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Osaka Math. J.
11 (1959), 63-69.

On Integral Basis of Algebraic Function Fields with Several Variables

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Let K be an algebraic function field with two variables and w a discrete valuation of rank 2 of K. Let L be a finite extension of K and $w_1, w_2, \cdots, w_g$ all the extensions of w in L. We denote the valuation rings of w in K and of w_i in L by $\mathfrak{o}_0$ and $\mathfrak{o}_i$. It is clear that $\mathfrak{o}=\bigcap_{i=1}^{g}\mathfrak{o}_i$ is the integral closure of $\mathfrak{o}_0$ in L. The structure of $\mathfrak{o}$ as an $\mathfrak{o}_0$-module will be determined in this paper. Let $(e_1^{(i)}, e_2^{(i)})$ be the value of ramification of w_i: $w_i(a)=(e_1^{(i)}, e_2^{(i)})w(a)$ for $a\in K$.[1] The main theorem given in this paper is that $\mathfrak{o}$ is a direct sum of $n_0(=\sum_i e_1^{(i)}f_i)$ $\mathfrak{o}_0$-modules of rank 1 and of $n-n_0$ $\mathfrak{o}_0$-modules of infinite rank where $n=[L:K]$. From this we can easily conclude that L/K has integral basis with respect to w in the classical sense when and only when $e_2^{(i)}=1$ for every i.[2] In order to get the theorem, some lemmas on the independence of valuations of rank 2 will be required, which are proved generalizing naturally the well-known proofs in case of rank 1. Then we construct concretely n linearly independent basis of L/K which are a generalization of the classical integral basis, having the following property: If $u_1, u_2, \cdots, u_n$ are those generalized integral basis and $\sum c_i u_i \in \mathfrak{o}$ with c_i in K, then $c_i u_i \in \mathfrak{o}$ for each i.

The above mentioned results will be inductively generalized in general case. Let K be an algebraic function field with several variables and w a discrete valuation of K. Let L be a finite extension of K. We must assume that there holds a fundamental equality with respect to the extensions of w in L: $\sum_i e_i f_i = n$ where e_i are the ramification indices of w_i and f_i are the relative degrees of w_i. This equality holds when rank $w+\dim w=n$. (See Roquette [4]. p. 43. Second Criterion.) In this case the main theorem is proved to be true, although we do not discuss the general case in this paper.

1) We express $w_i(a)$ and $w(a)$ in the normal exponential form and the order of value group will be determined by the last non-zero difference of components (contrary to the usual sense. [5]).

2) For the definition of integral basis, see [1].

1. Lemmas on valuations.[3]

Let us denote a discrete valuation w of rank 2 of a field K in the normal exponential form:

$$w(a) = (\alpha_1, \alpha_2) \qquad \text{for non zero elements } a \text{ of } K,$$

where α_i range over all rational integers. The order of the value group is defined such as $(\alpha_1, \alpha_2) > (\beta_1, \beta_2)$ if $\alpha_2 > \beta_2$ or if $\alpha_2 = \beta_2$ and $\alpha_1 > \beta_1$. From this valuation w, we get a valuation $w^{(2)}$ of rank 1 putting $w^{(2)}(a) = \alpha_2$.

Let $v_1, v_2, \cdots, v_n$ be a set of discrete valuations of rank 1 and of rank 2 satisfying the next condition:

$$(\mathrm{A}) \begin{cases} \text{i) if both } v_i \text{ and } v_j \text{ are of rank 1, then } v_i \neq v_j, \\ \text{ii) if } v_i \text{ is of rank 1 and } v_j \text{ of rank 2, then } v_i \neq v_j^{(2)}, \\ \text{iii) if both } v_i \text{ and } v_j \text{ are of rank 2, then } v_i \neq v_j \text{ and } v_i^{(2)} = v_j^{(2)}. \end{cases}$$

Lemma 1. *Let $v_1, v_2, \cdots, v_n$ be a set of discrete valuations of rank* 1 *and of rank* 2 *satisfying* (A) *and let $\mathfrak{o}_i$ be the valuation rings of v_i. Then $\mathfrak{o}_i \not\subseteq \mathfrak{o}_j$ for $i \neq j$.*

