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Osaka University
On Covering Surfaces

By Zenjiro Kuramoto

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Introduction

The purpose of the present paper is to investigate the behaviour of the boundary of a covering surface. In chapter I we shall consider the correspondence of the boundary points when the universal covering surface is mapped onto the unit-circle, and extend Fatou’s theorem in such a case. Chapter II is devoted to the study of accessible boundary points from measure theoretic views. Thus our study is a continuation of that due to M. Ohtsuka1). R. Nevanlinna discussed the Dirichlet

3) Note on the harmonic measure of the accessible boundary of a covering Riemann surface. ibid., 5, 1953, pp. 35-38.
problem in weak sense on a subsurface of an abstract Riemann surface. M. Ohtsuka proved that the Dirichlet problem for continuous boundary value is resolutive in Perron-Brelot’s sense which is stronger than that of R. Nevanlinna under the following three assumptions: A) the projection of accessible boundary points of the surface \( R \) is compact in the basic surface \( R \), B) the universal covering surface \( R^\infty \) of \( R \) is \( F \)-type, C) the connectivity of \( R \) is finite. We shall show that it is possible to remove conditions A) and C). Thus we see that the resolutivity of the problem has not so much relation with the measure of irregular points. If we suppose that \( R \) covers \( R \) a finite number of times, then we can prove two theorems concerning the Dirichlet problem: i) the irregular accessible boundary points are so rare that the projection of them may be contained in the \( F_\sigma \) set of capacity zero, ii) the regularity for Green’s function is equivalent to that for Dirichlet problem. Chapter III is related to R. S. Martin’s topology concerning harmonic functions. We shall study activity of points and minimality. R. S. Martin conjectured that the set of non minimal points would be non dense in the set of ideal boundary points. However we shall show by an example that his conjecture does not hold good and at the end of this chapter we shall prove miscellaneous theorems on the subsurfaces of abstract Riemann surfaces. In chapter IV we consider analytic functions on Riemann surfaces and the behaviour of covering surfaces. We shall give an example of Riemann surface belonging to the class \( O_{HR} \) and without Gross’s property. From this we see that the validity of the Gross’s property for every covering surface of a given Riemann surface does not depend upon the complexity of the boundary. It depends rather upon the “force” of the boundary, i.e., roughly speaking upon the size of the boundary.

Chapter I. Correspondence of boundaries

1. Image of boundary

Let \( R \) be an open abstract Riemann surface, and \( R_n (n = 1, 2, \ldots) \) be an exhaustion with relative boundary \( \Gamma_n \), then \( R - R_n \) is composed of a finite number of disjoint non compact subsurfaces \( G_n \), let \( G_n \) be a sequence of non compact subsurfaces such that \( G_t \supset G_{t+1} \ldots, \cap G_n = 0. \)

Two sequences \( \{G_n\} \) and \( \{G_m'\} \) are called equivalent if and only if, for a given number \( m \), there exists a number \( n \) such that \( G_m' \supset G_n \) and vice versa. We correspond an ideal boundary point (component) to an equivalent sequence, and denote the set of all ideal boundary point by \( B_f \). A-topology will be introduced on \( R + B_f \) by the completion of \( R \). It is clear that \( R + B_f \) is closed, compact and \( B_f \) is totally disconnected. For simplicity we denote \( R + B_f \) by \( R^* \) which has the topology being isomorphic in \( R \) to the original topology defined by the local parameter. When \( R \) is a covering surface over an abstract Riemann surface \( R^* \), \( R^* \) is a metric space, we introduce as O. Teichmüller and Mazurkiewicz a metric on \( R \) by \( \inf \delta(p \cdot p_2) \), where \( \delta(p \cdot p_2) \) is the diameter of the projection into \( R \) of \( p \sim p_2 \) which connects \( p \) with \( p_2 \) on \( R \), and define accessible boundary points of \( R \) by the completion with respect to this metric, it is well known that this definition is equivalent to the other definition by deciding the equivalency of two curves on \( R \) which determine an accessible boundary point (we abbreviate by A.B.P.). In this and next chapter we suppose \( R \) to be a null-boundary Riemann surface and discuss another Riemann surface \( R \) defined as the covering surface over \( R \).

**Lemma.** Let \( R \) be a covering surface over \( R_0 \), and does not cover a set of positive coapacity inner points of \( R_0 \), and the universal covering surface \( R^\circ \) of \( R \) mapped conformally onto the unit-circle \( U : \mid z \mid < 1 \). If a curve \( L \) on \( L^\circ \) contained in \( R \) and determining an A.B.P. ending at a point \( z_0 : \mid z_0 \mid = 1 \), determines an A.B.P. having its projection on \( B_f \), then the mapping function \( w = f(z) : w \in R^* \) has the same limit when \( z_0 \) tends to \( z_0 \) along Stolz’s path.

**Proof.** Let \( p_0 \) be the projection of the A.B.P. on \( R^* \) and \( V_n(p_0) \) be \( \Gamma_n \), the sequence of neighbourhoods of \( p_0 \) with a compact relative boundary. We define a super-harmonic function \( \omega_n(p) \) such as, \( \omega_n(p) \) is harmonic in \((R - R_0 - V_n(p_0)) \cup (\text{proj } R \cap R_0)\), \( \omega_n(p) = 0 \) if \( p \in \text{boundary of proj } R \) in \( R_0 \), where \( R_0 \) is a compact part of in which the boundary of the projection of has positive capacity, and \( \omega_n(p) = M_n \) if \( p \in \Gamma_n + V_n(p_0) \), and is normalized by \( \frac{1}{2\pi} \int_{\Gamma_n} \frac{\partial \omega_n}{\partial n} ds = 1 \). Since \( R \) is a null-boundary Riemann surface, this function is uniquely determined and \( \lim M_n = \infty \), we denote the domain \( \arg |z - z_0| < \frac{\pi}{2} - \delta, \mid z - z_0 \mid < 1 - r \) by \( \Delta_n \), and the part of \( L \) outside of \( |z| = r_0 \) by \( L_{r_0} \), then

\[
\omega_n(f(z)) \geq M_n \lambda^\delta, \quad \text{if } z \in \Delta_n; \quad n > i_0 \ldots
\]
where \( \lambda \delta > 0 \) and \( i_0 \) is the minimal number such as \( f(Lr) \in V_i(p) \). If \( f(z) \) had not limit \( p_0 \) in \( \Delta r_\delta \), then there would exist a sequence \( z_i \) such as \( \lim z_i = z_0 : z_i \in \Delta r_\delta \) and a number \( n_0 \) and a subsequence of \( i_{n_k} \) such as \( f(z_{n_k}) \in V_{n_k}(p) : i_{n_k} \geq i_0 \), therefore there exists a number \( N \) such that

\[
\omega_n(f(z_{i_k})) \leq N : i_k \geq i_{k_0} \quad n \geq n_0 \quad \ldots \tag{2}
\]

from (1) and (2) we have \( \geq \omega_n(f(z_n)) \geq M_n \lambda \delta : \lim M_n = \infty \), this is a contradiction.

**Remark.** When the A. B. P. lies on \( \mathcal{R} \), our assertion is trivial. From this lemma we can easily deduce the next proposition.

**Proposition.** Under the same condition as the lemma, the set \( E \) on \(|z| = 1\) where at least a curve ends along which \( f(z) \) has cluster set of capacity zero is a linear measure zero set on \(|z| = 1\).

**Proof.** We denote a closed set \( E' \subset E \) where \( f(z) \) tends uniformly in \( \mathcal{A} \)-topology, then \( f(E') \) is closed and capacity zero, then there exists a sequence \( V_n \) with compact relative boundary \( \gamma_n \) such as \( V_n \supset f(E') \), where \( V_n \) is the closure of \( V_n \). We denote continuous super-harmonic function such as \( \omega_n(p) \) is harmonic in \((R - R_0 - V_n) \cup (R_0 \cap \text{proj} \mathcal{R})\) and \( \omega_n(p) = 0 \) if \( p \in \text{boundary of proj} \mathcal{R} \cap R_0 \), and \( \omega_n(p) = M_n \) if \( p \in \gamma_n + V_n \),

\[
\frac{1}{2\pi} \int \frac{\partial \omega_n}{\partial n} ds = 1 \quad \text{then} \quad \lim M_n = \infty. \quad \text{We consider} \quad \omega_n(p) \quad \text{in} \quad |z| < 1, \quad
\]

\[
\omega_n(z) \geq \frac{1}{2\pi} \int_{E'} M_n \frac{r_n^2 - \rho^2}{r_n^2 - 2r_n \rho \cos(\theta - \phi) + \rho^2} d\theta, \quad \text{and let} \quad r_n \to 1 \quad \text{then} \quad \omega_n(z) \to \infty. \quad \text{This is a contradiction.}
\]

