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## ON THE EXISTENCE OF CHARACTERS OF DEFECT ZERO

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### 1. Introduction

Let  $G$  be a finite group of order  $g$ . Let  $p$  be a prime and let  $g = p^a g'$  with  $(p, g') = 1$ . An irreducible (complex) character of  $G$  is called  $p$ -defect zero if its degree is divisible by  $p^a$ . The following problem is still open (see Feit [6]).

*What are some necessary and sufficient conditions for the existence of characters of  $p$ -defect zero?*

In [15] we, have tried somewhat ring theoretical approaches to the problem (see also Iizuka and Watanabe [11]). Now, since a character of  $p$ -defect zero constitutes a  $p$ -block for itself, having the identity group as its defect group, we have the following consequences from the theory of defect groups of blocks. Namely if  $G$  possesses a character of  $p$ -defect zero, then

1. (Brauer [2])  $G$  contains an element of  $p$ -defect zero, i.e. one which is commutative with no non-trivial  $p$ -element of  $G$ .
2. (Brauer [2])  $G$  contains no non-trivial normal  $p$ -subgroup.
3. (Green [8]) There exist two Sylow  $p$ -subgroups  $S, T$  of  $G$  such that  $S \cap T = \{1\}$ . (This implies the second assertion above)

Furthermore the Theorem of Clifford shows that if  $G$  possesses a character of  $p$ -defect zero, then

4. (Clifford-Schur) Every proper normal subgroup possesses a character of  $p$ -defect zero.

Of course, the above four conditions are not sufficient in general for the existence of a character of  $p$ -defect zero (e.g.  $G = A_7$ , the alternating group on seven letters,  $p = 2$  or  $p = 3$ ).

However, in [12] Ito showed that if  $G$  is solvable and has an element of  $p$ -defect zero which is contained in  $O_{p'}(G)$ , the maximal normal  $p'$ -subgroup of  $G$ , then  $G$  possesses a character of  $p$ -defect zero. Also in [13] he showed that under certain circumstances the second condition implies the existence of a

character of  $p$ -defect zero.

In this paper, we shall generalize the Ito's result in [12] quoted above to arbitrary finite groups (see Theorem 1 below) and show in some cases the converse of the result may hold. In Appendix we shall give a *solvable group* which will enjoy all of the four conditions above, though fail to possess a character of  $p$ -defect zero (when  $p=2$ ).

## 2. Notations and preliminaries

$p$  denotes a fixed prime number and  $G$  a finite group of order  $g=p^a g'$  with  $(p, g')=1$ . We denote by  $v_p$  the exponential valuation of the rational number field determined by  $p$  with  $v_p(p)=1$ . For a subset  $T$  of  $G$ , we denote by  $|T|$  and by  $\langle T \rangle$  the cardinality of  $T$  and the subgroup of  $G$  generated by  $T$  respectively. If  $S, T$  are subgroups of  $G$ ,  $[S, T]$  denotes the commutator subgroup of  $S$  and  $T$ . If  $R$  is a (commutative) ring,  $RG$  denotes the group ring of  $G$  over  $R$  and  $Z(RG)$  the center of  $RG$ . By a character of  $G$ , we mean unless otherwise specified, an absolutely irreducible complex character of  $G$ .

For convenience of later references, we put down here the following well known facts due to Clifford (and Schur) (see Curtis-Reiner [5] §51 and 53.)

Let  $N$  be a normal subgroup of  $G$  and let  $\chi$  be an irreducible character of  $G$ .

[C-1]  $\chi_N = e(\varphi_1 + \varphi_2 + \cdots + \varphi_r)$ , where the  $\{\varphi_i\}$  are mutually  $G$ -conjugate distinct irreducible characters of  $N$  and  $e$  is a positive integer.  $r$  is equal to  $[G: I]$ , where  $I$  is the inertia group of  $\varphi_1$ , namely  $I = \{\sigma \in G \mid \varphi_1(\sigma^{-1}\tau\sigma) = \varphi_1(\tau) \text{ for all } \tau \in N\}$ .