Proof. When one of v_i and v_j is of rank 1, Lemma 1 is clear from the theory of valuations of rank 1 by (A) i) and ii). For, there exists an element a such that $v_i^{(l)}(a) = \alpha$ and $v_j^{(l)}(a) = \beta$ for any rational integers α and β. Here we denote, by $v_i^{(l)}$, $v_i^{(2)}$ when v_i is of rank 2 and v_i when v_i is of rank 1. Next suppose both v_i and v_j are of rank 2. Put $\mathfrak{o}'_i = \{a \in K \mid v_i(a) = (\alpha, 0), \alpha \geqq 0\}$ and $\mathfrak{p}'_i = \{b \in K \mid v_i(b) = (\beta, 0), \beta > 0\}$. If $\mathfrak{o}_i \subseteq \mathfrak{o}_j$, we have $\mathfrak{o}'_i \subseteq \mathfrak{o}'_j$ by (A) iii), and $\mathfrak{p}'_i = \mathfrak{o}'_i \cap \mathfrak{p}'_j$, since $\mathfrak{o}'_i \cap \mathfrak{p}'_j$ is a proper prime ideal of the semi-group $\mathfrak{o}'_i$[4] and $\mathfrak{p}'_i$ is the unique proper prime ideal of $\mathfrak{o}'_i$. If c is an element such that $v_i(c) = (\gamma, 0)$ with $\gamma < 0$, then $c^{-1} \in \mathfrak{p}'_i$ and hence $c^{-1} \in \mathfrak{p}'_j$. Therefore $v_j(c) = (\delta, 0)$ with $\delta < 0$, which implies $\mathfrak{o}'_i \supseteq \mathfrak{o}'_j$, and hence $\mathfrak{o}_i = \mathfrak{o}_j$ by (A) iii), that is, $v_i = v_j$.

Lemma 2. *Let $v_1, v_2, \cdots, v_n$ be a set of discrete valuations satisfying* (A). *Then there exists an element x such that*

$$v_1(1-x) > 0, \quad v_2(x) > 0, \cdots, v_n(x) > 0\,.$$

Proof. We shall prove Lemma 2 by induction. First assume $n=2$. By Lemma 1 there exists an element a such that $v_1(a) \geqq 0$ and $v_2(a) < 0$.

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3) The proofs in this paper are, as stated in the introduction, generalizations of the classical ones and we shall follow [3] for this purpose.

4) If we put $\mathfrak{g}_i = \{a \in K \mid v_i^{(2)}(a) = 0\}$, then $\mathfrak{g}_i = \mathfrak{g}_j$ by (A) iii) and $\mathfrak{g}_i = \{\mathfrak{o}_i', \mathfrak{o}_i'^{-1}\}$. Then $\mathfrak{o}_i' \cap \mathfrak{p}_j' = \phi$ implies that $\{\mathfrak{o}_i', \mathfrak{o}_i'^{-1}\} \cap \mathfrak{p}_j' = \phi$ which is a contradiction.

Put $x_1=1/a$ if $v_1(a)=0$ and $x_1=1/(1+a)$ if $v_1(a)>0$. Then $v_1(x_1)=0$ and $v_2(x_1)>0$. Similarly there exists x_2 such that $v_1(x_2)>0$ and $v_2(x_2)=0$. $x=x_1/(x_1+x_2)$ is then a required element. Assume that Lemma 2 is true for $n-1$, and there exist x_1 and x_2 such that

$$v_1(1-x_1)>0,\quad v_3(x_1)>0,\quad v_4(x_1)>0, \cdots, v_n(x_1)>0,$$

and

$$v_1(1-x_2)>0,\quad v_2(x_2)>0,\quad v_4(x_2)>0, \cdots, v_n(x_2)>0.$$

Then it is easy to show the following element x is a required one:

1) $x = x_1x_2$ if $v_2(x_1)\geqq 0$ and $v_3(x_2)\geqq 0$,
2) $x = x_1/(1+x_1(1-x_1))$ if $v_2(x_1)<0$,
3) $x = x_2/(1+x_2(1-x_2))$ if $v_2(x_1)\geqq 0$ and $v_3(x_2)<0$.