Let \( \mathcal{R} \) be a null-boundary Riemann surface, and \( \mathcal{R}'\infty \) be mapped onto \(|z| < 1\) conformally and \( D_0 \) be the normal polygon being a fundamental domain containing \( z = 0 \). We denote by \( D_0'(\rho) \) the part of \( D_0 \), which lies in \( 1 - \rho \leq |z| < 1 : 0 < \rho < 1 \); then \( D_0'(\rho) \) consists of a finite number of simply connected closed domain, we consider only such domains and \( D_n(\rho) \) be its equivalent ones and put

\[
\Delta(\rho) = \sum_{n=0}^{\infty} D_n(\rho),
\]

then \( \Delta(\rho) \) consists of a enumerable number of disjoint simply connected continua: \( \Delta_n(\rho) \) such that \( \Delta(\rho) = \sum \Delta_n(\rho) \). Since in a normal polygon, equivalent points on the boundary of \( D_0 \) are equidistant from \( z = 0 \), \( \Delta_n(\rho) \) is bounded by Jordan arcs \( \lambda_n^k \) \((k = 0, 1, 2, \ldots)\) and closed sets \( E_n \).
on $|z|=1$. We put $E(\rho) = \sum E_n$, $\lambda_n^k$ on $|z|=1$ which are fixed points of some substitution, if $\xi_n^k = \eta_n^k$ for one $k$, then $\Delta_n(\rho)$ is bounded by a single Jordan curve which touches $|z|=1$ at $\xi_n^k = \eta_n^k$. It is easily seen that if $\xi_n^k = \eta_n^k$ then $\lambda_n^k$ is contained between two circular arcs $C_n^k, C_n^{k'}$ through $\xi_n^k, \eta_n^k$ which meets $|z|=1$ with an angle $\alpha_n^k, \beta_n^k$ respectively. Since $\lambda_n^k (n=0,1,2,...)$ can be grouped into a finite number of equivalent classes, there exists $\alpha, \beta (0<\alpha, \beta, <\pi)$ such that for any $\lambda_n^k$ for which

$$\alpha \leq \alpha_n^k \leq \beta, \ \alpha \leq \beta_n^k \leq \beta \quad (n=0,1,2,3,...)$$

Since $\lambda_n^k$ is a $\sum S^\infty(l)$, where $l$ is the part of $|z| = \rho$ in $D_0$. and multiplier of the substitution $k = 1$, then $\lambda_n^k$ is rectifiable.

**Lemma.** For any substitution $T_i (i=1,2,...,i_0)$ of Fuchsian group

$$\delta_0 \leq \text{length } \lambda_n^k \text{length } T_\lambda(\lambda_n^k) \leq \delta_0.$$  

We denote by $ds$ the line-element of $\lambda_i$, and by $T(ds)$ its image, since $\lambda_n^k$ has a tangent almost everywhere on $\lambda_n^k$, let $C^k ds$ be the circular arc ending at two points $\xi$ and $\eta$ passing $ds$ at $P$ and $C ds$ the circular arc passing $ds$ at $P$ and ending at two fix points $\xi, \xi'$ of the substitution $T$, then

$$\frac{ds}{T(ds)} = \frac{\xi P}{T(\xi P)} \quad \text{or} \quad \frac{ds}{T(ds)} = \frac{\xi P}{T(\xi P)}'$$

but $\frac{\xi P}{T(\xi P)}$ has extremal values when $C ds$ coincides with $C'$ or $C$ and the intersecting angle of $\lambda_i$ and $|z|=1$ is contained between $\alpha$ and $\beta$, therefore we have

$$\delta_0 \frac{\xi \eta}{T(\xi \eta)} \frac{ds}{T(ds)} \frac{\xi \eta}{T(\xi \eta)} \leq \delta_0 \frac{\xi \eta}{T(\xi \eta)}$$

and if we denote by $\xi \eta$ the length of $|z|=1$ between $\xi, \eta$ then

$$\delta_0 \frac{\xi \eta}{T(\xi \eta)} \leq \frac{\lambda_n^k}{T(\lambda_n^k)} \leq \delta_0 \frac{\xi \eta}{T(\xi \eta)}.$$  

But $\Delta_n(\rho)$ are bounded by $(\xi_n, \eta_n)$ and its image transformed, one of them is an inner side of $\Delta_n(\rho)$ with respect to $|z|=1$, and the others are outer sides and have non common points on $|z|=1$, therefore the length of $\Delta_n(\rho)$ is finite. Since $\Delta_n$ does not cover a set of positive capacity on $R$, then we map $\Delta_n$ onto $|y|<1$ then from the proposition we see that $E_n$ is linear measure zero set, and also $\sum E_n$ is too, thus
Theorem 1.1. Let \( R \) be a null-boundary Riemann surface. When universal covering surface mapped onto the unit-circle, then the image of the boundary points is a set of linear measure zero.

Remark. This theorem is proved by M. Tsuji, our proof is slightly simpler.

2. Generalization of Fatou's theorem

Lemma. Let \( R \) be a null-boundary Riemann surface, we denote two points \( a \) and \( q \) of \( R \) and denote a harmonic function \( U_a(p) \) such that

\[
U_a(p) + \log p = 0, \quad \text{in the neighbourhood of } q
\]

\[
U_a(p) - \log p = \text{harmonic in the neighbourhood of } a
\]

then \( U_a(p) \) is uniquely determined and let \( U_a(p)^+ = 0 \) if \( U_a(p) \leq 0 \), and \( U_a(p) = U_a(p), \) if \( U_a(p) > 0 \), then for any points \( a \) and \( p \)

\[
U^+_a(p) - d - U^+_a(0) \leq U^+_0(p), \quad U^+_0(p) \leq U^+_a(p) + U^+_a(o) + d
\]

where \( d \) depends on \( 0 \) and \( p \) only.

Proof. We denote by \( D_0 \) the domain such that \( 0 \geq U_0(p) \) and \( C_0 \) the set such that \( U_0(p) = 0 \) and \( D_a, C_a \) respectively.

Case 1. \( a \in D_0 \). \( V(p) = U_a(p) - U_a(0) \) is harmonic in \( R - D_0 \) for fixed \( a \), then it takes its maximum in \( R - D_0 \), when \( p \) lies on \( C_0 \), where \( U_a(p) = U_a(p) - U_a(0) \), for varying \( a \). Let \( U_a(p_0) = U_a(p_0(a)) = d = \max \): \( a_0 \in C_0, p_0 \in C_0 \). We denote by \( D^+_a \) the domain such as \( U_a(p) - d - U_a \leq 0 \) and by \( C^+_a \) its boundary, then it is clear \( D^+_a \supset D_0 \), and \( U_a(p) - d - U_a(0) \leq 0 \), \( U^+_0(p) \geq 0 \) when \( p \in R \).

Case 2. \( a \in D_0 \). If \( p \in D_0 \), then \( \max U_a(p) - U_a(0) \) attains its maximum \( d \) when \( p, a \in C_0 \) then

\[
U_a(p) - d = U^+_a(p) - d - U^+_a(0) \leq U^+_0(p).
\]

In the case when \( p \in D_0 \), we can prove similarly, the latter part of the lemma can be proved in the same way.

Let \( R \) be a covering surface over \( R \) and of positive boundary Riemann surface, and \( f(z): p \in R; z \in R \) be the mapping function and \( G_n(z, 0) \) be the Green's function with its pole at 0 with respect to \( R_n \) and \( \bigcup R = R \), and by \( h_n(z, 0) \) be its conjugate. Put \( x_n = e^{-G_n(z, 0)} = re^{i\eta}, \lim x_n^n = x \).

In denoting by \( a_i, b_i \) the point where \( f(a_i) = 0, f(b_i) = q \) respectively then

\[
U_0(f(z)) = \frac{1}{2\pi} \int_0^{2\pi} U_0(f(re^{it})) dt + \sum \frac{G(z, b_i)}{g(z, a_i)} - \sum \frac{G(z, a_i)}{g(z, b_i)} + \log |c_k|,
\]

where \( c_k \) is the first non vanishing coefficient of the expansion of \( f(z) \) with respect to the local parameter defined in the neighbourhood of 0. Put

\[
m(r, f-a) = \frac{2\pi}{\int_0^{2\pi} U_0(f(re^{it})) dt}
\]

\[
N(r, f-a) = \int_0^r n(q, a) - m(0, a) dt
\]

from the lemma \(|m(r, f-a) - m(r, q)| \leq U_0(a) + d\)

\[
m(r, a) + N(r, a) = m(r, q) + N(r, q) + \varphi(r)
\]

where \(|\varphi(r)| \leq U_0(a) + d + |\log |c_k||, \text{ if}\]

\[
T(r, q) = \lim_{R \to R_0} (N(r, p) + m(r, q)) < \infty
\]

we call this mapping bounded type covering.

