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THE EXACTNESS OF GENERALIZED SKEW PRODUCTS

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0. Introduction

Recently, it appears several papers concerning ergodic properties of random maps i.e. skew products. See T. Morita [5], S. Pelikan [7], etc. The main proof tool in their considerations is the so-called Perron-Frobenius operator. In the present paper the autor proves the theorem about exactness of generalized skew products using the Pinkser algebra.

Let $\sigma: X \rightarrow X$ be the shift endomorphism in a space $X \subset \{1, \dots, s\}^{\mathbb{Z}}$ preserving μ . Let $(\sigma, \tilde{\mu})$ be the natural extension of (σ, μ) to the automorphism. The automorphism σ is the shift automorphism on the set $\tilde{X} \subset \{1, \dots, s\}^{\mathbb{Z}}$. Let $\tilde{A}_i = \{\tilde{x} \in \tilde{X}: \tilde{x}(0) = i\}$, $\tilde{\alpha} = \{\tilde{A}_1, \dots, \tilde{A}_s\}$ and $\tilde{\alpha}_m^n = \bigvee_{k=m}^n \sigma^k \tilde{\alpha}$.

DEFINITION. The endomorphism σ is called discrete if $\tilde{\mu}(\tilde{C}) > 0$, for some atom \tilde{C} in $\tilde{\alpha}_{-\infty}^0 \wedge \tilde{\alpha}_0^{\infty}$.

EXAMPLES: If σ is one-sided Markov shift then it is discrete. If σ is given by Lasota-Yorke type map, then it is also discrete (see [8]).

Let p be a Borel measure on $[0, 1]$ which is positive on open sets. Moreover, let T_1, \dots, T_s be piecewise monotonic and continuous transformations of $[0, 1]$ into itself so that there exists the partition $\beta_0 = \{I_1, I_2, \dots\}$ of finite entropy given by $I_i = (t_{i-1}, t_i)$ with $0 = t_0 < t_1 < \dots$, $\lim t_i = 1$, such that $T_j|_{(t_i, t_{i+1})}$ is continuous and strictly monotonic, for $j = 1, \dots, s$, $i = 0, 1, \dots$. We assume that the transformation

$$(1) \quad \bar{T}(x, y) = (\sigma(x), T_{x(0)}y)$$

preserves the product measure $\mu \times p$. Such a transformation is called generalized skew product [2]. The following theorem provides sufficient conditions for \bar{T} to be an exact transformation.

Theorem 1. *Let σ be a discrete endomorphism. If the transformations T_i are 1-1 p a.e., for $i = 1, \dots, s$, and T_i does not preserve the measure p for some i , then the transformation \bar{T} is exact or \bar{T}^m is not ergodic, for some $m \neq 0$.*

The above theorem will be applied to show the exactness of random transformations considered in [3] (see section 3). Moreover, section 4 contains conclusions about exactness of random perturbations of Lasota-Yorke maps.

1. Preliminary facts and lemmas which are used to prove Theorem 1

The following property results immediately from Theorem 2 [2].

Property 1. \bar{T} has not any one-sided generator with finite entropy.

Let $A_i = \{x \in X: x(0) = i\}$, $\alpha = \{A_1, \dots, A_s\}$. Moreover, let γ be a partition of the set $X \times [0, 1]$ defined as follows: $\gamma = \alpha \times \delta$, where δ is a countable partition of $[0, 1]$ into intervals such that $\beta_0 \leq \delta$ and $H(\delta) < \infty$. Here $\alpha \times \delta = \{A_i \times B: i = 1, \dots, s, B \in \delta\}$ and $H(\delta)$ denotes the entropy of δ .

