terms in Eq.(3.1.5). The bosonic part of this equation coincides with Eq.(3.1.31) by assumption.

Chapter 4

The Two Dimensional Models

In this chapter, we consider the fixed point theory of 2-dimensional $\mathcal{N}=2$ supersymmetric non-linear sigma model using the previous nonperturbative β function.

In the previous chapter, we obtained nonperturbative β function using Wilsonian renormalization group:

$$\beta(g_{i\bar{j}}) = \frac{1}{2\pi}R_{i\bar{j}} + \gamma\Big[\varphi^k g_{i\bar{j},k} + \varphi^{*\bar{k}} g_{i\bar{j},\bar{k}} + 2g_{i\bar{j}}\Big]. \tag{4.0.1}$$

The presence of the anomalous dimension reflects the nontrivial continuum limit of the fields.

When the anomalous dimension of the field vanishes, scale invariance is realized for NLσMs on Ricci-flat Kähler (Calabi-Yau) manifolds [24]. Calabi-Yau metrics have been explicitly constructed for some noncompact manifolds [27], in the case that the number of isometries is sufficient to reduce the Einstein equation to an ordinary differential equation.

However, when the anomalous dimension of the fields does not vanish, the condition of scale invariance is quite different. In this chapter, we study conformal field theories with anomalous dimensions by solving the condition of the fixed point: $\beta = 0$. We assume $\mathbf{U}(N)$ symmetry to reduce a set of partial differential equations to an ordinary differential equation. The conformal theories obtained have one free parameter corresponding to the anomalous dimension of the scalar fields. The geometry of the target manifolds depends strongly on the sign of the anomalous dimensions.

4.1 Fixed point of the U(N) symmetric WRG equation

In this section, we derive the action of the conformal field theory corresponding to the fixed point of the β function

$$\beta(g_{i\bar{j}}) = \frac{1}{2\pi}R_{i\bar{j}} + \gamma\Big[\varphi^k g_{i\bar{j},k} + \varphi^{*\bar{k}} g_{i\bar{j},\bar{k}} + 2g_{i\bar{j}}\Big] = 0. \tag{4.1.1}$$

Because the Ricci curvature $R_{i\bar{j}}$ takes the form of a second derivative of the metric $g_{i\bar{j}}$, the equation is composed of a set of coupled partial differential equations, and is very difficult to solve in general. Therefore we simplify the problem by assuming $\mathbf{U}(N)$ symmetry for the Kähler potential:

$$K[\varphi, \varphi^\dagger] \;=\; \sum_{n=1}^{\infty} g_n x^n \equiv f(x), \tag{4.1.2}$$

where x is the $\mathbf{U}(N)$ invariant combination

$$x \equiv \vec{\varphi}\cdot\vec{\varphi}^\dagger \tag{4.1.3}$$

of the N component scalar fields $\vec{\varphi} = (\varphi^1, \varphi^2, \cdots, \varphi^N)$. The coefficients g_n play the role of an infinite number of coupling constants that depend on the cutoff scale t. The Kähler potential gives the Kähler metric and Ricci tensor as follows[1]:

$$\begin{aligned} g_{i\bar{j}} &\equiv \partial_i \partial_{\bar{j}} K[\varphi, \varphi^\dagger] = f' \delta_{i\bar{j}} + f'' \varphi_i^* \varphi_{\bar{j}}, \qquad (4.1.4) \\ R_{i\bar{j}} &\equiv -\partial_i \partial_{\bar{j}} tr \ln g_{i\bar{j}} \\ &= -\Big[(N-1)\frac{f''}{f'} + \frac{2f'' + f'''x}{f' + f''x}\Big]\delta_{i\bar{j}} \\ &\quad -\Big[(N-1)\Big(\frac{f^{(3)}}{f''} - \frac{(f'')^2}{(f')^2}\Big) + \frac{3f^{(3)} + f^{(4)}x}{f' + f''x} - \frac{(2f'' + f'''x)^2}{(f' + f''x)^2}\Big]\varphi_i^* \varphi_{\bar{j}}, \qquad (4.1.5) \end{aligned}$$

where

$$f' = \frac{df}{dx}. \qquad (4.1.6)$$

We substitute this metric and Ricci tensor into the β function (4.0.1) and compare the coefficients of $\delta_{i\bar{j}}$ and $\varphi^i \varphi^{*\bar{j}}$ to find

$$\begin{aligned} \frac{\partial}{\partial t} f' &= \frac{1}{2\pi}\Big[(N-1)\frac{f''}{f'} + \frac{2f'' + f'''x}{f' + f''x}\Big] - 2\gamma(f' + f''x), \qquad (4.1.7) \\ \frac{\partial}{\partial t} f'' &= \frac{1}{2\pi}\Big[(N-1)\Big(\frac{f^{(3)}}{f''} - \frac{(f'')^2}{(f')^2}\Big) + \frac{3f^{(3)} + f^{(4)}x}{f' + f''x} - \frac{(2f'' + f'''x)^2}{(f' + f''x)^2}\Big] \\ &\quad -2\gamma(2f'' + f'''x). \qquad (4.1.8) \end{aligned}$$

Since the second equation (4.1.8) follows from the first equation by differentiation with respect to x, we discuss only the first equation.

Our differential equation (4.1.7) describes the renormalization group flow in the theory space specified by the infinite number of coupling constant in the Kähler potential. In fact, we can derive an infinite number of coupled differential equations relating the coupling constants g_n by inserting (4.1.2) into Eq.(4.1.7). We are especially interested in the fixed point of Eq.(4.1.7), which should give a scale invariant theory. The fixed-point theory is defined by the Kähler metric, which satisfies the following differential equation:

$$\begin{aligned} \frac{\partial}{\partial t} f' &= \frac{1}{2\pi}\Big[(N-1)\frac{f''}{f'} + \frac{2f'' + f'''x}{f' + f''x}\Big] - 2\gamma(f' + f''x) \\ &= 0. \qquad (4.1.9) \end{aligned}$$

To obtain the Lagrangian of the scale invariant field theory, we have to solve this differential equation.

Noting that this equation can be rewritten as

$$\frac{d}{dx}\Big[\ln(f')^{N-1}(f' + f''x)\Big] = 4\pi\gamma \frac{d}{dx}(f'x), \qquad (4.1.10)$$

we can integrate it easily to obtain

$$(f')^{N-1} F' = C_1 \exp[4\pi\gamma F], \qquad (4.1.11)$$

where

$$F \equiv f'x, \qquad (4.1.12)$$

[1] We use the convention $\varphi_i^* = \delta_{i\bar{j}} \varphi^{*\bar{j}}$ and $\varphi_{\bar{j}} = \delta_{i\bar{j}} \varphi^i$.

and C_1 is constant of integration. The normalization condition of the kinetic term,

$$g_{i\bar{j}}|_{x=0} = \delta_{i\bar{j}},$$

gives the initial condition

$$f'(0) = 1, \tag{4.1.13}$$

implying

$$F'(0) = 1, \tag{4.1.14}$$

which fixes $C_1 = 1$.

Integrating Eq.(4.1.11), we find the solution of the differential equation satisfy the following algebraic equation:

$$\frac{e^{aF}}{a} \sum_{r=0}^{N-1} (-1)^r \frac{(N-1)! F^{(N-1)-r}}{(N-1-r)! a^r} = \frac{1}{N} x^N + C_2. \tag{4.1.15}$$

Here we have introduced the constant $a \equiv -4\pi\gamma$; that is, we write the anomalous dimension of the scalar field using the free parameter a as

$$\gamma = -\frac{a}{4\pi}. \tag{4.1.16}$$

In NLσM, the anomalous dimension of the scalar field can take either a positive or negative value, and hence the parameter a can also take either sign [14]. Setting $x = 0$ in Eq.(4.1.15) and using the boundary condition (4.1.14), we obtain

$$C_2 = (-1)^{N-1} \frac{(N-1)!}{a^N}. \tag{4.1.17}$$

From the condition (4.1.15), we obtain the function $f' = F/x$. Because the metric (4.1.4) is determined by f' and f'', our Lagrangian is completely fixed by Eq.(4.1.15). Thus we have found that the Lagrangian of the scale invariant theory has the free parameter a corresponding to the anomalous dimension of the field.

4.2 Geometry of the target space of the scale invariant theory

In this section, we study the geometry of target space in the case that the theory is scale invariant.

4.2.1 One-dimensional target space

Equation (4.1.15) is very simple in the case that the target manifold is of one complex dimension. When $N = 1$, this equation reads

$$\frac{e^{aF}}{a} = x + \frac{1}{a}, \tag{4.2.1}$$

which gives

$$f' = \frac{1}{ax} \ln(1 + ax). \tag{4.2.2}$$

Using this form of f' in Eq.(4.1.4) gives the metric of the target space,

$$g_{i\bar{j}} = f' + f''x = \frac{1}{1 + ax}. \tag{4.2.3}$$

Note that this metric has only one component, and the indices i and $\bar{j}$ are 1. The scalar curvature is given by

$$R = \frac{a}{1+ax}. \tag{4.2.4}$$

The properties of this target manifold depend strongly on the sign of the parameter a.

Now, we investigate the properties of the target manifold for each sign of a.

1. When $a > 0$, the anomalous dimension is negative.
   Because the line element is given by [2]

   $$ds^2 = \frac{|dz|^2}{1+a|z|^2}, \tag{4.2.5}$$

   or in polar coordinates, with $z = re^{i\phi}$,

   $$ds^2 = \frac{1}{1+ar^2}\Big((dr)^2 + r^2(d\phi)^2\Big), \tag{4.2.6}$$

   the volume and the distance from the origin ($r = 0$) to infinity ($r = \infty$) are divergent, while the length of the circumference at infinity is finite. Therefore, the shape of the target manifold is that of a semi-infinite cigar. The volume integral of the scalar curvature is also finite, giving the Euler number

   $$\chi = \frac{1}{2\pi}\int dz d\bar{z}(\det g_{i\bar{j}})R = \frac{1}{2\pi}\int dz d\bar{z}\frac{1}{(1+a|z|^2)^2} = 1, \tag{4.2.7}$$

   which is equal to that of a disc.

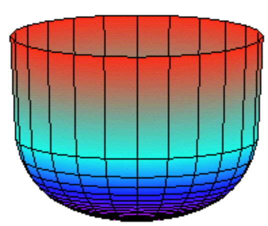

Figure 4.1: The target manifold for $a = +1$ embedded in 3-dimensional flat Euclidean spaces. It takes the form of a semi-infinite cigar with radius $\sqrt{\frac{1}{a}}$. Our metric (4.2.12) is the induced metric on this surface.

   Let us embed the manifold in 3-dimensional Euclidean spaces. When the hyperplane possessess rotational symmetry, the line element can be written in cylindrical coordinates as

   $$ds^2 = (dh)^2 + (d\rho)^2 + \rho^2(d\phi)^2, \tag{4.2.8}$$

   where the height is a function of the radius ρ:

   $$h = g(\rho). \tag{4.2.9}$$

[2] We use z for the coordinate of the manifold instead of φ throughout this section.

From Eq.(4.2.9), the line element can be rewritten

$$ds^2 = (1 + (g'(\rho))^2)(d\rho)^2 + \rho^2 (d\phi)^2, \tag{4.2.10}$$

where g' is the derivative of the function $g(\rho)$ with respect to ρ. Now we transform the line element for the target metric (4.2.3) to the form of Eq.(4.2.10) through the change of variable $r \to \rho$, with

$$\rho = \frac{r}{\sqrt{1 + ar^2}}, \tag{4.2.11}$$

which is a one-to-one mapping from the entire plane $0 \le r < \infty$ to the disc $0 \le \rho < \frac{1}{\sqrt{a}}$. Then, Eq.(4.2.6) can be rewritten as

$$ds^2 = \frac{1}{(1 - a\rho^2)^2}(d\rho)^2 + \rho^2 (d\phi)^2. \tag{4.2.12}$$

Comparing Eq.(4.2.10) with Eq.(4.2.12), we obtain the height function $h = g(\rho)$ as follows:

$$\begin{aligned} h &= \int_0^\rho d\rho g'(\rho) = \int_0^\rho d\rho \sqrt{\frac{1}{(1 - a\rho^2)^2} - 1} \\ &= -\frac{1}{2\sqrt{a}} \Big(\ln |\frac{\sqrt{2 - a\rho^2} - 1}{\sqrt{2 - a\rho^2} + 1}| + 2\sqrt{2 - a\rho^2} - \ln |\frac{\sqrt{2} - 1}{\sqrt{2} + 1}| - 2\sqrt{2} \Big) \\ &. \end{aligned} \tag{4.2.13}$$

Figure 4.1 shows the manifold embedded in 3-dimensional flat Euclidean spaces. The distance between any two points is measured along the shortest path on the surface in the Euclidean space.

