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Author(s)	Tanabe, Hiroki
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## ON ESTIMATES FOR DERIVATIVES OF SOLUTIONS OF WEIGHTED ELLIPTIC BOUNDARY VALUE PROBLEMS

HIROKI TANABE

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It is well known that any solution  $u$  of the parabolic differential equation

$$\partial u / \partial t - \partial^2 u / \partial x^2 = 0$$

belongs to a Gevrey's class, namely for any compact set  $K$  there exist constants  $M_0$  and  $M$  such that

$$\sup_{(x,t) \in K} |(\partial / \partial t)^m (\partial / \partial x)^n u(x, t)| \leq M_0 M^{m+n} \Gamma(n+1) \Gamma(2m+1)$$

for any non-negative integers  $m$  and  $n$ . In this paper this result is generalized to more general weighted elliptic boundary value problems of first order in  $t$  (cf. [2] for the definition of weighted ellipticity):

$$D_t u(x, t) + A(x, t, D_x) u(x, t) = f(x, t), \quad x \in \Omega, \quad (0.1)$$

$$B_j(x, t, D_x) u(x, t) = g_j(x, t), \quad x \in \partial\Omega, \quad j = 1, \dots, m, \quad (0.2)$$

where  $A(x, t, D_x)$  is a linear differential operator of order  $2m$  and  $\Omega$  is a bounded domain in the  $n$ -dimensional Euclidean space. The boundary system  $\{B_j(x, t, D_x)\}$  is assumed to consist of differential operators of order  $< 2m$ ; however, it need not be normal. Throughout this paper a function  $h$  (resp.  $\phi$ ) of  $(x, t)$  (resp.  $x$ ) is said to belong to Gevrey's class  $G(\sigma, \tau)$  (resp.  $G(\tau)$ ),  $\sigma, \tau \geq 1$ , if with some constants  $M_0$  and  $M$

$$\sup_{x \in \Omega, t} |D_t^l D_x^\kappa h(x, t)| \leq M_0 M^{l+|\kappa|} \Gamma(|\kappa| \sigma + 1) \Gamma(l \tau + 1)$$

$$(\text{resp. } \sup_{x \in \Omega} |D_x^\kappa \phi(x)| \leq M_0 M^{|\kappa|} \Gamma(|\kappa| \sigma + 1))$$

for any  $l$  and  $\kappa$ , and the boundary  $\partial\Omega$  is said to be of Gevrey's class  $G(\sigma)$  if some open part of  $\partial\Omega$  containing each point of  $\partial\Omega$  is mapped onto a part of a hyperplane by means of a one-to-one mapping of the class  $G(\sigma)$ . It will always be assumed that the elliptic boundary systems  $(\pm D_y^{2m} + A(x, t, D_x), \{B_j(x, t, D_x)\}, \Omega \times \{y: -\infty < y < \infty\})$  satisfy the

Complementing Condition ([3]) for each fixed  $t$ . Under this assumption it will be shown that any solution of (0.1)–(0.2) belongs to Gevrey's class  $G(\sigma, 2m\sigma)$  provided that all the coefficients of  $A$ ,  $\{B_j\}$  and  $f$ ,  $\{g_j\}$  belong to the class  $G(\sigma, 2m\sigma)$  and  $\partial\Omega$  is of the class  $G(\sigma)$ . This result gives an affirmative answer to the conjecture of J.L. Lions and E. Magenes [4].

In section 3 the property of the solution considered as a function of  $t$  with values in  $H_{2m}(\Omega)$  will be investigated. The main result in that section is that if all the coefficients of  $A$ ,  $\{B_j\}$ , and  $f$ ,  $\{g_j\}$  belong to Gevrey's class  $G(\tau)$ ,  $\tau \geq 1$ , as functions of  $t$ , then so does the solution  $u$  of (0.1)–(0.2) as a function of  $t$  with values in  $H_{2m}(\Omega)$ , namely with some constants  $L_0, L$

$$\|D_t^k u(t)\|_{2m} \leq L_0 L^k \Gamma(k\tau + 1)$$

for all integers  $k \geq 0$ , where  $\|\cdot\|_{2m}$  is the norm of  $H_{2m}(\Omega)$ . In this result the known functions need not belong to Gevrey's class in  $x$ .

In the last section it will be shown that  $u$  belongs to the class  $G(\sigma, 2m\sigma)$  with the aid of the result in section 3. As in [7] it will first be proved that the Cauchy data of  $u$  on the boundary belong to  $G(\sigma, 2m\sigma)$ . Unlike the case of analyticity Cauchy-Kowalevskii theorem and Holmgren's theorem cannot be used, therefore we estimate all derivatives of the solution following the technique of C.B. Morrey and L. Nirenberg [5].

It is quite probable that the same result remains valid for problems of arbitrary order in  $t$ :

$$A(x, t, D_x, D_t)u(x, t) = f(x, t), \quad x \in \Omega, \quad (0.3)$$

$$B_j(x, t, D_x, D_t)u(x, t) = g_j(x, t), \quad x \in \partial\Omega, \quad j = 1, \dots, m; \quad (0.4)$$

however, the computation in that case would be extremely lengthy, so we shall investigate only the simpler situation.

**1. Notations and assumptions.** We denote by  $\Omega$  a domain in the  $n$ -dimensional Euclidean space  $E_n$  and by  $\partial\Omega$  its boundary. Let  $(x, t) = (x_1, \dots, x_n, t)$  be the generic point in  $E_{n+1}$ . We write  $D_x = (D_1, \dots, D_n) = (-1)^{-1/2}(\partial/\partial x_1, \dots, \partial/\partial x_n)$ ,  $D_t = (-1)^{-1/2}\partial/\partial t$  and denote by  $D_x^\alpha$ ,  $\alpha = (\alpha_1, \dots, \alpha_n)$ , the  $x$ -derivative  $D_1^{\alpha_1} \dots D_n^{\alpha_n}$ .  $|\alpha|$  stands for the length of the multi-index of  $\alpha$ :  $|\alpha| = \alpha_1 + \dots + \alpha_n$ . For any non-negative integer  $k$  we denote by  $H_k(\Omega)$  the class of all complex valued functions whose distribution derivatives of order up to  $k$  are square integrable in  $\Omega$ , the norm of  $H_k(\Omega)$  being denoted by

$$\|u\|_{k,\Omega}^2 = \sum_{|\alpha| \leq k} \int_{\Omega} |D_x^\alpha u(x)|^2 dx.$$

$H_{k-1/2}(\partial\Omega)$  is to be the class of functions  $\phi$  which are the boundary values of functions belonging to  $H_k(\Omega)$ . In this class of functions we introduce the norm

$$\langle \phi \rangle_{k,\partial\Omega} = \inf \|v\|_{k,\Omega},$$

where the infimum is taken over all functions  $v$  in  $H_k(\Omega)$  which equal  $\phi$  on  $\partial\Omega$ .

$A(x, t, D_x)$  is a linear differential operator in  $x$  of order  $2m$  with coefficients defined in  $\bar{\Omega} \times \{t: -\infty < t < \infty\}$ :

$$A(x, t, D_x) = \sum_{|\alpha| \leq 2m} a_\alpha(x, t) D_x^\alpha.$$

For each  $j=1, \dots, m$ ,  $B_j(x, t, D_x)$  is a linear differential operator in  $x$  of order  $m_j$  with coefficients defined on  $\partial\Omega \times \{t: -\infty < t < \infty\}$ :

$$B_j(x, t, D_x) = \sum_{|\beta| \leq m_j} b_{j,\beta}(x, t) D_x^\beta.$$

$\{B_j(x, t, D_x)\}$  is a system of operators which defines boundary conditions, and in what follows we shall assume that all the coefficients of  $\{B_j\}$  are defined in the whole of  $\bar{\Omega} \times \{t: -\infty < t < \infty\}$ . Let  $y$  be an auxiliary real variable and we denote by  $Q$  the infinite cylinder:  $Q = \{(x, y): x \in \Omega, -\infty < y < \infty\}$ . For each fixed  $t$ ,  $\pm D_y^{2m} + A(x, t, D_x)$  are differential operators in  $(x, y)$  of order  $2m$  with coefficients defined in  $Q$ .

**Assumptions (I).** For each fixed  $t$ ,  $\pm D_y^{2m} + A(x, t, D_x)$  is an elliptic operator of order  $2m$  in  $Q$ .

(II) The order  $m_j$  of  $B_j$  is smaller than  $2m$  for each  $j$ .

(III) The Complementing Condition ([2]) is satisfied by the system  $(\pm D_y^{2m} + A(x, t, D_x), \{B_j(x, t, D_x)\}_{j=1}^m, Q)$  for each fixed  $t$ .

The assumption concerning the smoothness of the coefficients will be stated in each of the following sections and in the last section  $\partial\Omega$  will be required to satisfy a more restrictive assumption. By a solution of (0.1)–(0.2) we always mean a function  $u$  with the properties that (i)  $u(t) = u(x, \cdot) \in H_{2m}(\Omega)$  for each  $t$ , (ii)  $u(t)$  is continuous in  $t$  in the strong topology of  $H_{2m}(\Omega)$  and (iii)  $u$  satisfies (0.1)–(0.2).

