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THE SECOND LOWER LOEWY TERM OF THE PRINCIPAL INDECOMPOSABLE OF A MODULAR GROUP ALGEBRA

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1. Introduction

Let G be a finite group and consider a field \mathbb{K} of prime characteristic p . Let P be the projective cover of the trivial $\mathbb{K}G$ -module, which we denote by \mathbb{K} , and J the Jacobson radical $J(\mathbb{K}G)$ of the group algebra $\mathbb{K}G$. Let $e \in \mathbb{K}G$ be a primitive idempotent such that $P = e\mathbb{K}G$. We are concerned with the second term

$$eJ/eJ^2$$

of the lower Loewy series of P . It is a completely reducible $\mathbb{K}G$ -module, whose composition factors are just the irreducible $\mathbb{K}G$ -modules V such that there exists a nonsplit $\mathbb{K}G$ -module extension $0 \rightarrow V \rightarrow E \rightarrow \mathbb{K} \rightarrow 0$ (see [7, VII 16.8]).

Gaschütz (see [7, VII §15]) gives a complete description of eJ/eJ^2 for $\mathbb{K} = \mathbb{F}_p$, the field of p elements, and G a p -soluble group: Its composition factors are precisely the abelian complemented p -chief factors of G , counting the multiplicities. Later Willems shows [12] that for any G each complemented p -chief factor of G appears as a component of eJ/eJ^2 with multiplicity not less than that as a (complemented) chief factor of G . Okuyama and Tsushima [10] define a filtration of eJ/eJ^2 from a chief series of G , which provides a new proof of these results and makes explicit the relationship between the chief factors of G and the composition factors of eJ/eJ^2 .

In this paper we give a description of eJ/eJ^2 for any G and any field \mathbb{K} of characteristic p , which only depends on the knowledge of what occurs for certain almost simple sections of G , by means of the development of a reduction theorem of Kovács [8]. As an application we obtain the terms of the filtration of Okuyama and Tsushima corresponding to any chief factor of any G .

2. Notations and basic facts

We denote by $\text{Irr}(G, \mathbb{K})$ the set of irreducible $\mathbb{K}G$ -modules. If $V \in \text{Irr}(G, \mathbb{K})$, then, as P is the projective cover of \mathbb{K} ,

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$$H^1(G, V) \cong \text{Ext}_{\mathbb{K}G}(\mathbb{K}, V) \cong \text{Hom}_{\mathbb{K}G}(eJ, V) \cong \text{Hom}_{\mathbb{K}G}(eJ/eJ^2, V)$$

[5]. Therefore, if we denote by $\ell_2^G(V)$ the multiplicity of V as component of eJ/eJ^2 , then

$$\ell_2^G(V) = \dim_{\text{End}_{\mathbb{K}G}(V)} H^1(G, V).$$

($\text{End}_{\mathbb{K}G}(V)$ is a division ring, because of Schur's lemma [6, 10.5].) We set

$$\mathcal{C}(G, \mathbb{K}) = \{V; V \in \text{Irr}(G, \mathbb{K}), \ell_2^G(V) \neq 0\}$$

(here we identify isomorphic modules, that is $\mathcal{C}(G, \mathbb{K})$ consists actually of isomorphism classes of modules). On the other hand, $\text{Ext}_{\mathbb{K}G}(\mathbb{K}, V) \cong \text{E}(\mathbb{K}, V)$ [5], whence if $\xi \in H^1(G, V)$, then ξ represents an equivalence class of $\mathbb{K}G$ -module extensions

$$0 \rightarrow V \rightarrow E \rightarrow \mathbb{K} \rightarrow 0.$$

We put then $C_G(\xi) = C_G(E)$ and

$$\mathcal{C}_1(G, \mathbb{K}) = \{V \in \mathcal{C}(G, \mathbb{K}); \exists \xi \in H^1(G, V) \text{ such that } C_G(\xi) < C_G(V)\}.$$

Recall that $\mathcal{C}_1(G, \mathbb{F}_p)$ is the set of the abelian complemented p -chief factors of G [11, 2.4(1)].