[C-2] Let  $\phi$  be the homogenous component of  $\chi$  containing  $\varphi_1$ . Then  $\phi$  is an irreducible character of  $I$ . And furthermore.

(1)  $\phi_N = e\varphi_1$  (by definition of  $\phi$ )

(2) Let  $\Phi$  be a representation of  $I$  affording the character  $\phi$ . Then  $\Phi$  is a tensor product of two projective representations  $X, Y$  of  $I$  (over the field of complex numbers);  $\Phi = X \otimes Y$ , where the degree of  $X$  is equal to that of  $\varphi_1$  and  $Y$  may be viewed as a projective representation of  $\bar{I} = I/N$  whose degree is equal to  $e$ .

(3)  $e$  divides  $[I: N]$  (since the degree of an irreducible projective representation divides the order of the group)

[C-3] Since  $\chi(1) = e r \varphi_1(1)$  and  $e$  divides  $[I: N]$ , we have the inequalities

$$\begin{aligned} v_p(\chi(1)) &= v_p(e) + v_p(r) + v_p(\varphi_1(1)) \\ &\leq v_p([I: N]) + v_p([G: I]) + v_p(\varphi_1(1)) \\ &\leq v_p([G: N]) + v_p(|N|) = v_p(|G|) \quad (\text{Hence the inequalities hold if } \chi \\ &\text{is of } p\text{-defect zero}). \end{aligned}$$

**3. A generalization of Ito's result**

Let  $K$  be an algebraic number field containing the  $g$ -th roots of unity. In  $K$ , let  $p$  be a prime divisor of  $p$  with  $\mathfrak{o}$  the ring of  $\mathfrak{p}$ -integers and let  $k = \mathfrak{o}/\mathfrak{p}$  the residue class field. We denote by  $\alpha^*$  the image of an element  $\alpha$  of  $\mathfrak{o}$  under the map  $\mathfrak{o} \rightarrow k$ . The following Theorem was proved by Ito [12], in case  $G$  is solvable.

**Theorem 1.** *Let  $N$  be a normal subgroup of  $G$  whose order is prime to  $p$ . Suppose there exist  $v$  classes of conjugate elements of  $G$  of  $p$ -defect zero such that they are contained in  $N$ . Then  $G$  possesses at least  $v$  characters of  $p$ -defect zero which are linearly independent mod  $\mathfrak{p}$  on those classes.*

*Proof.* Suppose  $G$  possesses  $t$  characters  $\chi_1, \chi_2, \dots, \chi_t$  of  $p$ -defect zero,  $t \geq 0$ . Let  $\delta_i$  be the block idempotent of the  $p$ -block of  $kG$  (of  $p$ -defect zero) to which  $\chi_i$  belongs and let  $\psi_i$  the linear character of the center  $Z(kG)$  of the group ring  $kG$  defined by  $\chi_i$ , that is  $\psi_i(C) = \left( \frac{|C| \chi_i(\sigma)}{\chi_i(1)} \right)^*$ , for a conjugate class  $C$  of  $G$ ,  $C \ni \sigma$ . As is well known,

$$\psi_i(\delta_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \dots\dots\dots (*)$$