2. When $a < 0$, the anomalous dimension is positive.
In this case, the metric and scalar curvature read

$$g_{i\bar{j}} = \frac{1}{1 - |a|x}, \tag{4.2.14}$$

$$R = \frac{-|a|}{1 - |a|x}. \tag{4.2.15}$$

This metric is ill-defined at the boundary $|z| \sim \frac{1}{\sqrt{-a}}$. This is not merely a coordinate singularity, because the scalar curvature is divergent at the boundary. Although the volume integral is divergent, the distance to the boundary is finite. Now, let us embed this manifold in a flat space. Note that Eq.(4.2.13) is imaginary if $a < 0$. Thus the manifold is embedded as a space-like surface in a flat Minkowski space. Figure 4.2 shows the manifold embedded in a 3-dimensional flat Minkowski space.

4.2.2 Higher-dimensional target spaces

We now consider conformal field theories whose target spaces consist of more than two dimensions and investigate the properties of the target manifolds. For $N \ge 2$, we have to solve the algebraic equation (4.1.15), which for $N = 2$, for example, reads

$$e^{aF}(aF - 1) = \frac{1}{2}(ax)^2 - 1. \tag{4.2.16}$$

Figure 4.3 displays aF as a function of ax for $N = 2$ and $|a| = 1$.

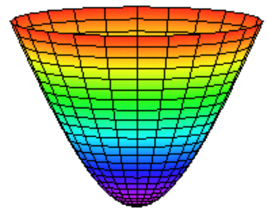

Figure 4.2: The target manifold for $a = -1$, embedded in a flat Minkowski space. The vertical axis has negative signature. In the asymptotic region $\rho \to \infty$, the surface approaches the lightcone.

The Kähler potential in the neighborhood of the origin is easily obtained by solving the equation (4.1.15)

$$f(x) = x - \frac{a}{2(N+1)}x^2 + \cdots . \tag{4.2.17}$$

The asymptotic behavior depend strongly on the sign of the parameter a, and for this reason, we discuss the two cases separately.

1. $a > 0$ case
   Figure 4.3 shows that the function f', which is the diagonal component of the target metric, goes to infinity as $x \to \infty$. When aF goes to infinity, the $r = 0$ term of Eq.(4.1.15) gives the dominant contribution on the left-hand side. To find the asymptotic behavior in this region, we retain only the dominant terms and solve

$$e^{aF}(aF)^{N-1} \approx \frac{1}{N}(ax)^N \tag{4.2.18}$$

   using the iteration method,

$$\begin{aligned} aF &\approx \ln\left(\frac{(ax)^N}{N}\right) - (N-1)\ln(aF) \\ &= \ln\left(\frac{(ax)^N}{N}\right) - (N-1)\ln\ln\left(\frac{(ax)^N}{N}\right) + \cdots , \end{aligned}$$

   where we have dropped terms that vanish as $ax \to \infty$. Then, we obtain the functions f' as

$$f' \approx \frac{1}{ax}\ln\frac{(ax)^N}{N} - \frac{N-1}{ax}\ln\ln\frac{(ax)^N}{N} + \cdots . \tag{4.2.19}$$

   The distance along a straight line in the radial direction is written

$$\begin{aligned} g_{i\bar{j}}dz^i dz^{*\bar{j}} &= (f'\delta_{i\bar{j}} + f'' z_{\bar{j}} z^*_i)dz^i dz^{*\bar{j}} \\ &\approx (f' + |r|^2 f'')|dr|^2 = \frac{N}{a|r|^2}|dr|^2. \end{aligned} \tag{4.2.20}$$

   Here, we have defined the complex radial coordinate r and the angle variables by

$$z^1 = u^1 r,\ \cdots\ ,\ z^{N-1} = u^{N-1}r,\ z^N = r. \tag{4.2.21}$$

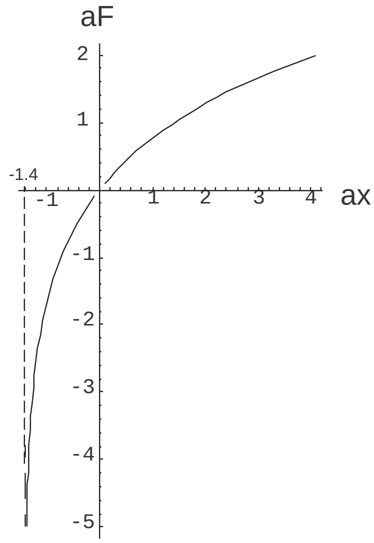

Figure 4.3: aF as a function of ax for $N=2$ and $|a|=1$. Because $x = zz^\dagger$ is positive, $ax \geq 0$ corresponds to $a>0$, while $ax \leq 0$ corresponds to $a<0$.

The asymptotic behavior (4.2.20) for any N is similar to that in the $N=1$ case, in which the metric in the asymptotic region is given by

$$g_{1\bar{1}} \approx \frac{1}{ax}. \tag{4.2.22}$$

The asymptotic behavior of the Kähler potential (4.1.2) can be found by integrating Eq.(4.2.19),

$$\begin{aligned} K(zz^*) &\approx \frac{N}{2a}(\log(azz^*))^2 + \cdots \\ &= \chi\chi^* + 2\sqrt{\frac{N}{a}}\Re e\chi \log\left(1+|u^1|^2+\cdots+|u^{N-1}|^2\right)+\cdots, \end{aligned} \tag{4.2.23}$$

where we have removed holomorphic and anti-holomorphic terms through a Kähler transformation and defined χ by

$$\chi = \sqrt{\frac{N}{a}} \log \sqrt{a}r. \tag{4.2.24}$$

For fixed values of the radius r, (4.2.23) is the Kähler potential for the Fubini-Study metric of the complex projective space CP^{N-1}, whose size is fixed by $\Re e\chi$. Therefore, our target space is the direct product of the complex line represented by χ and CP^{N-1} represented by $(u^1,\cdots,u^{N-1})$ in the asymptotic region.

2. $a<0$ case
Figure 4.3 shows that the allowed region $ax \leq 0$ is limited to the region $|\vec{z}| < (\frac{\sqrt{2}}{|a|})^{\frac{1}{2}}$ for $N=2$, as for $N=1$. By assuming $ax \sim -\sqrt{2}+\epsilon$ near the boundary, we can reduce Eq.(4.2.18) to

$$e^{aF}aF \sim -\sqrt{2}\epsilon,$$

which can be solved using the iteration method,

$$-aF = \ln\frac{1}{\sqrt{2}\epsilon} + \ln(-aF)$$

$$= \ln \frac{1}{\sqrt{2}\epsilon} + \ln\ln(\frac{1}{\sqrt{2}\epsilon}) + \cdots.$$

Because $\epsilon \ll 1$, the behavior of the function $f' = \frac{F}{ax}$ near the boundary is given by

$$f' \sim \frac{1}{ax}\ln(1 + \frac{ax}{\sqrt{2}}), \tag{4.2.25}$$

which leads to the curvature singularity at the boundary. Similarly, the allowed region in the z-plane for general N is

$$|\vec{z}| < (N!)^{\frac{1}{2N}}|a|^{-\frac{1}{2}}. \tag{4.2.26}$$

The asymptotic behavior of the function f' near the boundary,

$$f'(x) \sim \frac{1}{ax}\ln(1 + \frac{ax}{(N!)^{1/N}}),$$

leads to the curvature singularity at the boundary.

To summarize, we have found that the target spaces of the scale invariant theory with nontrivial anomalous dimension are noncompact and well-behaved at infinity for $a \geq 0$, while they have important curvature singularity at the boundary for $a < 0$.

4.3 The comparison with the perturbative result

In this section, we compare the above fixed point theory with the perturbative result. The perturbative one-loop β function for the target metric of non-linear sigma model is proportional to the Ricci tensor. However let's consider the non-linear sigma model coupled with dilaton on the curved space-time:

$$S = \int_{curved} d^2x \frac{k}{4\pi}\sqrt{h}h^{\mu\nu}g_{i\bar{j}}\partial_\mu\varphi^i\partial_\mu\varphi^{*\bar{j}} - \frac{1}{8\pi}\sqrt{h}R^{(2)}\Phi(\varphi,\varphi^*), \tag{4.3.1}$$

where $R^{(2)}$ denotes the scalar curvature of the space-time. The one-loop β function for above action has the additional effct of the dilaton:

$$\beta_{i\bar{j}} = \frac{d}{dt}g_{i\bar{j}} = R_{i\bar{j}} + 2\nabla_i\nabla_{\bar{j}}\Phi. \tag{4.3.2}$$

On flat space-time, a nontrivial dilaton gradient in target space is equivalent to assigning a non-trivial Weyl transformation law to target space coordinates. The solution of the eq.(4.3.2) is equivalent to the one obtained §.4.1.

To show the relation between the nonvanishing dilaton and the nonvanishing anomalous dimension, let's consider one complex target spaces. If we fix a conformal-flat gauge for target space metric $g_{i\bar{j}}$, we rewritten the action on flat space-time as follows:

$$S = \int_{flat} d^2x e^{\Psi(r)}(\partial_\mu r\partial^\mu r + \partial_\mu\theta\partial^\mu\theta) \tag{4.3.3}$$

The function $\Psi(r)$ does not depend on θ, because we consider the $U(1)$ symmetric theory. The perturbative one-loop β function for non-linear sigma model is obtained:

$$\beta_{ij} = \frac{1}{2\pi}R_{ij}. \tag{4.3.4}$$

From eq. (4.3.3), the Ricci tensor of target space is given

$$R_{rr} = R_{\theta\theta} = \frac{1}{2}\frac{\partial^2\Psi(r)}{\partial r^2}. \tag{4.3.5}$$

Then the β function for the field Ψ is written by

$$\frac{\partial\Psi(r)}{\partial t} = -\frac{1}{4\pi}e^{-\Psi(r)}\frac{\partial^2\Psi(r)}{\partial r^2} \tag{4.3.6}$$

Here we put most Weyl transformation law for r under the Weyl rescaling of the space-time metric by t^2:

$$r \to r - bt. \tag{4.3.7}$$

Here, b correspond to the anomalous dimension of the field r on space-time. (Because of $U(1)$ symmetry, θ is unchanged under Weyl rescaling.) Setting $\Psi(r) = \Psi_{fixedpoint}(r - bt)$, we obtain an equation for $\Psi_{fixedpoint(r)}$:

$$\frac{1}{4\pi}e^{-\Psi_{fixedpoint}}\Psi''_{fixedpoint} - b\Psi'_{fixedpoint} = 0. \tag{4.3.8}$$

The solution of this equation is

$$e^{\Psi_{fixedpoint}} = \frac{1}{e^{-\lambda(r-c)} - 4\pi b/\lambda}, \tag{4.3.9}$$

where λ, c are constants. If we choose

$$\lambda = 2, \qquad b = -\frac{1}{2\pi k}, \qquad e^{\Psi_{fixedpoint}} = \frac{1}{e^{-2(r-c)+\frac{1}{k}}},$$

we find the anomalous dimension b is the same as the previous $\gamma = -\frac{a}{4\pi}$ in §.4.1, and the parameter k in the action (4.3.1) correspond to the parameter a as follow:

$$a = \frac{2}{k}. \tag{4.3.10}$$

Chapter 5

The Three Dimensional Models

The three dimensional nonlinear sigma model is nonrenormalizable within the perturbative method. In the WRG approach, the renormalizability is equivalent to the existence of a nontrivial continuum limit, $\Lambda \to \infty$. When the ultraviolet (UV) cutoff Λ tends to infinity, we have to finetune the coupling constant to the critical value at the UV fixed point, so as to keep the observable quantities finite.