A $\mathbb{K}G$ -module V can be considered as a (faithful) $\mathbb{K}G/C_G(V)$ -module. We put

$$\mathcal{C}_0(G, \mathbb{K}) = \{V \in \mathcal{C}(G, \mathbb{K}); \ell_2^{G/C_G(V)}(V) \neq 0\}.$$

If $\mathbb{F} \subseteq \mathbb{K}$ is a field extension and M is an $\mathbb{F}G$ -module, then we set $M_{\mathbb{K}} = M \otimes_{\mathbb{F}} \mathbb{K}$ for the scalar extension.

If $V \in \text{Irr}(G, \mathbb{K})$, then a unique (up to isomorphisms) $\hat{V} \in \text{Irr}(G, \mathbb{F}_p)$ is determined such that V is a component of $\hat{V}_{\mathbb{K}}$. In this case $H^1(G, V) \neq 0$ if and only if $H^1(G, \hat{V}) \neq 0$, $C_G(V) = C_G(\hat{V})$ and $V \in \mathcal{C}_1(G, \mathbb{K})$ if and only if \hat{V} is isomorphic to a complemented chief factor of G [9, §1]. Therefore $V \in \mathcal{C}_\varepsilon(G, \mathbb{K})$ if and only if $\hat{V} \in \mathcal{C}_\varepsilon(G, \mathbb{F}_p)$, $\varepsilon = \emptyset, 0, 1$.

Proposition 2.1. *If $\mathbb{F} \subseteq \mathbb{K}$ is a field extension, let $V \in \text{Irr}(G, \mathbb{K})$ and $U \in \text{Irr}(G, \mathbb{F})$ be such that V is a component of $U_{\mathbb{K}}$. Then*

$$\dim_{\text{End}_{\mathbb{K}G}(V)} H^n(G, V) = \dim_{\text{End}_{\mathbb{F}G}(U)} H^n(G, U), \quad n = 1, 2, \dots$$

Proof. Let

$$\mathcal{P} : \dots \rightarrow P_{n+1} \rightarrow P_n \rightarrow \dots \rightarrow P_1 \rightarrow \mathbb{F} \rightarrow 0$$

be a minimal projective resolution of \mathbb{F} . Then $\dim_{\text{End}_{\mathbb{F}G}(U)} H^n(G, U)$ is the multiplicity of U as component of $P_{n+1}/P_n J(\mathbb{F}G)$.

On the other hand, $J(\mathbb{F}G)_{\mathbb{K}} \cong J(\mathbb{K}G)$ [7, VII 1.5]. As \mathbb{K} is of prime characteristic, $U_{\mathbb{K}}$ is a direct sum of pairwise non-isomorphic irreducible $\mathbb{K}G$ -modules [7, VII 1.15]. Then we have that the multiplicity of U as component of $P_{n+1}/P_n J(\mathbb{F}G)$ is equal to the multiplicity of V as component of $(P_{n+1}/P_n J(\mathbb{F}G))_{\mathbb{K}}$. And also we have that $\mathcal{P}_{\mathbb{K}}$ is a minimal projective resolution of \mathbb{K} .

We consider again dimensions in $\mathcal{P}_{\mathbb{K}}$ and have the claim. □

Corollary 2.2. *Let $V \in \text{Irr}(G, \mathbb{K})$. Then*

$$\ell_2^G(V) = \ell_2^G(\hat{V}), \quad \ell_2^{G/C_G(V)}(V) = \ell_2^{G/C_G(\hat{V})}(\hat{V}).$$

Denote by $\text{cm}^G(V)$ the multiplicity of \hat{V} as complemented chief factor in a chief series of G . As another immediate consequence we have the validity of the following equality, which appears in [1, 2.10(b)] for the case $\mathbb{K} = \mathbb{F}_p$:

Corollary 2.3. *Let $V \in \text{Irr}(G, \mathbb{K})$. Then we have*

$$\ell_2^G(V) = \text{cm}^G(V) + \ell_2^{G/C_G(V)}(V).$$

3. The second Loewy term

Recall that a *primitive* group is a group G with a maximal subgroup H such that $\text{core}_G(H) = 1$, $\text{core}_G(H)$ being the intersection of all conjugate of H in G . Then G has exactly either one minimal normal subgroup or two nonabelian minimal normal subgroups. If G has a single nonabelian minimal normal subgroup, then we say $G \in \mathcal{P}_2$.

