Biswas, I.
Osaka J. Math.
43 (2006), 263–274

INFINITESIMAL DEFORMATIONS OF THE TANGENT BUNDLE OF A MODULI SPACE OF VECTOR BUNDLES OVER A CURVE

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(Received October 18, 2004)

Abstract

Fix a line bundle ξ on a connected smooth complex projective curve X of genus at least three. Let $\mathcal{N}$ denote the moduli space of all stable vector bundles over X of rank n and determinant ξ. We assume that $n \geq 3$ and coprime to degree(ξ); If genus(X) ≤ 4, then we also assume that $n \geq 4$. We prove that $H^i(\mathcal{N}, \mathrm{End}(T\mathcal{N})) = H^i(X, \mathcal{O}_X)$ for $i = 0, 1$.

1. Introduction

Let X be a connected smooth complex projective curve of genus g, with $g \geq 3$. Fix an integer $n \geq 3$ and also fix a line bundle ξ over X such that the degree of ξ is coprime to n. We denote by $\mathcal{N}$ the moduli space of all stable vector bundles E of rank n over X with $\bigwedge^n E \cong \xi$. If $g \leq 4$, we assume that $n \geq 4$.

Our aim here is to prove that

$$H^i(\mathcal{N}, \mathrm{End}(T\mathcal{N})) = H^i(X, \mathcal{O}_X)$$

for $i = 0, 1$ (see Theorem 3.4).

The proof of Theorem 3.4 involves a method of computing cohomologies using the Hecke transformation initiated in [2]. In [2], the space of all infinitesimal deformations of $\mathcal{N}$ was computed to be of dimension $3g - 3$.

The moduli space $\mathcal{N}$ is a smooth complete Fano variety (anti-canonical line bundle is ample) of dimension $(g-1)(n^2-1)$ with Picard group isomorphic to $\mathbb{Z}$. Therefore, we obtain examples of smooth Fano varieties of Picard number one whose tangent bundle admits nontrivial infinitesimal deformations.

2. The universal projective bundle

Since n and degree(ξ) are mutually coprime, there is a universal vector bundle over $X \times \mathcal{N}$. Fix a universal vector bundle $\mathcal{E}$ over $X \times \mathcal{N}$. Let $\mathbb{P}(\mathcal{E})$ denote the projective bundle over $X \times \mathcal{N}$ parametrizing all hyperplanes in the fibers of $\mathcal{E}$. Any two universal bundles over $X \times \mathcal{N}$ differ by tensoring with a line bundle pulled back from $\mathcal{N}$.

2000 Mathematics Subject Classification. 14D20, 14F05.

Therefore, the universal projective bundle $\mathbb{P}(\mathcal{E})$ is unique although the universal vector bundle is not unique.

Let

$$\pi : \mathbb{P}(\mathcal{E}) \longrightarrow X \times \mathcal{N} \tag{2.1}$$

be the natural projection.

NOTATIONS. For a vector bundle V_1 (respectively, V_2) over A_1 (respectively, A_2), the notation $V_1 \boxtimes V_2$ will be used in denoting the vector bundle $p_1^* V_1 \otimes p_2^* V_2$ over $A_1 \times A_2$, where p_i is the projection to the factor A_i.

The holomorphic cotangent bundle of X will be denoted by K_X.

Proposition 2.1. *There is a canonical isomorphism*

$$H^i(\mathcal{N}, \mathrm{End}(T\mathcal{N})) = H^i(\mathbb{P}(\mathcal{E}), \pi^*(K_X \boxtimes T\mathcal{N}) \otimes T_\pi)$$

for every $i \geq 0$, where $T_\pi \subset T\mathbb{P}(\mathcal{E})$ is the relative tangent bundle for the projection π defined in (2.1).