Let  $\tau$  and  $\sigma$  be real numbers such that  $\tau \geq 1$ ,  $\sigma \geq 1$ .

**DEFINITION 1.** A function  $u(t)$ ,  $-\infty < t < \infty$ , with values in a Hilbert space  $X$  (in many cases in what follows  $X$  will be the set of all complex numbers) is said to belong to *Gevrey's class*  $G(\tau)$  if for any positive constant  $R$  there exist constants  $H_0$  and  $H$  such that

$$\sup_{-R < t < R} \|D_t^q u(t)\| \leq H_0 H^q \Gamma(\tau q + 1)$$

for all integers  $q \geq 0$ , where  $\| \cdot \|$  is the norm of  $X$ .

**DEFINITION 2.** A numerical valued function  $u(x, t)$  defined in  $\bar{\Omega} \times \{t: -\infty < t < \infty\}$  is said to belong to *Gevrey's class*  $G(\sigma, \tau)$  if for any positive constant  $R$  there exist constants  $H_0$  and  $H$  such that

$$\sup_{x \in \Omega, -R < t < R} |D_t^\kappa D_x^\kappa u(x, t)| \leq H_0 H^{q+|\kappa|} \Gamma(\sigma|\kappa| + 1) \Gamma(\tau q + 1).$$

for any  $\kappa$  and  $q \geq 0$ .

From now on we shall write  $\| \cdot \|_k, \langle \cdot \rangle_k$  omitting  $\Omega$  and  $\partial\Omega$  if there is no fear of confusion.

**2. Preliminary lemmas.** In this section we assume that all the coefficients of  $A$ ,  $\{B_j\}$  have derivatives in  $t$  of all orders which are continuous in  $\bar{\Omega} \times \{t: -\infty < t < \infty\}$  and that  $f, g_j, j=1, \dots, m$ , are infinitely differentiable functions of  $t$  with values in  $L^2(\Omega)$  and  $H_{2m-m_j}(\Omega)$  respectively. Let  $\rho$  be a positive number satisfying  $\rho \leq 1$ , and  $r, \delta$  be positive numbers such that  $r + \delta < \rho$ .  $\varphi$  is to be a smooth function such that  $\varphi(t) = 1$  for  $-r < t < r$ ,  $\varphi(t) = 0$  for  $|t| > r + \delta$ , and  $|\varphi'(t)| \leq K/\delta$  where  $K$  is a positive number independent of  $r$  and  $\delta$ . In what follows in this section we denote by  $C_1, C_2, \dots$  constants depending only on the assumptions stated in the preceding and the present sections. If  $h$  is a function of  $(x, t)$ , we denote by  $\hat{h}$  its Fourier transform with respect to  $t$ :

$$\hat{h}(x, \lambda) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-(-1)^{1/2} \lambda t} h(x, t) dt$$

and by  $h^{(q)}$  its derivative in  $t$  of order  $q$ :

$$h^{(q)}(x, t) = D_t^q h(x, t).$$

We shall use the following notations

$$e_q(f, r) = \left( \int_{-r}^r \|f^{(q)}(t)\|_0^2 dt \right)^{1/2}, \quad (2.1)$$

$$\begin{aligned} e_{j,q}(g_j, r + \delta) &= \left( \int_{-r-\delta}^{r+\delta} \|g_j^{(q)}(t)\|_{2m-m_j}^2 dt \right)^{1/2} \\ &+ \left( \int_{-\infty}^{\infty} (|\lambda|^{2m-m_j})^{1/2} \|(\varphi g_j^{(q)})^\wedge(\lambda)\|_0^2 d\lambda \right)^{1/2}, \end{aligned} \quad (2.2)$$

$$d_q(u, r) = \left( \int_{-r}^r \|u^{(q+1)}(t)\|_0^2 dt \right)^{1/2} + \left( \int_{-r}^r \|u^{(q)}(t)\|_{2m}^2 dt \right)^{1/2} \quad (2.3)$$

for  $q=0, 1, 2, \dots$ .

Let  $\eta$  be a smooth function of  $t$  such that  $\eta(t)=1$  for  $-1 < t < 1$ ,  $\eta(t)=0$  for  $|t| > 2$ . Let  $M_q, q=0, 1, \dots$ , be positive numbers such that for all  $\alpha, \beta, \kappa, j$  with  $|\alpha| \leq 2m, |\beta| \leq m_j, |\kappa| \leq 2m - m_j, j=1, \dots, m$ ,

$$|D_t^q a_\alpha(x, t)| \leq M_q, \quad (2.4)$$

$$|D_t^q D_x^\kappa b_{j,\beta}(x, t)| \leq M_q, \quad (2.5)$$

$$\int_{-\infty}^{\infty} |(\eta D_t^q b_{j,\beta})^\wedge(x, \lambda)| d\lambda \leq M_q, \quad (2.6)$$

$$\int_{-\infty}^{\infty} |\lambda|^{(2m-m_j)/2m} |(\eta D_t^q b_{j,\beta})^\wedge(x, \lambda)| d\lambda \leq M_q \quad (2.7)$$

in  $\Omega \times \{t: -1 < t < 1\}$  or  $\Omega$ .

**Lemma 2.1.** *If  $\rho$  is sufficiently small, then for any positive numbers  $r, \delta$  such that  $r + \delta < \rho$  and for any non-negative integer  $q$  the following inequality holds for any solution  $u$  of (0.1)–(0.2):*

$$\begin{aligned} d_q(u, r) &\leq C_1 \left\{ e_q(f, r + \delta) + \sum_{j=1}^m e_{j,q}(g_j, r + \delta) \right. \\ &\quad + \frac{1}{\delta} \left( \int_{-r-\delta}^{r+\delta} \|D_t^q u(t)\|_0^2 dt \right)^{1/2} + \sum_{p=0}^{q-1} \binom{q}{p} M_{q-p} d_p(u, r + \delta) \\ &\quad \left. + \frac{1}{\delta} \sum_{p=0}^{q-1} \binom{q}{p} M_{q-p} \left( \int_{-r-\delta}^{r+\delta} \|D_t^p u(t)\|_0^2 dt \right)^{1/2} \right\}. \end{aligned} \quad (2.8)$$

This lemma is essentially proved in [6] and [7]. However, for the sake of convenience we give below an outline of the proof.

**Lemma 2.2.** *If  $v$  is a solution of*

$$D_t v(x, t) + A(x, 0, D_x) v(x, t) = f(x, t), \quad x \in \Omega, \quad -\infty < t < \infty, \quad (2.9)$$

$$B_j(x, 0, D_x) v(x, t) = g_j(x, t), \quad x \in \partial\Omega, \quad -\infty < t < \infty, \quad j=1, \dots, m, \quad (2.10)$$

and if the support of  $v$  considered as a function of  $t$  with values in  $H_{2m}(\Omega)$  is compact, then

$$\begin{aligned} &\int_{-\infty}^{\infty} \|D_t v(t)\|_0^2 dt + \int_{-\infty}^{\infty} \|v(t)\|_{2m}^2 dt \\ &\leq C_2 \left\{ \int_{-\infty}^{\infty} \|f(t)\|_0^2 dt + \sum_{j=1}^m \int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/2m} \|\hat{g}_j(\lambda)\|_0)^2 d\lambda \right. \\ &\quad \left. + \sum_{j=1}^m \int_{-\infty}^{\infty} \|g_j(t)\|_{2m-m_j}^2 dt + \int_{-\infty}^{\infty} \|u(t)\|_0^2 dt \right\}. \end{aligned} \quad (2.11)$$

**Proof.** The Fourier transform  $\hat{v}$  of  $v$  with respect to  $t$  satisfies

$$\lambda \hat{v}(x, \lambda) + A(x, 0, D_x) \hat{v}(x, \lambda) = \hat{f}(x, \lambda), \quad x \in \Omega, \quad (2.12)$$

$$B_j(x, 0, D_x)\hat{\vartheta}(x, \lambda) = \hat{g}_j(x, \lambda), \quad x \in \partial\Omega, \quad j=1, \dots, m. \quad (2.13)$$

Following S. Agmon [1] let us consider the functions

$$w_{\pm}(x, y, \mu) = \zeta(y) \exp((-1)^{1/2} \mu y) \hat{\vartheta}(x, \pm \mu^{2m}),$$

where  $\zeta$  is a smooth function such that  $\zeta(y)=1$  for  $|y| \leq 1/2$  and  $\zeta(y)=0$  for  $|y| \geq 1$ , and  $\mu$  is an arbitrary real number. Due to (2.12) and (2.13)  $w_{\pm}$  satisfies

$$\begin{aligned} (\pm D_y^{2m} + A(x, 0, D_x))w_{\pm}(x, y, \mu) &= \zeta(y) \exp((-1)^{1/2} \mu y) \hat{f}(x, \pm \mu^{2m}) \\ &\pm \sum_{k=0}^{2m-1} \binom{2m}{k} D_y^{2m-k} \zeta(y) \mu^k \exp((-1)^{1/2} \mu y) \hat{\vartheta}(\pm \mu^{2m}, x), \quad x \in \Omega, \end{aligned} \quad (2.14)$$

$$B_j(x, 0, D_x)w_{\pm}(x, y, \mu) = \zeta(y) \exp((-1)^{1/2} \mu y) \hat{g}_j(x, \pm \mu^{2m}), \quad x \in \partial\Omega, \quad j=1, \dots, m. \quad (2.15)$$