A particular consequence of Kovács reduction theorem [8] is that, if $U \in \text{Irr}(G, \mathbb{F}_p)$ is faithful and $H^1(G, U) \neq 0$, then $G \in \mathcal{P}_2$ and $p \mid |\text{S}(G)|$ (where $\text{S}(G)$, the *socle* of G , is the product of the minimal normal subgroups of G). From the above proposition we have that this is also true for any faithful irreducible module in $\text{Irr}(G, \mathbb{K})$.

Proposition 3.1. *The following two assertions are equivalent:*

- (a) *There exists a faithful irreducible $\mathbb{K}G$ -module V such that $H^1(G, V) \neq 0$.*
- (b) *$G \in \mathcal{P}_2$ and $p \mid |\text{S}(G)|$.*

Proof. It suffices to show that (b) \implies (a). This follows from the fact that $\text{F}_p(G) = \bigcap \{C_G(V); V \in \mathcal{C}(G, \mathbb{K})\}$ [2, Theorem 1], as $\text{F}_p(G) = 1$ and $\text{S}(G)$ is contained in each nontrivial normal subgroup of G . □

Corollary 3.2. *Set $n_0(G) = \{C; C \triangleleft G, G/C \in \mathcal{P}_2, p \mid |S(G/C)|\}$. Then*

$$n_0(G) = \{C_G(V); V \in \mathcal{C}_0(G, \mathbb{K})\}.$$

Proof. By the definition of $\mathcal{C}_0(G, \mathbb{K})$ and Proposition 3.1, it is clear that if $V \in \mathcal{C}_0(G, \mathbb{K})$, then $C_G(V) \in n_0(G)$. Assume now that $C \in n_0(G)$. By Proposition 3.1 there exists a faithful irreducible $\mathbb{K}G/C$ -module V such that $H^1(G/C, V) \neq 0$, that is such that $\ell_2^{G/C}(V) \neq 0$. As the inflation map $H^1(G/C, V) \rightarrow H^1(G, V)$ is a monomorphism [5, VI 8.1], $V \in \mathcal{C}(G, \mathbb{K})$. As $C = C_G(V)$ we conclude that $V \in \mathcal{C}_0(G, \mathbb{K})$. \square

Let $C \in n_0(G)$. Then $S(G/C)$ is the only minimal normal subgroup of G/C and is nonabelian. Therefore it is the product of isomorphic nonabelian simple groups. Let S/C be a simple component of $S(G/C)$, $A = N_G(S/C)$ and $B = C_G(S/C)$. In these conditions we say $A/B \in \mathfrak{a}(C)$. Observe that A/B is an *almost simple* group, that is a group in \mathcal{P}_2 with simple socle (isomorphic to S/C).

If $H \leq G$ and V is a $\mathbb{K}G$ -module, then we set

$$V^H = \{v \in V; vh = v \ \forall h \in H\}$$

and write $V \downarrow_H$ for the $\mathbb{K}H$ -module obtained from V by restricting the action to $\mathbb{K}H$.

If W is a $\mathbb{K}H$ -module, then we set $W \uparrow^G = W \otimes_{\mathbb{K}H} \mathbb{K}G$.

Lemma 3.3. *Consider $C \in n_0(G)$, $A/B \in \mathfrak{a}(C)$ and assume that W is a faithful irreducible $\mathbb{K}A/B$ -module. Then*

- (a) $W \uparrow^G \in \text{Irr}(G, \mathbb{K})$, $W \cong (W \uparrow^G)^B$ and $C_G(W \uparrow^G) = C$.
- (b) $\ell_2^{A/B}(W) = \ell_2^{G/C}(W \uparrow^G)$.
- (c) $\ell_2^A(W) = \ell_2^G(W \uparrow^G)$ and $\text{cm}^A(W) = \text{cm}^G(W \uparrow^G)$.