In this chapter, first, we discuss that some $\mathcal{N} = 2$ supersymmetric nonlinear sigma models are renormalizable in three dimensions using the β function in the nonperturbative Wilsonian renormalization group method. Next, we construct a class of conformal field theories with $\mathbf{SU}(N)$ symmetry, defined at the fixed point of the nonperturbative β function.

5.1 Einstein-Kähler manifolds

In chapter 3, we obtain the WRG equations for the scalar part as follows:

$$\begin{aligned}
&\frac{d}{dt}\int d^3x g_{i\bar{j}}(\partial_\mu\varphi)^i(\partial^\mu\varphi^*)^{\bar{j}} \\
&= \int d^3x\Big[-\frac{1}{2\pi^2}R_{i\bar{j}} \\
&-\gamma\Big(\varphi^k g_{i\bar{j},k} + \varphi^{*\bar{k}} g_{i\bar{j},\bar{k}} + 2g_{i\bar{j}}\Big) - \frac{1}{2}\Big(\varphi^k g_{i\bar{j},k} + \varphi^{*\bar{k}} g_{i\bar{j},\bar{k}}\Big)\Big](\partial_\mu\varphi)^i(\partial^\mu\varphi^*)^{\bar{j}},
\end{aligned} \tag{5.1.1}$$

where the scalar fields $\varphi^n(x)$ are assumed to be independent of t through a suitable rescaling, which introduces the anomalous dimension γ. From this WRG equation, the β function of Kähler metric is

$$\begin{aligned}
\frac{d}{dt}g_{i\bar{j}} &= -\frac{1}{2\pi^2}R_{i\bar{j}} - \gamma\Big[\varphi^k g_{i\bar{j},k} + \varphi^{*\bar{k}} g_{i\bar{j},\bar{k}} + 2g_{i\bar{j}}\Big] - \frac{1}{2}\Big[\varphi^k g_{i\bar{j},k} + \varphi^{*\bar{k}} g_{i\bar{j},\bar{k}}\Big] \\
&\equiv -\beta(g_{i\bar{j}}).
\end{aligned} \tag{5.1.2}$$

Let us consider the theories whose target spaces are Einstein-Kähler manifolds. The Einstein-Kähler manifolds satisfy the condition

$$R_{i\bar{j}} = \frac{h}{a^2}g_{i\bar{j}}, \tag{5.1.3}$$

where a is the radius of the manifold, which is related to the coupling constant λ by

$$\lambda = \frac{1}{a}. \tag{5.1.4}$$

A special class of Einstein-Kähler manifolds is called the Hermitian symmetric space, if it is a symmetric coset space (G/H), namely, if the coset space is invariant under a parity operation. If the manifold is the Hermitian symmetric space, the positive constant h in eq.(5.1.3) is the eigenvalue of the quadratic Casimir operator in the adjoint representation of global symmetry G, as shown in Table 5.1.

G/H	Dimensions (complex)	h
$SU(N)/[SU(N-1)\otimes U(1)] = \mathrm{C}P^{N-1}$	$N-1$	N
$SU(N)/[SU(N-M)\otimes U(M)]$	$M(N-M)$	N
$SO(N)/[SO(N-2)\otimes U(1)] = Q^{N-2}$	$N-2$	$N-2$
$Sp(N)/U(N)$	$\frac{1}{2}N(N+1)$	$N+1$
$SO(2N)/U(N)$	$\frac{1}{2}N(N+1)$	$N-1$
$E_6/[SO(10)\otimes U(1)]$	16	12
$E_7/[E_6\otimes U(1)]$	27	18

Table 5.1: The values of h for Hermitian symmetric spaces

When the manifolds have the radius $a = \frac{1}{\lambda}$, the scalar part of the SNLσM Lagrangian can be represented in the following form:

$$\begin{aligned}\mathcal{L}_{scalar} &= g_{i\bar{j}}(\varphi,\varphi^*)\partial_\mu\varphi^i\partial^\mu\varphi^{*\bar{j}} \\ &\xrightarrow{\varphi,\varphi^*\approx 0} \frac{1}{\lambda^2}\delta_{i\bar{j}}\partial_\mu\varphi^i\partial^\mu\varphi^{*\bar{j}}.\end{aligned} \tag{5.1.5}$$

To normalize the kinetic term, we rescale the scalar fields as follows:

$$\varphi \to \tilde{\varphi} = \frac{1}{\lambda}\varphi. \tag{5.1.6}$$

Then, the Lagrangian (5.1.5) has the normalized kinetic term

$$\mathcal{L}_{scalar} = \tilde{g}_{i\bar{j}}(\lambda\tilde{\varphi},\lambda\tilde{\varphi}^*)\partial_\mu\tilde{\varphi}^i\partial^\mu\tilde{\varphi}^{*\bar{j}} \tag{5.1.7}$$

with

$$\tilde{g}_{i\bar{j}}|_{\tilde{\varphi},\tilde{\varphi}^*=0} = \delta_{i\bar{j}}. \tag{5.1.8}$$

Rescaling the WRG Eq.(5.1.2) and comparing the coefficient of $\partial_\mu\tilde{\varphi}^i\partial^\mu\tilde{\varphi}^{*\bar{j}}$, we have

$$\begin{aligned}\frac{\partial}{\partial t}\tilde{g}_{i\bar{j}}(\lambda\tilde{\varphi},\lambda\tilde{\varphi}^*) &= -\frac{1}{2\pi^2}\tilde{R}_{i\bar{j}} \\ &\quad -\gamma[\tilde{\varphi}^k\tilde{g}_{i\bar{j},k} + \tilde{\varphi}^{*\bar{k}}\tilde{g}_{i\bar{j},\bar{k}} + 2\tilde{g}_{i\bar{j}}] - \frac{1}{2}[\tilde{\varphi}^k\tilde{g}_{i\bar{j},k} + \tilde{\varphi}^{*\bar{k}}\tilde{g}_{i\bar{j},\bar{k}}],\end{aligned}$$

where $\tilde{R}_{i\bar{j}}$ is the rescaled Ricci tensor and can be written

$$\tilde{R}_{i\bar{j}} = h\lambda^2\tilde{g}_{i\bar{j}} \tag{5.1.9}$$

using the Einstein Kähler condition (5.1.3).

Because only λ depends on t, this differential equation can be rewritten as

$$\frac{\dot{\lambda}}{\lambda}\tilde{\varphi}^k\tilde{g}_{i\bar{j},k} + \frac{\dot{\lambda}}{\lambda}\tilde{\varphi}^{*\bar{k}}\tilde{g}_{i\bar{j},\bar{k}} = -\left(\frac{h\lambda^2}{2\pi^2} + 2\gamma\right)\tilde{g}_{i\bar{j}} - (\gamma + \frac{1}{2})[\tilde{\varphi}^k\tilde{g}_{i\bar{j},k} + \tilde{\varphi}^{*\bar{k}}\tilde{g}_{i\bar{j},\bar{k}}]. \tag{5.1.10}$$

The left-hand side vanishes for $\varphi,\varphi^*\approx 0$, so that the coefficient of $\tilde{g}_{i\bar{j}}$ must vanish on the right-hand side. Thus, we obtain the anomalous dimension of scalar fields (or chiral superfields) as

$$\gamma = -\frac{h\lambda^2}{4\pi^2}. \tag{5.1.11}$$

Comparing the coefficient of $\tilde{\varphi}^k \tilde{g}_{i\bar{j},k}$ (or $\tilde{\varphi}^{*\bar{k}} \tilde{g}_{i\bar{j},\bar{k}}$), we also obtain the β function of λ:

$$\beta(\lambda) \equiv -\frac{d\lambda}{dt} = -\frac{h}{4\pi^2}\lambda^3 + \frac{1}{2}\lambda. \tag{5.1.12}$$

We have a IR fixed point at

$$\lambda = 0, \tag{5.1.13}$$

and we also have a UV fixed point at

$$\lambda^2 = \frac{2\pi^2}{h} \equiv \lambda_c^2, \tag{5.1.14}$$

for positive h. *Therefore, if the constant h is positive, it is possible to take the continuum limit by choosing the cutoff dependence of the bare coupling constant as*

$$\lambda(\Lambda) \overset{\Lambda\to\infty}{\longrightarrow} \lambda_c - \frac{M}{\Lambda}, \tag{5.1.15}$$

where M is a finite mass scale. With this fine tuning, $\mathcal{N} = 2$ supersymmetric nonlinear σ models are renormalizable at least in our approximation, if the target spaces are Einstein-Kähler manifolds with positive curvature.

When the constant h is positive, the target manifold is compact Einstein-Kähler manifold [39]. In this case, the anomalous dimension at the fixed points are given by

$$\gamma_{IR} = 0 : \text{IR fixed point (Gaussian fixed point)} \tag{5.1.16}$$

$$\gamma_{UV} = -\frac{1}{2} : \text{UV fixed point} \tag{5.1.17}$$

At UV fixed point, the scaling dimension of the scalar fields (x_φ) is canonical plus anomalous dimension:

$$x_\varphi \equiv d_\varphi + \gamma_\varphi = 0. \tag{5.1.18}$$

Thus the scalar fields and the chiral superfields are dimensionless in the UV conformal theory as in the case of two dimensional field theories. Above the fixed point, the scalar fields have mass, and the symmetry restores [5, 40].

5.2 Renormalization group flows

In this section, we study the renormalization group flows for three examples, CP^N, Q^N and a new model. We will discuss both models with the other non-perturbative method in Chapter 6

5.2.1 CP^N and Q^N models

1. CP^N model: $SU(N+1)/[SU(N)\otimes U(1)]$
   Consider the following $SU(N+1)$ symmetric Kähler potential using $(N+1)$-dimensional homogeneous coordinates:

$$K[\Phi, \Phi^\dagger] = \frac{1}{\lambda^2}\ln(|\Phi^1|^2 + \cdots + |\Phi^N|^2 + |\Phi^{N+1}|^2). \tag{5.2.1}$$

   The complex projective space, CP^N, is defined by identifying two point related by

$$\Phi^i \sim a\Phi^i, \qquad (i = 1, \cdots, N+1), \tag{5.2.2}$$

where a is a complex chiral superfield, so that this is a complexified gauge symmetry. We obtain the Kähler potential for CP^N model by choosing a gauge $\Phi^{N+1} = 1$ as

$$K[\Phi, \Phi^\dagger] = \frac{1}{\lambda^2} \ln(1 + \vec{\Phi}\vec{\Phi}^\dagger), \tag{5.2.3}$$

where $\vec{\Phi}$ denotes a set of chiral superfields $\vec{\Phi} = (\Phi^1, \cdots, \Phi^N)$. Hereafter, we rescale scalar fields

$$\varphi \to \tilde{\varphi} = \frac{1}{\lambda}\varphi, \tag{5.2.4}$$

to normalize the kinetic term, and simply write the rescaled scalar fields $\tilde{\varphi}$ as φ.

From this Kähler potential, we can obtain the Kähler metric and Ricci tensor:

$$\begin{aligned} g_{i\bar{j}} &\equiv \partial_i \partial_{\bar{j}} K = \Big(\frac{\delta_{i\bar{j}}}{1 + \lambda^2 \vec{\varphi}\vec{\varphi}^*} - \frac{\lambda^2 \varphi^i \varphi^{*\bar{j}}}{(1 + \lambda^2 \vec{\varphi}\vec{\varphi}^*)^2} \Big), &(5.2.5)\\ R_{i\bar{j}} &\equiv -\partial_{\bar{j}} \partial_i (\ln \det g_{k\bar{l}}) = (N+1)\lambda^2 g_{i\bar{j}}. &(5.2.6)\end{aligned}$$

Equation (5.2.6) shows that this target manifold is Einstein-Kähler manifold with $h = N + 1$. Hence eqs. (5.1.11) and (5.1.12) give us immediately

$$\begin{aligned} \gamma &= -\frac{(N+1)\lambda^2}{4\pi^2}, &(5.2.7)\\ \beta(\lambda) &= -\frac{(N+1)\lambda^3}{4\pi^2} + \frac{1}{2}\lambda. &(5.2.8)\end{aligned}$$

This β function is consistent with the large N analysis [35].