Proof. For a vector bundle V, let $\mathrm{ad}(V) \subset \mathrm{End}(V)$ be the subbundle of trace zero endomorphisms. For a vector bundle $E \in \mathcal{N}$ over X we have

$$T_E^* \mathcal{N} = H^0(X, K_X \otimes \mathrm{ad}(E)).$$

The Serre duality gives $H^0(X, \mathrm{ad}(E))^* = H^1(X, K_X \otimes \mathrm{ad}(E))$. The vector bundle E being stable is simple. Hence from the Serre duality we have $H^1(X, K_X \otimes \mathrm{ad}(E)) = 0$. Now using the Leray spectral sequence for the projection $X \times \mathcal{N} \longrightarrow \mathcal{N}$ we conclude that

$$H^i(\mathcal{N}, \mathrm{End}(T\mathcal{N})) = H^i\left(\mathcal{N}, T\mathcal{N} \otimes \Omega^1_{\mathcal{N}}\right) = H^i(X \times \mathcal{N}, (K_X \boxtimes T\mathcal{N}) \otimes \mathrm{ad}(\mathcal{E})). \tag{2.2}$$

Since π in (2.1) defines a projective bundle, we have $R^i \pi_* T_\pi = 0$ if $i > 0$ and $R^0 \pi_* T_\pi = \mathrm{ad}(\mathcal{E})$. Therefore, using the Leray spectral sequence for the projection π in (2.1) we have

$$H^i(X \times \mathcal{N}, (K_X \boxtimes T\mathcal{N}) \otimes \mathrm{ad}(\mathcal{E})) = H^i(\mathbb{P}(\mathcal{E}), \pi^*(K_X \boxtimes T\mathcal{N}) \otimes T_\pi).$$

Combining this with (2.2) the proof of the proposition is complete. □

Let $\mathcal{M}$ denote the moduli space of all stable vector bundles over X of rank n and degree degree$(\xi) - 1$. Let

$$N_X^s \subset X \times \mathcal{M}$$

be the closed subvariety consisting of all pairs (x, V), where $\bigwedge^n V \cong \xi \otimes \mathcal{O}_X(-x)$.

We recall from [2] that the *Hecke map* is a rational morphism

$$\psi : \mathbb{P}(\mathcal{E}) \dashrightarrow N_X^s \tag{2.3}$$

which is defined as follows. Take any $(E, l_x) \in \mathbb{P}(\mathcal{E})$ (so $x \in X$ is some point and l_x is a quotient of the fiber E_x of dimension one). Consider the short exact sequence of sheaves on X

$$0 \longrightarrow V \longrightarrow E \longrightarrow l_x \longrightarrow 0. \tag{2.4}$$

If the vector bundle V is stable, then the map ψ sends the point $(E, l_x) \in \mathbb{P}(\mathcal{E})$ to the point of N_X^s defined by the pair (x, V); if V is not stable, then ψ is not defined on the point (E, l_x). We refer the reader to [2], [3] for all the properties of ψ that will be used here.

There is nonempty Zariski open subset $U \subset N_X^s$ such that

(i) the complement $N_X^s \setminus U$ is of codimension at least four, and

(ii) the map ψ is a projective fibration over U.

(See the codimension estimation in the proof of Proposition 5.4 of [3].)

Moreover, the complement $\mathbb{P}(\mathcal{E}) \setminus \psi^{-1}(U)$ is a Zariski closed subset of codimension at least four [2, Proposition 6.8]. The two assumptions that

(i) $g \geq 3$, and

(ii) $n \geq 4$ if $g \leq 4$

are needed for these codimension estimates.

Consider all possible extensions E of the type (2.4) with fixed V and x. They are parametrized by

$$P := P(V_x), \tag{2.5}$$

the projective space defined by all *lines* in the fiber V_x. For the map ψ defined in (2.3), the fiber $\psi^{-1}(x, V)$ is identified with P, provided $(x, V) \in U$.

Let

$$f : \psi^{-1}(U) \longrightarrow X \tag{2.6}$$

be the obvious projection. We will show that on $\psi^{-1}(U)$ there is a natural isomorphism

$$T_\pi \otimes f^* K_X \cong \Omega^1_\psi := T^*_\psi, \tag{2.7}$$

where π, ψ and f are defined in (2.1), (2.3) and (2.6) respectively.