**Theorem 1.2.** If \( p = f(z) \) is bounded type, we map \( R^n \) onto \( |z| < 1 \) conformally, \( \zeta = \varphi^{-1}(z) \) \( 0 = \varphi(0), z = \varphi(\zeta) \), then \( p = f(\varphi(\zeta)) = F(\zeta) \) has angular limits almost everywhere on \( |\zeta| = 1 \).

Proof. We can discuss the potential theory on \( R \), if \( \mu \) is a unit mass distribution on the positive capacity set \( E \) of \( R \), then we have

\[
T(r) = \int N(r, a) \mu(a) + O(1), \text{ we easily see that } T(r) \text{ is finite only and if}
\]

only when \( N(r, a) \) is finite everywhere. A system of equivalent points \( a_i \) of \( \zeta \)-plane \( (j = 1, 2, ...) \) corresponds to a point \( a_\zeta \in R \) and \( G(z, a_i) \geq \sum G_\zeta(z, a_i) \), where \( G_\zeta(z, a_i) \) is the Green's function of \( |\zeta| < r \) with the poles at \( a_i \), we have \( A \geq N(r, a_i) \geq G_\zeta(z, a_i) \) and a small circle in the neighbourhood of \( z = 0 \) in the \( x \)-plane and the neighbourhood of 0 in \( R \) corresponds conformally each other, therefore in ring domain \( \rho < |x| < \rho_2 \), \( N(r, a_i) \leq A \) for any point \( a_i \) in the ring. Hence \( F(\zeta) \to x \) is bounded type on \( |\zeta| < 1 \), by Fatou's theorem \( F(\zeta) \) has angular limits almost everywhere on \( |\zeta| = 1 \), on the \( x \)-plane, let \( l \) be an asymptotic path ending at \( \zeta_0; |\zeta_0| = 1 \) along which \( F(\zeta) \) converges to \( x_0 \) and the Green's function of \( R \) tends to zero, then it is easy to see that a curve \( L \) on \( R \) corresponds to \( l \), determines an A.B.P. of \( R \) relative to \( R_\zeta \).
Corollary. Let $\hat{R}$ be a covering surface over $R$ and $R$ is a bounded type covering over $\hat{R}$, then $\hat{R}$ is a bounded covering over $R$, too.

Proof. A system $a^i$ corresponds to a point $a_i$, which lie on $\alpha$, and $\sum G^\hat{R}(p, a^i) \leq G^R(a_i)$, then we easily have $T(\hat{R}) \leq T(R)$, where $G^\hat{R}, G^R$ are Green's function of $\hat{R}$ and $R$ respectively.

Chapter II. Accessible boundary point

1. Measures of accessible boundary points

Let $\hat{R}$ be a null-boundary Riemann surface with $A$-topology and $R$ be a covering surface over $\hat{R}$. When a curve $l$ converging to the boundary of $\hat{R}$ and its projection on $R$ converges to a point $p \in R^*$, then we say that $l$ determines an accessible boundary point (A. B. P.) relative to $\beta^*$. We consider non negative continuous super-harmonic functions such that $0 \leq v(p) \leq 1$, and $v(p) = 1$ when $p$ tends to an A.B.P. along every curve, we denote by $\mu(\hat{R}, \mathfrak{U})$ the lower envelope of $\{v(p)\}$ being harmonic on account of Perron-Brelot's theorem, and define $\mu(R^\infty, \mathfrak{U})$ similarly. Denoting by $\tilde{R}$ the projection of $\hat{R}$ over $R^*$, if the universal covering surface of $\tilde{R}$ is parabolic type, i.e., it cannot be mapped conformally onto the unit-circle, we take off a finite number of points $p_1, p_2, \ldots, p_n$ (when $R'$ is closed and its genus is zero, three points, and when $R'$ is closed and its genus is one, one point) and remove from $\tilde{R}$ all the points $p_{ij}; j = 1, 2, \ldots$, lying over them and denote the remaining surface by $\tilde{R}$ so that the universal covering surface of $(\tilde{R}' - \sum p_{ij})$ may be hyperbolic. As the preceding we define $\mu(\tilde{R}, \mathfrak{U})$, and $\mu(R^\infty, \mathfrak{U})$. In the following we assume that $R$ has at least one accessible point.

1. $R$ is a null-boundary Riemann surface, then $\tilde{R}$ is also too, since no bounded harmonic function exists on a null-boundary Riemann surface, $\mu(R, \mathfrak{U}) = \mu(\tilde{R}, \mathfrak{U}) = 1$. On the other hand if $R^\infty$ is parabolic, we see that $\mu(R^\infty, \mathfrak{U}) = 1$. If $R^\infty$ is hyperbolic, we also on account of theorem 1.1. that any curve determining an A. B. P. converges to the set on the periphery of the unit-circle, of linear measure zero, thus we have

6) See M. Ohtsuka, 1).
\[ \mu(R^\infty, \mathcal{U}) = \mu(\tilde{R}^\infty, \mathcal{U}) = 0. \]

2. \( R \) is a positive boundary Riemann surface, put \( R - \tilde{R} = \sum_j \sum_i p_{ij} \) and let \( G(p, p_{ij}) \) be the Green's function on \( R \) with its pole at \( p_{ij} \). For arbitrary point \( p_0 \in R \), the function \( g(p) = \sum \frac{G(p, p_{ij})}{(ij + i)^2} G(p_0, p_{ij}) \) represents a harmonic function tending to \( \infty \) at any points of \( p_{ij} \), accordingly for any positive number \( \varepsilon > 0 \) and \( v(p) \), one of the class defining \( \mu(R, \mathcal{U}) \), \( \min (1, v(p) + \varepsilon g(p)) \) is one of super-harmonic function defining \( \mu(R, \mathcal{U}) \), this follows \( \mu(R, \mathcal{U}) \geq \mu(\tilde{R}, \mathcal{U}) \). Conversely any \( v(p) \) on \( R \) can be considered as one of \( v(p) \) of \( \tilde{R} \) if the value 1 is supplement to \( \sum p_{ij} \). Hence \( \mu(\tilde{R}, \mathcal{U}) \geq \mu(R, \mathcal{U}) \) and the equality follows. Further there holds \( \mu(R, \mathcal{U}) \geq \mu(R^\infty, \mathcal{U}) \), because any \( v(p) \) of \( R \) is considered on \( R^\infty \). Since for any \( v(p) \) of \( R^\infty \) and \( \varepsilon > 0 \), \( \min (1, v(p) + \varepsilon g(p)) \) can be considered on \( \tilde{R}^\infty \), then we have \( \mu(R^\infty, \mathcal{U}) \geq \mu(\tilde{R}^\infty, \mathcal{U}) \).

**Theorem 2.1.**

\[ \mu(R, \mathcal{U}) = \mu(\tilde{R}, \mathcal{U}) \geq \mu(R^\infty, \mathcal{U}) \geq \mu(\tilde{R}^\infty, \mathcal{U}) \]

As in the case when \( R \) is a null-boundary Riemann surface, these relations are settled, in the sequel we suppose that \( R \) is a positive boundary Riemann surface and investigate when equality or inequality hold among these quantities.

We suppose \( R' \) the projection of \( R \) on \( R^* \) is a null-boundary Riemann surface and \( R^\infty \) is hyperbolic type, i.e., it can be mapped conformally onto the unit-circle. We map \( R^\infty \) onto \( \{\zeta \mid |\zeta| < 1\} \), then from Theorem 1.1. on \( \{\zeta \mid |\zeta| = 1\} \) the set of points at which at least a curve converging to the boundary of \( R' \) ends, is a zero linear measure set and if we map \( R^\infty \) onto \( \{|z| < 1\} \). On \( \{|z| = 1\} \), the set \( E_I \) of points at which at least a curve determining an A.B.P. lying on the ideal boundary of \( R' \) is also zero measure.

Let \( z_0 \) be a point at which a curve \( l \) (the image of \( L \) on \( R \)) determining an A.B.P. having its projection on \( R' \), then we can take a certain branch of \( f(z) \) which converges to a point \( \xi_0 \), \( |\xi_0| < 1 \) along \( l \), then \( f(z) \) has an angular limit \( \xi_0 \) also, this implies that any branch of \( f(z) \) has angular limit relative to \( R' \). Let us fix a certain branch \( f(z) \) mapping corresponding to \( z \rightarrow E' \) and denote by \( E \) the set at which a curve ends along which \( f(z) \) determines an A.B.P. with projection on \( R' \).