2. Q^N model: $SO(N+2)/[SO(N) \otimes SO(2)]$
Another example of the Einstein-Kähler manifold is the coset manifold $SO(N+2)/[SO(N) \otimes SO(2)]$ called Q^N. We consider the Kähler potential with homogeneous $(N+2)$-dimensional coordinates:

$$K[\Phi, \Phi^\dagger] = \frac{1}{\lambda^2} \ln(|\Phi^1|^2 + \cdots + |\Phi^N|^2 + |\Phi^{N+1}|^2 + |\Phi^{N+2}|^2). \tag{5.2.9}$$

Now, we impose two conditions, identification and $O(N)$ symmetric conditions:

$$\begin{aligned} \Phi^i &\sim a\Phi^i, \qquad (i = 1, \cdots, N+2), &(5.2.10)\\ (\Phi^1)^2 + \cdots &+ (\Phi^N)^2 + (\Phi^{N+1})^2 + (\Phi^{N+2})^2 = 0, &(5.2.11)\end{aligned}$$

on the Kähler potential. By these condition, the dimensions of target space becomes N, and the Kähler potential for Q^N can be rewritten as [1]

$$K[\Phi, \Phi^\dagger] = \frac{1}{\lambda^2} \ln \Big(1 + \vec{\Phi}\vec{\Phi}^\dagger + \frac{1}{4}\vec{\Phi}^2 \vec{\Phi}^{\dagger 2}\Big), \tag{5.2.12}$$

where $\vec{\Phi} = (\Phi^1, \cdots, \Phi^N)$.

Hereafter we use rescaled fields (5.2.4). From this Kähler potential, the Kähler metric and Ricci tensor are given by

$$\begin{aligned} g_{i\bar{j}} &= \frac{\delta_{i\bar{j}}}{1 + \lambda^2 \vec{\varphi}\vec{\varphi}^* + \frac{1}{4}\lambda^4 \vec{\varphi}^2 \vec{\varphi}^{*2}} \\ &\quad + \frac{\lambda^2 \varphi^i \varphi^{*\bar{j}} \left(1 + \lambda^2 \vec{\varphi}\vec{\varphi}^*\right) - \lambda^2 \left(\varphi_i^* \varphi_{\bar{j}} + \frac{1}{2}\lambda^2 \vec{\varphi}^2 \varphi_i^* \varphi^{*\bar{j}} + \frac{1}{2}\lambda^2 \vec{\varphi}^{*2} \varphi^i \varphi_{\bar{j}}\right)}{\left(1 + \lambda^2 \vec{\varphi}\vec{\varphi}^* + \frac{1}{4}\lambda^4 \vec{\varphi}^2 \vec{\varphi}^{*2}\right)^2}, &(5.2.13)\\ R_{i\bar{j}} &= N\lambda^2 g_{i\bar{j}}. &(5.2.14)\end{aligned}$$

[1] We choose the same gauge as the one in the next subsection.

Equation (5.2.14) shows that this manifold is also an Einstein-Kähler manifold with $h = N$. Employing the same argument as in the case of the $\mathrm{C}P^N$ model, we obtain the anomalous dimension and β function for the coupling constant:

$$\gamma = -\frac{N\lambda^2}{4\pi^2}, \tag{5.2.15}$$

$$\beta(\lambda) = -\frac{N\lambda^3}{4\pi^2} + \frac{1}{2}\lambda. \tag{5.2.16}$$

Next example shows that there are renormalization group flows which connect $\mathrm{C}P^N$ and Q^N models.

5.2.2 A new model

1. Construction

Again, we consider the Kähler potential with homogeneous $(N+2)$-dimensional coordinates:

$$K[\Phi, \Phi^\dagger] = \frac{1}{\lambda^2} \ln(|\Phi^1|^2 + \cdots + |\Phi^N|^2 + |\Phi^{N+1}|^2 + |\Phi^{N+2}|^2). \tag{5.2.17}$$

As in $\mathrm{C}P^N$ and Q^N models, we identify two points related by

$$\Phi^i \sim a\Phi^i. \qquad (i = 1, \cdots, N+2) \tag{5.2.18}$$

Now we deform the $O(N)$ symmetric condition to

$$b[(\Phi^1)^2 + \cdots + (\Phi^N)^2] + (\Phi^{N+1})^2 + (\Phi^{N+2})^2 = 0, \tag{5.2.19}$$

where b is an arbitrary complex parameter.

(a) $b = 0$ case:
The deformed condition (5.2.19) is rewritten as

$$(\Phi^{N+1})^2 + (\Phi^{N+2})^2 = 0. \tag{5.2.20}$$

We fixed Φ^{N+1} and Φ^{N+2} by using the two conditions (5.2.18) and (5.2.19) as follows:

$$\Phi^{N+1} = \frac{1}{\sqrt{2}}, \tag{5.2.21}$$

$$\Phi^{N+2} = \pm\frac{i}{\sqrt{2}}. \tag{5.2.22}$$

Substituting these values for the Kähler potential (5.2.17), we obtain the Kähler potential of $\mathrm{C}P^N$:

$$K[\Phi, \Phi^\dagger] = \frac{1}{\lambda^2} \ln(1 + |\Phi^1|^2 + \cdots + |\Phi^N|^2). \tag{5.2.23}$$

Thus, the target space is double cover of $\mathrm{C}P^N$ located at $\Phi^{N+2} = \pm\frac{i}{\sqrt{2}}$. This target manifold has isometry $SU(N+1)$.

(b) $b \neq 0$ case:
Using the two conditions, we can choose a gauge

$$\Phi^{N+1} + i\Phi^{N+2} = \sqrt{2}, \tag{5.2.24}$$

and

$$\Phi^{N+1} - i\Phi^{N+2} = \frac{-b}{\sqrt{2}}\Big((\Phi^1)^2 + \cdots + (\Phi^N)^2\Big). \tag{5.2.25}$$

Then, the Kähler potential is rewritten

$$K[\Phi, \Phi^\dagger] = \frac{1}{\lambda^2}\ln\Big(1 + |\Phi^1|^2 + \cdots + |\Phi^N|^2 + \frac{|b|^2}{4}|\sum_{i=1}^{N}(\Phi^i)^2|^2\Big). \tag{5.2.26}$$

If we take $|b| = 1$, this Kähler potential is equal to that of Q^N model. Thus for this special value of b, the target manifold has isometry $SO(N+2)$.

(c) $b = \infty$ case:
The $O(N)$ symmetric condition reduces to

$$(\Phi^1)^2 + \cdots + (\Phi^N)^2 = 0. \tag{5.2.27}$$

The remaining fields Φ^{N+1} and Φ^{N+2} can take arbitrary values. Using the identification condition, we can fix

$$\Phi^{N-1} + i\Phi^N = \sqrt{2}. \tag{5.2.28}$$

Then we obtain the Kähler potential as follow:

$$\begin{aligned} &K[\Phi, \Phi^\dagger] \\ &= \frac{1}{\lambda^2}\ln\Big(1 + |\Phi^1|^2 + \cdots + |\Phi^{N-2}|^2 + |\Phi^{N+1}|^2 + |\Phi^{N+2}|^2 + \frac{1}{4}|\sum_{i=1}^{N-2}(\Phi^i)^2|^2\Big). \end{aligned} \tag{5.2.29}$$

2. Strong-weak duality
We have very interesting duality for $N = 2$. For $b = 0$, the target space is the double cover of $\mathrm{C}P^2$. For $|b| = 1$, the target space is Q^2, which is isomorphic to $\mathrm{C}P^1 \times \mathrm{C}P^1$.

For $b = \infty$, if we choose

$$\Phi^1 = \frac{1}{\sqrt{2}}, \qquad \Phi^2 = \pm\frac{i}{\sqrt{2}}, \tag{5.2.30}$$

the Kähler potential (5.2.29) reads

$$K[\Phi, \Phi^\dagger] = \frac{1}{\lambda^2}\ln(1 + |\Phi^3|^2 + |\Phi^4|^2), \tag{5.2.31}$$

which is the Kähler potential of $\mathrm{C}P^2$. Therefore, the target space is again the double cover of $\mathrm{C}P^2$, and coincides with that of $b = 0$ case exactly.

Let us replace the coordinates Φ^1, Φ^2 with Φ^3, Φ^4 in the constraint (5.2.19). With this operation, the deformation parameter b is replaced by $1/b$

$$b \quad \leftrightarrow \quad \frac{1}{b}. \tag{5.2.32}$$

Although the Kähler potential (5.2.26) for $N = 2$ has completely different form for the deformation parameter b and $1/b$, these two theories are equivalent. Thus, this model for $N = 2$ has strong-weak duality. The strong coupling region of the new model with (5.2.26) corresponds to the weak coupling region of the dual model. At the self-dual point $|b| = 1$, we have a model on $Q^2 \simeq \mathrm{C}P^1 \times \mathrm{C}P^1$. At $b = 0, \infty$, the target space of this theory is the double cover of $\mathrm{C}P^2$.

3. Renormalization group flow

Now, we will see the renormalization group flow for general value of b. We use the Kähler potential

$$K[\Phi, \Phi^\dagger] = \frac{1}{\lambda^2} \ln\left(1 + \vec{\Phi}\vec{\Phi}^\dagger + g\vec{\Phi}^2\vec{\Phi}^{\dagger 2}\right), \tag{5.2.33}$$

where $g = \frac{|b|^2}{4}$ in eq.(5.2.26). This Kähler potential gives the following Kähler metric and Ricci tensor:

$$\begin{aligned} g_{i\bar{j}} &= \frac{\delta_{i\bar{j}}}{1+\lambda^2\vec{\varphi}\vec{\varphi}^* + g(t)\lambda^4\vec{\varphi}^2\vec{\varphi}^{*2}} \\ &\quad + \frac{4g(t)\lambda^2\varphi^i\varphi^{*\bar{j}}(1+\lambda^2\vec{\varphi}\vec{\varphi}^*) - \lambda^2(\varphi_i^*\varphi_{\bar{j}} + 2g(t)\lambda^2\vec{\varphi}^2\varphi_i^*\varphi^{*\bar{j}} + 2g(t)\lambda^2\vec{\varphi}^{*2}\varphi^i\varphi_{\bar{j}})}{(1+\lambda^2\vec{\varphi}\vec{\varphi}^* + g(t)\lambda^4\vec{\varphi}^2\vec{\varphi}^{*2})^2}, \\ R_{i\bar{j}} &= (N+1)\lambda^2 g_{i\bar{j}} - \left[\frac{4g(t)\lambda^2\delta_{i\bar{j}}}{1+4g(t)\lambda^2\vec{\varphi}\vec{\varphi}^* + g(t)\lambda^4\vec{\varphi}^2\vec{\varphi}^{*2}}\right. \\ &\quad \left. + \frac{16g^2(t)\lambda^4\varphi^i\varphi^{*\bar{j}}\vec{\varphi}\vec{\varphi}^* - 16g^2(t)\lambda^2\varphi_i^*\varphi_{\bar{j}} - 8g^2(t)\lambda^4\left(\vec{\varphi}^2\varphi_i^*\varphi^{*\bar{j}} + \vec{\varphi}^{*2}\varphi^i\varphi_{\bar{j}}\right)}{(1+4g(t)\lambda^2\vec{\varphi}\vec{\varphi}^* + g(t)\lambda^4\vec{\varphi}^2\vec{\varphi}^{*2})^2}\right]. \end{aligned} \tag{5.2.34}$$

Here we use the rescaled fields as before. Note that Eq.(5.2.34) shows that this manifold is not an Einstein Kähler manifold unless g takes specific values: $g = 0, \frac{1}{4}$.

Substituting these metric and Ricci tensor for eq.(5.1.2), we obtain

$$\gamma = -\frac{\lambda^2}{4\pi^2}[(N+1) - 4g], \tag{5.2.35}$$

$$\beta(\lambda) = -\frac{\lambda^3}{4\pi^2}[(N+1) + 8g(2g-1)] + \frac{\lambda}{2}, \tag{5.2.36}$$

$$\beta(g) = \frac{4\lambda^2}{\pi^2}g^2(4g-1). \tag{5.2.37}$$

Figure 5.1 shows renormalization group flow. If we use the perturbation theory, the flow diagram is reliable only in the vicinity of the origin. Because we do not use the perturbation theory to derive the β function, our flow diagram is reliable in the entire region. The nontrivial UV fixed points of the flow are indicated by points A and B. Any points on the g-axis ($\lambda = 0$) are IR fixed points. The curve BAE shows the critical line, along which the direction of the flow is tangential to the line. The lines FBG($g = 0$) and CAD($g = 1/4$) both represent the renormalized trajectories. The critical line intersects with the renormalized trajectories at UV fixed points. We can define the continuum theories by using these UV fixed points. In this sense, NLσMs are renormalizable in three dimensions, at least in our truncated WRG equation.