Let  $C_1, C_2, \dots, C_u$  be the set of conjugate classes of  $G$  of  $p$ -defect zero and let the first  $v$  of them contained in  $N$ . We denote by  $c_i$  the sum in  $kG$  of all elements of  $C_i$ . Let  $U = \bigoplus_{i=1}^u kc_i$  and  $T = \bigoplus_{i=1}^t k\delta_i$ , the subspace of  $Z(kG)$  spanned by  $\{c_1, c_2, \dots, c_u\}$  and  $\{\delta_1, \delta_2, \dots, \delta_t\}$  respectively. Then as is well known, both are ideals of  $Z(kG)$ ,  $T$  is contained in  $U$  and  $U = T \oplus J(U)$ , where  $J(U)$  is the ideal of  $Z(kG)$  consisting of all nilpotent elements of  $U$  (Brauer [2], see also Iizuka and Watanabe [11]). Let  $\rho$  be the projection of  $U$  onto  $T$ . Thus  $\rho(u) = 0$  ( $u \in U$ ), if and only if  $u \in J(U)$ , or  $u$  is nilpotent. Let  $V = \bigoplus_{i=1}^v kc_i \subset U$ . Then  $\rho$  is one to one on  $V$ , since  $V$ , being contained in the center of the semisimple algebra  $kN$  by our assumptions, contains no nilpotent element other than zero. Since the  $\{\psi_1, \psi_2, \dots, \psi_t\}$  form a  $k$ -basis of the dual space of  $T$  by (\*) and the dimension of the space  $\rho(V)$  is  $v$ , we may choose  $v$  functions from  $\{\psi_1, \psi_2, \dots, \psi_t\}$  such that their restrictions on  $\rho(V)$  form a  $k$ -basis of the dual space of it. Assume they are  $\psi_1, \psi_2, \dots, \psi_v$ , after a suitable change of indexes if necessary. Since  $\psi_i(J(U)) = 0$  (for every linear character  $\psi_i$  of  $Z(kG)$ ), the above  $\{\psi_1, \psi_2, \dots, \psi_v\}$  are actually a  $k$ -basis of the dual space of  $V$  when restricted on  $V$ . Therefore it follows that  $\det(\psi_i(C_j)) \not\equiv 0$  ( $1 \leq i, j \leq v$ ), or  $\det \left( \frac{h_j \chi_i(\sigma_j)}{\chi_i} \right) = \left( \prod_{i=1}^v \frac{h_i}{x_i} \right) \det(\chi_i(\sigma_j)) \not\equiv 0 \pmod{\mathfrak{p}}$ , where  $x_i =$

$\chi_i(1), h_j = |C_j|$  and  $\sigma_j \in C_j$ . Moreover, we know  $\prod_{i=1}^v \frac{h_i}{x_i} \in \mathfrak{o}$  since  $\nu_p(h_i) = a = \nu_p(x_i)$  by our assumptions. Thus we may conclude that  $\det(\chi_i(\sigma_j)) \not\equiv 0 \pmod{\mathfrak{p}}$  ( $1 \leq i, j \leq v$ ), which implies  $\{\chi_1, \chi_2, \dots, \chi_v\}$  are linearly independent on  $\{C_1, C_2, \dots, C_v\} \pmod{\mathfrak{p}}$ . This completes the proof of Theorem 1.

As a direct consequence of the above Theorem, we have

**Corollary 2.** *If  $O_p'(G)$  contains an element of  $p$ -defect zero, then  $G$  possesses a character of  $p$ -defect zero.*

For a while, we shall show that under certain conditions the converse of Corollary 2 is true. First we note,

**Lemma 3.** *If  $G$  possesses a character of  $p$ -defect zero, then so does any normal subgroup of  $G$ .*

Proof. Clear from [C-3] of §2.

We have

**Proposition 4.** *Suppose  $G$  is  $p$ -solvable and possesses a character of  $p$ -defect zero. Then  $O_p'(G)$  contains an element of  $p$ -defect zero (in  $G$ ) if  $G$  satisfies one of the following conditions.*

- (1) *A Sylow  $p$ -subgroup of  $G$  is abelian*
- (2) *A Sylow  $p$ -complement of  $G$  is abelian*
- (3)  *$G$  is metabelian*

Proof. Clearly we may assume (by virtue of Lemma 3), that  $G$  contains no proper normal subgroup of index prime to  $p$ . Furthermore, since  $G$  possesses a character of  $p$ -defect zero,  $G$  contains no non-trivial normal  $p$ -subgroup. Then, since  $G$  is  $p$ -solvable, Lemma 1.2.3. of Hall-Higman [9] shows that  $G = S_p O_p'(G)$  in any one of the above cases, where  $S_p$  denotes a Sylow  $p$ -subgroup of  $G$ . Hence our assertion is clear.