To construct the isomorphism in (2.7), consider the exact sequence

$$0 \longrightarrow \lambda_x \longrightarrow V_x \longrightarrow E_x \longrightarrow l_x \longrightarrow 0 \tag{2.8}$$

of fibers at x for the exact sequence (2.4) of sheaves. Now, the fiber of T_π at the point $(E, l_x) \in \psi^{-1}(U)$ is $\mathrm{Hom}(V_x/\lambda_x, l_x)$. Similarly, the fiber of T_ψ at $(E, l_x) \in \psi^{-1}(U)$ is $\mathrm{Hom}(\lambda_x, V_x/\lambda_x)$. There is a certain canonical isomorphism of the vector space λ_x with $l_x \otimes T_x^*$, where T_x is the fiber at x of the holomorphic tangent bundle TX (so $T_x^* = (K_X)_x$). The isomorphism in question is obtained from the observation that $E \otimes \mathcal{O}_X(-x)$ is a subsheaf of V with the image of the fiber $(E \otimes \mathcal{O}_X(-x))_x$ in V_x coinciding with the subspace λ_x in (2.8); note that the Poincaré adjunction formula identifies T_x^* with $\mathcal{O}_X(-x)_x$. Therefore,

$$T_\psi^*\big|_{(E,l_x)} = \mathrm{Hom}(V_x/\lambda_x, \lambda_x) = \mathrm{Hom}\left(V_x/\lambda_x, l_x \otimes T_x^*\right) = T_\pi|_{(E,l_x)} \otimes T_x^*.$$

This gives the isomorphism in (2.7).

Let

$$\overline{\pi}\colon \mathbb{P}(\mathcal{E}) \longrightarrow \mathcal{N} \tag{2.9}$$

be the following composition of maps:

$$\mathbb{P}(\mathcal{E}) \xrightarrow{\pi} X \times \mathcal{N} \longrightarrow \mathcal{N}.$$

Using the isomorphism in (2.7) and an Hartog type result for higher degree cohomologies, [1, Proposition 1.11], the Leray spectral sequence for the projection ψ gives that the converging limit $\mathbb{L}^*$ of

$$E_2^{p,q} := H^p\left(U, R^q\psi_*\left(\overline{\pi}^* T\mathcal{N} \otimes \Omega^1_\psi\right)\right) \tag{2.10}$$

has the property that $\mathbb{L}^i$ is $H^i(\mathbb{P}(\mathcal{E}), \pi^*(K_X \boxtimes T\mathcal{N}) \otimes T_\pi)$ for $i \leq 2$, where $\overline{\pi}$ is defined in (2.9); recall that the codimensions of $N_X^s \setminus U$ and $\mathbb{P}(\mathcal{E}) \setminus \psi^{-1}(U)$ are at least four and hence we get isomorphisms of cohomologies of degree up tp two.

In view of Proposition 2.1 and the above observation on the converging limit of $E_2^{p,q}$, to compute $H^i(\mathcal{N}, \mathrm{End}(T\mathcal{N}))$, $i \leq 1$, we need to compute the direct images $R^q\psi_*(\overline{\pi}^* T\mathcal{N} \otimes \Omega^1_\psi)$, $q \leq 2$, over U. This will be carried out in the next section.

3. The direct images over U and their cohomology

Let

$$h\colon U \longrightarrow X \tag{3.1}$$

be the natural projection. So h sends any pair (x, V) to x.

Lemma 3.1. *Over U,*

$$R^j\psi_*\left(\overline{\pi}^* T\mathcal{N} \otimes \Omega^1_\psi\right) = \begin{cases} \mathcal{O}_U & \text{for} \quad j = 0 \\ 0 & \text{for} \quad j = 2 \end{cases}$$

and $R^1\psi_*(\overline{\pi}^*T\mathcal{N}\otimes\Omega^1_\psi)$ *fits in an exact sequence*

$$0\longrightarrow T_h\longrightarrow R^1\psi_*\left(\overline{\pi}^*T\mathcal{N}\otimes\Omega^1_\psi\right)\longrightarrow h^*TX\longrightarrow 0$$

where T_h *is the relative tangent bundle for the projection* h *defined in* (3.1).

Proof. Take any point $(x, V)\in U$. Let

$$p\colon X\times P\longrightarrow P \tag{3.2}$$

be the natural projection, where $P = P(V_x)$ is the projective space defined in (2.5). Let

$$p_X\colon X\times P\longrightarrow X \tag{3.3}$$

be the other projection.