Proof. (a) We may assume that $C = 1$. Then $G \in \mathcal{P}_2$. Set $N = S(G)$. Let $V \in \text{Irr}(G, \mathbb{K})$ be a component of the head $H(W \uparrow^G) := W \uparrow^G / (W \uparrow^G)J$ of $W \uparrow^G$. By Nakayama's theorem [6, V 16.6], W is a submodule of $S(V \downarrow_A)$, and so $W \downarrow_N$ is a submodule of $V \downarrow_N$.

Let $\{g_1, \dots, g_n\}$ be a transversal of A in G , with $g_1 = 1$. Then, by putting $S_i = S^{g_i}$, $B_i = B^{g_i}$, we have $N = S_1 \times \dots \times S_n$, $B_i = C_G(S_i)$. Set moreover for $1 \leq i \leq n$

$$V_i = V^{B_i}, \quad U_i = V_1 + \dots + V_{i-1} + V_{i+1} + \dots + V_n, \quad M_i = V_i \cap U_i.$$

We have

$$S_i \leq \bigcap_{j \neq i} B_j \leq \bigcap_{j \neq i} C_G(V_j) = C_G(U_i), \quad B_i \leq C_G(V_i)$$

and hence $N \leq S_i B_i \leq C_G(M_i)$. Therefore $M_i \subseteq V^N$. As V is an irreducible $\mathbb{K}G$ -module and $N \trianglelefteq G$, either $V^N = V$ or $V^N = 0$. Assume that $V^N = V$. As $W \downarrow_N$ is

a submodule of $V \downarrow_N$, $N \leq C_A(W)$. Then $B < BN \leq C_A(W)$, contradicting the fact that W is a faithful A/B -module.

So we have that $V^N = 0$. In particular $M = 0$, that is $V_1 + \cdots + V_n$ is a direct sum. As $Wg_i \subseteq V_i$, we have that also $Wg_1 + \cdots + Wg_n$ is a direct sum, and hence

$$W \uparrow^G \cong Wg_1 \oplus \cdots \oplus Wg_n \leq V.$$

As $\dim_{\mathbb{K}} V \leq \dim_{\mathbb{K}} W \uparrow^G$, we have that $V \cong W \uparrow^G$. Clearly $W \cong (W \uparrow^G)^B$. And $C_G(V) = \text{core}_G(C_A(W)) = \text{core}_G(B) = 1$.

(b) By Shapiro's lemma [3, 6.3], $H^1(A/C, W) \cong H^1(G/C, W \uparrow^G)$. By [8, 3.5], $\text{End}_{\mathbb{K}A/C}(W) \cong \text{End}_{\mathbb{K}G/C}(W \uparrow^G)$. Therefore $\ell_2^{A/C}(W) = \ell_2^{G/C}(V)$.

Assume that \hat{W} appears as a chief factor of A between C and B . Then $S \leq C_A(\hat{W}) = C_A(W) = B = C_A(S/C)$, a contradiction. In particular $\text{cm}^{A/C}(W) = 0$. Therefore $\ell_2^{A/B}(W) = \ell_2^{A/C}(W)$, and hence $\ell_2^{A/B}(W) = \ell_2^{G/C}(V)$.

(c) Again by Shapiro's lemma, $\ell_2^A(W) = \ell_2^G(W \uparrow^G)$. From (b) and [1, 2.10(b)] we have that $\text{cm}^A(W) = \text{cm}^G(W \uparrow^G)$. \square

We now deduce the validity of [8] for any field \mathbb{K} of prime characteristic p :

Theorem 3.4 (Kovács Reduction.). *Consider $V \in \mathcal{C}_0(G, \mathbb{K})$, $A/B \in \mathfrak{a}(C)$ and set $N/C = \mathbf{S}(G/C)$. Let $W = V^{B \cap N}$. Then $W \in \mathcal{C}_0(A, \mathbb{K})$, $C_A(W) = B$, $\ell_2^{G/C}(V) = \ell_2^{A/B}(W)$ and $V \cong W \uparrow^G$.*

Proof. As $V \in \mathcal{C}_0(G, \mathbb{K})$, $\hat{V} \in \mathcal{C}_0(G, \mathbb{F}_p)$. Moreover $C := C_G(V) = C_G(\hat{V}) \in \mathfrak{n}_0(G)$. By [8], $U := \hat{V}^{B \cap N} \in \mathcal{C}_0(A, \mathbb{F}_p)$, $C_A(U) = B$ and $\ell_2^{G/C}(\hat{V}) = \ell_2^{A/B}(U)$.