Lemma 3. *Let $w_1, w_2, \cdots, w_g$ be g distinct discrete valuations of rank 2 of K. If $w_1(a)=0$ with some element a, then there exists an element a' such that $w_1(a-a')>0$, $w_2(a')>0, \cdots, w_g(a')>0$.*

Proof. We may assume that $w_1^{(2)}=w_2^{(2)}=\cdots=w_i^{(2)}$ and $w_1^{(2)}\neq w_{i+j}^{(2)}$ for $j\geqq 1$. All the distinct valuations in $w_1, \cdots, w_i, w_{i+1}^{(2)}, \cdots, w_g^{(2)}$ will be denoted $v_1(=w_1)$, $v_2, \cdots, v_n$. Then $v_1, v_2, \cdots, v_n$ satisfy (A) and hence there exists an element x by Lemma 2 such that $v_1(1-x)>0$, $v_2(x)>0, \cdots, v_n(x)>0$. Put $a'=ax^m$ with a sufficiently large number m. Then we have

$$v_1(a-a') = v_1(a)+v_1(1-x^m)\geqq v_1(1-x)>0$$

and

$$v_k(a') = v_k(a)+mv_k(x) \qquad \text{for } k\neq 1.$$

When v_k is of rank 1, $v_k(a)+mv_k(x)>0$ with a sufficiently large number m. When v_k is of rank 2, $v_k^{(2)}(a)=v_1^{(2)}(a)=0$, and hence we have also $v_k(a')>0$ with a sufficiently large numer m.

Lemma 4. *When $v_1, v_2, \cdots, v_n$ are a set of discrete valuations satisfying* (A), *there exist x_i for every i such that*

$$v_i(x_i) = \begin{cases} 1 & \text{if } v_i \text{ is of rank } 1, \\ (1, 0) & \text{if } v_i \text{ is of rank } 2, \end{cases}$$
$$v_j(x_i) = 0 \qquad \text{for } j\neq i.$$

Proof. We may prove the existence of x_1. If v_1 is of rank 1, we choose an element a such that $v_1(a)=1$ and $v_i^{(l)}(a)=0$ for $i\neq 1$. This can be done by the theory of valuations of rank 1. If v_1 is of rank 2, we choose an element a with $v_1(a)=(1, 0)$, when it is seen $v_i^{(2)}(a)=0$ with v_i of rank 2. Let x be an element in Lemma 2 and put

$$x_1 = ax^m + (x-1)^m$$

with a sufficiently large natural number m. Then

$$v_1(x_1) = v_1(ax^m + (x-1)^m) = v_1(ax^m) = v_1(a) + mv_1(x) = v_1(a),$$

and

$$v_i(x_1) = v_i(ax^m + (x-1)^m) = v_i((x-1)^m) = 0\,,$$

since $v_i(ax^m) = v_i(a) + mv_i(x) > 0$ with sufficiently large m.

Lemma 5. *Let $v_1, v_2, \cdots, v_n$ be a set of discrete valuations satisfying* (A) *and let α_i be any rational integers. Then there exists an element x such that*

$$v_i(x) = \begin{cases} \alpha_i & \text{if } v_i \text{ is of rank } 1\,, \\ (\alpha_i, 0) & \text{if } v_i \text{ is of rank } 2\,. \end{cases}$$

Proof. We may put $x = \Pi x_i^{\alpha_i}$ with x_i in Lemma 4.

Lemma 6. *Let $w_1, w_2, \cdots, w_g$ be g distinct discrete valuations of rank* 2. *When l, m and l' are any rational integers, there exist such elements $x_{lm}^{(i)}$ for every i that $w_i(x_{lm}^{(i)}) = (l, m)$ and $w_j(x_{lm}^{(i)}) \geqq (l', m)$ for $j \neq i$.*

Proof. We may prove Lemma 6 for $i=1$. We define $v_1, v_2, \cdots, v_n$ from $w_1, w_2, \cdots, w_g$ as in Lemma 3. Assume that $v_1, v_2, \cdots, v_i$ are of rank 2 and that $v_{i+1}, \cdots, v_n$ are of rank 1. Then there exists an element a such that $v_1^{(2)}(a) = \cdots = v_i^{(2)}(a) = m$ and $v_{i+j}^{(2)}(a) > m$ for $j \geqq 1$. When $v_k(a) = (\alpha_k, m)$ for $1 \leqq k \leqq i$, put $\beta_1 = l - \alpha_1$ and $\beta_k > l' - \alpha_k$ for $2 \leqq k \leqq i$. By Lemma 5 we can choose an element b such that $v_k(b) = (\beta_k, 0)$ for $1 \leqq k \leqq i$ and $v_{i+j}(b) \geqq 0$ for $j \geqq 1$. $x_{lm}^{(1)} = ab$ is a required element.

