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ON THE EXISTENCE OF CHARACTERS OF DEFECT ZERO

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1. Introduction

Let G be a finite group of order g. Let p be a prime and let $g=p^a g'$ with $(p, g')=1$. An irreducible (complex) character of G is called p-defect zero if its degree is divisible by p^a. The following problem is still open (see Feit [6]).

What are some necessary and sufficient conditions for the existence of characters of p-defect zero?

In [15] we, have tried somewhat ring theoritical approaches to the problem (see also Iizuka and Watanabe [11]). Now, since a character of p-defect zero constitutes a p-block for itself, having the identity group as its defect group, we have the following consequences from the theory of defect groups of blocks. Namely if G posseses a character of p-defect zero, then

1. (Brauer [2]) *G contains an element of p-defect zero, i.e. one which is commutative with no non-trivial p-element of G.*
2. (Brauer [2]) *G contains no non-trivial normal p-subgroup.*
3. (Green [8]) *There exist two Sylow p-subgroups S,T of G such that $S\cap T=\{1\}$. (This implies the second assertion above)*

Furthermore the Theorem of Clifford shows that if G possesses a character of p-defect zero, then

4. (Clifford-Schur) *Every proper normal subgroup possesses a character of p-defect zero.*

Of course, the above four conditions are not sufficient in general for the existence of a character of p-defect zero (e.g. $G=A_7$, the alternating group on seven letters, $p=2$ or $p=3$).

However, in [12] Ito showed that if G is solvable and has an element of p-defect zero which is contained in $O_{p'}(G)$, the maximal normal p'-subgroup of G, then G possesses a character of p-defect zero. Also in [13] he showed that under certain circumstances the second condition implies the existence of a character of p-defect zero.

In this paper, we shall generalize the Ito's result in [12] quoted above to arbitary finite groups (see Theorem 1 below) and show in some cases the converse of the result may hold. In Appendix we shall give *a solvable group* which will enjoy all of the four conditions above, though fail to possess a character of p-defect zero (when $p=2$).

2. Notations and preliminaries

p denotes a fixed prime number and G a finite group of order $g=p^a g'$ with $(p, g')=1$. We denote by ν_p the exponential valuation of the rational number field determined by p with $\nu_p(p)=1$. For a subset T of G, we denote by $|T|$ and by $\langle T\rangle$ the cardinality of T and the subgroup of G generated by T respectively. If S, T are subgroups of G, $[S, T]$ denotes the commutator subgroup of S and T. If R is a (commutative) ring, RG denotes the group ring of G over R and $Z(RG)$ the center of RG. By a character of G, we mean unless otherwise specified, an absolutely irreducible complex character of G.

For convenience of later references, we put down here the following well known facts due to Clifford (and Schur) (see Curtis-Reiner [5] §51 and 53.)

Let N be a normal subgroup of G and let χ be an irreducible character of G.

[C–1] $\chi_N=e(\varphi_1+\varphi_2+\cdots+\varphi_r)$, *where the* $\{\varphi_i\}$ *are mutually G-conjugate distinct irreducible characters of N and e is a positive integer. r is equal to* $[G:I]$, *where I is the inertia group of* φ_1, *namely* $I=\{\sigma\in G\,|\,\varphi_1(\sigma^{-1}\tau\sigma)=\varphi_1(\tau)$ *for all* $\tau\in N\}$.