The theory has different symmetry at the right- and left-side of the critical line. First, we consider continuous symmetry. The global symmetry on the renormalized trajectories, FBG($g = 0$) and CAD($g = 1/4$), is enhanced to $G = \mathbf{SU}(N+1)$ and $G = \mathbf{SO}(N+2)$, respectively. In other region of the flow diagram, the global symmetry is $\mathbf{SO}(N)\otimes\mathbf{U}(1)$. The global symmetry is realized manifestly in the right of the critical line BAE. At the left of the critical line, however, the enhanced global symmetries are spontaneously broken and there are Numbu-Goldstone bosons, although the $\mathbf{SO}(N)\otimes\mathbf{U}(1)$ symmetry remains manifest. Next, we consider a discrete transformation

$$\begin{aligned} \psi(t, x_1, x_2) &\rightarrow \psi'(t, x_1, x_2) = \gamma^2\psi(t, x_1, -x_2), \\ x_2 &\rightarrow x_2' = -x_2. \end{aligned} \tag{5.2.38}$$

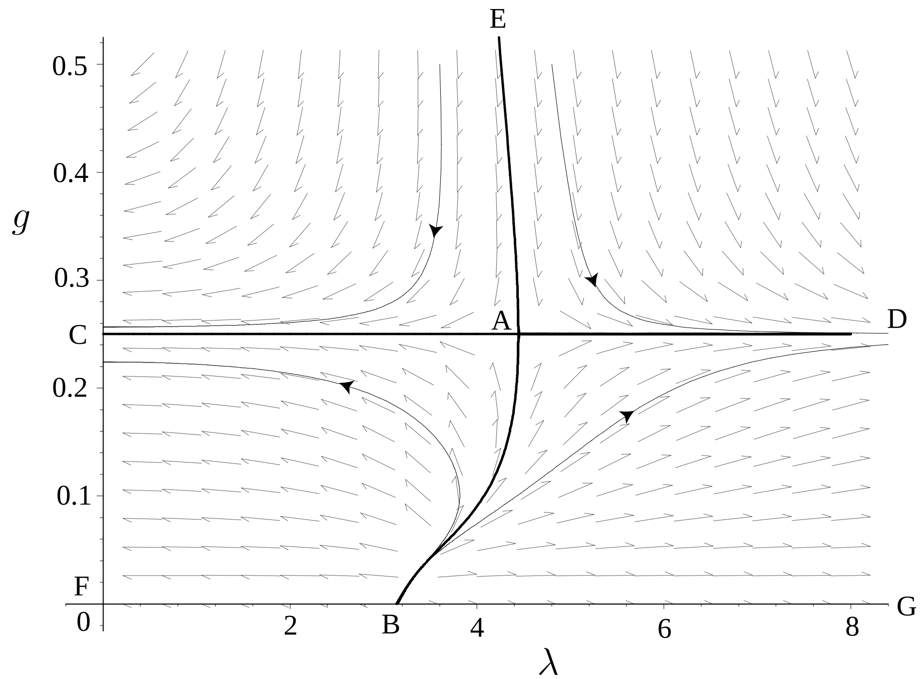

Figure 5.1: Renormalization group flows (The arrows point toward the infrared region.)

The Lagrangian (3.1.5) is invariant under this transformation. This transformation forbids fermion mass terms, so that it is broken spontaneously at the massive phase. At the right of the critical line, the fermion is massive and the discrete symmetry is spontaneously broken. At the left of the critical line, the discrete symmetry protects the fermion to be massless, and the supersymmetry keeps the bosons also massless [40].

5.3 The SU(N) symmetric solution of WRG equation

In this section, we investigate the conformal field theories defined as the fixed point of the β function

$$\begin{aligned}\beta &= \frac{1}{2\pi^2}R_{i\bar{j}} + \gamma\Big[\varphi^k g_{i\bar{j},k} + \varphi^{*\bar{k}} g_{i\bar{j},\bar{k}} + 2g_{i\bar{j}}\Big] + \frac{1}{2}\Big[\varphi^k g_{i\bar{j},k} + \varphi^{*\bar{k}} g_{i\bar{j},\bar{k}}\Big] \\ &= 0. \end{aligned} \tag{5.3.1}$$

To simplify, we assume $\mathbf{SU}(N)$ symmetric Kähler potential

$$K[\Phi,\Phi^\dagger] = \sum_{n=1}^{\infty} g_n(\vec{\Phi}\cdot\vec{\Phi}^\dagger)^n \equiv f(x), \tag{5.3.2}$$

where the chiral superfields $\vec{\Phi}$ have N components, g_n plays the role of the coupling constant and $x \equiv \vec{\Phi}\cdot\vec{\Phi}^\dagger$. Using the function $f(x)$, we derive the Kähler metric and Ricci tensor as follow:

$$g_{i\bar{j}} \equiv \partial_i\partial_{\bar{j}} K[\Phi,\Phi^\dagger] = f'\delta_{i\bar{j}} + f''\varphi_i^*\varphi_{\bar{j}},$$

$$\begin{aligned} R_{i\bar{j}} &\equiv -\partial_i \partial_{\bar{j}} tr \ln g_{i\bar{j}} \\ &= -\Big[(N-1)\frac{f''}{f'} + \frac{2f'' + f'''x}{f' + f''x}\Big]\delta_{i\bar{j}} \\ &\quad -\Big[(N-1)\Big(\frac{f^{(3)}}{f''} - \frac{(f'')^2}{(f')^2}\Big) + \frac{3f^{(3)} + f^{(4)}x}{f' + f''x} - \frac{(2f'' + f'''x)^2}{(f' + f''x)^2}\Big]\varphi_i^* \varphi_{\bar{j}}, \end{aligned} \tag{5.3.3}$$

where

$$f' = \frac{df}{dx}. \tag{5.3.4}$$

To normalize the kinetic term, we set

$$f'|_{x\approx 0} = 1 \Rightarrow g_1 = 1. \tag{5.3.5}$$

We substitute these metric and Ricci tensor for the β function (5.3.1) and compare the coefficients of $\delta_{i\bar{j}}$ and $\varphi^i \varphi^{*\bar{j}}$ respectively.

$$\frac{\partial}{\partial t} f' = \frac{1}{2\pi^2}\Big[(N-1)\frac{f''}{f'} + \frac{2f'' + f'''x}{f' + f''x}\Big] - 2\gamma(f' + f''x) - f''x, \tag{5.3.6}$$

$$\begin{aligned} \frac{\partial}{\partial t} f'' &= \frac{1}{2\pi^2}\Big[(N-1)\Big(\frac{f^{(3)}}{f''} - \frac{(f'')^2}{(f')^2}\Big) + \frac{3f^{(3)} + f^{(4)}x}{f' + f''x} - \frac{(2f'' + f'''x)^2}{(f' + f''x)^2}\Big] \\ &\quad -2\gamma(2f'' + f'''x) - (f'''x + f''). \end{aligned} \tag{5.3.7}$$

The second equation (5.3.7) is equivalent to the derivative of the first equation with respect to x, so that we use only the first equation.

To obtain a conformal field theory, we must solve the differential equation

$$\frac{\partial}{\partial t} f' = \frac{1}{2\pi^2}\Big[(N-1)\frac{f''}{f'} + \frac{2f'' + f'''x}{f' + f''x}\Big] - 2\gamma(f' + f''x) - f''x = 0. \tag{5.3.8}$$

The function $f(x)$ is a polynomial of infinite degree, and it is hard to solve it analytically. So we truncate the function $f(x)$ at order $O(x^4)$. From the normalization (5.3.5), the function $f(x)$ is

$$f(x) = x + g_2 x^2 + g_3 x^3 + g_4 x^4. \tag{5.3.9}$$

We substitute it for WRG eq.(5.3.6) and expand it around $x \approx 0$, then the equation (5.3.8) can be written

$$\begin{aligned} \frac{\partial}{\partial t} f' &= \frac{1}{2\pi^2}\Big[2(N+1)g_2 + \Big(6(N+2)g_3 - 4(N+3)g_2^2\Big)x \\ &\quad -\Big(18(N+7)g_2 g_3 - 8(N+7)g_2^3 - 12(N+2)g_4\Big)x^2\Big] \\ &\quad -2\gamma(1 + 4g_2 x + 9g_3 x^2) - (2g_2 x + 6g_3 x^2) + O(x^3) \\ &= 0. \end{aligned} \tag{5.3.10}$$

We choose the coupling constants and the anomalous dimension, which satisfy this equation.

$$\gamma = \frac{N+1}{2\pi^2} g_2, \tag{5.3.11}$$

$$g_3 = \frac{2(3N+5)}{3(N+2)} g_2^2 + \frac{2\pi^2}{3(N+2)} g_2, \tag{5.3.12}$$

$$\begin{aligned} g_4 &= 3g_2 g_3 - \frac{2(N+7)}{3(N+3)} g_2^3 + \frac{\pi^2}{N+3} g_3 \\ &= \frac{1}{3(N+2)(N+3)}\Big((16N^2 + 66N + 62)g_2^3 + 2\pi^2(6N+14)g_2^2 + 2\pi^4 g_2\Big). \end{aligned} \tag{5.3.13}$$

Note that all coupling constant is written in terms of g_2 only. Similarly, we can fix all coupling constant g_n using g_2 order by order. The function $f(x)$ with such coupling constants describe the conformal field theory and has one free parameter g_2. In other words, if we fix the value of g_2, we obtain a conformal field theory.

5.3.1 The explicit example of the novel conformal field theories

For general g_2, the power series (5.3.9) is a complicated function. In this subsection, we take a specific value to g_2, for which $f(x)$ takes an especially simple form.

We take

$$g_2 = -\frac{1}{2} \cdot \frac{2\pi^2}{N+1} \equiv -\frac{1}{2}a, \tag{5.3.14}$$

where

$$a \equiv \frac{2\pi^2}{N+1}. \tag{5.3.15}$$

We can express all other coupling constants from eqs.(5.3.12),(5.3.13)

$$\begin{aligned} g_3 &= \frac{1}{3}a^2, \\ g_4 &= -\frac{1}{4}a^3, \\ &\vdots \end{aligned}$$

These coupling constants show the function $f(x)$ is

$$f(x) = \frac{1}{a}\ln(1+ax), \tag{5.3.16}$$

and this is the Kähler potential of CP^N model. In fact, the function (5.3.16) satisfies the condition (5.3.8) exactly.

From this discussion, one of the novel $\mathbf{SU}(N)$ symmetric conformal field theory is equal to the UV fixed point theory of CP^N model for the specific value of g_2. In this case, the symmetry of this theory enhances to $\mathbf{SU}(N+1)$ because the CP^N model has the isometry $\mathbf{SU}(N+1)$.

Chapter 6

Other Nonperturbative Analysis

In §.5.1, we found the UV fixed point of some 3-dimensional non-linear sigma models In WRG approach, it is equivalent to the renormalizability of these models. To derive the WRG equation, we use the approximation to drop the higher derivative terms with more than three derivatives. However, eq.(5.1.18) shows the scalar fields is dimensionless around UV fixed point. Then there would be the possibility that third derivative interaction becomes marginal operator and disturbs the existence of UV fixed point.

In this chapter, to reconfirm the existence of the UV fixed point, we consider three-dimensional non-linear sigma model using large-N expansion which is the other nonperturbative method. This analysis will make clear the phase structure of CP^N and Q^N models.

6.1 CP^{N-1} model

Let's consider CP^{N-1} model using large-N method. In section 5.2, we show the model has the nontrivial UV fixed point and the phase transition. This model is also investigated in [35] using large-N expansion and is found that the β function has no next-to-leading correction.