It is easy to show that

$$\begin{aligned} &\|(\pm D_y^{2m} + A(x, 0, D_x))w_{\pm}\|_{0,Q} \\ &\leq C_3 \{ \|\hat{f}(\pm \mu^{2m})\|_0 + (1 + |\mu|^{2m-1}) \|\hat{u}(\pm \mu^{2m})\|_0 \}, \quad (2.16) \\ &\langle B_j(x, 0, D_x)w_{\pm} \rangle_{2m-m_j, \partial Q}^2 = \langle \zeta \exp((-1)^{1/2} \mu y) \hat{g}_j(\pm \mu^{2m}) \rangle_{2m-m_j, \partial Q}^2 \\ &\leq \|\zeta \exp((-1)^{1/2} \mu y) \hat{g}_j(\pm \mu^{2m})\|_{2m-m_j, Q}^2 \\ &\leq C_4 \sum_{k=0}^{2m-m_j} (1 + |\mu|)^{2k} \|\hat{g}_j(\pm \mu^{2m})\|_{2m-m_j-k}^2. \end{aligned}$$

Hence with the aid of the well known inequality

$$\|w\|_{2m-m_j-k} \leq c_0 \|w\|_{2m-m_j}^{(2m-m_j-k)/(2m-m_j)} \|w\|_0^{k/(2m-m_j)},$$

we get

$$\begin{aligned} &\langle B_j(x, 0, D_x)w_{\pm} \rangle_{2m-m_j, \partial Q} \\ &\leq C_5 \{ \|\hat{g}_j(\pm \mu^{2m})\|_{2m-m_j} + (1 + |\mu|)^{2m-m_j} \|\hat{g}_j(\pm \mu^{2m})\|_0 \}. \end{aligned} \quad (2.17)$$

As is easily seen

$$\|w_{\pm}\|_{2m,Q}^2 \geq \sum_{k=0}^{2m} |\mu|^{2k} \|\hat{\vartheta}(\pm \mu^{2m})\|_{2m-k}^2. \quad (2.18)$$

Using (2.16), (2.17) and (2.18) in the Agmon-Douglis-Nirenberg inequality

$$\begin{aligned} &\|w_{\pm}\|_{2m,Q} \leq C_6 \{ \|(\pm D_y^{2m} + A(x, 0, D_x))w_{\pm}\|_{0,Q} \\ &\quad + \sum_{j=1}^m \langle B_j(x, 0, D_x)w_{\pm} \rangle_{2m-m_j, \partial Q} + \|w_{\pm}\|_{0,Q} \} \end{aligned}$$

which can be applied to  $w_{\pm}$  in  $Q$  by assumption and then putting  $\lambda = \pm \mu^{2m}$ , we get

$$\begin{aligned}
& |\lambda| \{ \|\hat{\phi}(\lambda)\|_0 + \|\hat{\phi}(\lambda)\|_{2m} \\
& \leq C_7 \{ \|\hat{f}(\lambda)\|_0 + \sum_{j=1}^m |\lambda|^{(2m-m_j)/2m} \|\hat{g}_j(\lambda)\|_0 \\
& + \sum_{j=1}^m \|\hat{g}_j(\lambda)\|_{2m-m_j} + \|\hat{u}(\lambda)\|_0 \}
\end{aligned} \tag{2.19}$$

for any real number  $\lambda$ . Integrating the squares of both sides of (2.19) over  $-\infty < \lambda < \infty$  and then applying Plancherel's theorem, we get (2.11).

**Lemma 2.3.** *Let  $v$  be a solution of (0.1)–(0.2). If the support of  $v$  considered as a function of  $t$  is contained in a sufficiently small neighbourhood of the origin, then the same estimate as (2.11) holds replacing  $C_2$  by another constant if necessary.*

*Proof.* The lemma is easily proved considering  $\psi(t)v(x, t)$  where  $\psi$  is a smooth function which has a small compact support and identically equals 1 on the support of  $v$ .

**Lemma 2.4.** *If  $\rho$  is sufficiently small, then for the solution  $u$  of (0.1)–(0.2)*

$$\begin{aligned}
d_0(u, r) & \leq C_8 \left\{ e_0(f, r + \delta) \right. \\
& \left. + \sum_{j=1}^m e_{j,0}(g_j, r + \delta) + \frac{1}{\delta} \int_{-r-\delta}^{r+\delta} \|u(t)\|_0^2 dt \right\}
\end{aligned}$$

whenever  $r + \delta < \rho$ .

*Proof.* The lemma is easily proved if we apply Lemma 2.3 to  $\varphi(t)u(x, t)$ .

Lemma 2.1 can be obtained if we differentiate both sides of (0.1)–(0.2)  $q$  times in  $t$  and applying Lemma 2.4 to  $D_t^q u$ .

**Lemma 2.5.** *If  $\alpha \geq 1$  and  $\beta > 0$ , then*

$$\Gamma(\alpha + \beta) \geq \Gamma(\alpha)\Gamma(\beta + 1). \tag{2.20}$$

*If  $\alpha \geq 1$  and  $\beta \geq 1$ , then*

$$2^{\alpha+\beta-1} \Gamma(\alpha + 1)\Gamma(\beta + 1) \geq \Gamma(\alpha + \beta + 1). \tag{2.21}$$

*If  $0 \leq \alpha' \leq \alpha$  and  $0 \leq \beta' \leq \beta$ , then*

$$\frac{\Gamma(\alpha' + \beta' + 1)}{\Gamma(\alpha' + 1)\Gamma(\beta' + 1)} \leq \frac{\Gamma(\alpha + \beta + 1)}{\Gamma(\alpha + 1)\Gamma(\beta + 1)} \frac{1 + \alpha + \beta}{1 + \alpha' + \beta'}. \tag{2.22}$$

*If  $\tau \geq 1$ , then for any pair of non-negative integers  $p$  and  $q$  satisfying  $p \leq q$*



$$\binom{q}{p} \leq \frac{\tau \Gamma(\tau q + 1)}{\Gamma(\tau p + 1) \Gamma(\tau(q - p) + 1)}. \quad (2.23)$$

Proof. (2.20), (2.21) and (2.22) are all simple consequences of

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt.$$

(2.22) implies

$$\binom{q}{p} = \frac{\Gamma(q+1)}{\Gamma(p+1)\Gamma(q-p+1)} \leq \frac{\Gamma(\tau q+1)}{\Gamma(\tau p+1)\Gamma(\tau(q-p)+1)} \frac{\tau q+1}{q+1}. \quad (2.24)$$

(2.23) is a direct consequence of (2.24).

**3. Estimates for derivatives in  $t$ .** In this section we assume that

(V) the coefficients of  $A$  and the derivatives in  $x$  of the coefficients of  $B_j$  of order up to  $2m - m_j$ ,  $j = 1, \dots, m$ , all belong to Gevrey's class  $G(\tau)$  as functions of  $t$  uniformly. Hence there exist positive constants  $M_0$  and  $M$  such that

$$M_q \leq M_0 M^q \Gamma(\tau q + 1) \quad (3.1)$$

for all integers  $q \geq 0$  (cf. (2.4)~(2.7) for the meaning of  $M_q$ );

(VI)  $f$  and  $g_j$ ,  $j = 1, \dots, m$ , belong to Gevrey's class  $G(\tau)$  when they are considered as functions of  $t$  with values in  $L^2(\Omega)$  and  $H_{2m-m_j}(\Omega)$  respectively. Hence there exist positive constants  $N_0$  and  $N$  such that

$$\|f^{(q)}(t)\|_0 \leq N_0 N^q \Gamma(\tau q + 1), \quad -1 \leq t \leq 1, \quad (3.2)$$

$$\|g_j^{(q)}(t)\|_{2m-m_j} \leq N_0 N^q \Gamma(\tau q + 1), \quad -1 \leq t \leq 1, \quad j = 1, \dots, m. \quad (3.3)$$

We introduce the notation

$$N_{\rho, q}(u) = \Gamma(\tau q + 1)^{-1} \sup_{\rho/2 \leq r < \rho} d_q(u, r) (\rho - r)^{q+1} \quad (3.4)$$

for  $q = 0, 1, 2, \dots$ .