[C–2] *Let ϕ be the homogenous component of χ containing φ_1. Then ϕ is an irreducible character of I. And furthermore.*

(1) $\phi_N=e\varphi_1$ (*by definition of* ϕ)

(2) *Let Φ be a representation of I affording the character ϕ. Then Φ is a tensor product of two projective representations X, Y of I (over the field of complex numbers)*; $\Phi=X\otimes Y$, *where the degree of X is equal to that of φ_1 and Y may be viewed as a projective representation of $\bar{I}=I/N$ whose degree is equal to e.*

(3) *e divides $[I\colon N]$ (since the degree of an irreducible projective representation divides the order of the group)*

[C–3] *Since* $\chi(1)=er\,\varphi_1(1)$ *and e divides $[I\colon N]$, we have the inequalities*

$$\begin{aligned}\nu_p(\chi(1)) &= \nu_p(e)+\nu_p(r)+\nu_p(\varphi_1(1))\\ &\leqq \nu_p([I\colon N])+\nu_p([G\colon I])+\nu_p(\varphi_1(1))\\ &\leqq \nu_p([G\colon N])+\nu_p(|N|) = \nu_p(|G|)\end{aligned}$$ *(Hence the iqualities hold if χ is of p-defect zero).*

3. A generalization of Ito's result

Let K be an algebraic number field containing the g-th roots of unity. In K, let $\mathfrak{p}$ be a prime divisor of p with $\mathfrak{o}$ the ring of $\mathfrak{p}$-integers and let $k=\mathfrak{o}/\mathfrak{p}$ the residue class field. We denote by α^* the image of an element α of $\mathfrak{o}$ under the map $\mathfrak{o}\to k$. The following Theorem was proved by Ito [12], in case G is solvable.

Theorem 1. *Let N be a normal subgroup of G whose order is prime to p. Suppose there exist v classes of conjugate elements of G of p-defect zero such that they are contained in N. Then G possesses at least v characters of p-defect zero which are linearly independent mod $\mathfrak{p}$ on those classes.*

Proof. Suppose G possesses t characters $\chi_1,\chi_2,\cdots,\chi_t$ of p-defect zero, $t\geqq 0$. Let δ_i be the block idempotent of the p-block of kG (of p-defect zero) to which χ_i belongs and let ψ_i the linear character of the center $Z(kG)$ of the group ring kG defined by χ_i, that is $\psi_i(C)=\left(\frac{|C|\chi_i(\sigma)}{\chi_i(1)}\right)^*$, for a conjugate class C of G, $C\ni\sigma$. As is well known,

$$\psi_i(\delta_j) = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases} \qquad \cdots\cdots\cdots\cdots\cdots(*)$$

Let $C_1, C_2, \cdots, C_u$ be the set of conjugate classes of G of p-defect zero and let the first v of them contained in N. We denote by c_i the sum in kG of all elements of C_i. Let $U=\oplus\sum_{i=1}^{u} kc_i$ and $T=\oplus\sum_{i=1}^{t} k\delta_i$, the subspace of $Z(kG)$ spanned by $\{c_1, c_2, \cdots, c_u\}$ and $\{\delta_1, \delta_2, \cdots, \delta_t\}$ respectively. Then as is well known, both are ideals of $Z(kG)$, T is contained in U and $U=T\oplus J(U)$, where $J(U)$ is the ideal of $Z(kG)$ consisting of all nilpotent elements of U (Brauer [2], see also Iizuka and Watanabe [11]). Let ρ be the projection of U onto T. Thus $\rho(u)=0$ $(u\in U)$, if and only if $u\in J(U)$, or u is nilpotent. Let $V=\oplus\sum_{i=1}^{v} kc_i\subset U$. Then ρ is one to one on V, since V, being contained in the center of the semisimple algebra kN by our assumptions, contains no nilpotent element other than zero. Since the $\{\psi_1, \psi_2, \cdots\psi_t\}$ form a k-basis of the dual space of T by $(*)$ and the dimension of the space $\rho(V)$ is v, we may choose v functions from $\{\psi_1, \psi_2, \cdots, \psi_t\}$ such that their restrictions on $\rho(V)$ form a k-basis of the dual space of it. Assume they are $\psi_1, \psi_2, \cdots, \psi_v$, after a suitable change of indexes if necessary. Since $\psi_i(J(U))=0$ (for every linear character ψ_i of $Z(kG)$), the above $\{\psi_1, \psi_2, \cdots, \psi_v\}$ are actually are k-basis of the dual space of V when restricted on V. Therefore it follows that det $(\psi_i(C_j))\neq 0$ $(1\leqq i, j\leqq v)$, or det $\left(\frac{h_j\chi_i(\sigma_j)}{\chi_i}\right)=\left(\prod_{i=1}^{v}\frac{h_i}{x_i}\right)$ det $(\chi_i(\sigma_j))\not\equiv 0$ mod $\mathfrak{p}$, where $x_i=$