On $X\times P$ we have the exact sequence of sheaves

$$0\longrightarrow p_X^*V\longrightarrow\mathcal{F}\longrightarrow\mathcal{O}_P(-1)|_{x\times P}\longrightarrow 0.$$

For any point $\lambda_x\in P$, the restriction of this exact sequence to $X\times\{\lambda_x\}$ is an exact sequence of sheaves on X of the form (2.4). So for any line $\lambda_x\in P$ in the fiber V_x, the vector bundle $\mathcal{F}|_{X\times\{\lambda_x\}}$ over X and the hyperplane in $\mathcal{F}|_{x\times\lambda_x}$ given by the image of V_x has the property that the image, under the Hecke map ψ in (2.3), of this pair is the point of N^s_X defined by (x, V).

We have $T_E\mathcal{N} = H^1(X, \mathrm{ad}(E))$ and $p_*\,\mathrm{ad}(\mathcal{F}) = 0$. Therefore, to prove the lemma it suffices to show the following assertions are valid:

$$H^j\left(P, R^1p_*\,\mathrm{ad}(\mathcal{F})\otimes\Omega^1_P\right) = \begin{cases}\mathbb{C} & \text{for}\quad j=0\\ 0 & \text{for}\quad j=2\end{cases} \tag{3.4}$$

and there is a natural exact sequence

$$0\longrightarrow H^1(X,\mathrm{ad}(V))\longrightarrow H^1\left(P, R^1p_*\,\mathrm{ad}(\mathcal{F})\otimes\Omega^1_P\right)\longrightarrow T_xX\longrightarrow 0. \tag{3.5}$$

We start proving (3.4) and (3.5) by first noting that the exact sequence (2.4) gives

the following diagram of homomorphisms of sheaves on X:

(3.6)

$$\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & \mathrm{Hom}(E, V) & & & & \mathrm{End}(l_x) & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \mathrm{End}(E) & \longrightarrow & \mathrm{Hom}(E, l_x) & \longrightarrow & \mathrm{Hom}(E, l_x)_x & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & \mathrm{End}(V) & & & & \mathrm{Hom}(V_x/(l_x \otimes K_x), l_x) & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & \mathrm{Hom}(l_x \otimes K_x, V)_x & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

where K_x denotes the fiber at x of the canonical line bundle K_X. To explain the vertical homomorphisms in (3.6), recall that just after (2.8) we noted that the vector space λ_x is identified with $l_x \otimes T^*_x$. The vertical homomorphism $\mathrm{End}(V) \longrightarrow \mathrm{Hom}(l_x \otimes K_x, V)_x$ in the left-hand side of (3.6) is obtained by combining this isomorphism with the restriction homomorphism $\mathrm{End}(V_x) \longrightarrow \mathrm{Hom}(\lambda_x, V_x)$. Similarly, the vertical homomorphism

$$\mathrm{Hom}(E, l_x)_x \longrightarrow \mathrm{Hom}(V_x/(l_x \otimes K_x), l_x)$$

in the right-hand side of (3.6) is obtained by combining this isomorphism $\lambda_x = l_x \otimes T^*_x$ with the homomorphism

$$\mathrm{Hom}(E_x, l_x) \longrightarrow \mathrm{Hom}(V_x, l_x)$$

constructed using the homomorphism $V_x \longrightarrow E_x$ in (2.8).

The above diagram (3.6) of morphisms of sheaves over X evidently gives the following diagram

(3.7)

$$\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & \mathrm{Hom}\left(\mathcal{F}, p_X^* V\right) & & & & \mathrm{End}\left(\mathcal{O}_P(-1) \otimes \widetilde{T}_x\right)\big|_{x \times P} & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \mathrm{End}(\mathcal{F}) & \longrightarrow & \mathrm{Hom}\left(\mathcal{F}, \mathcal{O}_P(-1) \otimes \widetilde{T}_x\right)\big|_{x \times P} & \longrightarrow & \mathrm{Hom}\left(\widetilde{V}_x/\mathcal{O}_P(-1), \mathcal{O}_P(-1) \otimes \widetilde{T}_x\right)\big|_{x \times P} & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & \mathrm{End}\left(p_X^* V\right) & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & \mathrm{Hom}\left(\mathcal{O}_P(-1), p_X^* V\right)\big|_{x \times P} & & & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

of sheaves over $X \times P$, where $\widetilde{V}_x$ and $\widetilde{T}_x$ are the trivial vector bundles over P with fiber V_x and T_x respectively.