Let \( R_\lambda \) be an exhaustion of \( R : \bigcup R_\lambda = R \) and denote by \( \Delta_{y_0}(\theta) \), See M. Ohtsuka, 2).
the set \( \frac{1}{n} \leq |z - \theta| < \frac{1}{m} \), \( 0 < |\arg(z - \theta)| < \frac{\pi}{2} \) and by \( \delta(f(z)) : z \in \Delta_{m,n}(\theta) \) the diameter of the set \( (f(z) : z \in \Delta_{m,n}(\theta)) \) with respect to A-topology of \( R^* \), then
\[
E = \frac{1}{\kappa} \left[ I \sum \sum \sum (\delta f(\Delta_{m,n,l}(\theta))) \leq \frac{1}{\kappa}, f(\Delta_{m,n,l}) < R_\kappa \right],
\]
because \( \delta(f(\Delta_{m,n,l}(\theta))) \) is continuous with respect to \( \theta \), for fixed \( \lambda, l, m, n \) and \( \kappa \), \( E \) is a Borel set on \( |z| = 1 \).

Let \( \mu(R^\omega, \mathcal{A}) \) be the lower envelope of the super-harmonic function \( |v(z)| \) such as \( 0 \leq v(z) \leq 1 \), and \( \text{im} v(z) = 1 \) along every path determining an A.B.P. relative to \( R \) and \( \mu(R^\omega, \mathcal{A}(E)) \), \( \mu(R^\omega, \mathcal{A}(E)) \) be the lower envelope of the super-harmonic functions such that \( \lim v(p) = 1 \) along every path determining an A.B.P. lying on the boundary of \( R \), and on \( R \) respectively. We have \( \mu(R^\omega, \mathcal{A}(E)) \leq \mu(R^\omega, \mathcal{A}) \) by Löwner's lemma and moreover \( \mu(R^\omega, \mathcal{A}) \) is zero as the solution of the Dirichlet problem on the \( \xi \)-plane, thus we have
\[
\mu(R^\omega, \mathcal{A}(E)) \leq \mu(R^\omega, \mathcal{A}) \leq \mu(R^\omega, \mathcal{A}(E)) + \mu(R^\omega, \mathcal{A}(E)) = \mu(R^\omega, \mathcal{A}(E)).
\]
On the other hand we denote by \( \omega(E) \) the harmonic measure of measurable set \( E \) then, since \( v(R^\omega, \mathcal{A}) = 1 \), when \( z \) converges radially to \( E \), then we have easily the next equality \( \mu(R^\omega, \mathcal{A}) = \omega(z, E) \).

**Theorem 2.2.** When \( R \) is a covering surface over a null-boundary Riemann surface \( R \) if the universal covering surface of \( R \) is hyperbolic type then \( \mu(R^\omega, \mathcal{A}) = \omega(z, E) \).

**Theorem 2.3.** Let \( R \) be a null-boundary Riemann surface with A-topology and \( R \) be a positive boundary Riemann surface given as a covering surface over \( \bar{R} \), if the universal covering surface of the projection of \( R \) is parabolic type, i.e. cannot be mapped onto the unit-circle, take a finite number of points \( p_i \) \( (i = 1, 2, \ldots, n) \) and remove from \( R \) all the points \( (p_{ij}) \) \( (j = 1, 2, 3, \ldots) \) lying over \( p_i \) so that the projection of the remaining surface \( \hat{R} \) may has an universal covering surface of hyperbolic type. Let \( G(p, p_{ij}) \) be the Green's function of \( R \) with the pole at \( p_{ij} \). In the case when \( R \) covers \( p_i \) \( (i = 1, 2, \ldots, n) \) so rarey that \( \sum \sum G(p, p_{ij}) < \infty \), if we map \( \hat{R}^\omega \) onto \( |\eta| < 1 \) conformally, then we have
\[
\mu(R^\omega, \mathcal{A}) = \mu(\hat{R}^\omega, \mathcal{A}).
\]

---

8) This theorem is proved by M. Ohtsuka 2) under the condition that \( R \) covers \( p_i \) a finite number of times.
Proof. We map $R^\infty$ onto $|z|<1$, then a system $z_{i,k}$ $(k = 1, 2, \ldots)$ corresponds to a point $p_{i,k}$ and $\sum_j G(p, p_{i,k}) = \sum_k \sum_j \sum_i \log \left| \frac{1 - z_{i,k}z}{z_{i,k} - z} \right| < \infty$ is equivalent to $\sum_k (1 - |z_{i,k}|) < \infty$.

a) A curve $l$ in $|z|<1$ being the image of the curve $L$ on $R$ not passing through $p_{i,k}$ and determining an A.B.P. not lying on $p_i (i = 1, 2, \ldots, n)$ converges to a point on $|z|=1$.

We map $(R - \sum_{i,k} p_i)^\infty$ onto $|\xi|<1$ if $l$ does not converge to a point, then converges to an arc $\gamma'$; where $\mes \gamma' > 0$. Since the mapping $f(z) \to R$ is one-valued function, there exists a branch $\xi = f(z)$ tending to a point $\xi_0 : |\xi_0|<1$ corresponding to $p_0$ along $l$. Let us take off radial segments $S_{i,k}$ from $z_{i,k}$ to the periphery $|z|=1$ and denote it by $(U_z - \sum_{i,k} S_{i,k})$ being simply connected domain, we map it onto $|\xi|<1 : z = \varphi(\xi)$. Since the boundary of $(U_z - \sum_{i,k} S_{i,k})$ is rectifiable, $\gamma$ is transformed onto a positive measure set $\gamma''$ on which $\xi = f(\varphi(\xi))$ has angular limits on the $\xi$-plane almost everywhere on $\gamma''$, therefore there exists a positive measure set $\gamma'''$ on which $\xi = f(\varphi(\xi))$ has angular limit $\xi_0 : |\xi_0|<1$, because the radius ending at $\varphi'''$ is transformed to a curve in $|z|<1$, intersecting $l$ infinitely many times on which $f(z)$ tends to $\xi_0$, this is a contradiction.

b) The set $E_z$ on $|z|=1$ at which least an image of the curve not passing through $p_i$ and determining A.B.P.s not lying on $p_i$ ends is linearly measurable$^9$.

In fact we map $R^\infty$ onto $|\eta|<1$, and denote by $E_{\eta}$ the set at which least a curve determining an A.B.P. ends. Let $L$ be the above mentioned curve on $R$, since $L$ does not pass through $p_{i,k}, L$ is contained in $R$ and the image $l_\eta$ in $\eta$-plane of $L$ converges. Since the projection of $R$ is $(R - \sum_{i,k} p_i), \xi = \xi(\eta)$ has angular limits relative to $R$ at every point of $E_{\eta}$, then $E_{\eta}$ is a Borel set. As $R^\infty$ can be considered as the universal covering surface of the domain $(\text{Unit-circle} - \sum_{i,k} z_{i,k})$ and $l_\eta$ corresponds one-to-one manner to a system $l_\eta$, then $E_{\eta}$ can be thought to be the image of $E_{l_\eta}$ of the function $z = z(\eta)$, then $E_{\eta}$ is an analytic set. Next, let $l_\eta$ be a curve in $R^\infty$ determining an A.B.P. lying on $p_i (i = 1, 2, \ldots, n)$ and $E_{l_\eta}$ be the set at least such a $l_\eta$ converges, then it is easy to see that $l_\eta$ converges and its outer measure of the set where at least a $l_\eta$ converges is zero in regarding that the mapping $z \to \xi$ and Löwner's lemma.

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9) See M. Ohtsuka, 2).
c) Let $E_z$ be the measurable set of positive measure on $|z| = 1$, and $E_\eta$ be the set on $|\eta| = 1$, where $z = z(\eta)$ has angular limits contained in $E_z$, then $E_\eta$ is a set of inner measure positive.

Let $E'_z$ be a closed set of positive measure of $E_z$ where $\eta = \eta(\xi)$ tends uniformly, then the set on $|\eta| = 1$ where $z = z(\eta)$ has angular limits contained in $E'_z$, is also measurable denoted by $E_\eta'$. On the other hand $(U - \sum \mathcal{S}_i)$ has a rectifiable curve as its boundary then $E_\eta'$ on $|\xi| = 1$ of positive measure and consider the composed function $E_\eta'$ is transformed onto the set $\xi = \eta(z(\xi)}$ in $U - \sum \mathcal{S}_i$ and let $E''_\eta$ be the image of $E''_\eta$ then $E''_\eta \subset E_\eta$, in mapping $z = z(\eta)$: $|\eta| < 1$. We denote any super-harmonic function $\nu(\eta)$ such as $0 \leq \nu(\eta) \leq 1$, we consider this function on $|\xi| < 1$ then clearly $\nu(\eta) \geq \omega(E''_\eta)$, where $\omega(E''_\eta)$ is the harmonic measure of $E''_\eta$ with respect to $|\xi| = 1$, this follows that $\text{mes } E_\eta > 0$.