2. The structure of $\mathfrak{o}$.

Let K be an algebraic function field with two variables and w a discrete valuation of rank 2 of K. We denote the valuation ring and the valuation ideal of w by $\mathfrak{o}_0$ and $\mathfrak{p}_0$. Let L be a finite extension of K and $w_1, w_2, \cdots, w_g$ all the extensions of w in L. We denote the valuation rings and the valuation ideals of w_i by $\mathfrak{o}_i$ and $\mathfrak{p}_i$. Let $(e_1^{(i)}, e_2^{(i)})$ be defined such as $w_i(a) = (e_1^{(i)}, e_2^{(i)}) w(a)$ for $a \in K$. $e_i = e_1^{(i)} e_2^{(i)}$ is the ramification index. $f_i = [\mathfrak{o}_i/\mathfrak{p}_i : \mathfrak{o}_0/\mathfrak{p}_0]$ is the relative degree of w_i. Then $\sum_{i=1}^{g} e_i f_i = [L : K]$ by Roquette [4]. Let $\bar{t}_{ij}$ $(j = 1, 2, \cdots, f_i)$ be $\mathfrak{o}_0/\mathfrak{p}_0$-basis of $\mathfrak{o}_i/\mathfrak{p}_i$ and let t_{ij} be representatives of $\bar{t}_{ij}$ chosen by Lemma 3 such as

$$w_k(t_{ij}) \geqq 0 \qquad \text{for every } k\,. \tag{1}$$

By lemma 6 we can choose $x_{lm}^{(i)}$ such that

(2) $$w_i(x_{lm}^{(i)}) = (l, m) \text{ and } w_j(x_{lm}^{(i)}) \geqq (e_1^{(j)}, m) \text{ for } j \neq i,$$

where $l=0, 1, \cdots, e_1^{(i)}-1$ and $m=0, 1, \cdots, e_2^{(i)}-1$.

Lemma 7. *If* $a = \sum_{i,j,l,m} c_{ijlm} t_{ij} x_{lm}^{(i)}$ *with* $c_{ijlm} \in K$ *such that* $w_i^{(2)}(a) \geqq e_2^{(i)}$ *for every* i *and* $w^{(2)}(c_{ijlm}) \geqq 0$, *then* $w^{(2)}(c_{ijlm}) > 0$.

Proof. Assume that $w^{(2)}(c_{ijlm})=0$ for some c_{ijlm} and put

(3) $$b = {\sum_{i,j,l,m}}' c_{ijlm} t_{ij} x_{lm}^{(i)}$$

where $\sum'$ denotes the sum of all c_{ijlm} such that $w^{(2)}(c_{ijlm})=0$. It is clear $w_i^{(2)}(b) \geqq e_2^{(i)}$ for every i. Let m_0 be the smallest of m with $c_{ijlm} \neq 0$ in (3), and let $c = c_{i_0 j_0 l_0 m_0}$ be one of c_{ijlm_0} having the smallest value with respect to w. Then

$$bc^{-1} = \sum_{\substack{m \geqq m_0 \\ i,j,l}} b_{ijlm} t_{ij} x_{lm}^{(i)} \text{ with } b_{ijlm} = c_{ijlm} c^{-1},$$

where $w_i^{(2)}(bc^{-1}) \geqq e_2^{(i)}$, $b_{ijlm_0} \in \mathfrak{o}_0$ and $b_{i_0 j_0 l_0 m_0} = 1$. We shall show that all $b_{i_0 jlm_0} \in \mathfrak{p}_0$. We have, by (1) and (2),

$$w_{i_0}^{(2)}(b_{ijlm} t_{ij} x_{lm}^{(i)}) > m_0 \qquad \text{for } m > m_0,$$

and

$$w_{i_0}(b_{ijlm_0} t_{ij} x_{lm_0}^{(i)}) \geqq (e_1^{(i_0)}, m_0) \qquad \text{for } i \neq i_0.$$