Lemma 1. *If \bar{T} is ergodic, then almost every atom of the partition $\gamma_{\rightarrow\infty} = \bigvee_{i=0}^{\infty} \bar{T}^{-i} \gamma$ has the form $x \times U$, where U is a nonempty interval.*

Proof. The atoms of the partition $\gamma_{\rightarrow\infty}$ have form $x \times U$, where $U \in \lim_{n \rightarrow \infty} (\delta \vee T_{x(0)}^{-1} \delta \vee \dots \vee T_{x(0)}^{-1} \circ \dots \circ T_{x(n)}^{-1} \delta)$. Therefore $\text{card}(U) \leq 1$ or U is a nonempty interval. By Property 1, the set V of atoms $x \times U$ such that U is non-empty interval has positive measure. We note that if $\bar{T}_{\gamma_{\rightarrow\infty}}(x, U) = (\sigma(x), U^*)$ then $T_{x(0)} U \subset U^*$ so that $\bar{T}_{\gamma_{\rightarrow\infty}} V \subset V$. By ergodicity of $\bar{T}_{\gamma_{\rightarrow\infty}}$ we get $\mu \times p_{\gamma_{\rightarrow\infty}}(V) = 1$. \square

Let (\tilde{T}, \tilde{m}) be the natural extension of $(\bar{T}, \mu \times p)$ to the automorphism. The automorphism \tilde{T} is defined on the set $M \subset \tilde{X} \times [0, 1]^N$ of pairs $(\tilde{x}, \mathfrak{Y})$, where \tilde{x} is a two sided sequence and $\mathfrak{Y} = (y_0, y_1, \dots)$ where $T_{\tilde{x}(i)}(y_i) = y_{i-1}$, for $i = 1, 2, \dots$. Therefore \tilde{T} is given by $\tilde{T}(\tilde{x}, \mathfrak{Y}) = (\tilde{\sigma}(\tilde{x}), \tilde{T}_{\tilde{x}(0)}(\mathfrak{Y}))$ where $\tilde{T}_{\tilde{x}(0)}(\mathfrak{Y}) = (T_{x(0)} y_0, y_0, y_1, \dots)$.

Hence the automorphism $(\tilde{\sigma}, \tilde{\mu})$ is a factor of (\tilde{T}, \tilde{m}) . Now, let $\beta_n = \{[0, \frac{1}{n}], (\frac{1}{n}, \frac{2}{n}], \dots, (\frac{n-1}{n}, 1]\} \vee \beta_0^n$, $n = 1, 2, \dots$ and $\beta_0^n = \{I_1, \dots, I_n, J_{n+1}\}$, $J_{n+1} = (t_n, 1]$. Let $\tilde{\gamma}_n$ be the extension of the partition $\gamma_n = \alpha \times \beta_n$ on the space M . Let $\tilde{\gamma}_{n_{\infty}} = \bigvee_{-\infty}^{\infty} \tilde{T}^i \tilde{\gamma}_n$, $\tilde{\gamma}_{n_{\infty}}^0 = \bigvee_0^{\infty} \tilde{T}^{-i} \tilde{\gamma}_n$, $\tilde{\gamma}_{n_0}^{\infty} = \bigvee_0^{\infty} \tilde{T}^i \tilde{\gamma}_n$ and let $\tilde{T}_n = \tilde{T}_{\tilde{\gamma}_{n_{\infty}}}$ be the factor automorphism.

Lemma 2. *The pair $\{\tilde{\gamma}_{n_{\infty}}^0, \tilde{\gamma}_{n_0}^{\infty}\}$ is discrete i.e. there exists an atom \tilde{D} in $\tilde{\gamma}_{n_{\infty}}^0 \wedge \tilde{\gamma}_{n_0}^{\infty}$ such that $\tilde{m}(\tilde{D}) > 0$.*