Now consider the long exact sequence of direct images, with respect to the projection p defined in (3.2), for the horizontal short exact sequence in (3.7). Since

$$H^0(X, \mathrm{Hom}(E, V)) = 0$$

(the vector bundle E being stable is simple and V is a proper subsheaf of E) and also $H^0(X, \mathrm{End}(V)) = \mathbb{C}$, the long exact sequence of direct images becomes

(3.8)
$$0 \to \mathbb{C} \to \mathrm{Hom}\left(\mathcal{F}|_{x\times P}, \mathcal{O}_P(-1) \otimes \widetilde{T}_x\right) \to R^1 p_* \mathrm{Hom}\left(\mathcal{F}, p_X^* V\right) \to R^1 p_* \mathrm{End}(\mathcal{F}) \to 0$$

(as $\mathrm{Hom}\left(\mathcal{F}, \mathcal{O}_P(-1) \otimes \widetilde{T}_x\right)\big|_{x\times P}$ is supported on $x \times P$, we have $R^1\left(p_* \mathrm{Hom}\left(\mathcal{F}, \mathcal{O}_P(-1) \otimes \widetilde{T}_x\right)\big|_{x\times P}\right) = 0$).

Note that the image of $H^0(X, \mathrm{End}(E))$ in $\mathrm{Hom}(E, l_x)_x$ for the horizontal short exact sequence in (3.6) coincides with the subspace $\mathrm{End}(l_x)$ in the right-hand side vertical exact sequence in (3.6). Therefore, the long exact sequence (3.8) gives

(3.9)
$$0 \to \mathrm{Hom}\left(\widetilde{V}_x/\mathcal{O}_P(-1), \mathcal{O}_P(-1) \otimes \widetilde{T}_x\right) \to R^1 p_* \mathrm{Hom}\left(\mathcal{F}, p_X^* V\right) \to R^1 p_* \mathrm{End}(\mathcal{F}) \to 0.$$

The initial cohomologies of $R^1 p_* \mathrm{ad}(\mathcal{F}) \otimes \Omega_P^1$ (see (3.4), (3.5)) will be computed using the long exact sequence of cohomologies corresponding to the short exact sequence of sheaves on P obtained by tensoring (3.9) with Ω_P^1.

Our next step in the proof of Lemma 3.1 will be to prove the following proposition.

Proposition 3.2. *The initial cohomologies of* $R^1 p_* \mathrm{Hom}(\mathcal{F}, p_X^* V) \otimes \Omega_P^1$ *are as follows*:

$$H^j\left(P, R^1 p_* \mathrm{Hom}\left(\mathcal{F}, p_X^* V\right) \otimes \Omega_P^1\right) = \begin{cases} \mathbb{C} & \text{for} \quad j = 0 \\ H^1(X, \mathrm{End}(V)) & \text{for} \quad j = 1 \\ 0 & \text{for} \quad j = 2 \end{cases}.$$

Proof. To prove this proposition, first consider the long exact sequence of direct images on P for the left-hand side vertical exact sequence of sheaves in (3.7). Since V is stable, after using the exact sequence

(3.10)
$$0 \longrightarrow \mathrm{End}(\mathcal{O}_P(-1)) \longrightarrow \mathrm{Hom}\left(\mathcal{O}_P(-1), \widetilde{V}_x\right) \longrightarrow \mathrm{Hom}\left(\mathcal{O}_P(-1), \widetilde{V}_x/\mathcal{O}_P(-1)\right) \longrightarrow 0$$

over P, just as in (3.9) the long exact sequence of direct images for the left-hand side vertical exact sequence becomes

(3.11)
$$0 \to \mathrm{Hom}\left(\mathcal{O}_P(-1), \widetilde{V}_x/\mathcal{O}_P(-1)\right) \to R^1 p_* \mathrm{Hom}\left(\mathcal{F}, p_X^* V\right) \to R^1 p_* \mathrm{End}\left(p_X^* V\right) \to 0.$$