Proof of the theorem. $\mu(R^\infty, \mathcal{S}) \leq \mu(R^\infty, E_z)$ implies that $\mu(R^\infty, \mathcal{S})$ has angular limits zero almost everywhere on the complementary set of $E_z$, on the other hand $\mu(\tilde{R}^\infty, \mathcal{S}) = \mu(\tilde{R}^\infty, E_\eta) = \omega(E_\eta, \eta)$ and $\mu(\tilde{R}^\infty, E_\eta)$ is one-valued harmonic function in $\tilde{R}$, this is also harmonic in $R$, there $\mu(\tilde{R}^\infty, E_\eta)$ is one-valued in $|z| < 1$ and has angular limits almost everywhere on $|z| = 1$. In denoting by $E_{1, \pi}$ the subset of $E_z$ being measurable where $\mu(\tilde{R}^\infty, E_\eta)$ has angular limits less than 1 and denoting by $E_{1, n}$ the image of $E_{1, \pi}$, then $\mu(\tilde{R}^\infty, E_{1, n}) = 0$ follows in connection c) that $\mu(\tilde{R}^\infty, E_{1, n}) = 0$. Thus we have $\mu(\tilde{R}^\infty, E_\eta) \geq \mu(\tilde{R}^\infty, E_\eta) \geq \mu(R, \mathcal{S})$, because $\mu(\tilde{R}^\infty, E_\eta)$ is the lower envelope of super-harmonic functions having limits 1 along any curve ending at $E_z$. Since any super-harmonic determining $\mu(R^\infty, \mathcal{S})$ can be considered as the function on $\tilde{R}^\infty$ if we supplement the value 1 on $z_{i\xi}$ then we have $\mu(\tilde{R}^\infty, \mathcal{S}) \leq \mu(R^\infty, \mathcal{S})$. Finally we have

$$\mu(\tilde{R}^\infty, E_\eta) \geq \mu(\tilde{R}^\infty, E_z) \geq \mu(R^\infty, \mathcal{S}) \geq \mu(\tilde{R}^\infty, \mathcal{S}) \geq \mu(\tilde{R}^\infty, E_\eta).$$

2. On the behaviour of Green's function in the neighbourhood of the ideal boundary

Theorem 2.1. Let $G(p, p_0)$ be the Green's function of $R$ with its pole at $p_0$, $D_\lambda$ be the closed domain such as $G(p, p_0) \geq \lambda$ where $\lambda > 0$ and $R_\lambda$ be an exhaustion of $R$. We denote by $B_\lambda$ the boundary elements defined by $\lim (R - R_\lambda) \cap D_\lambda$, then $B_\lambda$ is outer harmonic measure zero.

Proof. Let $v_n$ be continuous super-harmonic function in $R$ such as $0 \leq v_n(p) \leq 1$, and $v(p) = 1$ if $p \in D_\lambda \cap (R - R_\lambda)$ and its lower envelope
of \( v^*(p) \) be denoted by \( \mu^*(p) \), since \( \frac{1}{\lambda} \min [\lambda, G(p, p_0)] \) is one of such functions, and \( \mu^*(p) \leq \mu^*_{n+1}(p) \) and \( \lim_n \mu^*(p) = \mu^*(p) \) is harmonic in \( R \).

\( G(p, p_0) - \lambda \mu^*(p) \) is harmonic and positive in \( R_n \) and has a logarithmic singularity \( p_0 \) then

\[
G(p, p_0) - \lambda \mu^*(p) \geq G(p, p_0)_{R_n}
\]

where \( G(p, p_0)_{R_n} \) is the Green's function of the surface \( R_n \), let \( n \to \infty \)

\[
G(p, p_0) - \lambda \mu^*(p) \geq G(p, p_0),
\]

because \( G(p, p_0) \) is the lower envelope of positive harmonic function with the same singularity as \( G(p, p_0) - \lambda \mu^*(p) \), this follows that \( \lim \mu^*(p) = 0 \).

We extract a subsequence \( v^*_{n+1}(p), v^*_n(p) \ldots \) of which the lower envelope is \( \mu^*(p) \), thus we have the theorem.

**Corollary.** Let \( \mu^*_{n+1}(p) \) be the lower envelope of non negative super-harmonic function \( v^*_{n+1}(p) \) in \( D_n \) such as \( 0 \leq v^*_{n+1}(p) \leq 1 \), and \( \lim_{P \to R_n} v^*_{n+1}(p) = 1 \), then \( \mu^*(p) = 0 \).

In fact, we can consider \( v^*_{n+1}(p) \) is one of \( v^*_{n+1}(p) \), then we have the conclusion.

**Corollary.** There exists a continuous positive super-harmonic function satisfying the following conditions 1) \( 0 \leq U(p) < \infty \) in \( R \), 2) \( \lim_{P \to R} U(p) = \infty \).

Proof. Let \( B = \sum_{n=2}^{\infty} B(n) \), since for every \( \lambda \), \( B(n) \) is outer harmonic measure zero, then we can extract a sequence of continuous superharmonic function \( U_1^{n}(p) \) such that \( \max (U_1^{n}(p)) \leq \frac{1}{2^n} \), if \( p \in R_n \), \( U_1^{n}(p) = 1 \) if \( p \in R_{n'} \) and \( p \in D_1(n) \), where \( n' = n'(n) \). Put \( U(z) = \sum U_1^{n}(p) \), then \( U(p) \leq n + \frac{1}{2^n} \) if \( p \in R_n \) and uniformly convergent in every \( R_n \) then \( U(p) \) is continuous and superharmonic in \( R \) and \( \lim_{P \to R} U(p) = \infty \).

We consider the case when \( \mu(R, \mathcal{U}) \) is equal to \( \mu(\mathcal{R}^\infty, \mathcal{U}) \).

**Theorem 2.5.** Let \( R \) be a positive boundary Riemann surface and \( \mathcal{R} \) be the projection of \( R \) over \( \mathcal{R} \), we take off points \( p_i \) (\( i = 1, 2, \ldots, n \)) so that the universal covering surface of \( (R - \sum p_i) \) is hyperbolic type, and denote by \( G(p, p_{i+1}) \) the Green's function with its pole at \( p_{i+1} \) lying over \( p_i \).

When \( R^\infty \) is mapped conformally on \( |z| < 1 \), we denote by \( D_0 \) the normal polygon of Fuchsian Group containing \( z = 0 \), with arcs \( \alpha_j \) (\( j = 1, 2, \ldots \)) on \( |z| = 1 \) and by \( \{T_j\} \) (\( i = 1, 2, \ldots \)) the substitutions of Fuchsian group, if
\[
\sum_j \sum_i G(p, p_{ij}) < \infty, \quad \text{and} \\
\text{mes}\left(\sum_i T_i \left(\sum_j \alpha^j\right)\right) = 2\pi, \quad \text{then} \\
\mu(R, \mathcal{U}) = \mu(\tilde{R}, \mathcal{U}) = \mu(\tilde{R}^\infty, \mathcal{U}) = \mu(R^\infty, \mathcal{U}).
\]

Proof. We denote the set by \(E\) where at least a curve determining an A.B.P ends, we have seen that \(E\) is measurable and \(\mu(R^\infty, \mathcal{U}) = \mu(\tilde{R}, \mathcal{U}) = o(z, E)\). For any given positive number \(\varepsilon > 0\), there exists a number \(\rho\) and \(k\) such that

\[
\text{mes}\left(\sum_j T_i \left(\sum \alpha^j\right)\right) > 2\pi - \varepsilon, \quad \frac{1}{k} \leq \frac{\text{mes} T_i(l)}{\text{mes} l} < k: i = 1, 2, \ldots, p
\]

for any line element \(l\) on \(\alpha_i\), we can extract subarcs denoted by \(\alpha^j\) having no common ends with \(\alpha_i\) such that \(\text{mes}\left(\sum_j \alpha^j - \sum_j \alpha^j\right) < \frac{\varepsilon}{2kp}\), then we have \(\text{mes}\left(\sum_i T_i \left(\sum_j \alpha^j\right) - \sum_i T_i \left(\sum_j \alpha^j\right)\right) < 2\varepsilon\), if \(\mu(R, R) \geq \mu(R^\infty, \mathcal{U})\) we have \(\mu(\tilde{R}, 0) > \mu(\tilde{R}^\infty, 0) + \delta: \delta > 0\), since \(\mu(\tilde{R}, \mathcal{U})\) is one-valued harmonic function in \(|z| < 1\), denoting by \(H_{\frac{1}{n}}\) the set where \(\mu(\tilde{R}, \mathcal{U})\) has angular limits larger than \(\frac{1}{n}\), then \(\text{mes}\left(\frac{(\pi - E) \cap H_{\frac{1}{n}} \cap \sum_i T_i \left(\sum \alpha^j\right)\right} > \delta\) and \(\text{mes}\left(\left(\pi - E\right) \cap \sum_i T_i \left(\sum \alpha^j\right)\right) > \delta - \varepsilon\).