Proof. To see this, let $\bar{\alpha} = \{\bar{A}_1, \dots, \bar{A}_s\}$ where $\bar{A}_i = \{(\tilde{x}, \mathfrak{Y}): \tilde{x}(0) = i\}$. Let $\tilde{C} \in \bar{\alpha}_{n_{\infty}}^0 \wedge \bar{\alpha}_{n_0}^{\infty}$ be an atom of positive \tilde{m} measure and let $C = \{x \in X: (\tilde{x}, \mathfrak{Y}) \in \tilde{C} \text{ for some } \mathfrak{Y}\}$. There exists an atom $\tilde{D} \in \tilde{\gamma}_{n_{\infty}}^0 \wedge \tilde{\gamma}_{n_0}^{\infty}$ such that $\bigcup_{y \in [0, 1]} D_y = C$. Here

$D_y = \{x: (x, y) \in D\}$ and $D = \{(x, y): (\tilde{x}, \tilde{y}) \in \tilde{D}\}$. By Lemma 1 we have $\tilde{m}(\tilde{D}) = \mu \times p(D) = \int_C p(U_x) d\mu(x) > 0$, where U_x is the union of sets U such that $x \times U$ is an atom of $\gamma_{n-\infty}$ and $x \times U \subset D$. □

2. Proof of Theorem 1

Assume that \tilde{T}^m is ergodic for every $m \neq 0$. Then the same conditions holds for \tilde{T} and \tilde{T}_n , $n=1, 2, \dots$. By Theorem 1 [8] and by Lemma 2 the transformation \tilde{T}_n is K -automorphism, for $n=1, 2, \dots$. Let $\mathcal{A}_n = \sigma(\tilde{\gamma}_{n-\infty})$. Then $\tilde{T}^{-1}\mathcal{A}_n = \mathcal{A}_n$ and $\mathcal{A}_n \uparrow \mathcal{B}$ where \mathcal{B} is the σ -algebra of \tilde{m} measurable sets. By Theorem 13 ([6] p. 69) $P(\tilde{T}) \cap \mathcal{A}_n \uparrow P(\tilde{T})$ where $P(\tilde{T})$ denotes Pinsker algebra for \tilde{T} . The equalities $P(\tilde{T}) \cap \mathcal{A}_n = P(\tilde{T}_n) = \{M, \phi\}$ imply $P(\tilde{T}) = \{M, \phi\}$. Therefore \tilde{T} is K -automorphism and hence \tilde{T} is the exact endomorphism. □

3. An application to some class of random maps

Now we proceed to consider the exactness of generalized skew products considered in [3]. Let $\{T_\varepsilon\}_{\varepsilon \in (a,b)}$ be the one-parameter family of transformations of the interval $[0, 1]$ into itself such that

$$(2) \quad T_\varepsilon^{-1}(y) = (1-\varepsilon)y + \varepsilon g(y),$$

where $g \in C^2[0, 1]$, $g(0)=0$, $g(1)=1$, and $a=(1-\sup g')^{-1}$, $b=(1-\inf g')^{-1}$. Moreover, assume that there exists exactly one point y_0 , for which $g'(y_0)=1$.

Let T be an infinite interval exchange transformation of $[0, 1]$ of the following type:

- (i) there exists a partition $\beta_0 = \{I_1, I_2, \dots\}$ given by $I_i = (t_{i-1}, t_i)$ with $0 = t_0 < t_1 < \dots$, $\lim t_i = 1$, $H(\beta_0) < \infty$
- (ii) there exist real constants a_i , so that for $t \in I_i$, $T(t) = t + a_i$
- (iii) the only accumulation point of $\{t_{i-1} + a_i\} \cup \{t_i + a_i\}$ is 1
- (iv) T is one-to-one.

The above transformations have been considered in [1]. Let σ be the one-sided (p_1, \dots, p_s) -Bernoulli shift and let p be the Lebesgue measure. Using the endomorphism σ we will randomly perturb the automorphism T by s elements of the family (2). Namely we take s functions $T_{\varepsilon_i}, \dots, T_{\varepsilon_s}, \varepsilon_i \neq \varepsilon_j$ for $i \neq j$, and we define the transformation

$$(3) \quad \tilde{T}(x, y) = (\sigma(x), T_{\varepsilon_{x(0)}}T(y)).$$

In addition we postulate that \tilde{T} preserves the product measure, which is equivalent to $\sum_{i=1}^s \varepsilon_i p_i = 0$. For the rest of the paper we shall denote $T_{\varepsilon_i}T$ by T_i , $i=1, \dots, s$. The transformation \tilde{T} has the following property.