Hence

(4) $$w_{i_0}\left(\sum_{j,l} b_{i_0 jlm_0} t_{i_0 j} x_{lm_0}^{(i_0)}\right) \geqq (e_1^{(i_0)}, m_0).$$

We have also

$$w_{i_0}(b_{i_0 jlm_0} t_{i_0 j} x_{lm_0}^{(i_0)}) > (0, m_0) \qquad \text{for } l > 0,$$

and hence by (4)

$$w_{i_0}\left(\sum_j b_{i_0 j0m_0} t_{i_0 j} x_{0m_0}^{(i_0)}\right) > (0, m_0),$$

from which we get

$$w_{i_0}\left(\sum_j b_{i_0 j0m_0} t_{i_0 j}\right) > 0,$$

that is, $b_{i_0 j0m_0} \in \mathfrak{p}_0$ for all j. Therefore

(5) $$w_{i_0}(b_{i_0 j0m_0}) \geqq (e_1^{(i_0)}, 0).$$

Next we have, by (1), (2) and (5),

$$w_{i_0}(b_{i_0 jlm_0} t_{i_0 j} x_{lm_0}^{(i_0)}) > (1, m_0) \qquad \text{for } l \neq 1,$$

and hence, by (4),

$$w_{i_0}(\sum_j b_{i_0 j 1 m} t_{i_0 j} x_{1 m_0}^{(i_0)}) > (1, m_0)\,,$$

that is,

$$w_{i_0}(\sum_j b_{i_0 j 1 m_0} t_{i_0 j}) > 0\,.$$

Thus $b_{i_0 j 1 m_0} \in \mathfrak{p}_0$ for all j. Continuing the same procedure, we get that $b_{i_0 i l m_0} \in \mathfrak{p}_0$ for all j and l. But this contradicts $b_{i_0 j_0 l_0 m_0} = 1 \notin \mathfrak{p}_0$.

Lemma 8. *If* $a = \sum_{i,j,l} c_{jjl0} t_{ij} x_{l0}^{(i)}$ *with* $c_{ijl0} \in K$ *is an element of* $\mathfrak{o}$, *then all* $c_{ijl0} \in \mathfrak{o}_0$.

Proof. Assume that $c_{ijl0} \notin \mathfrak{o}_0$ for some c_{ijl0}. Let $c = c_{i_0 j_0 l_0 0}$ be one of c_{ijl0} having the smallest value with respect to w. Then by assumption $c \notin \mathfrak{o}_0$ and hence $c^{-1} \in \mathfrak{p}_0$. Then

$$ac^{-1} = \sum_{i,j,l} b_{ijl0} t_{ij} x_{l0}^{(i)}$$

with $b_{ijl0} \in \mathfrak{o}_0$ and $b_{i_0 j_0 l_0 0} = 1$. We have

$$w_{i_0}(ac^{-1}) \geqq w_{i_0}(c^{-1}) \geqq (e_1^{(i_0)}, 0)\,.$$

As in the proof of Lemma 7, we can show $b_{i_0 j l 0} \in \mathfrak{p}_0$ for all j and l, which is a contradiction.

Theorem 1. *If* $a = \sum_{i,j,l,m} c_{ijlm} t_{ij} x_{lm}^{(i)} \in \mathfrak{o}$ *with* $c_{ijlm} \in K$, *then* $c_{ijlm} t_{ij} x_{lm}^{(i)} \in \mathfrak{o}$.