We introduce the auxiliary field and rewrite the Kähler potential (5.2.3) as follow:

$$\mathcal{L} = \int d^4\theta(\Phi^i\Phi^{\dagger\bar{i}}e^V - cV). \qquad (6.1.1)$$
$$i = 1, \cdots, N$$

Here, $V(\theta, \bar{\theta}, x)$ is $U(1)$ gauge superfield and is defined by dimensional reduction from 4-dimensional $\mathcal{N} = 1$ to 3-dimensions. This gauge superfield is rewritten by the component fields in the Wess-Zumino gauge as follow:

$$V = \bar{\theta}\gamma^\mu\theta v_\mu - \bar{\theta}\theta M + \frac{1}{2}\theta\theta\bar{\theta}\bar{\lambda}^c + \frac{1}{2}\theta^c\theta^c\bar{\theta}^c\lambda - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}D,$$

where the scalar field M corresponds to the fourth component of 4-dimensional vector field. In order to make the Lagrangian of order N in the large N limit, we take the coefficient Fayet-Illiopoulous D-term $c = N/g^2$ and keep g^2 fixed when we take the limit of $N \to \infty$.

This model has global $SU(N)$ symmetry coming from the isometry of the target manifold. Furthermore, we assign $U(1)$ charge

$$[\Phi] = 1, \qquad [\Phi^\dagger] = -1. \qquad (6.1.2)$$

The action is invariant under this gauged $U(1)$ symmetry.

The action (6.1.1) is rewritten using the component fields:

$$\begin{aligned}\mathcal{L} &= \partial_\mu\varphi^{*i}\partial^\mu\varphi^i + i\bar{\psi}\partial_\mu\gamma^\mu\psi^i + F^iF^{*i} - [i(\varphi^{*i}\partial_\mu\varphi^i - \varphi^i\partial_\mu\varphi^{*i}) - \bar{\psi}^i\gamma_\mu\psi^i]v^\mu + v^\mu v_\mu\varphi^{*i}\varphi^i \\ &\quad -M^2\varphi^{*i}\varphi^i - M\bar{\psi}^i\psi^i - D\varphi^{*i}\varphi^i + \frac{N}{g^2}D + (\varphi^i\bar{\psi}^i\bar{\lambda}^c + \varphi^{*i}\lambda^c\psi^i) \qquad (6.1.3)\end{aligned}$$

Since the gauge field V or its component fields do not have kinetic term, they are auxiliary field and do not propagate in the tree approximation. If we eliminate all auxiliary fields using their equation of motion, we obtain the constraints

$$\varphi^i\varphi^{*i} = \frac{N}{g^2}, \qquad M = -\frac{g^2}{N}\bar{\psi}^i\psi^i, \qquad v^\mu = -\frac{g^2}{N}\bar{\psi}^i\gamma^\mu\psi^i, \qquad \varphi^i\bar{\psi}^i = \varphi^{*i}\psi^i = 0. \qquad (6.1.4)$$

The first equation means fields φ^i is constrained on the $(2N-1)$ dimensional sphere S^{2N-1}. Furthermore, the gauge transformation of gauge field v_μ eliminate a common phase of φ^i. Thus, the target manifold reduces to the complex projective space CP^{N-1}. In previous chapters, we used the Lagrangian written in terms of independent field, obtained by eliminating auxiliary field and dependent fields by solving the constraints.

To investigate the phase structure, let us calculate the effective potential. The partition function of this model can be written as

$$Z = \int D\Phi^i D\Phi^{\dagger\bar{i}} DV e^{i\int d^3x\mathcal{L}}, \qquad (6.1.5)$$

where the Lagrangian $\mathcal{L}$ is given eq.(6.1.1). We divide the dynamical field into the vacuum expectation value and the fluctuation, $\varphi^i = \tilde{\varphi}^i + \varphi'^i$. The fluctuation field has the constraint

$$\int d^3x\varphi'^i(x) = 0. \qquad (6.1.6)$$

Integrated over the fluctuation field φ'^i, we obtain the effective action:

$$\begin{aligned}Z &= \int DV e^{iS_{eff}}, \\ S_{eff} &= -\frac{N}{i}Tr\ln(\nabla_B + \bar{\lambda}\nabla_F^{-1}\lambda) + \frac{N}{i}Tr\ln\nabla_F + \frac{N}{g^2}D - \tilde{\varphi}^{*i}(M^2+D)\tilde{\varphi}^i + (\lambda\text{-dependent terms}).\end{aligned}$$

Here,

$$\nabla_B = D^\mu D_\mu + (M^2 + D), \qquad \nabla_F = iD_\mu\gamma - M, \qquad D_\mu = \partial_\mu - iv_\mu.$$

We denote the vacuum expectation values of each auxiliary field as follow:

$$\langle M(x)\rangle = M_0, \qquad \langle D(x)\rangle = D_0, \qquad \langle\text{the others}\rangle = 0.$$

Then we obtain the effective potential.

$$\begin{aligned}\frac{V}{N} &= \int\frac{d^3k}{(2\pi)^3}\ln(k_\mu k^\mu + M_0^2 + D_0) - \int\frac{d^3k}{(2\pi)^3}tr\ln(-k^\mu\gamma_\mu - M_0) \\ &\quad + \frac{1}{N}\tilde{\varphi}^{i*}(M_0^2 + D_0)\tilde{\varphi}^i - \frac{1}{g^2}D_0 \\ &= -\frac{1}{6\pi}|M_0^2 + D_0|^{\frac{3}{2}} + \frac{1}{6\pi}|M_0|^3 + \frac{1}{N}(M_0^2 + D_0)|\tilde{\varphi}^i|^2 + (\frac{\Lambda}{2\pi^2} - \frac{1}{g^2})D_0. \\ &= -\frac{1}{6\pi}|M_0^2 + D_0|^{\frac{3}{2}} + \frac{1}{6\pi}|M_0|^3 + \frac{1}{N}(M_0^2 + D_0)|\tilde{\varphi}^i|^2 + \frac{m}{4\pi}D_0\end{aligned}$$

Here, we use the renormalization group invariant mass m which is given by the renormalized coupling constant and the renormalization point μ as follow:

$$\begin{aligned} \frac{\mu}{g_R^2} &= \frac{1}{g^2} - \frac{1}{2\pi^2}\Lambda + \frac{\mu}{4\pi}, \\ m &\equiv \mu(1 - \frac{4\pi}{g_R^2}). \end{aligned} \tag{6.1.7}$$

We consider the stationary condition of the effective potential (6.1.7) for two cases $M_0^2 + D_0 > 0$ and $M_0^2 + D_0 < 0$ separately:
The case of $M_0^2 + D_0 > 0$

$$\begin{aligned} \frac{1}{N}\frac{\partial V}{\partial M_0} &= -2M_0\Big(\frac{1}{4\pi}|M_0^2 + D_0|^{\frac{1}{2}} - \frac{1}{4\pi}|M_0|^{\frac{1}{N}}|\tilde{\varphi}^i|^2\Big) = 0, \\ \frac{1}{N}\frac{\partial V}{\partial D_0} &= -\frac{1}{4\pi}|M_0^2 + D_0|^{\frac{1}{2}} + \frac{1}{N}|\tilde{\varphi}^i|^2 + \frac{m}{4\pi} = 0. \end{aligned}$$

The case of $M_0^2 + D_0 < 0$

$$\begin{aligned} \frac{1}{N}\frac{\partial V}{\partial M_0} &= -2M_0\Big(-\frac{1}{4\pi}|M_0^2 + D_0|^{\frac{1}{2}} - \frac{1}{4\pi}|M_0|^{\frac{1}{N}}|\tilde{\varphi}^i|^2\Big) = 0, \\ \frac{1}{N}\frac{\partial V}{\partial D_0} &= \frac{1}{4\pi}|M_0^2 + D_0|^{\frac{1}{2}} + \frac{1}{N}|\tilde{\varphi}^i|^2 + \frac{m}{4\pi} = 0. \end{aligned}$$

We put $\langle D_0 \rangle = 0$ to protect supersymmetry at stationary point because we expect supersymmetric vacuum has the lowest energy. Then we find the following two cases satisfying above stationary conditions.

1. symmetric phases ($|\tilde{\varphi}^i|^2 = 0, M_0 = m$)

   This phase is SU(N) symmetric and also keeps gauged $U(1)$ symmetry. The N scalar and spinor fields have mass m because of the vacuum expectation value of M_0.

   From eq.(6.1.4), the field M is meson, which is bound state of fermion and anti-fermion. In fact, we calculate the two point function of all fields in Appendix.A, and find the field M have twice mass of the dynamical fields.

2. broken phases ($|\tilde{\varphi}^i|^2 = \frac{N}{4\pi}|m|, M_0 = 0$)

   In this phase, global $SU(N)$ symmetry is broken down to $SU(N-1)$ and there are $N-1$ massless Nambu-Goldstone bosons and their superpartners. Furthermore, gauged $U(1)$ symmetry is also broken because the dynamical fields, which have $U(1)$ charge, have nonvanishing vacuum expectation values.

To derive β function, we calculate the next-to-leading correction of the gap equation in symmetric phase. The β function is determined by the UV behavior and it is same in both phases. We summarize the Feynmann rule in symmetric phase in Appendix A. In this notation, only the propagators of auxiliary field have the factor $1/N$.

The gap equation is given by equation of motion of the auxiliary field D.

$$\frac{N}{g^2} = \int \frac{d^3p}{(2\pi)^3}\langle \varphi^i(p)\varphi^{*i}(0)\rangle. \tag{6.1.8}$$

In the leading order of $1/N$, the gap equation is written as follow:

$$\frac{N}{g^2} = \int \frac{d^3p}{(2\pi)^3}\frac{iN}{p^2 - m^2}. \tag{6.1.9}$$

We renormalize the coupling constant to absorb the divergence using UV cutoff and obtain the leading order β function of $1/N$ expansion.

$$\beta(g_R) = -\frac{1}{8\pi} g_R^3 + \frac{1}{2} g_R \tag{6.1.10}$$

We calculate the next-to-leading Feynman diagrams, shown Fig.6.1, and find the next-to-leading correction of β function vanishes.

Finally, let's compare the result with the WRG result of CP^{N-1} case:

$$\beta(\lambda) = -\frac{N}{4\pi^2}\lambda^3 + \frac{1}{2}\lambda. \tag{6.1.11}$$

We define the 't Hooft coupling $g^2 \equiv N\lambda^2$, and find both β functions are same. This β function is shown in Fig.6.2. From the eq.(6.1.7), the region where the coupling constant is smaller than the critical point g_c corresponds to broken phase.

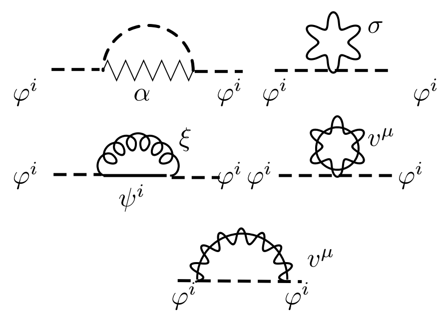

Figure 6.1: The Feynman diagrams contributing to the next-to-leading order correction of φ propagator

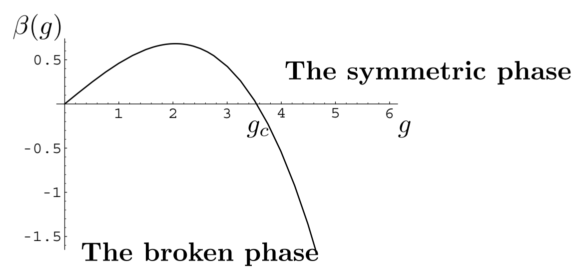

Figure 6.2: The β function of the coupling constant g.

6.2 Q^{N-2} model

Similarly to CP^{N-1} model, let's consider Q^{N-2} model using large-N expansion. The β function given by the WRG approach is rewritten using 't Hooft coupling as follow:

$$\begin{aligned} \beta(\lambda) &= -\frac{N-2}{4\pi^2}\lambda^3 + \frac{1}{2}\lambda, \\ \Rightarrow \beta(g) &= -\frac{g^3}{4\pi^2}(1 - \frac{2}{N}) + \frac{1}{2}g. \end{aligned} \tag{6.2.1}$$

This result shows this model has the next-to-leading correction of β function in $1/N$ expansion.