Firstly, $H^1(P, \Omega^1_P) = \mathbb{C}$ and $H^i(P, \Omega^1_P) = 0$ for all $i \neq 1$ (note that $H^1(P, \Omega^1_P)$ has a canonical generator given by the hyperplane class). Secondly, $H^i(P, \mathcal{O}_P(1)\otimes\Omega^1_P) = 0$ for $i \leq 2$ [4, Theorem 4.3] (see [4, Theorem 4.3 (a)] for $i = 0$, [4, Theorem 4.3 (d)] for $i = 1$ and [4, Theorem 4.3 (b)] for $i = 2$). Combining these two facts with the long exact sequence of cohomologies for the short exact sequence of sheaves on P obtained by tensoring (3.10) with Ω^1_P, we have

$$H^j\left(P, \operatorname{Hom}\left(\mathcal{O}_P(-1), \widetilde{V}_x/\mathcal{O}_P(-1)\right)\otimes\Omega^1_P\right) = \begin{cases} \mathbb{C} & \text{for} \quad j = 0 \\ 0 & \text{for} \quad j = 1 \\ 0 & \text{for} \quad j = 2 \end{cases}.$$

Finally, $R^1 p_* \operatorname{End}(p_X^* V)$ is the trivial vector bundle over P with fiber $H^1(X, \operatorname{End}(V))$. Therefore, the proposition follows from the long exact sequence of cohomologies for the exact sequence of sheaves on P obtained by tensoring (3.11) with Ω^1_P. □

We need another proposition for the proof of Lemma 3.1.

Proposition 3.3. *For the vector bundle* $\operatorname{Hom}\left(\widetilde{V}_x/\mathcal{O}_P(-1), \mathcal{O}_P(-1)\right)\otimes\Omega^1_P$ *over* P, *where* $V \in U$,

$$H^j\left(P, \operatorname{Hom}\left(\widetilde{V}_x/\mathcal{O}_P(-1), \mathcal{O}_P(-1)\right)\otimes\Omega^1_P\right) = \begin{cases} 0 & \textit{for} \quad j = 0 \\ 0 & \textit{for} \quad j = 1 \\ \mathbb{C} & \textit{for} \quad j = 2 \\ 0 & \textit{for} \quad j = 3 \end{cases}.$$

Proof. To prove the proposition consider the exact sequence of vector bundles (3.12)

$$0 \longrightarrow \operatorname{Hom}\left(\widetilde{V}_x/\mathcal{O}_P(-1), \mathcal{O}_P(-1)\right)\otimes\Omega^1_P \longrightarrow \operatorname{Hom}\left(\widetilde{V}_x, \mathcal{O}_P(-1)\right)\otimes\Omega^1_P \longrightarrow \Omega^1_P \longrightarrow 0$$

over P obtained by tensoring the exact sequence

$$0 \longrightarrow \operatorname{Hom}\left(\widetilde{V}_x/\mathcal{O}_P(-1), \mathcal{O}_P(-1)\right) \longrightarrow \operatorname{Hom}\left(\widetilde{V}_x, \mathcal{O}_P(-1)\right) \longrightarrow \mathcal{O}_P \longrightarrow 0$$

(this is the dual of (3.10)) with Ω^1_P. As $n > 2$, we have

$$H^k\left(P, \mathcal{O}_P(-1)\otimes\Omega^1_P\right) = 0$$

for $k \leq 3$ [4, Theorem 4.3] (see [4, Theorem 4.3 (c)] for $k = 1$ and [4, Theorem 4.3 (b)] for $k = 2, 3$). (Note that if $n = 2$, then $H^1(P, \mathcal{O}_P(-1)\otimes\Omega^1_P) \neq 0$.) Now the proposition follows from the long exact sequence of cohomologies corresponding to the exact sequence in (3.12) of vector bundles over P. □

Continuing with the proof of Lemma 3.1, in view of Proposition 3.2 and Proposition 3.3, the initial part of the long exact sequence of cohomologies for the exact sequence (3.9) tensored with Ω^1_P becomes

(3.13)
$$\begin{aligned} 0 \to 0 \to \mathbb{C} \to H^0\left(P, R^1p_*\operatorname{End}(\mathcal{F})\otimes\Omega^1_P\right) \to 0 \to H^1(X, \operatorname{End}(V)) \\ \to H^1\left(P, R^1p_*\operatorname{End}(\mathcal{F})\otimes\Omega^1_P\right) \to T_x \to 0 \to H^2\left(P, R^1p_*\operatorname{End}(\mathcal{F})\otimes\Omega^1_P)\right) \to 0, \end{aligned}$$

where T_x, as before, is the tangent line at $x \in X$. Now (3.4) follows immediately from (3.13).