**Theorem 3.1.** *Under the assumptions (I)~(VI) any solution of (0.1)–(0.2) considered as a function of  $t$  with values in  $H_{2m}(\Omega)$  belongs to Gevrey's class  $G(\tau)$ .*

Proof. Let us multiply both sides of (2.8) with  $\delta = (\rho - r)(q + 1)^{-1}$  by  $\Gamma(\tau q + 1)^{-1}(\rho - r)^{q+1}$ . If we notice  $\delta = (\rho - r - \delta)/q$  we get

$$\begin{aligned}
& \Gamma(\tau q + 1)^{-1} d_q(u, r)(\rho - r)^{q+1} \\
& \leq C_9 e(1 + q^{-1}) \left\{ \Gamma(\tau q + 1)^{-1} e_q(f, r + \delta)(\rho - r - \delta)^{q+1} \right. \\
& \quad + \Gamma(\tau q + 1)^{-1} \sum_{j=1}^m e_{j,q}(g_j, r + \delta)(\rho - r - \delta)^{q+1} \\
& \quad + q \Gamma(\tau q + 1)^{-1} d_{q-1}(u, r + \delta)(\rho - r - \delta)^q \\
& \quad + \tau M_0 \sum_{p=0}^{q-1} \Gamma(\tau p + 1)^{-1} M^{p-q} d_p(u, r + \delta)(\rho - r - \delta)^{q+1} \\
& \quad + \tau M_0 q \sum_{p=0}^{q-1} \Gamma(\tau p + 1)^{-1} M^{p-q} d_{p-1}(u, r + \delta)(\rho - r - \delta)^q \\
& \quad \left. + M_0 q M^q \left( \int_{-r-\delta}^{r+\delta} \|u(t)\|_0^2 dt \right)^{1/2} (\rho - r - \delta)^q \right\}. \tag{3.5}
\end{aligned}$$

From (3.2) it follows that

$$\Gamma(\tau q + 1)^{-1} e_q(f, r + \delta)(\rho - r - \delta)^{q+1} \leq \sqrt{2\rho} \rho N_0 (\rho N)^q. \tag{3.6}$$

Noting  $|\lambda|^{(2m-m_j)/2m} \leq |\lambda| + 1$  and

$$\Gamma(\tau q + \tau + 1) \leq 2^{\tau q + \tau - 1} \Gamma(\tau q + 1) \Gamma(\tau + 1) \tag{3.7}$$

which follows from (2.21), we can easily show

$$\begin{aligned}
& \Gamma(\tau q + 1)^{-1} e_{j,q}(g_j, r + \delta)(\rho - r - \delta)^{q+1} \\
& \leq \sqrt{2\rho} \{2^{\tau-1} \Gamma(\tau + 1) N_0 \rho N (2^\tau \rho N)^q + (2\rho + Kq) N_0 (\rho N)^q\}. \tag{3.8}
\end{aligned}$$

Using (3.4) and

$$\begin{aligned}
& \Gamma(\tau q + 1) = \tau q \Gamma(\tau q) = \tau q \Gamma(\tau(q-1) + 1 + \tau - 1) \\
& \geq \tau q \Gamma(\tau(q-1) + 1) \Gamma(\tau) = \Gamma(\tau + 1) q \Gamma(\tau(q-1) + 1)
\end{aligned}$$

which follows from (2.20), we get

$$q \Gamma(\tau q + 1)^{-1} d_{q-1}(u, r + \delta)(\rho - r - \delta)^q \leq \Gamma(\tau + 1)^{-1} N_{\rho, q-1}(u). \tag{3.9}$$

Similarly

$$\Gamma(\tau p + 1)^{-1} M^{q-p} d_p(u, r + \delta)(\rho - r - \delta)^{q+1} \leq (\rho M)^{q-p} M_{\rho, p}(u), \tag{3.10}$$

$$\begin{aligned}
& \tau q \Gamma(\tau p + 1)^{-1} M^{q-p} d_{p-1}(u, r + \delta)(\rho - r - \delta)^q \\
& \leq \Gamma(\tau)^{-1} q p^{-1} (\rho M)^{q-p} N_{\rho, p-1}(u). \tag{3.11}
\end{aligned}$$

From (3.5)~(3.11) and  $qp^{-1} \leq e^{q-p}$  it follows that

$$\begin{aligned}
& N_{\rho, q}(u) \leq C_{10} e(1 + q^{-1}) \left[ \sqrt{2\rho} \{2m + 1 + 2^{\tau-1} m \Gamma(\tau + 1) N\} \rho N_0 (2^\tau \rho N)^q \right. \\
& \quad + \sqrt{2\rho} m K N_0 q (\rho N)^q + \Gamma(\tau + 1)^{-1} N_{\rho, q-1}(u) \\
& \quad + \tau M_0 \sum_{p=0}^{q-1} (\rho M)^{q-p} N_{\rho, p}(u) + \Gamma(\tau)^{-1} M_0 \sum_{p=0}^{q-1} (e \rho M)^{q-p} N_{\rho, p-1}(u) \\
& \quad \left. + M_0 q (\rho M)^q \left( \int_{-r-\delta}^{\rho} \|u(t)\|_0^2 dt \right)^{1/2} \right]. \tag{3.12}
\end{aligned}$$

We want to show that there exist constants  $H_0, H \geq 0$  such that for any non-negative integer  $q$

$$N_{\rho,q}(u) \leq H_0 H^q. \quad (3.13)$$

With the aid of (3.12) we can proceed by induction without difficulty to verify (3.13) provided that  $H_0$  and  $H$  are so large that

$$\begin{aligned} 12 C_{10} e \sqrt{2\rho} \{2m+1+2^{\tau-1} m \Gamma(\tau+1) N\} \rho N_0 &\leq H_0, \\ 12 C_{10} e \sqrt{2\rho} m K N_0 &\leq H_0, \quad 12 C_{10} e M_0 \left( \int_{-\rho}^{\rho} \|u(t)\|_0^2 dt \right)^{1/2} \leq H_0, \\ \max(2^{\tau} \rho N, e \rho N, 2 e \rho M) &\leq H, \quad 12 C_{10} e \Gamma(\tau+1)^{-1} \leq H, \\ 24 C_{10} e \tau M_0 \rho M &\leq H, \quad 24 \Gamma(\tau)^{-1} C_{10} e^2 M_0 \rho M \leq H^2. \end{aligned}$$

The proof of the theorem has been completed.

Next we show that  $u$  belongs to the same class in the space  $H_{4m}(\Omega)$  under the following more restrictive assumptions:

(V') the derivatives in  $x$  of the coefficients of  $A$  of order up to  $2m$  and those in  $x$  of the coefficients of  $B_j$  of order up to  $4m-m_j$ ,  $j=1, \dots, m$ , all belong to Gevrey's class  $G(\tau)$  as functions of  $t$  uniformly;

(VI')  $f$  and  $g_j$ ,  $j=1, \dots, m$ , belong to Gevrey's class  $G(\tau)$  when considered as functions of  $t$  with values in  $H_{2m}(\Omega)$  and  $H_{4m-m_j}(\Omega)$  respectively.

**Theorem 3.2.** *Under the assumptions (I)~(IV), (V'), (VI') any solution of (0.1)–(0.2) belongs to Gevrey's class  $G(\tau)$  when considered as a function of  $t$  with values in  $H_{4m}(\Omega)$ .*

**Proof.** By assumption and Theorem 3.1 there exist constants  $\bar{M}_0, \bar{M}, \bar{N}_0, \bar{N}$  and  $L_0, L$  such that for all  $q=1, 2, \dots$

$$\sup |D_t^q D_x^{\kappa} a_{\alpha}(x, t)| \leq \bar{M}_0 \bar{M}^q \Gamma(\tau q + 1), \quad |\kappa| \leq 2m, \quad (3.14)$$

$$\sup |D_t^q D_x^{\kappa} b_{j,\beta}(x, t)| \leq \bar{M}_0 \bar{M}^q \Gamma(\tau q + 1), \quad |\kappa| \leq 4m - m_j, \quad j=1, \dots, m, \quad (3.15)$$

$$\|f^{(q)}(t)\|_{2m} \leq \bar{N}_0 \bar{N}^q \Gamma(\tau q + 1), \quad (3.16)$$

$$\|g^{(q)}(t)\|_{4m-m_j} \leq \bar{N}_0 \bar{N}^q \Gamma(\tau q + 1), \quad j=1, \dots, m, \quad (3.17)$$

$$\|D_t^q u(t)\|_{2m} \leq L_0 L^q \Gamma(\tau q + 1). \quad (3.18)$$

We want to show that there exist constants  $\bar{L}_0$  and  $\bar{L}$  such that

$$\|D_t^q u(t)\|_{4m} \leq \bar{L}_0 \bar{L}^q \Gamma(\tau q + 1) \quad (3.19)$$

for all integers  $q \geq 0$ . Supposing that (3.19) is true for  $q=0, 1, \dots, l-1$ , let us prove that the same is true for  $q=l$ . In view of the Agmon-

Douglis-Nirenberg inequality concerning the system  $(A(x, t, D_x), \{B_j(x, t, D_x)\}, \Omega)$

$$\begin{aligned} \|D_t^l u(t)\|_{4m} &\leq C_{11} \{ \|A(x, t, D_x) D_t^l u(t)\|_{2m} \\ &+ \sum_{j=1}^m \langle B_j(x, t, D_x) D_t^l u(t) \rangle_{4m-m_j} + \|D_t^l u(t)\|_0 \}. \end{aligned} \quad (3.20)$$