Let (complement of \(E\)) \(\cap \alpha_i^j = \alpha_i^j\), and we can take a closed subset denoted by \(F^\infty_n\) in \(\sum \alpha_i^j\) such that \(\sum_i T_i \left(F^\infty_n\right) > \frac{\delta}{2}\). On the other hand let \(\omega_{F^\infty_n}(z)\) be the harmonic measure of \(\sum_i T_i \left(F^\infty_n\right)\), then \(\omega_{F^\infty_n}(0) \geq \frac{\delta}{2}\) and one-valued on \(R\). We consider non negative super-harmonic function \(v^*(p)\) in \(\tilde{R}\) such that \(\lim v^*(p) = 1\) when \(p\) tends to \(F^\infty_n\) along every curve and denote by \(\mu^*(\tilde{R}, p)\) their lower envelope, then we have \(\mu^*(\tilde{R}, \mathcal{U}) \geq \omega_{F^\infty_n}(p)\). Since \(F^\infty_n\) is a set on arcs \(\alpha_1 \cdots \alpha_2\), the behaviour of \(\mu^*(\tilde{R}, \mathcal{U})\) is the same as that of Green’s function i.e., if \(\lim G(p, p_0) = 0\) when \(p\) tends along every curve converging out of \(F^\infty_n\) then \(\lim \mu^*(R, p) = 0\) along these curves. But we have seen in corollary of theorem 2.4 that there exists a non negative continuous super-harmonic function \(V(p)\) such that \(\lim V(p) = \infty\), where \(B = \sum B_\lambda: G(p, p_0) > \lambda\). Let \(v(p)\) be of the class defining \(\mu(\tilde{R}, \mathcal{U})\) and consider next function

\[
S(p) = \text{Min}(\varepsilon V(p) + v(p) - \frac{1}{n} o(p, 1)).
\]

case 1. \(p\) tends to an A.B.P. then this curve ends out side of \(\sum T(F^\infty_n)\).
we have to consider in $D_0$, since $F_n^\infty$ is closed then $\omega_{F_n^\infty}(p) = 0$ thus $S(p) = 1$.

case 2. $p$ tends to $F_n^\infty$ on the $z$-plane, since $v(p) \geq \frac{1}{n}$ along Stolz’s path converging to $F_n^\infty$, $S(p) \geq 0$.

case 3. $p$ tends to the boundary of $R$, except $F_n^\infty$, then $\varepsilon V(p) - \frac{1}{n} \omega_{F_n^\infty}(p) \geq 0$. Hence $S(p)$ has angular limits $\geq 0$ almost everywhere on $|z| = 1$ and bounded, therefore $S(p) \geq 0$ in $R$, finally $S(p)$ is a one of super-harmonic function but $\varepsilon$ is arbitrary we have, the lower envelope of $S(p) = \mu(R, \mathcal{U}) - \frac{1}{n} \omega_{F_n^\infty}(p) \leq \mu(R, \mathcal{U})$. This is a contradiction, then we have $\mu(\tilde{R}, \mathcal{U}) = \mu(\tilde{R}^\infty, \mathcal{U})$ any by the preceding theorems we have the conclusion.

**Corollary.** The connectivity of $R$ is finite and universal covering surface of the projection of $R$ on $\mathbb{R}$ is hyperbolic then we have $\mu(R, \mathcal{U}) = \mu(R^\infty, \mathcal{U})$.

**Theorem 2.6.** Let $R$ be a positive boundary Riemann surface and the universal covering surface of the projection of $R$ over $\mathbb{R}$ be hyperbolic type. If $\mu(R, \mathcal{U}) = \mu(R^\infty, \mathcal{U})$ then we have $\mu(R, \mathcal{F}) = \mu(R^\infty, \mathcal{F})$ for every closed subset $\mathcal{F}$ of $\mathcal{U}$.

We denote by $\mathcal{F}_n$ the set of of $R + \mathcal{U}$ such as all points of $R$ with distance $\leq \frac{1}{n}$ from $\mathcal{F}$, then $\mathcal{F} = \bigcap \mathcal{F}_n$ and denote by $F_n$ the image of $\mathcal{U} \cap \mathcal{F}_n$ on $|z| = 1$: $(|z| < 1) \equiv R^\infty$, then $F_n$ is measurable. Since the image $\mathcal{U}$ on $|z| = 1$ with its projection lying on the boundary of $R$ has outer zero measure where the mapping function has not necessarily angular limits, we can suppose the projectionof $\mathcal{U}$ lies on $R$. Let $R_m$ be the exhaustion of $R$, $\Gamma_m$ be its relative boundary, and $\alpha_n$ be the relative boundary of $\mathcal{F}_n$, we define

$$
\omega^m_{m+n}(p): \quad (i = 1, 2, \ldots) \text{ satisfying the following conditions,} \\
\omega^m_{m+n}(p) \text{ is non negative continuous super-harmonic function in } R_{m+n} \\
\omega^m_{m+n}(p) = 1, \quad \text{if } p \in \mathcal{F}_n \cap (R_{m+n} - R_m), \quad \omega^m_{m+n}(p) \text{ is harmonic in } R_{m+n} - (\mathcal{F}_n \cap (R_{m+n} - R_m)), \quad \omega^m_{m+n}(p) = 0 \quad \text{if } p \in \Gamma_m - (\Gamma_{m+n} \cap \mathcal{F}_n).
$$

$$
\omega^m_{m+n}(p) \leq \omega^m_{m+n+i}(p). \quad \text{ Let } \omega^m_n(p) \text{ be the lower envelope of } \omega^m_{m+n+i}: \quad i = 1, 2, \ldots \quad \omega^m_n(p) \text{ is super-harmonic therefore } \mu(R, \mathcal{F}_n) \leq \omega^m_n(p), \quad \mu(R, \mathcal{F}) < \lim \omega^m_n(p) \quad \text{for every } n.
$$

If $\mu(R, \mathcal{F}) = \mu(R^\infty, \mathcal{F})$, we would have $\mu(\mathcal{U}_z \cap C\mathcal{F}_z) > 0$, where $\mathcal{U}$ means the image of $\mathcal{U}$ and $C\mathcal{F}$ means the complement of the image $\mathcal{F}$ with respect to $|z| = 1$. 


Let $f(z)$ be the mapping function from the $z$-plane onto $R + \mathcal{R}$, $D_{\delta,F_i}$ the domain in $|z|<1$ such that it contains the endpart of the angular domain: $|\arg (1-e^{-i\theta}z)|<\frac{\pi}{2}-\delta$ $(\delta>0)$ at every point $e^{i\theta}$ of closed set $F_i$ and $C_{\nu'}$ be the ring domain $1-\delta'<|z|<1$.

Since A.B.P corresponding to $\mathcal{A} \cap C_{\beta}$ which has positive distance from ones corresponding to $\mathcal{A}$, then for given positive number $\varepsilon > 0$ and $\delta$, there exists a closed subset $F_i$ of $\mathcal{A}_z \cap C_{\beta \alpha}$ such that $\text{mes} (C_{\beta \alpha} - F_i) < \varepsilon$, and $\text{dist} (f(z), C_{\beta \alpha}) \geq \frac{1}{2n}$ if $z \in D_{\delta,F_i} \cap C_{\gamma'}$. The boundary $\Gamma_n$ of $R_m$ is transformed onto a simple closed Jordan curve $\gamma_m$ in $|z|<1$. Denote by $r_m$ the distance $\gamma_m$ from $z = 0$, and let $m_0$ be the smallest number such as $r_{m_0} > 1-\delta$, then for $\Gamma_n : (m \geq m_0)$, the image of $\Gamma_n$ is contained in $C_{\nu'}$ and the part of the image contained in $D_{\delta,F_i}$ are the arcs where $\omega^m_{m+1} : i = 1, 2, ...$ takes value 0 except a set of linear measure zero.

Let $C'_{r_m}$ be the circle $|z|<r_m$, and $D_{\delta,F_i}^*$ be the set $D_{\delta,F_i} \cup C'_{r_m}$, and $U_m(z)$ be the harmonic function in $D_{\delta,F_i}^*$ such that $0 \leq U_m(z) \leq 1$, $U_m(z) = 1$ when $z$ lies on the boundary of $D_{\delta,F_i}^*$ except on the boundary on $|z|=1$ and vanishes on the boundary lying on $|z|=1$. Since the boundary of $D_{\delta,F_i}^*$ is rectifiable $U_m(z) \leq 1$ and $U_m(z) = 0$ when $z$ tends to $F_i$ almost everywhere. It is clear that $U_m(z) \geq \omega_{m+1}^m(z) \geq \mu(R, \mathcal{F})$ if $z \in D_{\delta,F_i} \cap R_m^\alpha$ for every $i$ and $n$, but $\varepsilon$ and $\delta$ are arbitrary, therefore $\mu(R, \mathcal{F})$ has angular limits 0 almost everywhere on $\mathcal{A}_z \cap C_{\beta \alpha}$. On the other hand $\mu(R, \mathcal{A}) > \mu(R, \mathcal{F})$, it follows $\mu(R, \mathcal{F})$ has angular limits zero almost everywhere on $C_{\beta \alpha}$, thus $\mu(R, \mathcal{F})$ has angular limit zero almost everywhere on $C_{\beta \alpha}$ and so $\mu(R, \mathcal{F}) = \mu(R, \mathcal{F})$.