$\chi_i(1)$, $h_j=|C_j|$ and $\sigma_j\in C_j$. Moreover, we know $\prod_{i=1}^{v}\frac{h_i}{x_i}\in\mathfrak{o}$ since $\nu_p(h_i)=a=\nu_p(x_i)$ by our assumptions. Thus we may conclude that $\det(\chi_i(\sigma_j))\not\equiv 0 \bmod \mathfrak{p}$ $(1\leqq i, j\leqq v)$, which implies $\{\chi_1, \chi_2, \cdots, \chi_v\}$ are linearly independent on $\{C_1, C_2, \cdots, C_v\}$ mod $\mathfrak{p}$. This completes the proof of Theorem 1.

As a direct consequence of the above Theorem, we have

Corollary 2. *If $O_{p'}(G)$ contains an element of p-defect zero, then G possesses a character of p-defect zero.*

For a while, we shall show that under certain conditions the converse of Corollary 2 is true. First we note,

Lemma 3. *If G possesses a character of p-defect zero, then so does any normal subgroup of G.*

Proof. Clear from [C–3] of §2.

We have

Proposition 4. *Suppose G is p-solvable and possesses a character of p-defect zero. Then $O_{p'}(G)$ contains an element of p-defect zero (in G) if G satisfies one of the following conditions.*

(1) *A Sylow p-subgroup of G is abelian*

(2) *A Sylow p-complement of G is abelian*

(3) *G is metabelian*

Proof. Clearly we may assume (by virtue of Lemma 3), that G contains no proper normal subgroup of index prime to p. Furthermore, since G possesses a character of p-defect zero, G contains no non-trivial normal p-subgroup. Then, since G is p-solvable, Lemma 1. 2. 3. of Hall-Higman [9] shows that $G=S_pO_{p'}(G)$ in any one of the above cases, where S_p denotes a Sylow p-subgroup of G. Hence our assertion is clear.

In case $p=2$, we have the following,

Proposition 5. *Suppose a Sylow 2-subgroup of G is a generalized quaternion. Then if G possesses a character of p-defect zero, $O_{2'}(G)$ contains an element of 2-defect zero.*

Proof. Let $N=O_{2'}(G)$. First of all, we note that G/N contains a central (hence unique) element of order 2 by Brauer-Suzuki [4] and by Brauer [3]. Hence every subgroup of G/N whose order is divisible by 2 contains a non-trivial normal 2-subgroup. In what follows, we use the same notations and terminologies as in [C–1]–[C–3] of §2, letting $N=O_{2'}(G)$ and χ a character of 2-defect zero. To prove the proposition, it is sufficient to show that e is odd. Indeed, since $|N|$ is prime to 2, *i.e.* prime to the characteristic of the field

$k=\mathfrak{o}/\mathfrak{p}$, where of course $\mathfrak{p}$ is a prime divisor of 2 in K, there exists a $\sigma\in N$ such that $\varphi_1(\sigma)+\varphi_2(\sigma)+\cdots\varphi_r(\sigma)\not\equiv 0 \bmod \mathfrak{p}$. Then it follows $\chi(\sigma)\not\equiv 0 \bmod \mathfrak{p}$, provided e is odd, which asserts that σ is of 2-defect zero.