Let $U_{\mathbb{K}} \cong W_1 \oplus \cdots \oplus W_r$, where each W_i is irreducible. Then $W_i \in \mathcal{C}_0(A, \mathbb{K})$ and $C_A(W_i) = B$. Let now $\hat{V}_{\mathbb{K}} \cong V_1 \oplus \cdots \oplus V_s$, with each V_i irreducible and $V_1 \cong V$. Then we have

$$V_1 \oplus \cdots \oplus V_s \cong \hat{V}_{\mathbb{K}} \cong (U \uparrow^G)_{\mathbb{K}} \cong U_{\mathbb{K}} \uparrow^G \cong W_1 \uparrow^G \oplus \cdots \oplus W_r \uparrow^G.$$

By Lemma 3.3 (a) each $W_i \uparrow^G$ is irreducible. Therefore, by the Krull-Remak-Schmidt theorem [6, I 12.3], we have that $r = s$ and, after rearranging the indices if necessary, $V_i \cong W_i \uparrow^G$, $1 \leq i \leq r$. Moreover, as $U = \hat{V}^{B \cap N}$, $U_{\mathbb{K}} \cong (\hat{V}_{\mathbb{K}})^{B \cap N}$, and therefore $W_i \cong V_i^{B \cap N}$. Finally, by Corollary 2.2, $\ell_2^{G/C}(V) = \ell_2^{G/C}(\hat{V}) = \ell_2^{A/B}(U) = \ell_2^{A/B}(W_1)$. \square

This reduction theorem allows us to reduce also the study of $\mathcal{C}(G, \mathbb{K})$ to the almost simple case:

Theorem 3.5. *Consider $C \in \mathfrak{n}_0(G)$ and $A/B \in \mathfrak{a}(C)$. Then the map*

$$\uparrow^G: \{W \in \mathcal{C}_0(A, \mathbb{K}); C_A(W) = B\} \rightarrow \{V \in \mathcal{C}_0(G, \mathbb{K}); C_G(V) = C\}$$

is bijective. Moreover $\ell_2^{A/B}(W) = \ell_2^{G/C}(W \uparrow^G)$, $\ell_2^A(W) = \ell_2^G(W \uparrow^G)$ and $\text{cm}^A(W) = \text{cm}^G(W \uparrow^G)$.

Proof. By Lemma 3.3, (a) (b) we have a well-defined injective map. It is surjective by Theorem 3.4. \square

Now we can give the following first explicit description of eJ/eJ^2 .

Theorem 3.6. *Let $C \in \mathfrak{n}_0(G)$ and $A/B \in \mathfrak{a}(C)$. Let $\{W_1 \cdots W_m\}$ be a complete set of representatives of the isomorphism classes of faithful modules in $\mathcal{C}(A/B, \mathbb{K})$. We set:*

$$\begin{aligned} \mathbf{M}(C) &:= \ell_2^{A/B}(W_1) \cdot W_1 \uparrow^G \oplus \cdots \oplus \ell_2^{A/B}(W_m) \cdot W_m \uparrow^G \\ \mathbf{R}(C) &:= \ell_2^A(W_1) \cdot W_1 \uparrow^G \oplus \cdots \oplus \ell_2^A(W_m) \cdot W_m \uparrow^G. \end{aligned}$$

Then we have:

$$\begin{aligned} eJ/eJ^2 &\cong \left(\bigoplus_{V \in \mathcal{C}(G, \mathbb{K})} \text{cm}^G(V) \cdot V \right) \oplus \left(\bigoplus_{C \in \mathfrak{n}_0(G)} \mathbf{M}(C) \right) \\ &\cong \left(\bigoplus_{V \in \mathcal{C}(G, \mathbb{K}) \setminus \mathcal{C}_0(G, \mathbb{K})} \text{cm}^G(V) \cdot V \right) \oplus \left(\bigoplus_{C \in \mathfrak{n}_0(G)} \mathbf{R}(C) \right). \end{aligned}$$