**Theorem 2.7.** Suppose that $R^\alpha$ is hyperbolic type. Take a finite number of $\{p_i\}$ from $R$ and remove from $R$ all the points lying over them and denote the remaining surface by $\widetilde{R}$, then there holds

$$\mu(R^\alpha, \mathcal{A}) = \mu(\widetilde{R}^\alpha, \mathcal{A}).$$

This is proved by M. Ohtsuka in regarding that the mapping function from $R^\alpha$ onto $\widetilde{R}$ has angular limits and the set where at least a curve determining A.B.P. lying on $\{p_i\}$ is outer harmonic measure zero.

Till now we have investigated the case when harmonic measures have equal values, in the following we show by example there are cases when inequalities hold.
3. Counter examples of the preceding theorems

Example. Let \( B_{2n}, B_{2n+1} : n = 1, 2, \ldots \) be the system closed domains in \(|z| < 1\) such as

\[
B_{2n} : 1 - \frac{1}{4n+3} \leq r \leq 1 - \frac{1}{4n+4}, \quad \frac{3\pi}{4} \leq \theta \leq \frac{\pi}{4} \quad \text{(containing } -\frac{\pi}{2})
\]

\[
B_{2n+1} : 1 - \frac{1}{4n} \leq r \leq 1 - \frac{1}{4n+1}, \quad -\frac{3\pi}{4} \geq \theta \geq -\frac{5\pi}{4} \quad \text{(containing } \frac{\pi}{2})
\]

We can construct the holomorphic function\(^{10}\) \( f(z) : |z| < 1 \) by Runge’s theorem such that

\[
|f(z) - 1| < \frac{1}{n} \quad \text{if } z \in B_{2n}, \quad |f(z)| < \frac{1}{n} \quad \text{if } z \in B_{2n+1}.
\]

It is clear that \( f(z) \) is not bounded in \(|z| < 1\) and \( w = f(z) = \infty \) is an asymptotic value, therefore there exists an asymptotic path \( l \) along which \( f(z) \) tends to infinity, this path determines an A.B.P. lying on \( w = \infty \), let \( p_0 \) be the starting point on \( l \) where \(|f(z)| = M_0\) and this path lies in the unit-circle except \( \sum_{n=0}^{\infty} (B_{2n} + B_{2n+1}) \) and denote by \( p_r \) the first point when \( p \) lies on \(|z| = r\) in converging to the boundary \(|z| = 1\) from \( p_r \) and by \( l_r \) the endpoint from \( p_r \).

Let \( \{v(z)\} \) be non negative continuous super-harmonic function in \(|z| < 1\) such as \( 0 \leq v(z) \leq 1 \) and \( \lim v(z) = 1 \) when \( z \) tends to \(|z| = 1\) along \( l \), take a \( v(z) \), then there is \( r_0 \) such that \( v(z) \geq 1 - \delta \), if \( z \in l_{r_0} \) for every \( \delta \) where \( l_{r_0} \) has a branch passing left half plane and let \( n_0 \) be the number such as \( 1 - \frac{1}{4n_0+3} \geq r_0 \), and \( A \) and \( E \) be the point \( \left( r = 1 - \frac{1}{4n_0+1}, \theta = \frac{5\pi}{4}\right) \). \( B = \left( r = \frac{1}{4n_0+3}, \theta = \frac{3\pi}{4}\right) = E \). \( C = \left( r = 1 - \frac{1}{4n_0+3}, \theta = \frac{5\pi}{4}\right) \) and \( D = \left( r = 1 - \frac{1}{4n_0+3}, \theta = 0\right) \) and \( U(z) \) be a harmonic function such as \( U(z) = 1 \), if \( z \in AB \) and on the circular arc \( BC \), and vanishes on the circular arc \( CDE \), or on the segment \( EF \) then \( V(0) \geq U(0) = \delta_0 > 0 \), for every \( v(z) \).\(^{11}\) Hence \( \mu(R, w = \infty) \) being the lower envelope of \( \{v(z)\} \) is \( \geq \delta_0 > 0 \).

We remove all points where \( f(z) = 0, 1 \) and \( 2 \) from the unit-circle, then the universal covering surface of the projection of the remaining surface \( R' \) is hyperbolic type then it is clear \( \mu(R', \infty) > 0 \) and \( \mu(R'^{\infty}, \infty) = \mu(\tilde{R}^{\infty}, \infty) = 0 \). Hence by theorem 2.6. \( \mu(R', \mathbb{H}) \geq \mu(R'^{\infty}, \mathbb{H}) \).

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10) L. Bieberbach: Funktionentheorie II.
11) When \( l_{r_0} \) has right half plane, similar fact occurs.
1) If we consider the $R'$ to be given surface $R$, then $\mu(R, A) \equiv \mu(R', A)$. 

2) If we consider $R$ given surface $R'' = R$, then $\mu(R'', A) \geq \mu(\tilde{R}'', A)$. 

We easily see that these surfaces do not satisfy the conditions of preceding theorems.

4. Type of covering surfaces

1) If there holds $u(p) \leq 0$ on $R$ whenever $u(p)$ is upper bounded continuous sub-harmonic function on $R$ such that $\lim_{P \to A} u(p) \leq 0$, we call $R$ a surface of $D$-type.

2) Let $R''$ be hyperbolic hyperbolic type, when the function corresponding to the mapping onto the unit circle $R'' \to R \to R^*$ has limits along Stolz's path almost everywhere on $|z| = 1$, where $R'' \equiv (|z| < 1)$.

We call in this case $R$ $F$-type covering.

3) If $T(r)$ of the mapping function from $R$ to $\overline{R}$ is bounded, we say $R$ is bounded covering, we have easily

$$D\text{-type} \rightarrow \mu(R, A) = 1 \rightarrow B\text{-type} \rightarrow F\text{-type} \rightarrow Z\text{-type},$$

and if the universal covering surface of the projection of $R$ is hyperbolic, we have $F\text{-type} \rightarrow \mu(R'', A) = 1$.

In the sequel we investigate type of covering surface among covering surfaces over the same basic surface. We have at once from theorem 1.2. the next

**Theorem 2.8.** When $\hat{R}$ is a covering surface over $R$ and $R$ is of bounded type over $R$, then $\hat{R}$ is of bounded type over $R$, too.

**Theorem 2.9.** When the universal covering surface of the projection of $R$ over $\overline{R}$ is hyperbolic type, if $R''$ is $F$-type and if $\hat{R}$ is a covering surface over $R$, then $\hat{R}$ is also $F$-type relative to $R$.

**Proof.** We map $R'', R''', \hat{R}''$, onto $|\xi| < 1$, $|z| < 1$ and $|\zeta| < 1$ and denote by $E_{1\xi}, E_{1\zeta}$ and $E_{1\zeta}$ the set where at least a curve determining an A.B.P. lying on the boundary of $R$, since $E_{1\xi}, E_{1\zeta}, E_{1\zeta}$ are of outer harmonic measure zero, we may suppose without loss of generality that A.B.P's lie on $R$. Let $\mathbb{A}_\xi, \mathbb{A}_\zeta, \mathbb{A}_\zeta$ the set being measurable where the mapping function onto $\overline{R}$ has angular limits on $\overline{R}$, then $\mu(R'', A) = 1$. If $\text{mes}(C\mathbb{A}_\zeta) > 0$, where $C\mathbb{A}_\zeta$ means the complementary set of $\mathbb{A}_\zeta$ and $z = z(\zeta)$, $0 = z(0)$, $\xi(z(\zeta))$ be the mapping function onto the $z$-plane and $\xi$-plane respectively, they have angular limits almost every-
where on \(|z|<1\) and \(\xi\)-plane. Denote by \(l_\xi\) the radial path ending at \(C\mathfrak{M}_\xi\) along which \(z = z(\xi)\). \(\xi = \xi(\zeta)\) has angular limits on \(z, \xi\)-planes.

\(z = z(\xi)\) does not converge to \(z_0: |z_0| < 1, \) because if it were so, \(l_\xi\) determines an A.B.P. on \(R\), therefore \(z = z(\xi)\) converges to \(z_0: |z_0| = 1\) and further the image \(l_\xi\) of \(l_\xi\) does not intersect infinitely many times with the angular endpart \(\arg |z - e^{i\theta}| < \frac{\pi}{2} - \delta: \delta > 0, e^{i\theta} \in \mathfrak{N}_z\), because if \(l_\xi\) intersects it infinitely, since \(\xi = \xi(z)\) tends in this angular domain to a point \(\xi_0: |\xi_0| < 1\) with regardings that \(\xi = \xi(\zeta) = \xi(z(\xi))\) has limit \(\xi_0: |\xi_0| < 1\) on the \(\xi\)-plane, it follows that \(\xi = \xi(z)\) determines an A.B.P along \(l_\xi\), this is a contradiction, thus \(z = z(\xi)\) has no angular limit or converges to \(z : z \in \mathfrak{N}_z\) or converges tangentially to \(z : z \in \mathfrak{N}_z\).