In case  $p=2$ , we have the following,

**Proposition 5.** *Suppose a Sylow 2-subgroup of  $G$  is a generalized quaternion. Then if  $G$  possesses a character of  $p$ -defect zero,  $O_2'(G)$  contains an element of 2-defect zero.*

Proof. Let  $N = O_2'(G)$ . First of all, we note that  $G/N$  contains a central (hence unique) element of order 2 by Brauer-Suzuki [4] and by Brauer [3]. Hence every subgroup of  $G/N$  whose order is divisible by 2 contains a non-trivial normal 2-subgroup. In what follows, we use the same notations and terminologies as in [C-1]–[C-3] of §2, letting  $N = O_2'(G)$  and  $\chi$  a character of 2-defect zero. To prove the proposition, it is sufficient to show that  $e$  is odd. Indeed, since  $|N|$  is prime to 2, *i.e.* prime to the characteristic of the field

$k = v/p$ , where of course  $p$  is a prime divisor of 2 in  $K$ , there exists a  $\sigma \in N$  such that  $\varphi_1(\sigma) + \varphi_2(\sigma) + \dots + \varphi_r(\sigma) \not\equiv 0 \pmod{p}$ . Then it follows  $\chi(\sigma) \not\equiv 0 \pmod{p}$ , provided  $e$  is odd, which asserts that  $\sigma$  is of 2-defect zero.

Now, let  $\hat{I}$  be a representation group of  $\bar{I}$  having the kernel  $M$  isomorphic to the second cohomology group  $H^2(\bar{I}, C)$ , where  $C$  is the field of complex numbers.

$$1 \rightarrow M \rightarrow \hat{I} \rightarrow \bar{I} \rightarrow 1 \text{ (exact)}$$

We note that  $|M|$  is not divisible by 2. In fact, since a Sylow 2-subgroup of  $\bar{I}$  is a generalized quaternion or cyclic, the 2-part of  $|M|$  vanishes (see Huppert [10] §25). In particular we have  $v_2(|\bar{I}|) = v_2(|\hat{I}|)$ . The projective representation  $Y$  of  $\hat{I}$  can be lifted to a (linear) representation  $\hat{Y}$  of  $\hat{I}$ . Then  $\hat{Y}$  is a representation of  $\hat{I}$  of 2-defect zero, since  $v_2(\deg \hat{Y}) = v_2(\deg Y) = v_2(|\bar{I}|) = v_2(|\hat{I}|)$  by [C-3]. Suppose  $|\bar{I}|$  is divisible by 2. Then it contains a non-trivial normal 2-subgroup as is remarked at the beginning. Since  $M$  is a central subgroup of  $\hat{I}$  of order prime to 2 it follows from the exact sequence written above that  $\hat{I}$  contains a non-trivial normal 2-subgroup. This is a contradiction, since  $\hat{I}$  possesses a character of 2-defect zero. Therefore  $|\bar{I}|$  is odd. Since  $e$  divides  $|\bar{I}|$ ,  $e$  is also odd, completing the proof.

In case  $G$  is solvable,  $O_p(G)$  is always larger than  $\{1\}$  unless  $G$  contains a non-trivial normal  $p$ -subgroup. Hence it seems to be natural to ask whether the converse of Corollary 2 is true for a solvable group  $G$ . The answer is “no”, as is shown in Ito [12].

#### 4. Appendix

The purpose of this section is to give a solvable group which will enjoy all of the four conditions described in the introduction, though possess no character of  $p$ -defect zero (when  $p=2$ ).

EXAMPLE. Let  $F=GF(3)$ . Let  $V$  be the 2-dimensional column vector space over the field  $F$ ;  $V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \mid a, b \in F \right\}$ . Let  $G$  be the semidirect product of  $GL(V) = GL(2, 3)$  and  $V$ , i.e. the 2-dimensional affine group over  $F$ . Hence  $G$  consists of all pairs  $(\sigma, \alpha)$ , where  $\sigma \in GL(2, 3)$  and  $\alpha \in V$ , with the multiplication given by

$$(\sigma, \alpha)(\sigma', \alpha') = (\sigma\sigma', \sigma\alpha' + \alpha)$$

We have easily  $(\tau, \beta)(\sigma, \alpha)(\tau, \beta)^{-1} = (\tau\sigma\tau^{-1}, (1 - \tau\sigma\tau^{-1})\beta + \tau\alpha)$