For proving (3.5), consider the short exact sequence

$$0 \longrightarrow H^1(X, \operatorname{End}(V)) \longrightarrow H^1\left(P, R^1p_*\operatorname{End}(\mathcal{F})\otimes\Omega^1_P\right) \longrightarrow T_x \longrightarrow 0 \tag{3.14}$$

contained in the exact sequence in (3.13). We will show that (3.14) gives an exact sequence

$$0 \longrightarrow H^1(X, \operatorname{ad}(V)) \longrightarrow H^1\left(P, R^1p_*\operatorname{ad}(\mathcal{F})\otimes\Omega^1_P\right) \longrightarrow T_x \longrightarrow 0. \tag{3.15}$$

To construct (3.15) from (3.14) first observe that

$$R^1p_*\operatorname{End}(\mathcal{F})\otimes\Omega^1_P = \left(R^1p_*\operatorname{ad}(\mathcal{F})\oplus R^1p_*\mathcal{O}_{X\times P}\right)\otimes\Omega^1_P$$

and hence

(3.16)
$$H^1\left(P, R^1p_*\operatorname{End}(\mathcal{F})\otimes\Omega^1_P\right) = H^1\left(P, R^1p_*\operatorname{ad}(\mathcal{F})\otimes\Omega^1_P\right)\oplus H^1\left(P, R^1p_*\mathcal{O}_{X\times P}\otimes\Omega^1_P\right).$$

Now, $R^1p_*\mathcal{O}_{X\times P}$ is the trivial vector bundle over P with fiber $H^1(X, \mathcal{O}_X)$. Since

$$H^1\left(P, \Omega^1_P\right) = \mathbb{C},$$

we conclude that

$$H^1\left(P, R^1p_*\mathcal{O}_{X\times P}\otimes\Omega^1_P\right) = H^1(X, \mathcal{O}_X).$$

Consequently, from (3.16) we obtain an inclusion

$$H^1(X, \mathcal{O}_X) \subset H^1\left(P, R^1p_*\operatorname{End}(\mathcal{F})\otimes\Omega^1_P\right). \tag{3.17}$$

On the other hand, since $H^1(X, \mathcal{O}_X) \subset H^1(X, \operatorname{End}(V))$, using (3.14) we have another inclusion

$$H^1(X, \mathcal{O}_X) \subset H^1\left(P, R^1p_*\operatorname{End}(\mathcal{F})\otimes\Omega^1_P\right). \tag{3.18}$$

To obtain the exact sequence (3.15) from (3.14) we need to show that the two subspaces $H^1(X, \mathcal{O}_X)$ in $H^1(P, R^1p_* \operatorname{End}(\mathcal{F}) \otimes \Omega^1_P)$ constructed in (3.17) and (3.18) actually coincide.

The subspace $H^1(X, \mathcal{O}_X)$ in (3.17) must lie inside the kernel of the projection to T_x in (3.14). Indeed, this is an immediate consequence of the fact that there is no nonzero homomorphism of $\mathcal{O}_X$ into TX (by moving the point x over X we get a homomorphism to TX of the trivial vector bundle over X with fiber $H^1(X, \mathcal{O}_X)$). Therefore, the subspace in (3.17) is contained in the subspace $H^1(X, \operatorname{End}(V))$ in (3.14).

We have $H^1(X, \operatorname{End}(V)) = H^1(X, \operatorname{ad}(V)) \oplus H^1(X, \mathcal{O}_X)$. To obtain (3.15) it suffices to show that the projection of the subspace $H^1(X, \mathcal{O}_X)$ in (3.17) to $H^1(X, \operatorname{ad}(V))$ vanishes (we have already shown that the subspace $H^1(X, \mathcal{O}_X)$ in (3.17) is contained in $H^1(X, \operatorname{End}(V))$).