Now, let $\hat{I}$ be a representation group of $\bar{I}$ having the kernel M isomorphic to the second cohomology group $H^2(\bar{I}, C)$, where C is the field of complex numbers.

$$1 \to M \to \hat{I} \to \bar{I} \to 1 \text{ (exact)}$$

We note that $|M|$ is not divisible by 2. In fact, since a Sylow 2-subgroup of $\bar{I}$ is a generalized quaternion or cyclic, the 2-part of $|M|$ vanishes (see Huppert [10] §25) In particular we have $\nu_2(|\bar{I}|)=\nu_2(|\hat{I}|)$. The projective representation Y of $\hat{I}$ can be lifted to a (linear) representation $\hat{Y}$ of $\hat{I}$. Then $\hat{Y}$ is a representation of $\hat{I}$ of 2-defect zero, since $\nu_2(\deg \hat{Y})=\nu_2(\deg Y)=\nu_2(|\bar{I}|)=\nu_2(|\hat{I}|)$ by [C–3]. Suppose $|\bar{I}|$ is divisible by 2. Then it contains a non-trivial normal 2-subgroup as is remarked at the beginning. Since M is a central subgroup of $\hat{I}$ of order prime to 2 it follows from the exact sequence written above that $\hat{I}$ contains a non-trivial normal 2-subgroup. This is a contradiction, since $\hat{I}$ possesses a character of 2-defect zero. Therefore $|\bar{I}|$ is odd. Since e divides $|\bar{I}|$, e is also odd, completing the proof.

In case G is solvable, $O_{p'}(G)$ is always larger than $\{1\}$ unless G contains a non-trivial normal p-subgroup. Hence it seems to be natural to ask whether the converse of Corollary 2 is true for a solvable group G. The answer is "no", as is shown in Ito [12].

4. Appendix

The purpose of this section is to give a solvable group which will enjoy all of the four conditions described in the introduction, though possess no character of p-defect zero (when $p=2$).

EXAMPLE. Let $F=GF(3)$. Let V be the 2-dimensional column vector space over the field F; $V=\left\{\begin{pmatrix} a \\ b\end{pmatrix} \mid a, b\in F\right\}$. Let G be the semidirect product of $GL(V)=GL(2, 3)$ and V, *i.e.* the 2-dimensional affine group over F. Hence G consists of all pairs (σ, α), where $\sigma\in GL(2, 3)$ and $\alpha\in V$, with the multiplication given by

$$(\sigma, \alpha)(\sigma', \alpha') = (\sigma\sigma', \sigma\alpha'+\alpha)$$

We have easily $(\tau, \beta)(\sigma, \alpha)(\tau, \beta)^{-1}=(\tau\sigma\tau^{-1}, (1-\tau\sigma\tau^{-1})\beta+\tau\alpha)$

We identify $\sigma\in GL(2, 3)$ with $(\sigma, 0)$ and $\alpha\in V$ with $(1, \alpha)$ as usual. Then $\tau\alpha\tau^{-1}=\tau\alpha$. G is a solvable group of order $2^4 3^3$. We write simply GL for $GL(2, 3)$ and SL for $SL(2, 3)$. If K is a subgroup of GL, we denote by $\hat{K}$ the semidirect product of K and V.

First we note

(1) *Every proper normal subgroup of GL is contained in SL. On the other hand, every non-trivial normal subgroup of G contains V. In particular, it follows every proper normal subgroup of G is contained in* $\widehat{SL}$.

Proof. The first assertion is well known and elementary. To show the second, let N be any normal subgroup of G. Then $[N, V]$ is a GL-submodule of the irreducible GL-module V, so that $[N, V]=V$ or $\{0\}$, implying $N \supset V$ or $N=\{1\}$. (cf. Proposition 2.3 [1]).