We identify  $\sigma \in GL(2, 3)$  with  $(\sigma, 0)$  and  $\alpha \in V$  with  $(1, \alpha)$  as usual. Then  $\tau\alpha\tau^{-1} = \tau\alpha$ .  $G$  is a solvable group of order  $2^4 \cdot 3^3$ . We write simply  $GL$  for  $GL(2, 3)$  and  $SL$  for  $SL(2, 3)$ . If  $K$  is a subgroup of  $GL$ , we denote by  $\tilde{K}$  the semidirect product of  $K$  and  $V$ .

First we note

(1) *Every proper normal subgroup of  $GL$  is contained in  $SL$ . On the other hand, every non-trivial normal subgroup of  $G$  contains  $V$ . In particular, it follows every proper normal subgroup of  $G$  is contained in  $\widehat{SL}$ .*

Proof. The first assertion is well known and elementary. To show the second, let  $N$  be any normal subgroup of  $G$ . Then  $[N, V]$  is a  $GL$ -submodule of the irreducible  $GL$ -module  $V$ , so that  $[N, V]=V$  or  $\{0\}$ , implying  $N \supset V$  or  $N=\{1\}$ . (cf. Proposition 2.3 [1]).

(2) *Every element of  $V$  other than the identity is of 2-defect zero in  $\widehat{SL}$ . In particular  $\widehat{SL}$  possesses a character of 2-defect zero.*

Proof. Let  $\alpha=(1, \alpha) \in V$ . If  $\alpha$  is of positive defect in  $\widehat{SL}$ , then there exists an involution  $(\tau, \beta) \in \widehat{SL}$  such that  $(\tau, \beta)(1, \alpha)(\tau, \beta)^{-1}=(1, \tau\alpha)=(1, \alpha)$ . Then  $\tau$  is an involution of  $SL$  and  $\tau\alpha=\alpha$ . However  $SL$  contains only one involution, namely  $\tau=\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ ,  $\tau\alpha=\alpha$  implies  $\alpha=0$ .

From the aboves and Lemma 3, we have

(3) *Every proper normal subgroup of  $G$  possesses a character of 2-defect zero.*

By a simple caluculation, we find

(4)  *$G$  contains an element of 2-defect zero, e.g.  $(\tau, \beta)$ , where  $\tau=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  and  $\beta=\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in V$ .*

(5) *Let  $S$  be a Sylow 2-subgroup of  $GL$ . Then  $S=(S, 0)$  is a Sylow 2-subgroup of  $G$ . Let  $(\tau, \beta) \in G$ . where  $\tau \notin S$  and  $\beta \neq 0$ . Then  $S \cap S^{(\tau, \beta)}=\{1\}$ .*

Proof. Let  $Q$  be a Sylow 2-subgroup of  $SL$ . Then  $S \cap S^\tau=Q$  if  $\tau \notin S$ , since  $GL \triangleright Q$  and  $S$  is a self-normalizing subgroup of  $GL$ . Let  $(\sigma, 0) \in S$ , where  $\sigma \neq 1$ . Then we have  $(\tau, \beta)(\sigma, 0)(\tau, \beta^{-1})=(\tau\sigma\tau^{-1}, (1-\tau\sigma\tau^{-1})\beta) \in S=(S, 0)$  if and only if  $\tau\sigma\tau^{-1} \in S$  and  $(1-\tau\sigma\tau^{-1})\beta=0$ . If  $\tau\sigma\tau^{-1} \in S$ , then  $\tau\sigma\tau^{-1} \in S \cap S^\tau=Q \subset SL$ . Hence  $(1-\tau\sigma\tau^{-1})\beta \neq 0$  as is remarked in (2). Thus we have shown that  $G$  satisfies all of the four conditions described in the introduction. Hence it remains only to show that  $G$  possesses no character of 2-defect zero.