Property 2. ([3]) \tilde{T} is weakly mixing, for $s \geq 3$. If $T=I$ where $I(x)=x$,

then \bar{T} is mixing for $s \geq 2$ ([4]).

By Theorem 1 and by Property 2 we get

Theorem 2. *The endomorphism \bar{T} is exact for $s \geq 3$ (or $s \geq 2$ if $T=I$).*

As the conclusion of Theorem 2 we obtain the convergence of the iterations of some double stochastic operators which arise from Frobenius-Perron operator for \bar{T} . The Frobenius-Perron operator for \bar{T} is given by the formula

$$P_{\bar{T}}(r(x)f(y)) = \sum_{i=1}^s p_i r(ix)(T_i^{-1})'(y)f(T_i^{-1}y),$$

where $ix=(i, x(0), x(1), \dots)$. Here $r \in L_1(\mu)$ and $f \in L_1(m)$. For $r=1$ we get

$$P_{\bar{T}}f(y) = \sum_{i=1}^s p_i (T_i^{-1})'(y)f(T_i^{-1}y).$$

Conclusion 1. $\lim_{n \rightarrow \infty} P_{\bar{T}}^n f = \int_0^1 f dm$ in L_1 norm, for every $f \in L_1(m)$.

REMARK. The results of this paper are still true if it is only assumed that $g \in C^2[0, 1]$, $g(0)=0$, $g(1)=1$, $g(y) \neq y$ for every $y \in (0, 1)$ and there exists $y_0 \in (0, 1)$ such that $g'(y_0)=1$ and $g'(y) \neq 1$ in some neighbourhood of the point y_0 . In this case the family of transformations T_ε should be defined by (2) for ε from suitable smaller interval.

4. Exactness of random perturbations of Lasota-Yorke type maps

Let T be piecewise monotone and C^2 . Piecewise monotone and C^2 means that there is a partition of $[0, 1]$, $0=a_0 < \dots < a_k=1$, so that for each $i=0, 1, \dots, k-1$, $T|_{(a_i, a_{i+1})}$ is monotone and extends to a C^2 map on $[a_i, a_{i+1}]$. Assume that T preserves the Lebesgue measure m . Denote by \bar{T} the transformation defined for T by equality (3) and let $T_i = T_{\varepsilon_i} T$ for $i=1, \dots, s$.

Theorem 3. *If, for all $x \in [0, 1]$*

$$\sum_{i=1}^s \frac{p_i}{|T_i'(x)|} < 1,$$

then \bar{T} is exact, for $s \geq 3$.

Proof. By Theorem 1 [7] some power of Frobenius-Perron operator $P_{\bar{T}} f = \sum_{i=1}^s p_i P_{T_i} f$ is quasicompact on $BV[0, 1]$. By Property 2 all iterations of \bar{T} are ergodic. Due to the uniqueness of absolutely continuous invariant measure we obtain $\lim_{n \rightarrow \infty} P_{\bar{T}}^n f = \int f dm$ in L_1 norm for every $f \in L_1(m)$. Therefore $\lim_{n \rightarrow \infty} P_{\bar{T}}^n(rf) = \int r(x)f(y) d\mu \times m$ in L_1 norm for every $r \in L_1(\mu)$ and $f \in L_1(m)$ which implies the exactness of \bar{T} . \square

Conclusion 2. *If $\inf |T'(x)| > 1$, then there exists $\eta > 0$ such that for $|\varepsilon_i| < \eta$, $i=1, \dots, s$, the transformation \bar{T} is exact.*

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