Let's consider Q^{N-2} model by using large-N expansion. This model has two constraints eq.(5.2.10) and eq.(5.2.11), then we introduce the two kinds of auxiliary fields as follow:

$$\mathcal{L} = \int d^4\theta(\Phi^i\Phi^{\dagger\bar{i}}e^V - cV) + \frac{1}{2}\Big(\int d^2\theta\Phi_0\Phi^i\Phi^i + \int d^2\bar{\theta}\Phi_0^\dagger\Phi^{\dagger i}\Phi^{\dagger i}\Big), \tag{6.2.2}$$

where $V(\theta,\bar{\theta},x)$ is $U(1)$ gauge superfield and $\Phi_0, \Phi_0^\dagger$ are chiral- and antichiral- superfields respectively. Components of the new auxiliary field Φ_0 is defined by

$$\Phi_0(y) \quad = \quad A_0(y) + \sqrt{2}\theta\psi_0^i(y) + \theta\theta F_0(y)$$

The action in terms of component fields is given by

$$\begin{aligned}
\mathcal{L} \quad = \quad & \partial_\mu\varphi^{*i}\partial^\mu\varphi^i + i\bar{\psi}\partial_\mu\gamma^\mu\psi^i + F^iF^{*i} - [i(\varphi^{*i}\partial_\mu\varphi^i - \varphi^i\partial_\mu\varphi^{*i}) - \bar{\psi}^i\gamma_\mu\psi^i]v^\mu + v^\mu v_\mu\varphi^{*i}\varphi^i \\
& -M^2\varphi^{*i}\varphi^i - M\bar{\psi}^i\psi^i - D\varphi^{*i}\varphi^i + \frac{N}{g^2}D + (\varphi^i\bar{\psi}^i\lambda^c + \varphi^{*i}\lambda^c\psi^i) \\
& +\frac{1}{2}(F_0\varphi^i\varphi^i + F_0^*\varphi^{*i}\varphi^{*i}) - (\bar{\psi}_0^c\psi^i\varphi^i + \bar{\psi}^i\psi_0^c\varphi^{*i}) \\
& +A_0(\varphi^iF^i - \frac{1}{2}\bar{\psi}^{ci}\psi^i) + A_0^*(\varphi^{*i}F^{*i} - \frac{1}{2}\bar{\psi}^i\psi^{ci}).
\end{aligned} \tag{6.2.3}$$

Similarly to CP^{N-1} case, integrate out the fluctuation field and put the vacuum expectation values as follows:

$$\begin{aligned}
\langle M(x)\rangle \quad &= \quad M_0, \quad \langle D(x)\rangle = D_0, \\
\langle F_0(x)\rangle \quad &= \quad F_0, \quad \langle A_0(x)\rangle = A_0, \quad \langle\text{the others}\rangle = 0.
\end{aligned}$$

Then we obtain the effective potential.

$$\begin{aligned}
\frac{V}{N} \quad = \quad & \int\frac{d^3k}{(2\pi)^3}\ln(k_\mu k^\mu + M_0^2 + D_0 + |A_0|^2 - |F_0|^2) - \int\frac{d^3k}{(2\pi)^3}tr\ln(-k^\mu\gamma_\mu - M_0 - |A_0|^2) \\
& +\frac{1}{N}\tilde{\varphi}^{i*}(M_0^2 + D_0 + A_0A_0^*)\tilde{\varphi}^i - \frac{1}{g^2}D_0 - \frac{1}{2N}(F_0\varphi^i\varphi^i + F_0^*\varphi^{*i}\varphi^{*i}) \\
= \quad & -\frac{1}{6\pi}\frac{1}{2}\Big[|M_0^2 + D_0 + |A_0|^2 + |F_0||^{\frac{3}{2}} + |M_0^2 + D_0 + |A_0|^2 - |F_0||^{\frac{3}{2}}\Big] \\
& +\frac{1}{6\pi}\frac{1}{2}\Big[|M_0 + |A_0||^3 + |M_0 - |A_0||^3\Big] - (\frac{1}{g^2} - \frac{1}{2\pi^2}\Lambda)D_0 \\
& +\frac{1}{N}\Big(M_0^2\varphi^{*i}\varphi^i + A_0^*A_0\varphi^{*i}\varphi^i + D_0\varphi^{*i}\varphi^i - \frac{1}{2}(F_0\varphi^i\varphi^i + F_0^*\varphi^{*i}\varphi^{*i})\Big), \\
= \quad & -\frac{1}{6\pi}\frac{1}{2}\Big[|M_0^2 + D_0 + |A_0|^2 + |F_0||^{\frac{3}{2}} + |M_0^2 + D_0 + |A_0|^2 - |F_0||^{\frac{3}{2}}\Big] \\
& +\frac{1}{6\pi}\frac{1}{2}\Big[|M_0 + |A_0||^3 + |M_0 - |A_0||^3\Big] + \frac{m}{4\pi}D_0 \\
& +\frac{1}{N}\Big(M_0^2\varphi^{*i}\varphi^i + A_0^*A_0\varphi^{*i}\varphi^i + D_0\varphi^{*i}\varphi^i - \frac{1}{2}(F_0\varphi^i\varphi^i + F_0^*\varphi^{*i}\varphi^{*i})\Big).
\end{aligned}$$

We use the same renormalization of coupling constant and the same invariant mass as CP^{N-1} case. If we eliminate auxiliary fields using their equation of motion, we obtain the constraint eqs.(6.1.4) and following equations.

$$\varphi^i\varphi^i = \varphi^{*i}\varphi^{*i} = 0, \quad \varphi^i\psi^i = \varphi^{*i}\bar{\psi}^i = 0, \quad A_0 = -\frac{g^2}{2N}\bar{\psi}^{ci}\psi^i \quad A_0^* = -\frac{g^2}{2N}\bar{\psi}^i\psi^{ci}. \tag{6.2.4}$$

Now we obtain the stationary conditions of the effective potential. If the all contents of absolute value are positive, the stationary points satisfy the following equations.

$$\begin{aligned}
\frac{1}{N}\frac{\partial V}{\partial M_1} &= -\frac{1}{4\pi}M_1\left[|M_1^2+M_2^2+D_1|^{\frac{1}{2}}+|M_1^2+M_2^2+D_2|^{\frac{1}{2}}+2\sqrt{2}|M_1|\right] \\
&\quad +\frac{1}{N}M_1(\varphi_1^2+\varphi_2^2)=0 \\
\frac{1}{N}\frac{\partial V}{\partial M_2} &= -\frac{1}{4\pi}M_1\left[|M_1^2+M_2^2+D_1|^{\frac{1}{2}}+|M_1^2+M_2^2+D_2|^{\frac{1}{2}}+2\sqrt{2}|M_2|\right] \\
&\quad +\frac{1}{N}M_2(\varphi_1^2+\varphi_2^2)=0 \\
\frac{1}{N}\frac{\partial V}{\partial D} &= -\frac{1}{8\pi}\left[|M_1^2+M_2^2+D_1|^{\frac{1}{2}}+|M_1^2+M_2^2+D_2|^{\frac{1}{2}}\frac{m}{8\pi}+\frac{1}{2N}(\varphi_1^2+\varphi_2^2)=0\right. \\
\frac{1}{N}\frac{\partial V}{\partial F_0} &= \frac{1}{16\pi}\left[e^{-2i\theta}|M_1^2+M_2^2+D_1|^{\frac{1}{2}}-e^{-2i\theta}|M_1^2+M_2^2+D_2|^{\frac{1}{2}}\right]-\frac{1}{2N}\varphi^i\varphi^i=0 \\
\frac{1}{N}\frac{\partial V}{\partial F_0^*} &= \frac{1}{16\pi}\left[e^{2i\theta}|M_1^2+M_2^2+D_1|^{\frac{1}{2}}-e^{2i\theta}|M_1^2+M_2^2+D_2|^{\frac{1}{2}}\right]-\frac{1}{2N}\varphi^{*i}\varphi^{*i}=0
\end{aligned}$$

where we defined new parameters as follow:

$$\begin{aligned}
F_0 &= |F_0|e^{2i\theta}, \quad \frac{1}{\sqrt{2}}(M+|A_0|)=M_1, \quad \frac{1}{\sqrt{2}}(M-|A_0|)=M_2, \\
D-|F_0| &= D_1, \quad D+|F_0|=D_2, \quad \varphi^i e^{i\theta}=\frac{1}{\sqrt{2}}(\varphi_1+i\varphi_2).
\end{aligned}$$

Then we find three kinds of phases.

1. Chern-Simons phases

   The case of $M_0=m$, $|A_0|=0$, $|m^2+D_1|^{\frac{1}{2}}=\frac{4\pi}{N}\varphi_1^2$, $|m^2+D_2|^{\frac{1}{2}}=\frac{4\pi}{N}\varphi_2^2+m$.

   In this phase, both global $SO(N)$ and gauged $U(1)$ symmetries are preserved. N dynamical scalar and spinor fields obtain same masses m due to the vacuum expectation value of M_0. Similarly to CP^{N-1} case, the auxiliary fields have twice masses of the dynamical fields. The auxiliary field with the vacuum expectation value M corresponds to the bound state of the fermion and anti-fermion.

   The gauge field parts of the effective action have the Chern-Simons interaction term in this phase.

$$NS_{eff}\sim\int\frac{d^3p}{(2\pi)^3}-\frac{1}{2}v^{\mu}(-p)\frac{N}{4\pi}(p^2\eta_{\mu\nu}-p_{\mu}p_{\nu}-2mi\epsilon_{\mu\nu\rho}p^{\rho})I(p)v^{\mu}, \tag{6.2.5}$$

   where,

$$I(p)=\frac{\sqrt{-p^2}}{\arctan(\sqrt{\frac{-p^2}{4m}})}. \tag{6.2.6}$$

2. Higgs phases

   The case of $M_0=0$, $|A_0|=m$, $|m^2+D_1|^{\frac{1}{2}}=\frac{4\pi}{N}\varphi_1^2$, $|m^2+D_2|^{\frac{1}{2}}=\frac{4\pi}{N}\varphi_2^2+m$.

   In this phase, global $SO(N)$ symmetry is protected and N dynamical fields have masses due to the vacuum expectation value of A_0. However gauged $U(1)$ symmetry is broken because the superfields Φ_0, which has $U(1)$ charge, has nonvanishing vacuum expectation value.

   The auxiliary field A_0 has the nonvanishing vacuum expectation value. The gauge bosons acquire masses through the Higgs mechanism and the imaginary part of A_0 is removed from the theory. From eq. of motion for A_0, we find the fields A_0^* corresponds to the pair of fermion and fermion:

$$A_0^*=\frac{g^2}{4N}\bar{\psi}^{ci}\psi^i. \tag{6.2.7}$$

3. broken phases

   The case of $M_0 = 0$, $|A_0| = 0$, $|D_1|^{\frac{1}{2}} = \frac{4\pi}{N}\varphi_1^2 + m$, $|D_2|^{\frac{1}{2}} = \frac{4\pi}{N}\varphi_2^2 + m$.

   In this phase, both global $SO(N)$ and gauged $U(1)$ symmetries are broken. There are $N-2$ massless Nambu-Goldstone bosons and their superpartners.

From now on, we calculate the β function in Chern-Simon phase.

Similarly to CP^{N-1} model, the gap equation is given by

$$\frac{N}{g^2} = \int \frac{d^3p}{(2\pi)^3}\langle\varphi^i(p)\varphi^{*i}(0)\rangle. \tag{6.2.8}$$

The gap equation of the leading order of $1/N$ expansion is

$$\frac{N}{g^2} = \int \frac{d^3p}{(2\pi)^3}\frac{iN}{p^2 - m^2}. \tag{6.2.9}$$

Then the β function of the coupling constant is given by

$$\beta(g_R) = -\frac{1}{8\pi}g_R^3 + \frac{1}{2}g_R. \tag{6.2.10}$$

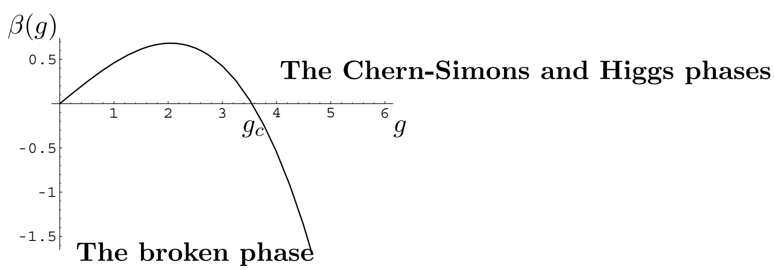

Figure 6.3: The β function of the coupling constant g.