Proof. First we shall prove $w^{(2)}(c_{ijlm}) \geqq 0$. Assume on the contrary that $w^{(2)}(c_{ijlm}) < 0$ for some c_{ijlm}. Let $c_{i_0 j_0 l_0 m_0} = c$ be one of c_{ijlm} having the smallest value with respect to w. By assumption $c^{-1} \in \mathfrak{p}_0$. Then

$$ac^{-1} = \sum_{i,j,l,m} b_{ijlm} t_{ij} x_{lm}^{(i)}$$

with $b_{ijlm} = c_{ijlm} c^{-1} \in \mathfrak{o}_0$ and $b_{i_0 j_0 l_0 m_0} = 1$, and also $w_i^{(2)}(ac^{-1}) \geqq w_i^{(2)}(c^{-1}) \geqq e_2^{(i)}$. Then, by Lemma 7, we have $w^{(2)}(b_{ijlm}) > 0$ for all i, j, l and m, which contradicts $b_{i_0 j_0 l_0 m_0} = 1$. Thus we have $c_{ijlm} t_{ij} x_{lm}^{(i)} \in \mathfrak{o}$ for $m > 0$ and for all i, j, l. We may now assume that $a = \sum_{i,j,l} c_{ijl0} t_{ij} x_{l0}^{(i)}$. In this case, by Lemma 8, we can say that $c_{ijl0} \in \mathfrak{o}_0$ and hence $c_{ijl0} t_{ij} x_{lm}^{(i)} \in \mathfrak{o}$, which completes the proof.

Corollary 1. $t_{ij} x_{lm}^{(i)}$ *are linearly independent over* K.

Proof. If $0 = \sum c_{ijlm} t_{ij} x_{lm}^{(i)}$, then $c_{ijlm} t_{ij} x_{lm}^{(i)} \in \mathfrak{o}$ by Theorem 1. Since we may suppose that c_{ijlm} take any small values with respect to w, $c_{ijlm} t_{ij} x_{lm}^{(i)} \in \mathfrak{o}$ imply $c_{ijlm} = 0$.

Theorem 2. *If we put* $\mathfrak{o}_{ijlm}=\mathfrak{o}\cap\{ct_{ij}x_{lm}^{(i)}$ *with* $c\in K\}$, *then* $\mathfrak{o}=\sum_{i,j,l,m}\mathfrak{o}_{ijlm}$ *(direct).*

Proof. This is a direct consequence of Theorem 1 and Corollary 1.

Corollary 1. *If* $e_2^{(i)}=1$ *for every* i, *then* $\mathfrak{o}$ *is a finite* $\mathfrak{o}_0$*-module and has* n *linearly independent basis with respect to* $\mathfrak{o}_0$.

Proof. Note $\mathfrak{o}_{ijl0}=\mathfrak{o}_0 t_{ij}x_{lm}^{(i)}$.

Corollary 2. *If* $e_2^{(i)}\neq 1$ *for some* i, *then* $\mathfrak{o}$ *is not a finite* $\mathfrak{o}_0$*-module.*

Proof. It is sufficient to show that $\mathfrak{o}_{ijlm}$ is not a finite $\mathfrak{o}_0$-module if $m>0$. Assume on the contrary that $\mathfrak{o}_{ijlm}$ is a finite $\mathfrak{o}_0$-module. Then there exists a minimal value of $w(c)$ where $ct_{ij}x_{lm}^{(i)}\in\mathfrak{o}_{ijlm}$. For, if $c_1t_{ij}x_{lm}^{(i)}, c_2t_{ij}x_{lm}^{(i)},\cdots, c_pt_{ij}x_{lm}^{(i)}$ constitute $\mathfrak{o}_0$-basis of $\mathfrak{o}_{ijlm}$, take $w(c)=\operatorname{Min}_{1\leq s\leq p}(w(c_s))$. If it is $w(c_0)$, then $w^{(2)}(c_0)\geqq 0$, since

$$w_i(c_0t_{ij}x_{lm}^{(i)}) = w_i(c_0)+(l, m)\geqq 0\,.$$

Therefore $w_k(c_0t_{ij}x_{lm}^{(i)})\geqq(\alpha, 1)$ for every k with some integer α, since $m\geqq 1$. Let c' be an element of K such that $w(c')=(\alpha', 0)$ with $\alpha'<0$. Then

$$w_k(c'c_0t_{ij}x_{lm}^{(i)})\geqq(\alpha+\beta_k,\ 1)>0 \qquad \text{for all } k\,,$$

where $\beta_k=e_1^{(k)}\alpha'$, that is $c'c_0t_{ij}x_{lm}^{(i)}\in\mathfrak{o}_{ijlm}$ and $w(c'c_0)<w(c_0)$, which contradicts the minimality of c_0.

(Received March 17, 1959)

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