Proof. By Corollary 2.3,

$$\begin{aligned} eJ/eJ^2 &\cong \bigoplus_{V \in \mathcal{C}(G, \mathbb{K})} \ell_2^G(V) \cdot V \\ &\cong \left(\bigoplus_{V \in \mathcal{C}(G, \mathbb{K})} \text{cm}^G(V) \cdot V \right) \oplus \left(\bigoplus_{V \in \mathcal{C}(G, \mathbb{K})} \ell_2^{G/C_G(V)}(V) \cdot V \right). \end{aligned}$$

Now,

$$\bigoplus_{V \in \mathcal{C}(G, \mathbb{K})} \ell_2^{G/C_G(V)}(V) \cdot V \cong \bigoplus_{V \in \mathcal{C}_0(G, \mathbb{K})} \ell_2^{G/C_G(V)}(V) \cdot V$$

(by the definition of $\mathcal{C}_0(G, \mathbb{K})$)

$$\cong \bigoplus_{C \in \mathfrak{n}_0(C)} \left(\bigoplus_{\substack{V \in \mathcal{C}_0(G, \mathbb{K}) \\ C_G(V) = C}} \ell_2^{G/C}(V) \cdot V \right)$$

(as $\mathcal{C}_0(G, \mathbb{K}) = \bigcup_{C \in \mathfrak{n}_0(G)} \{V; V \in \mathcal{C}_0(G, \mathbb{K}), C_G(V) = C\}$ by Corollary 3.2)

$$\cong \bigoplus_{C \in \mathfrak{n}_0(G)} \mathbf{M}(C)$$

X(by Theorem 3.5).

On the other hand, if $V \in \mathcal{C}(G, \mathbb{K}) \setminus \mathcal{C}_0(G, \mathbb{K})$, then $\ell_2^G(V) = \text{cm}^G(V)$. Therefore

$$eJ/eJ^2 \cong \left(\bigoplus_{V \in \mathcal{C}(G, \mathbb{K}) \setminus \mathcal{C}_0(G, \mathbb{K})} \text{cm}^G(V) \cdot V \right) \oplus \left(\bigoplus_{V \in \mathcal{C}_0(G, \mathbb{K})} \ell_2^G(V) \cdot V \right)$$

and

$$\bigoplus_{V \in \mathcal{C}_0(G, \mathbb{K})} \ell_2^G(V) \cdot V \cong \bigoplus_{C \in \mathfrak{n}_0(G)} \left(\bigoplus_{\substack{V \in \mathcal{C}_0(G, \mathbb{K}) \\ C_G(V) = C}} \ell_2^G(V) \cdot V \right) \cong \bigoplus_{C \in \mathfrak{n}_0(G)} \mathbf{R}(C). \quad \square$$

If $H \leq G$, then we put

$$\mathfrak{h}_G(H) = e\mathbf{I}(H)\mathbb{K}G + eJ^2,$$

where $\mathbf{I}(H) = \{\sum_{h \in H} a_h h; \sum_{h \in H} a_h = 0, a_h \in \mathbb{K}\}$ is the augmentation ideal of $\mathbb{K}H$.

Observe that $\mathfrak{h}_G(H)$ is a $\mathbb{K}G$ -module and $eJ^2 \subseteq \mathfrak{h}_G(H) \subseteq eJ$ since $e\mathbf{I}(G) = eJ$.