Consider $\psi^{-1}(U)$, where ψ and U are as before (see (2.6)). Let F denote the trivial vector bundle over $\psi^{-1}(U)$ with fiber $H^1(X, \mathcal{O}_X)$. Moving over the points of $\psi^{-1}(U)$, the projection of the subspace $H^1(X, \mathcal{O}_X)$ in (3.17) to $H^1(X, \operatorname{ad}(V))$ gives a homomorphism of vector bundles

$$\gamma' : F \longrightarrow \psi^* T_h,$$

where T_h is the relative tangent bundle for the projection h defined in (3.1). Indeed, this follows immediately from the fact that the fiber of T_h at any point $(y, V) \in U$ is $H^1(X, \operatorname{ad}(V))$. Since the projection $\psi|_{\psi^{-1}(U)}$ is proper with connected fibers, any homomorphism from F to $\psi^* T_h$ is obtained from a homomorphism to T_h from the trivial vector bundle over U with fiber $H^1(X, \mathcal{O}_X)$. Let

$$\gamma : F' \longrightarrow T_h \tag{3.19}$$

be the homomorphism of vector bundles, where F' is the trivial vector bundle over U with fiber $H^1(X, \mathcal{O}_X)$, from which the above homomorphism γ' is obtained.

Let N'' be a moduli space of stable vector bundles over X of rank n and fixed determinant. We have

$$H^0(\mathcal{N}'', T\mathcal{N}'') = 0 \tag{3.20}$$

[2, Theorem 1]. From this it follows that the homomorphism γ in (3.19) vanishes. Consequently, the two subspaces $H^1(X, \mathcal{O}_X)$ in $H^1(P, R^1p_* \operatorname{End}(\mathcal{F}) \otimes \Omega^1_P)$ constructed in (3.17) and (3.18) coincide. Therefore, the exact sequence (3.14) gives the exact sequence (3.15). This completes the proof of Lemma 3.1. □

Now we are in a position to prove the main result.

Theorem 3.4. *For the moduli space* $\mathcal{N}$,

$$H^i(\mathcal{N}, \mathrm{End}(T\mathcal{N})) = \begin{cases} \mathbb{C} & \text{for} \quad i = 0 \\ H^1(X, \mathcal{O}_X) & \text{for} \quad i = 1 \end{cases}.$$

Proof. If we fix a point $y \in X$, then N_X^s (see (2.3)) gets identified with the product $X \times \mathcal{N}'$, where $\mathcal{N}'$ is the moduli space of all stable vector bundle bundles W over X of rank n and $\bigwedge^n W = \xi \otimes \mathcal{O}_X(-y)$. In this identification of N_X^s with $X \times \mathcal{N}'$, the relative tangent bundle T_h is the pullback of $T\mathcal{N}'$ to $X \times \mathcal{N}'$.

We have

$$H^1\left(\mathcal{N}', \mathcal{O}_{\mathcal{N}'}\right) = 0$$

and $H^0(\mathcal{N}', \mathcal{O}_{\mathcal{N}'}) = \mathbb{C}$. Consequently, $H^i(U, \mathcal{O}_U) = H^i(X, \mathcal{O}_X)$ for $i = 0, 1$. Therefore, in view of Proposition 2.1, Lemma 3.1 and the fact that $E_2^{p,q}$ in (2.10) converges to $H^i(\mathbb{P}(\mathcal{E}), \pi^*(K_X \boxtimes T\mathcal{N}) \otimes T_\pi)$ for $i \leq 2$, it suffices to show that

$$H^0\left(U,\, R^1\psi_*\left(\overline{\pi}^* T\mathcal{N} \otimes \Omega^1_\psi\right)\right) = 0. \tag{3.21}$$

Consider the exact sequence of cohomologies

$$0 \longrightarrow H^0(U, T_h) \longrightarrow H^0\left(U, R^1\psi_*\left(\overline{\pi}^* T\mathcal{N} \otimes \Omega^1_\psi\right)\right) \longrightarrow H^0(U, h^* TX)$$

obtained from the exact sequence of sheaves in Lemma 3.1. We have $H^0(X, TX) = 0$ and hence $H^0(U, h^*TX) = 0$. Also, from (3.20) it follows that $H^0(U, T_h) = 0$. Therefore, (3.21) follows from the above exact sequence of cohomologies. This completes the proof of the theorem. □

REMARK 3.5. Since $H^2(U, \mathcal{O}_U) = 0 = H^0(X, TX)$, it follows from (2.10) and Lemma 3.1 that

$$H^1(U, T_h) \hookrightarrow H^2(\mathcal{N}, \mathrm{End}(T\mathcal{N})).$$

As a consequence, we have $\dim H^2(\mathcal{N}, \mathrm{End}(T\mathcal{N})) \geq 3g - 3$.

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