Differentiating both sides of (0.1)–(0.2) we get

$$\begin{aligned} A(x, t, D_x) D_t^l u(x, t) &= -D_t^{l+1} u(x, t) + D_t^l f(x, t) \\ &- \sum_{k=0}^{l-1} \binom{l}{k} A^{(l-k)}(x, t, D_x) D_t^k u(x, t), \quad x \in \Omega, \end{aligned} \quad (3.21)$$

$$\begin{aligned} B_j(x, t, D_x) D_t^l u(x, t) &= D_t^l g_j(x, t) \\ &- \sum_{k=0}^{l-1} \binom{l}{k} B^{(l-k)}(x, t, D_x) D_t^k u(x, t), \quad x \in \partial\Omega, \quad j=1, \dots, m, \end{aligned} \quad (3.22)$$

where  $A^{(l-k)}$  and  $B_j^{(l-k)}$  are differential operators obtained by differentiating the corresponding coefficients of  $A$  and  $B_j$   $l-k$  times with respect to  $t$  respectively. In view of (3.14) and an elementary calculation we get

$$\|A^{(l-k)}(x, t, D_x) D_t^k u(t)\|_{2m} \leq C_{12} \bar{M}_0 \bar{M}^{l-k} \Gamma(\tau(l-k) + 1) \|D_t^k u(t)\|_{4m},$$

and hence with the aid of (3.16), (3.18), (3.21) and the induction hypothesis we obtain

$$\begin{aligned} \|A(x, t, D_x) D_t^l u(t)\|_{2m} &\leq L_0 L^{l+1} \Gamma(\tau(l+1) + 1) \\ &+ \bar{N}_0 \bar{N}^l \Gamma(\tau l + 1) + C_{13} \bar{M}_0 \bar{L} \sum_{k=0}^{l-1} \bar{M}^{l-k} \bar{L}^k. \end{aligned} \quad (3.23)$$

Estimating  $\langle B_j(x, t, D_x) D_t^l u(t) \rangle_{4m-m_j}$  in a similar manner and using (2.23) we can show without difficulty that (3.19) holds provided that  $\bar{L}_0$  and  $\bar{L}$  are sufficiently large.

**4. Estimates for derivatives in all variables.** In addition to the assumptions in section 1 we assume in this section that

(VII) all the functions  $a_\alpha$ ,  $|\alpha| \leq 2m$ ,  $b_{j,\beta}$ ,  $|\beta| \leq m_j$ ,  $f$  and  $g_j$ ,  $j=1, \dots, m$ , belong to Gevrey's class  $G(\sigma, \tau)$ ;

(VIII)  $\tau = 2m\sigma$ ;

(IV')  $\Omega$  is a bounded domain of the class  $G(\sigma)$  in the sense that each point of  $\partial\Omega$  is contained in some open subset of  $\partial\Omega$  which can be mapped onto a subset of a hyperplane by means of a one-to-one mapping of Gevrey's class  $G(\sigma)$ .

Under the assumptions above we show that any solution of (0.1)–(0.2) belongs to the class  $G(\sigma, \tau) = G(\sigma, 2m\sigma)$ . In this section we denote by  $C_{13}, C_{14}, \dots$  constants depending only on the assumptions stated so

far. By (IV') we may suppose that the origin is located on a part of  $\partial\Omega$  which is contained in the hyperplane  $x_n=0$ . First we prove that the Cauchy data of  $u$  are in Gevrey's class  $G(\sigma, \tau)$  near the origin.

We shall employ the following semi-norms and norms:

$$|v|_i^2 = |v|_{i,\Omega}^2 = \sum_{|\kappa|=i} \int_{\Omega} |D_{\kappa}^{\kappa} v(x)|^2 dx, \quad (4.1)$$

$$|v|_{i,r}^2 = \sum_{|\kappa|=i} \int_{|x|<r, x_n>0} |D_{\kappa}^{\kappa} v(x)|^2 dx, \quad (4.2)$$

$$||v|_{k,r}^2 = \sum_{i=0}^k |v|_{i,r}^2. \quad (4.3)$$

We may choose constants  $c_0$  and  $c_1$  in such a manner that

$$|v|_i \leq c_0 |v|_j^{i/j} |v|_0^{(j-i)/j} + c_1 |v|_0, \quad (4.4)$$

$$|v|_{i,r} \leq c_0 |v|_{j,r}^{i/j} |v|_{0,r}^{(j-i)/j} + c_1 r^{-i} |v|_{0,r}. \quad (4.5)$$

for  $0 < i < j < 2m$  and  $0 < r$ . From now on we shall distinguish the normal variable  $x_n$  from tangential space variables  $x' = (x_1, \dots, x_{n-1})$  and by  $\nabla^p$  we denote any derivative of order  $p$  in  $x'$ . We denote by  $A^{\#}$  and  $B_j^{\#}$  the principal parts of  $A$  and  $B_j$  respectively:

$$A^{\#}(x, t, D_x) = \sum_{|\alpha|=2m} a_{\alpha}(x, t) D_{\alpha}^{\alpha}, \quad (4.6)$$

$$B_j^{\#}(x, t, D_x) = \sum_{|\beta|=m_j} b_{j,\beta}(x, t) D_{\beta}^{\beta}, \quad j=1, \dots, m. \quad (4.7)$$

Let  $\rho_0(<1)$  be a positive number such that

$$\{(x', x_n) : |x| < \rho_0, x_n > 0\} \subset \Omega. \quad (4.8)$$

Let  $\eta_1$  be a smooth function such that  $\eta_1(t) \equiv 1$  for  $|t| \leq 1$ ,  $\eta_1(t) = 0$  for  $|t| \geq 2$ . For  $p, q=0, 1, 2, \dots$  we denote by  $M_{p,q}$  constants such that for all  $\alpha, \beta, \kappa, j$  with  $|\alpha| \leq 2m$ ,  $|\beta| \leq m_j$ ,  $|\kappa| \leq 2m - m_j$ ,  $j=1, \dots, m$ ,

$$|D_t^q \nabla^p a_{\alpha}(x, t)| \leq M_{p,q}, \quad (4.9)$$

$$|D_x^{\kappa} D_t^q \nabla^p b_{j,\beta}(x, t)| \leq M_{p,q}, \quad (4.10)$$

$$\int_{-\infty}^{\infty} |(\eta_1 D_t^q \nabla^p b_{j,\beta})^{\wedge}(x, \lambda)| d\lambda \leq M_{p,q}, \quad (4.11)$$

$$\int_{-\infty}^{\infty} |\lambda|^{(2m-m_j)/2m} |(\eta_1 D_t^q \nabla^p b_{j,\beta})^{\wedge}(x, \lambda)| d\lambda \leq M_{p,q} \quad (4.12)$$

in  $\Omega \times (-\infty, \infty)$  or  $\Omega$ . We shall use the following notations:

$$\begin{aligned} d_{p,q}(u, r) &= \max \left\{ \left( \int_{-r}^r |D_t^{q+1} \nabla^p u(t)|_{0,r}^2 dt \right)^{1/2} \right. \\ &\quad \left. + \left( \int_{-r}^r |D_t^q \nabla^p u(t)|_{2m,r}^2 dt \right)^{1/2} \right\}, \\ e_{p,q}(f, r) &= \max \left( \int_{-r}^r |D_t^q \nabla^p f(t)|_{0,r}^2 dt \right)^{1/2}, \end{aligned}$$

for  $p, q = 0, 1, 2, \dots$ ,  $0 < r < \rho_0$ , with the maximum taken over all derivatives  $\nabla^p$  of order  $p$ . Let  $\varphi$  be a function stated in the preceding section. Then as in [7, pp. 181-187] we get

**Lemma 4.1.** *If  $\rho_1$  is sufficiently small, then for any  $\delta > 0$ ,  $r > 0$  such that  $r + \delta < \rho_1$*

$$\begin{aligned} d_{p,q}(u, r) \leq & C_{13} \left[ e_{p,q}(f, r + \delta) \right. \\ & + \sum_{j=1}^m \left( \int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/2m} |(\varphi D_t^q \nabla^p g_j)^\wedge(\lambda)|_{0,r+\delta})^2 d\lambda \right)^{1/2} \\ & + \sum_{j=1}^m \left( \int_{-r-\delta}^{r-\delta} |D_t^q \nabla^p g_j(t)|_{2m-m_j, r+\delta}^2 dt \right)^{1/2} \\ & + \sum_{j=1}^m \delta^{m_j-2m} \left( \int_{-r-\delta}^{r-\delta} |D_t^q \nabla^p g_j(t)|_{0, r+\delta}^2 dt \right)^{1/2} \\ & + (r + \delta + \varepsilon) d_{p,q}(u, r + \delta) + \varepsilon^{1-2m} \delta^{-2m} \left( \int_{-r-\delta}^{r+\delta} |D_t^q \nabla^p u(t)|_{0, r+\delta}^2 dt \right)^{1/2} \\ & + \sum' \left( \frac{q}{q'} \right) \left( \frac{p}{p'} \right) M_{p-p', q-q'} d_{p', q'}(u, r + \delta) \\ & + \delta^{-2m} \sum' \left( \frac{q}{q'} \right) \left( \frac{p}{p'} \right) M_{p-p', q-q'} \left( \int_{-r-\delta}^{r+\delta} |D_t^{q'} \nabla^{p'} u(t)|_{0, r+\delta}^2 dt \right)^{1/2} \Big], \quad (4.13) \end{aligned}$$

where  $\sum'$  means that the summation extends over all  $(p', q')$  satisfying  $0 \leq p' \leq p$ ,  $0 \leq q' \leq q$  except  $(p, q) = (p', q')$ , and  $\varepsilon$  is an arbitrary positive number.