Suppose the contrary and let  $\chi$  be any character of 2-defect zero. Then the degree of  $\chi$  must be 16, since  $(2^4 3)^2 > |G|=2^4 3^3$ . Then by Corollary (2E) of Fong [7],  $\chi$  is induced by a linear character of a Sylow 2-complement of  $G$ . Let  $\sigma=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  and  $H=\widehat{\langle \sigma \rangle}=\langle \sigma \rangle \cdot V$ . Then  $H$  is a Sylow- 2-complement of  $G$

and by a simple calculation, we have  $[H, H] = [\langle \sigma \rangle, V] = \left\{ \begin{pmatrix} a \\ 0 \end{pmatrix} \mid a \in F \right\}$  and so  $H/[H, H] = \langle \sigma \rangle \times V/[\langle \sigma \rangle, V]$ . Hence any linear character of  $H$  is of the form  $\psi = \varphi \times (1, \eta)$ , where  $\varphi$  and  $\eta$  are linear characters of  $\langle \sigma \rangle$  and  $V$  respectively with  $(1, \eta) \begin{pmatrix} a \\ b \end{pmatrix} = \eta(b)$  for  $\begin{pmatrix} a \\ b \end{pmatrix} \in V$ . Let  $\tau = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$  and  $x = (\tau, 0) \in H$ . Let  $y = (1, \begin{pmatrix} a \\ b \end{pmatrix}) \in x^{-1}Hx \cap H$ . Then  $xyx^{-1} = (1, \begin{pmatrix} -a \\ b \end{pmatrix})$ , so that  $\psi(y) = \eta(b) = \psi(xyx^{-1})$  for any  $y \in x^{-1}Hx \cap H$ . Hence  $\psi^G$  is not irreducible for any linear character  $\psi$  of  $H$  by the criterion given by Shoda [13] originally (see Curtis-Reiner [5] pp. 329) Thus  $G$  posseses no character of 2-defect zero.

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### References

- [1] H. Bass: *K-theory and stable algebra*: Publ. I.H.E.S. No. **22** (1964).
- [2] R. Brauer: *Zur Darstellungstheorie der Gruppen endlicher Ordnung I*, Math. Z. **63** (1956), 406–444.
- [3] ———: *Some applications of the theory of blocks of characters of finite groups II*, J. Algebra **1** (1964), 307–334.
- [4] ——— and M. Suzuki: *On finite groups of even order whose 2-Sylow group is a quaternion group*, Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 1757–1759.
- [5] C.W. Curtis and I. Reiner: *Representations of Finite Groups and Associative Algebras*, Wiley-Interscience, New York 1962.
- [6] W. Feit: *Representations of Finite Groups*, Yale Univ. mimeographed, 1969.
- [7] P. Fong: *Solvable groups and modular representation theory*, Trans. Amer. Math. Soc. **103** (1962), 484–494.
- [8] J.A. Green: *Blocks of modular representations*, Math. Z. **79** (1962), 100–115.
- [9] P. Hall and G. Higman: *On the  $p$ -length of  $p$ -soluble groups and reduction theorems for Burnside's problem*, Proc. London Math. Soc. (3) **6** (1956), 1–42.
- [10] B. Huppert: *Endliche Gruppen I*, Springer Verlag, Berlin-New York, 1967.
- [11] K. Iizuka and A. Watanabe: *On the number of blocks of irreducible characters of a finite group with a given defect group*, Kumamoto J. Sci. (Math.) **9** (1973), 55–61.
- [12] N. Ito: *On the characters of soluble groups*, Nagoya Math. J. **3** (1951), 31–48.
- [13] ———: *Note on the characters of solvable groups*, Nagoya J. Math. **39** (1970), 23–28.
- [14] K. Shoda: *Über die monomialen Darstellungen einer endliche Gruppen*, Proc. Phy.-Math. Soc. (3) **15** (1933), 249–257.
- [15] Y. Tsushima: *On the block of defect zero*, Nagoya Math. J. **44** (1971), 57–61.