The filtration of eJ/eJ^2 given by Okuyama and Tsushima [10] for $\mathbb{K} = \mathbb{F}_p$ and p -soluble G is a particular case of the following second description we give of eJ/eJ^2 :

Theorem 3.7. *Let $1 = G_0 \leq G_1 \leq \dots \leq G_{n-1} \leq G_n = G$ be a chief series of G and consider the associated filtration of eJ/eJ^2 :*

$$eJ^2 = \mathfrak{h}_G(G_0) \subseteq \mathfrak{h}_G(G_1) \subseteq \dots \subseteq \mathfrak{h}_G(G_{n-1}) \subseteq \mathfrak{h}_G(G_n) = eJ.$$

Then we have:

$$\mathfrak{h}_G(G_i)/\mathfrak{h}_G(G_{i-1}) \cong \begin{cases} 0 & \text{if } G_i/G_{i-1} \text{ is a } p'\text{-chief factor or a frattini } p\text{-chief factor} \\ (G_i/G_{i-1})_{\mathbb{K}} & \text{if } G_i/G_{i-1} \text{ is a complemented } p\text{-chief factor} \\ \mathbf{M}(C_G(G_i/G_{i-1})) & \text{otherwise.} \end{cases}$$

Proof. We proceed with the induction on n . If $n = 0$, the result is trivial. Assume $n > 0$, take $N = G_1$ and consider $\overline{G} = G/N$.

As eJ/eJ^2 is completely reducible, $eJ/eJ^2 \cong eJ/\mathfrak{h}_G(N) \oplus \mathfrak{h}_G(N)/eJ^2$. Now

$$eJ/\mathfrak{h}_G(N) = \mathfrak{h}_G(G)/\mathfrak{h}_G(N) \cong \mathfrak{h}_{\overline{G}}(\overline{G})/\mathfrak{h}_{\overline{G}}(\overline{N}) = \overline{eJ}/\overline{eJ}^2.$$

Therefore

$$(*) \quad eJ/eJ^2 \cong \bar{e}\bar{J}/\bar{e}\bar{J}^2 \oplus h_G(N)/eJ^2.$$

As $h_{\bar{G}}(\bar{G}_i)/h_{\bar{G}}(\bar{G}_{i-1}) \cong h_G(G_i)/h_G(G_{i-1})$, the result is true by the inductive hypothesis for the factors G_i/G_{i-1} , $i > 1$.

Assume that N is a p -group or a p' -group. Then $N \leq F_p(G) \leq C_G(V)$ for each $V \in \mathcal{C}(G, \mathbb{K})$, and hence $\mathcal{C}(G, \mathbb{K}) = \mathcal{C}(\bar{G}, \mathbb{K})$ and $n_0(G) = n_0(\bar{G})$.

If N is a Frattini p -chief factor or a p' -factor, then $\text{cm}^G(V) = \text{cm}^{\bar{G}}(V)$ for each $V \in \text{Irr}(G, \mathbb{K})$. Then, by Theorem 3.6, we have in this case that $eJ/eJ^2 \cong \bar{e}\bar{J}/\bar{e}\bar{J}^2$. From (*) we conclude that $h_G(N)/eJ^2 = 0$.

If N is a complemented p -chief factor, from Theorem 3.6 $eJ/eJ^2 \cong \bar{e}\bar{J}/\bar{e}\bar{J}^2 \oplus N_{\mathbb{K}}$, and by (*) we have that $h_G(N)/eJ^2 \cong N_{\mathbb{K}}$.

Assume that N is nonabelian and p is a divisor of $|N|$. Let $C = C_G(N)$. Then $G/C \in \mathcal{P}_2$, as NC/C is the only minimal normal subgroup of G/C . We have that, if $i > 1$, then $N \leq G_{i-1} \leq C_G(G_i/G_{i-1})$, and hence $C \neq C_G(G_i/G_{i-1})$, as N is nonabelian. Therefore $n_0(G) = n_0(\bar{G}) \cup \{C\}$. On the other hand $\mathcal{C}(\bar{G}, \mathbb{K}) \subseteq \mathcal{C}(G, \mathbb{K})$, as the inflation map $H^1(\bar{G}, V) \rightarrow H^1(G, V)$ is injective, and $\text{cm}^G(V) = \text{cm}^{\bar{G}}(V)$ for each $V \in \text{Irr}(G, \mathbb{K})$. Consequently $eJ/eJ^2 \cong \bar{e}\bar{J}/\bar{e}\bar{J}^2 \oplus M(C)$ and so $h_G(N)/eJ^2 \cong M(C)$.

As $h_G(N)/eJ^2 = h_G(G_1)/h_G(G_0)$, this completes the proof. \square

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