By assumption there exist constants  $N_0, N, M_0$  and  $M$  such that for any pair of integers  $p, q \geq 0$

$$\sup |D_t^q \nabla^p f(x, t)| \leq N_0 N^{p+q} \Gamma(\sigma p + \tau q + 1), \quad (4.14)$$

$$\sup |D_x^* D_t^q \nabla^p g_j(x, t)| \leq N_0 N^{p+q} \Gamma(\sigma p + \tau q + 1), \quad (4.15)$$

$$|\kappa| \leq 2m - m_j, \quad j = 1, \dots, m,$$

$$M_{p,q} \leq M_0 M^{p+q} \Gamma(\sigma p + 1) \Gamma(\tau q + 1). \quad (4.16)$$

Then as in section 3 we get

$$e_{p,q}(f, r + \delta) \leq C_{14} N_0 N^{p+q} \Gamma(\sigma p + \tau q + 1), \quad (4.17)$$

$$\begin{aligned} & \left( \int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/2m} |(\varphi D_t^q \nabla^p g_j)^\wedge(\lambda)|_{0,r+\delta})^2 d\lambda \right)^{1/2} \\ & \leq C_{15} \left\{ N_0 N^{p+q+1} \Gamma(\sigma p + \tau q + \tau + 1) \right. \\ & \quad \left. + (1 + K \delta^{-1}) N_0 N^{p+q} \Gamma(\sigma p + \tau q + 1) \right\}, \quad (4.18) \end{aligned}$$

$$\left( \int_{-r-\delta}^{r+\delta} |D_t^q \nabla^p g_j(t)|_{2m-m_j, r+\delta}^2 dt \right)^{1/2} \leq C_{16} N_0 N^{p+q} \Gamma(\sigma p + \tau q + 1), \quad (4.19)$$

$$\left( \int_{-r-\delta}^{r+\delta} |D_i^q \nabla^p g_j(t)|_{0, r+\delta}^2 dt \right)^{1/2} \leq C_{17} N_0 N^{p+q} \Gamma(\sigma p + \tau q + 1). \quad (4.20)$$

With the aid of (4.13), (4.16)~(4.20) and (2.23) we get

**Lemma 4.2.** *Under the assumptions of the preceding lemma the following inequality holds for any pair of integers  $p \geq 2m$  and  $q \geq 0$ :*

$$\begin{aligned} d_{p,q}(u, r) &\leq C_{18} \left[ N_0 N^{p+q+1} \Gamma(\sigma p + \tau q + \tau + 1) \right. \\ &\quad + \delta^{-2m} N_0 N^{p+q} \Gamma(\sigma p + \tau q + 1) + (r + \delta + \varepsilon) d_{p,q}(u, r + \delta) \\ &\quad + \delta^{-2m} \varepsilon^{1-2m} d_{p-2m,q}(u, r + \delta) \\ &\quad + \tau \sigma M_0 \sum' \frac{\Gamma(\tau q + 1)}{\Gamma(\tau q' + 1)} \frac{\Gamma(\sigma p + 1)}{\Gamma(\sigma p' + 1)} M^{p-p'+q-q'} d_{p',q'}(u, r + \delta) \\ &\quad + \frac{\tau \sigma M_0}{\delta^{2m}} \sum'_{p' \geq 2m} \frac{\Gamma(\tau q + 1)}{\Gamma(\tau q' + 1)} \frac{\Gamma(\sigma p + 1)}{\Gamma(\sigma p' + 1)} M^{p-p'+q-q'} d_{p'-2m,q'}(u, r + \delta) \\ &\quad + \frac{\tau \sigma M_0}{\delta^{2m}} \sum'_{p' < 2m} \frac{\Gamma(\tau q + 1)}{\Gamma(\tau q' + 1)} \frac{\Gamma(\sigma p + 1)}{\Gamma(\sigma p' + 1)} M^{p-p'+q-q'} \\ &\quad \left. \times \left( \int_{-r-\delta}^{r+\delta} |D_i^{q'} \nabla^{p'} u(t)|_{0, r+\delta}^2 dt \right)^{1/2} \right]. \quad (4.21) \end{aligned}$$

We introduce the notation

$$N_{\rho,p,q}(u) = \Gamma(\sigma p + \tau q + 1)^{-1} \sup_{\rho/2 \leq r < \rho} d_{p,q}(u, r) (\rho - r)^{p+q+2m} \quad (4.22)$$

for  $p, q = 0, 1, 2, \dots$ .

**Lemma 4.3.** *If  $\rho (< \rho_1)$  is sufficiently small, there exist constants  $H_0$  and  $H$  such that*

$$N_{\rho,p,q}(u) \leq H_0 H^{p+q} \quad (4.23)$$

for any pair of integers  $p, q \geq 0$ .

**Proof.** Suppose  $p > 2m$  and  $\delta = (\rho - r)/(1 + \sigma p + \tau q)$ . Let us multiply both sides of (4.21) by  $\Gamma(\sigma p + \tau q + 1)^{-1} (\rho - r)^{p+q+2m}$ . Noting  $\delta = (\rho - r - \delta)/(1 + \sigma p + \tau q)$ ,  $\rho - r = \{1 + (\sigma p + \tau q)^{-1}\}(\rho - r - \delta)$  and  $\{1 + (\sigma p + \tau q)^{-1}\}^{p+q+2m} \leq e^2$  we get

$$\begin{aligned} &\Gamma(\sigma p + \tau q + 1)^{-1} d_{p,q}(u, r) (\rho - r)^{p+q+2m} \\ &\leq C_{19} (I + II + III + IV + V + VI + VII), \end{aligned} \quad (4.24)$$

where

$$\begin{aligned} I &= \Gamma(\sigma p + \tau q + 1)^{-1} N_0 N^{p+q+1} \Gamma(\sigma p + \tau q + \tau + 1) (\rho - r)^{p+q+2m}, \\ II &\leq e^2 (\sigma p + \tau q)^{2m} N_0 N^{p+q} (\rho - r - \delta)^{p+q}, \\ III &= e^2 (\rho + \varepsilon) \Gamma(\sigma p + \tau q + 1)^{-1} d_{p,q}(u, r + \delta) (\rho - r - \delta)^{p+q+2m}, \end{aligned}$$

$$\begin{aligned}
IV &= e^2 \varepsilon^{1-2m} (\sigma p + \tau q)^{2m} \Gamma(\sigma p + \tau q + 1)^{-1} d_{p-2m, q}(u, r + \delta) (\rho - r - \delta)^{p+q}, \\
V &= e^2 \tau \sigma M_0 \sum' \frac{\Gamma(\tau q + 1) \Gamma(\sigma p + 1)}{\Gamma(\tau q' + 1) \Gamma(\sigma p' + 1)} \frac{M^{p-p'+q-q'}}{\Gamma(\sigma p + \tau q + 1)} \\
&\quad \times d_{p', q'}(u, r + \delta) (\rho - r - \delta)^{p+q+2m}, \\
VI &= e^2 \tau \sigma M_0 (\sigma p + \tau q)^{2m} \sum_{p' \geq 2m}' \frac{\Gamma(\tau q + 1) \Gamma(\sigma p + 1)}{\Gamma(\tau q' + 1) \Gamma(\sigma p' + 1)} \\
&\quad \times \frac{M^{p-p'+q-q'}}{\Gamma(\sigma p + \tau q + 1)} d_{p'-2m, q'}(u, r + \delta) (\rho - r - \delta)^{p+q}, \\
VII &= e^2 \tau \sigma M_0 (\sigma p + \tau q)^{2m} \sum_{p' < 2m}' \frac{\Gamma(\tau q + 1) \Gamma(\sigma p + 1)}{\Gamma(\tau q' + 1) \Gamma(\sigma p' + 1)} \\
&\quad \times \frac{M^{p-p'+q-q'}}{\Gamma(\sigma p + \tau q + 1)} \left( \int_{-\rho}^{\rho} |D_{\varepsilon'}^q \nabla^{p'} u(t)|_{0, r+\delta}^2 dt \right)^{1/2} (\rho - r - \delta)^{p+q}.
\end{aligned}$$

Since by Lemma 2.5

$$\Gamma(\sigma p + \tau q + \tau + 1) \leq 2^{\sigma p + \tau q + \tau - 1} \Gamma(\tau + 1) \Gamma(\sigma + 1) \Gamma(\sigma p + \tau q + 1), \quad (4.25)$$

we get

$$I \leq 2^{\tau-1} \Gamma(\tau + 1) \rho^{2m} N_0 N (2^{\sigma} \rho N)^p (2^{\tau} \rho N)^q. \quad (4.26)$$