(2) *Every element of V other than the identity is of 2-defect zero in* $\widehat{SL}$. *In particular* $\widehat{SL}$ *possesses a character of 2-defect zero.*

Proof. Let $\alpha=(1, \alpha)\in V$. If α is of positive defect in $\widehat{SL}$, then there exists an involution $(\tau, \beta)\in\widehat{SL}$ such that $(\tau, \beta)(1, \alpha)(\tau, \beta)^{-1}=(1, \tau\alpha)=(1, \alpha)$. Then τ is an involution of SL and $\tau\alpha=\alpha$. However SL contains only one involution, namely $\tau=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$, $\tau\alpha=\alpha$ implies $\alpha=0$.

From the aboves and Lemma 3, we have

(3) *Every proper normal subgroup of G possesses a character of 2-defect zero.*

By a simple caluculation, we find

(4) *G contains an element of 2-defect zero, e.g.* (τ, β), *where* $\tau=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ *and* $\beta=\begin{pmatrix}0\\ 1\end{pmatrix}\in V$.

(5) *Let S be a Sylow 2-subgroup of GL. Then* $S=(S, 0)$ *is a Sylow 2-subgroup of G. Let* $(\tau, \beta)\in G$. *where* $\tau\notin S$ *and* $\beta\neq 0$. *Then* $S\cap S^{(\tau,\beta)}=\{1\}$.

Proof. Let Q be a Sylow 2-subgroup of SL. Then $S\cap S^{\tau}=Q$ if $\tau\notin S$, since $GL\triangleright Q$ and S is a self-normalizing subgroup of GL. Let $(\sigma, 0)\in S$, where $\sigma\neq 1$. Then we have $(\tau, \beta)(\sigma, 0)(\tau, \beta^{-1})=(\tau\sigma\tau^{-1}, (1-\tau\sigma\tau^{-1})\beta)\in S=(S, 0)$ if and only if $\tau\sigma\tau^{-1}\in S$ and $(1-\tau\sigma\tau^{-1})\beta=0$. If $\tau\sigma\tau^{-1}\in S$, then $\tau\sigma\tau^{-1}\in S\cap S^{\tau}=Q\subset SL$. Hence $(1-\tau\sigma\tau^{-1})\beta\neq 0$ as is remarked in (2). Thus we have shown that G satisfies all of the four conditions described in the introduction. Hence it remains only to show that G possesses no character of 2-defect zero.

Suppose the contrary and let χ be any character of 2-defect zero. Then the degree of χ must be 16, since $(2^4 3)^2>|G|=2^4 3^3$. Then by Corollary (2E) of Fong [7], χ is induced by a linear character of a Sylow 2-complement of G. Let $\sigma=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ and $H=\widehat{\langle\sigma\rangle}=\langle\sigma\rangle\cdot V$. Then H is a Sylow- 2-complement of G

and by a simple calcuulation, we have $[H, H]=[\langle\sigma\rangle, V]=\left\{\begin{pmatrix} a \\ 0 \end{pmatrix} \mid a\in F\right\}$ and so $H/[H, H]=\langle\sigma\rangle\times V/[\langle\sigma\rangle, V]$. Hence any linear character of H is of the form $\psi=\varphi\times(1, \eta)$, where φ and η are linear characters of $\langle\sigma\rangle$ and V respectively with $(1, \eta)\begin{pmatrix} a \\ b \end{pmatrix}=\eta(b)$ for $\begin{pmatrix} a \\ b \end{pmatrix}\in V$. Let $\tau=\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $x=(\tau, 0)\notin H$. Let $y=(1, \begin{pmatrix} a \\ b \end{pmatrix})\in x^{-1}Hx\cap H$. Then $xyx^{-1}=(1, \begin{pmatrix} -a \\ b \end{pmatrix})$, so that $\psi(y)=\eta(b)=\psi(xyx^{-1})$ for any $y\in x^{-1}Hx\cap H$. Hence ψ^G is not irreducible for any linear character ψ of H by the criterion given by Shoda [13] originally (see Curtis-Reiner [5] pp. 329) Thus G posssses no character of 2-defect zero.

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