In the next-to-leading order of $1/N$, we calculate eight Feynmann diagrams in Fig.6.4. We introduce the UV cutoff Λ then the gap equation is given by

$$\begin{aligned}\frac{1}{g^2} &= \frac{1}{2\pi^2}\Big((\Lambda - \frac{\pi}{2}m) - \frac{2}{N}(\Lambda - \frac{\pi}{2}2m)\Big)\\ &= \frac{\Lambda}{2\pi}(1 - \frac{2}{N}) - \frac{m}{4\pi}(1 - \frac{4}{N}).\end{aligned} \tag{6.2.11}$$

In three dimensional Q^{N-2} model, the coupling constant g has the canonical dimension $\frac{1}{2}$. Then the β function of such coupling constant depends on the subtraction of finite quantity [36]. We define the renormalized coupling constant g_R by

$$\frac{1}{g^2} - (1 - \frac{2}{N})\frac{\Lambda - \mu}{2\pi^2} = \frac{\mu}{g_R^2}, \tag{6.2.12}$$

and obtain the next-to-leading order β function.

$$\beta = \frac{1}{2}g_R - \frac{1}{4\pi^2}(1 - \frac{2}{N})g_R^3 \tag{6.2.13}$$

This β function coincides with the WRG result and is shown in Fig.6.3.

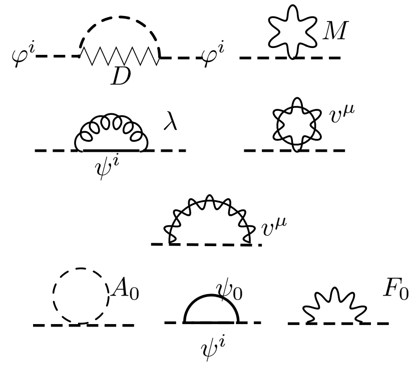

Figure 6.4: The Feynmann diagrams contributing to the next-to-leading order correction of φ propagetor

Chapter 7

Conclusion

We discussed the lower dimensional supersymmetric nonlinear sigma model by means of nonperturbative method. In particular, we used the Wilsonian renormalization group method.

First, we derived the Wilsonian renormalization group flow equation for 2- and 3-dimensional $\mathcal{N}=2$ supersymmetric nonlinear sigma models. From the point of view of Wilsonian renormalization group this model corresponds to the next-to-leading order approximation of derivative expansion for scalar fields. In general, we must consider both the local potential term and the nonlinear sigma model term in the next-to-leading order approximation. By imposing $\mathcal{N}=2$ supersymmetry, we can safely drop the local potential term. The β function of the target metric consists of the part from one-loop diagram and the part of field rescaling.

Next, we constructed the fixed point theory of the WRG equation. To find the conformal field theory, we have assumed $\mathbf{U}(N)$ symmetry to reduce the coupled partial differential equations to an ordinary differential equation. The new class of conformal field theories have one parameter, a, corresponding to the anomalous dimension of the scalar field. These conformal field theories are well behaved for positive a, while they have curvature singularities at the boundary for $a < 0$. We obtained the Lagrangian explicitly for $N = 1$. The target space in this case is a semi-infinite cigar, with one dimension compactified to a circle. This theory has been discussed in another context by Witten as a model of a two-dimensional black hole [15], and it was subsequently generalized by Kiritsis, Kounnas and Lust to bescribe consistent backgrounds of superstrings in the presence of a dilaton [38]. Their nonvanishing dilaton field assigns the same Weyl transformation law as our nonvanishing anomalous dimension for the target coordinates.

Third, we considered three dimensional nonlinear sigma model within WRG approach. The model is nonrenormalizable in perturbative theory.

We examined the sigma models whose target spaces are the Einstein-Kähler manifolds. We have shown that the theories whose target spaces are compact Einstein-Kähler manifolds with positive scalar curvature have two fixed points. One of them is the Gaussian IR fixed point and the other is the nontrivial UV fixed point. We can define the continuum limit at this UV fixed point by the fine-tuning of the bare coupling constant. In this sense, NLσMs on Einstein-Kähler manifolds with positive scalar curvature are renormalizable in three dimensions. At this point, the scaling dimension of all superfields is zero, as in the two dimensional theories. On the other hand, the theories whose target spaces are Einstein-Kähler manifolds with negative scalar curvature (for example D^2 with the Poincaré metric) have only an Gaussian IR fixed point, and cannot have a continuum limit.

We also studied a new model with two parameters, whose target space is not an Einstein-Kähler manifold. This theory has two nontrivial fixed points, corresponding to the UV fixed points of the $\mathrm{C}P^N$ and Q^N models. In the theory spaces of this model, there are a critical surface and two renormalized trajectories, and the theory has four phases. We have also shown that the model possesses strong-weak duality for $N = 2$. In order to study the phase structure, we have to introduce the auxiliary fields [44]. This is left for future work.

We constructed a class of the $\mathbf{SU}(N)$ symmetric conformal field theory using the WRG equation. This class has one free parameter, g_2, corresponding to the anomalous dimension of the scalar fields. If we choose a specific value of this parameter, we recover the conformal field theory defined at the UV fixed point of the $\mathrm{C}P^N$ model and the symmetry is enhanced to $\mathbf{SU}(N+1)$.

Finally, we discussed the three dimensional nonlinear sigma model by using another nonperturbative method, large-N expansion. We investevated CP^{N-1} and Q^{N-2} model and found that the β function in the next-to-leading order coincides with the WRG result. We also studied the phase structure of both model. The CP^{N-1} model has two phases; $SU(N)$ symmetric massive and $SU(N)$ broken massless phases. On the other hand, the Q^{N-2} model has three phases; Chern-Simons, Higgs and $SO(N)$ broken phases. The global $SO(N)$ symmetry remains unbroken in both Chern-Simons and Higgs phases, while the gauged $U(1)$ symmetry is broken in Higgs phase.

As a conclusion, we investigated the supersymmetric nonlinear sigma model and found that the WRG analysis is very powerful to reveal various nonperturbative aspects of field theories in two- and three-dimensions.

Appendix A

The Feynmann rules

We give the Feynmann rules of CP^{N-1} and Q^{N-2} models in symmetric phase.

The CP^{N-1} model

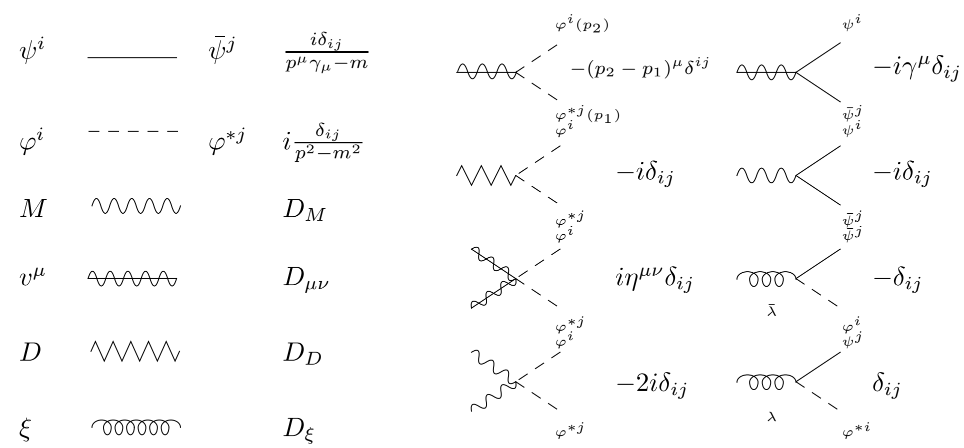

Figure A.1: The Feynman rules of CP^{N-1} model in Minkowski spaces

Here,

$$
\begin{aligned}
D_M &= \frac{4\pi i}{N}\frac{1}{p^2-4m^2}I(p^2), \\
D_{\mu\nu} &= \frac{4\pi}{N}\frac{1}{p^2-4m^2}[-i\eta_{\mu\nu} + (1+\alpha+\frac{4m^2\alpha}{p^2})\frac{ip_\mu p_\nu}{p^2} + \frac{2m\epsilon_{\mu\nu\lambda}p^\lambda}{p^2}]I(p^2), \\
D_D &= \frac{4\pi i}{N}I(p^2), \\
D_\xi &= -\frac{8\pi i}{N}\frac{1}{p^\mu\gamma_\mu + 2m}I(p^2), \\
I(p) &= \frac{\sqrt{-p^2}}{\arctan(\sqrt{\frac{-p^2}{4m}})}.
\end{aligned}
$$

The Q^{N-2} model

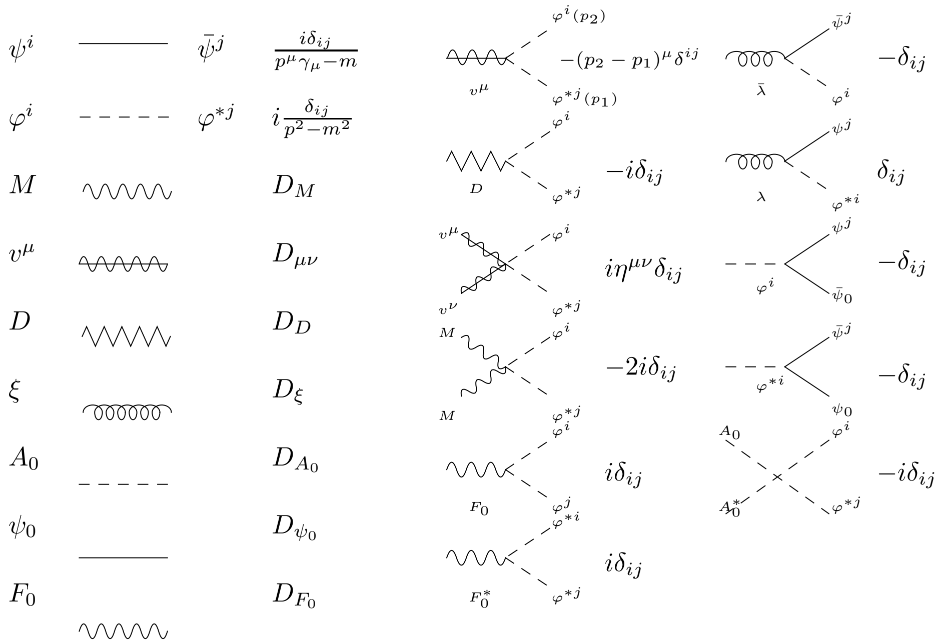

Figure A.2: The Feynman rule of Q^{N-2} model in Minkowski spaces. We show only the parts which are concerned the next-to-leading correction of φ^i propagator.

Here,

$$
\begin{aligned}
D_M &= \frac{4\pi i}{N}\frac{1}{p^2-4m^2}I(p^2),\\
D_{\mu\nu} &= \frac{4\pi}{N}\frac{1}{p^2-4m^2}[-i\eta_{\mu\nu}+(1+\alpha+\frac{4m^2\alpha}{p^2})\frac{ip_\mu p_\nu}{p^2}+\frac{2m\epsilon_{\mu\nu\lambda}p^\lambda}{p^2}]I(p^2),\\
D_D &= \frac{4\pi i}{N}I(p^2),\\
D_\xi &= -\frac{8\pi i}{N}\frac{1}{p^\mu\gamma_\mu+2m}I(p^2),\\
D_{A_0} &= \frac{8\pi i}{N}\frac{1}{p^2-4m^2}I(p^2),\\
D_{\psi_0} &= \frac{8\pi i}{N}\frac{1}{p^\mu\gamma_\mu+2m}I(p^2)\\
D_{F_0} &= \frac{8\pi i}{N}I(p^2).
\end{aligned}
$$

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