It is easy to show

$$II \leq e^2 N_0 (\sigma p + \tau q)^{2m} (\rho N)^{p+q}, \quad (4.27)$$

$$III \leq (\rho + \varepsilon) e^2 N_{\rho, p, q}(u). \quad (4.28)$$

It follows from Lemma 2:5 that

$$\Gamma(\sigma(p-2m) + \tau q + 1) \leq \Gamma(2m\sigma - 2m + 1)^{-1} \Gamma(\sigma p + \tau q + 1 - 2m) \quad (4.29)$$

for  $p \geq 2m$ , and hence

$$\begin{aligned}
&(\sigma p + \tau q)^{2m} \Gamma(\sigma(p-2m) + \tau q + 1) \Gamma(\sigma p + \tau q + 1)^{-1} \\
&\leq (\sigma p + \tau q)^{2m} \Gamma(2m\sigma - 2m + 1)^{-1} \Gamma(\sigma p + \tau q + 1 - 2m) \Gamma(\sigma p + \tau q + 1)^{-1} \\
&= (\sigma p + \tau q)^{2m} \Gamma(2m\sigma - 2m + 1)^{-1} \{(\sigma p + \tau q)(\sigma p + \tau q - 1) \cdots \\
&\cdots (\sigma p + \tau q + 1 - 2m)\}^{-1} \leq (2m)^{2m} \Gamma(2m\sigma - 2m + 1)^{-1}.
\end{aligned} \quad (4.30)$$

From (4.22) and (4.30) it follows that

$$IV \leq e^2 (2m)^{2m} \varepsilon^{1-2m} \Gamma(2m\sigma - 2m + 1)^{-1} N_{\rho, p-2m, q}(u). \quad (4.31)$$

With the aid of

$$\frac{\Gamma(\tau q + 1) \Gamma(\sigma p + 1) \Gamma(\sigma p' + \tau q' + 1)}{\Gamma(\tau q' + 1) \Gamma(\sigma p' + 1) \Gamma(\sigma p + \tau q + 1)} \leq e^{\sigma(p-p') + \tau(q-q')} \quad (4.32)$$

which follows from Lemma 2.5, we get



$$V \leq e^2 \tau \sigma M_0 \sum' (e^\sigma \rho M)^{p-p'} (e^\tau \rho M)^{q-q'} N_{\rho, p', q'}(u). \quad (4.33)$$

(4.29) with  $p, q$  replaced by  $p', q'$  implies

$$\begin{aligned} \Gamma(\sigma p' + \tau q' + 1) &= (\sigma p' + \tau q')(\sigma p' + \tau q' - 1) \cdots \\ &\cdots (\sigma p' + \tau q' + 1 - 2m) \Gamma(\sigma p' + \tau q' + 1 - 2m) \\ &\geq (\sigma p' + \tau q' + 1 - 2m)^{2m} \Gamma(2m\sigma - 2m + 1) \Gamma(\sigma(p' - 2m) + \tau q' + 1) \end{aligned} \quad (4.34)$$

for  $p' \geq 2m$ . If  $p \geq p' \geq 2m$  and  $q \geq q'$

$$\begin{aligned} \frac{\sigma p + \tau q}{\sigma p' + \tau q' + 1 - 2m} &= 1 + \frac{\sigma(p - p') + \tau(q - q') + 2m - 1}{\sigma p' + \tau q' + 1 - 2m} \\ &\leq 1 + \sigma(p - p') + \tau(q - q') + 2m - 1 \\ &\leq \exp(\sigma(p - p') + \tau(q - q') + 2m - 1). \end{aligned} \quad (4.35)$$

With the aid of (4.34) and (4.35)

$$\begin{aligned} VI &\leq \sigma \tau M_0 e^{2m(2m-1)+2} \Gamma(2m\sigma - 2m + 1)^{-1} \\ &\times \sum'_{p' \geq 2m} (e^{(2m+1)\sigma} \rho M)^{p-p'} (e^{(2m+1)\tau} \rho M)^{q-q'} N_{\rho, p'-2m, q'}(u). \end{aligned} \quad (4.36)$$

By Theorem 3.2 there exist constants  $R_0$  and  $R$  such that

$$\|D_i^q u(t)\|_{4m, \alpha} \leq R_0 R^q \Gamma(\tau q + 1) \quad (4.37)$$

for any integer  $q \geq 0$ . We may assume  $R \geq 2M$ . Hence

$$\left( \int_{-\rho}^{\rho} |D_i^q \nabla^{p'} u(t)|_{0, r+\delta}^2 dt \right)^{1/2} \leq \sqrt{2\rho} R_0 R^q \Gamma(\tau q' + 1). \quad (4.38)$$

Noting

$$(\sigma p + \tau q)^{2m} \frac{\Gamma(\sigma p + 1)}{\Gamma(\sigma p' + 1)} \frac{\Gamma(\tau q + 1)}{\Gamma(\sigma p + \tau q + 1)} \leq \frac{(\sigma p + \tau q + 1)^{2m}}{\Gamma(\sigma p' + 1)}$$

which is also a simple consequence of Lemma 2.5, we easily obtain

$$\begin{aligned} VII &\leq 2\sqrt{2\rho} e^2 \tau \sigma M_0 R_0 \sum_{p'=0}^{2m-1} \Gamma(\sigma p' + 1)^{-1} M^{-p'} \\ &\times (\sigma p + \tau q + 1)^{2m} \rho^{p+q} M^p R^q. \end{aligned} \quad (4.39)$$

Using (4.26), (4.27), (4.28), (4.31), (4.33), (4.36), (4.39) and then choosing  $\rho$  and  $\varepsilon$  sufficiently small, we obtain

$$\begin{aligned} N_{\rho, p, q}(u) &\leq C_{20} [N_0 N(2^\sigma \rho N)^p (2^\tau \rho N)^q + N_0 (\sigma p + \tau q)^{2m} (\rho N)^{p+q} \\ &+ N_{\rho, p-2m, q}(u) + M_0 \sum' (e^\sigma \rho M)^{p-p'} (e^\tau \rho M)^{q-q'} N_{\rho, p', q'}(u) \\ &+ M_0 \sum'_{p' \geq 2m} (e^{(2m+1)\sigma} \rho M)^{p-p'} (e^{(2m+1)\tau} \rho M)^{q-q'} N_{\rho, p'-2m, q'}(u) \\ &+ M_0 R_0 (\sigma p + \tau q + 1)^{2m} \rho^{p+q} M^p R^q]. \end{aligned} \quad (4.40)$$

(4.37) implies that (4.23) is true for  $0 \leq p \leq 2m$ ,  $q=0, 1, 2, \dots$  with some constants  $H_0$  and  $H$ . If  $H_0$  and  $H$  are so large that

$$\begin{aligned} 6 C_{20} N_0 &\leq H_0, \quad 6 C_{20} N_0 N \leq H_0, \quad 6 C_{20} M_0 R_0 \leq H_0, \\ e^\tau \rho N &\leq H, \quad 6 C_{20} \leq H^{2m}, \quad 2 e^{(2m+1)\tau} \rho M \leq H, \\ 12 C_{20} M_0 \rho M (2 e^\tau + e^\sigma) &\leq H, \\ 12 C_{20} M_0 \rho M (2 e^{(2m+1)\tau} + e^{(2m+1)\sigma}) &\leq H^{2m+1}, \\ (\sigma p + \tau q)^{2m} &\leq (\rho^{-1} N^{-1} H)^{p+q}, \\ (\sigma p + \tau q)^{2m} &\leq (\rho^{-1} R^{-1} H)^{p+q} \end{aligned}$$

for all  $p$  and  $q$ , then with the aid of (4.40) we can first verify that (4.23) is true for  $q=0$ ,  $p=0, 1, 2, \dots$  and then that the same is valid for all  $p$  and  $q$  by means of the induction argument concerning  $p+q$ . Thus the proof of Lemma 4.3 is completed.

So far we have not used  $\tau=2m\sigma$ . Especially if  $\tau=\sigma=1$ , (4.23) implies the analyticity of the Cauchy data of  $u$ , and hence with the aid of Holmgren's theorem and Cauchy-Kowalevskii theorem it follows that  $u$  is analytic near the origin ([7]).

In what follows we denote the normal variable by  $y$  (i.e.  $y=x_n$ ), and introduce the notation

$$\bar{N}_{p,k,q}(u) = \max \left( \int_{-\rho/2}^{\rho/2} |D_t^q \nabla^p D_y^k u(t)|_{0,\rho/2}^2 dt \right)^{1/2} \quad (4.41)$$

for  $p, k, q=0, 1, 2, \dots$  with the maximum taken over all derivatives  $\nabla_p$  of order  $p$ .

**Lemma 4.4.** *There exist constants  $\bar{L}_0, \bar{L}$  and  $\theta \leq 1/2$  such that*

$$\bar{N}_{p,q,k}(u) \leq \bar{L}_0 \bar{L}^{\sigma p + \sigma k + \tau q} \theta^{\sigma p + \tau q} \Gamma(\sigma p + \tau k + \tau q + 1) \quad (4.42)$$

for all  $p, q, k \geq 0$ .  $\bar{L}_0, \bar{L}$  and  $\theta$  may depend on  $\rho$ , but are independent of  $p, q, k$ .

*Proof.* From (4.23) it follows that there exist constants  $L_0$  and  $L$  such that for  $p \geq 0$ ,  $q \geq 0$ ,  $0 \leq k \leq 2m$

$$\left( \int_{-\rho/2}^{\rho/2} |D_t^q \nabla^p u(t)|_{k,\rho/2}^2 dt \right)^{1/2} \leq L_0 L^{\sigma p + \sigma k + \tau q} \Gamma(\sigma p + \sigma k + \tau q + 1),$$

$L_0$  and  $L$  being allowed to depend on  $\rho$ . Hence

$$\bar{N}_{p,k,q}(u) \leq L_0 L^{\sigma p + \sigma k + \tau q} \Gamma(\sigma p + \sigma k + \tau q + 1) \quad (4.43)$$

for  $p \geq 0$ ,  $q \geq 0$  and  $0 \leq k \leq 2m$ . Due to the ellipticity of  $A$  we can solve

(0.1) with respect to  $D_y^{2m}u$  near the origin to obtain

$$D_y^{2m}u = \sum_{k=0}^{2m-1} \sum_{|\beta|=2m-k} c_{\beta,k} D_x^\beta D_y^k u + \sum_{k=0}^{2m-1} \sum_{|\beta| \leq 2m-k-1} c_{\beta,k} D_x^\beta D_y^k u + c D_l u + af. \quad (4.44)$$

By assumption there exist constants  $\bar{M}_0$  and  $\bar{M}$  such that if  $h$  stands for any of the functions  $c_{\beta,k}$ ,  $c$ ,  $a$ ,  $f$ , then

$$\sup |D_t^q D_x^\gamma h| \leq \bar{M}_0 \bar{M}^{|\sigma|+|\tau|+q} \Gamma(\sigma|\gamma|+1) \Gamma(\tau q+1) \quad (4.45)$$

for any  $q$  and  $\gamma$ . Hence with some constants  $\bar{R}_0$  and  $\bar{R}$

$$\begin{aligned} & \left( \int_{-\rho/2}^{\rho/2} |D_t^q \nabla^p D_y^l (af)(t)|_{0,\rho/2}^2 dt \right)^{1/2} \\ & \leq \bar{R}_0 \bar{R}^{\sigma p + \tau q + \sigma l} \Gamma(\sigma p + \tau q + \sigma l + \tau + 1) \end{aligned} \quad (4.46)$$

for any  $q, p, l=0, 1, 2, \dots$ . (4.42) is valid for  $0 \leq k \leq 2m$  if

$$L_0 \leq \bar{L}_0, \quad L \leq \bar{L}\theta. \quad (4.47)$$

We show by induction that (4.41) is valid for all  $p, q, k$  if  $\bar{L}_0$  and  $\bar{L}$  are so large and  $\theta$  is so small that (4.47) as well as the following inequalities are all true:

$$\bar{L}\theta \geq 2\bar{M}e^2, \quad (4.48)$$

$$64\sigma\tau(\tau+1)C_0\bar{M}_0\theta^\sigma \leq 1, \quad (4.49)$$

$$32\sigma\tau(\tau+1)\bar{M}_0\theta^\tau \leq 1, \quad (4.50)$$

$$32\sigma\tau(\tau+1)\bar{M}_0 \sum_{k=0}^{2m-1} \sum_{|\beta| \leq 2m-k-1} \Gamma(\sigma(2m-k-|\beta|)+1)^{-1} \leq \bar{L}^\sigma, \quad (4.51)$$

$$4\bar{R}_0 \leq \bar{L}_0, \quad \bar{R} \leq \bar{L}\theta, \quad (4.52)$$

where  $C_0$  is the number of  $\beta$  with  $|\beta| \leq 2m$ . To see this we first differentiate both sides of (4.44) to obtain

$$\begin{aligned} D_t^q \nabla^p D_y^{2m+l} u &= \sum_{k=0}^{2m-1} \sum_{|\beta|=2m-k} \binom{q}{q'} \binom{p}{p'} \binom{l}{l'} \\ &\times D_t^{q-q'} \nabla^{p-p'} D_y^{l-l'} c_{\beta,k} \cdot D_t^{q'} D_x^\beta \nabla^{p'} D_y^{k+l'} u + \dots \end{aligned} \quad (4.53)$$

Suppose (4.42) is true for  $0 \leq k \leq 2m+l-1$ . When we estimate the right side of (4.53), we use (2.23) for  $\binom{q}{q'}$  and  $\binom{p}{p'}$ , and for  $\binom{l}{l'}$  use

$$\binom{l}{l'} \leq \frac{(\tau+1)\Gamma(\sigma l + \tau + 1)}{\Gamma(\sigma l' + \tau + 1)\Gamma(\sigma(l-l') + 1)} \quad (4.54)$$

which also follows from Lemma 2.5. Hence with the aid of (4.45),

the induction hypothesis and the inequalities

$$\begin{aligned} & \frac{\Gamma(\tau q + 1) \Gamma(\sigma p + 1) \Gamma(\sigma l + \tau + 1)}{\Gamma(\tau q' + 1) \Gamma(\sigma p' + 1) \Gamma(\sigma l' + \tau + 1)} \\ & \leq \exp(2\sigma(p - p') + 2\tau(q - q') + \sigma(l - l')), \\ & \Gamma(\sigma(p' + |\beta|) + \sigma(l' + k) + \tau q' + 1) \\ & \leq \Gamma(\sigma p' + \sigma l' + \tau q' + \tau + 1) \Gamma(\sigma(2m - k - |\beta|) + 1)^{-1} \end{aligned}$$

which are consequences of Lemma 2.5, we get

$$\begin{aligned} \bar{N}_{p,q,l+2m}(u) & \leq \Gamma(\sigma p + \tau q + \sigma l + \tau + 1) \\ & \times \left\{ \sigma \tau (\tau + 1) \bar{M}_0 \bar{L}_0 e^{2\sigma p + 2\tau q + \sigma l} \bar{M}^{\sigma p + \sigma l + \tau q} \bar{L}^\tau \right. \\ & \times \sum_{k=0}^{2m-1} \sum_{|\beta|=2m-k} \sum_{p'=0}^p \sum_{q'=0}^q \sum_{l'=0}^l \left( \frac{\bar{L}\theta}{e^2 \bar{M}} \right)^{\sigma p' + \tau q'} \left( \frac{\bar{L}}{e \bar{M}} \right)^{\sigma l'} \theta^{\sigma(2m-k)} \\ & + \sigma \tau (\tau + 1) \bar{M}_0 \bar{L}_0 e^{2\sigma p + 2\tau q + \sigma l} \bar{M}^{\sigma p + \sigma l + \tau q} \\ & \times \sum_{k=0}^{2m-1} \sum_{|\beta| \leq 2m-k-1} \sum_{p'=0}^p \sum_{q'=0}^q \sum_{j'=0}^l \left( \frac{\bar{L}\theta}{e^2 \bar{M}} \right)^{\sigma p' + \tau q'} \left( \frac{\bar{L}}{e \bar{M}} \right)^{\sigma l'} \\ & \times \frac{\bar{L}^{\sigma(|\beta|+k)}}{\Gamma(\sigma(2m-k-|\beta|)+1)} + \sigma \tau (\tau + 1) \bar{M}_0 \bar{L}_0 e^{2\sigma p + 2\tau q + \sigma l} \bar{M}^{\sigma p + \sigma l + \tau q} \bar{L}^\tau \theta^\tau \\ & \left. \times \sum_{p'=0}^p \sum_{q'=0}^q \sum_{l'=0}^l \left( \frac{\bar{L}\theta}{e^2 \bar{M}} \right)^{\sigma p' + \tau q'} \left( \frac{\bar{L}}{e \bar{M}} \right)^{\sigma l'} + \bar{R}_0 \bar{R}^{\sigma p + \tau q + \sigma l} \right\}. \end{aligned}$$

If (4.48) is true, we easily get

$$\begin{aligned} \bar{N}_{p,q,l+2m}(u) & \leq \Gamma(\sigma p + \tau q + \sigma l + \tau + 1) \\ & \times \{ 16 C_0 \sigma \tau (\tau + 1) \bar{M}_0 \bar{L}_0 \bar{L}^{\sigma p + \sigma l + \tau q + \tau} \theta^{\sigma p + \tau q + \sigma} \\ & + 8 \sigma \tau (\tau + 1) \bar{M}_0 \bar{L}_0 \sum_{k=0}^{2m-1} \sum_{|\beta| \leq 2m-k-1} \Gamma(\sigma(2m-k-|\beta|)+1)^{-1} \\ & \times \bar{L}^{\sigma p + \sigma l + \tau q + \tau - \sigma} \theta^{\sigma p + \tau q} \\ & + 8 \sigma \tau (\tau + 1) \bar{M}_0 \bar{L}_0 \bar{L}^{\sigma p + \tau q + \sigma l + \tau} \theta^{\sigma p + \tau q + \tau} + \bar{R}_0 \bar{R}^{\sigma p + \tau q + \sigma l} \}. \end{aligned} \quad (4.55)$$

Thus if (4.48)~(4.52) are all true, it is immediately seen that the right side of (4.55) is dominated by

$$\bar{L}_0 \bar{L}^{\sigma p + \sigma(l+2m) + q} \theta^{\sigma p + \tau q} \Gamma(\sigma p + \sigma(l+2m) + \tau q + 1).$$

Thus the proof of Lemma 4.4 is completed.

The interior estimates of the derivatives of the solution is easier to be obtained, and hence we conclude

**Theorem 4.1.** *Under the assumptions (I), (II), (III), (VII), (VIII) and (IV') any solution of (0.1)–(0.2) belongs to Gevrey's class  $G(\sigma, \tau)$ .*

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