For given numbers \(\varepsilon, \delta\) there exist numbers \(\varepsilon', \delta'\) such that \(z = z(\zeta)\) is not contained in \(D_{\xi, \delta, F,z} \cap \mathfrak{N}_z\) if \(\zeta \in D_{\xi, \varepsilon', F'} \cap \mathfrak{N}_z\), where \(F, F'\) are closed sets of \(\mathfrak{N}_z, \mathfrak{M}_\xi\) and mes \(|\mathfrak{N}_z - F| < \varepsilon, \) mes \(|\mathfrak{M}_\xi - F'| < \varepsilon', \)

\(D_{\zeta, \delta, F,z}, D_{\xi, \delta, F'}\) are angular domains on \(z, \zeta\)-plane arg \(\arg |1-\zeta e^{-i\theta}| < \frac{\pi}{2} - \delta\) for any point of \(e^{i\theta} \in F,\) arg \(\arg |1-\xi e^{-i\theta}| < \frac{\pi}{2} - \delta'\) \(e^{i\theta} \in F',\) respectively, \(\mathfrak{M}_{\delta, \zeta}, C_\beta, \tilde{C}_\beta\) are the rings \(1-\delta < |z| < 1, 1-\delta' < |\zeta| < 1\) and a circle \(1-\delta > |z|, \) \(\lim \varepsilon = 0, \lim \delta = 0.\)

Let \(U_{\delta, \varepsilon}(z)\) be the harmonic function in \(D_{\zeta, \delta, F,z} \cup \mathfrak{N}_z\) such as \(U_{\delta, \varepsilon}(z) = 0\) on the boundary of \(D_{\zeta, \delta, F,z} \cup \mathfrak{N}_z\) except its boundary on \(|z| = 1\) and 1 on the boundary lying on \(|z| = 1\), and \(\omega(\zeta)\) be the harmonic measure of \(F',\) then we have \(1-\omega(\zeta) > U_{\delta, \varepsilon}(z)\) when \(U_{\delta, \varepsilon}(z)\) is considered on \(\zeta\)-plane, since \(\mathfrak{N}_z, C_{\mathfrak{M}_\xi}\) are measurable and \(\varepsilon, \delta\) are arbitrary and \(\lim \varepsilon = 0, \lim \delta = 0.\)

This implies the conclusion.

Corollary. We can weaken the conditions of the preceding theorem in such a way the condition that the universal covering surface of the projection \(R\) is hyperbolic is changed by the condition that \(\sum_{i,j} G(p, p_\ast) < \infty,\) where \(p_\ast\) are points of \(R\) lying on \(p_i\) of \(R,\) so that the universal covering surface of \((R - \Sigma p_i)\) is hyperbolic.

Corollary. Let \(R\) be a positive boundary Riemann surface and covers \(p_i\) so rarely as \(\sum G(p, p_\ast) < \infty,\) then \(\mu(R^\infty, \mathfrak{N}) \leq \mu(R^\infty, \mathfrak{N})\)

Proof. We denote by \(p_\ast, \delta; (k = 1, 2, \ldots)\) the points lying on \(p_i,\) then we have
where $G(p, p_{ij})$ are Green's function of $R$ and $\hat{R}$. Let $\tilde{R}$ and $\check{R}$ be the remaining surface after removing $p_{ij}$ and $p_{kl}$ from $\hat{R}$ or $R$ then we have from theorem 4.4.

$$\mu(\hat{R}^{\infty}, \mathcal{A}) = \mu(\check{R}^{\infty}, \mathcal{A}) \geq \mu(\tilde{R}^{\infty}, \mathcal{A}) = \mu(R^{\infty}, \mathcal{A}).$$

**Example.** A covering surface over a D-type covering surface is not always D-type. In the preceding theorem we take as $B_{2n}$ and $B_{2n+1}$ the domains shown in the figure 1.

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**5. Dirichlet problem**

**5.1. Dirichlet problem on a covering surface over a null-boundary Riemann surface**

**Lemma.** Let $\hat{R}$ be a null-boundary Riemann surface and $R$ be a $F$-type covering surface over $\hat{R}$ and $\mathcal{A}$ be a closed set of $\mathcal{A}$, the upper class $\bar{U}^{\infty}_{\hat{R}}$ is defined by all non negative continuous super-harmonic functions such that $\lim_{p \to \mathcal{A}} u(p) = 1$, then the lower envelope $\bar{R}^{\infty}_{\mathcal{A}}(p)$ is harmonic. Similarly the lower class $\bar{B}_{\hat{R}}$ and $\bar{H}^{R}_{\mathcal{A}}(p)$ are defined for upper bounded continuous sub-harmonic function such that $\lim_{p \to \mathcal{A}} v(p) \leq 0$. Further it is clear that
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Lemma. Let $\mathcal{F}$ be a closed subset of $\mathcal{A}$ of $F$-type covering surface then there holds

$$\tilde{H}_n^R(p) = H_n^R(p).$$

Proof. Mapping $R^n$ onto $|z|<1$, since the set $E$, the image of the A.B.P.s lying on the boundary of $R$ has outer harmonic measure zero i.e. $\mu(R, \mathcal{A} \cap E) = 0$. We may suppose that A.B.Ps lie on $R$. Let $\mathcal{F}_n$ be the set of $R+\mathcal{A}$ which has distance from $\mathcal{F} \leq \frac{1}{n}$ and $F_n$ be the image of $\mathcal{F}_n \cap \mathcal{A}$ on $|z|=1$, then $\mathcal{F} = \bigcap_n \mathcal{F}_n$ and $F = \bigcap_n F_n$ and they are measurable.

Let $R_m$ be an exhaustion of $R$ with relative boundary $\Gamma_m$ and $\gamma_n$ be the boundary of $\mathcal{F}_n$ and $\omega_{m+\epsilon}(p)$ be the non negative continuous super-harmonic function in $R$ such that $\omega_{m+\epsilon} \leq 1$ and harmonic in $(R_m, (\mathcal{F}_n \cap (R_{m+1} - R_m)))$ and 1, if $p \in \mathcal{F}_n \cap (R - R_m)$ and vanishes on $C_{\mathcal{F}_n} \cap \Gamma_{m+\epsilon}$, $\omega_{m+\epsilon}(p)$ is increasing with respect to $\epsilon$ and $\omega_{m+\epsilon}(\lim_{\epsilon \to 0} \omega_{m+\epsilon}(p)$ is contained in $U_n$ class and harmonic in $R_m$ and $\omega_{m+\epsilon}(p)$ is decreasing with respect to $m$. $\lim_{n \to \infty} \omega_{m}(p)$ is harmonic in $R - F_n$, then $\tilde{H}_n^R(p) = H_n^R(p)$ for every $n$. On the other hand at any point of $\mathcal{A} \cap CF_n$, $f(z)$ does not tend to $F_n$ in angular domain, because $\text{dist}(\mathcal{F}_n \cap R_n, \mathcal{F}_n) \geq \frac{1}{2n}$.

Let $D_{\epsilon, \delta}$ be the domain with $\text{arg} |ze^{-i\theta}| < \frac{\pi}{2} - \delta$, $\epsilon \theta \in \kappa$, $r_{m+\epsilon} = \text{dist}(\gamma_{m+\epsilon})$ in $|z|<z=0$, $\mathcal{C}_{r_{m}}$ be the ring $r_{m}<|z|<1$, and $\mathcal{C}_{r_{m}}$ be open circle $|z|<r'_{m}$. For any given number $\epsilon > 0$, $\delta > 0$ we can find a closed subset $F_{2n}^\epsilon$ of $\mathcal{A}$ and $\mathcal{A} \cap CF_{2n}^\epsilon$, such that, $\text{mes}(\mathcal{A} - F_{2n}^\epsilon) < \epsilon$, if $z \in D_{\mathcal{P}_{2n}^\epsilon \cap \mathcal{C}_{r_{m}}}$, then $f(z)$ is contained in the complement of $F_n$.

Let $U_n(z)$ be harmonic in $DF_{2n}^\epsilon \cup \mathcal{P}_{r_{m}}$ and $\leq 1$ and 1 on this domain not lying on $|z|=1$ and vanishes on the boundary on $|z|=1$, then

$$U_n \geq m_{m+\epsilon}$$ and harmonic in $DF_{2n}^\epsilon \cup \mathcal{P}_{r_{m}}$,

$U_n \geq m_{m+\epsilon}$ for every $\epsilon$, there holds $U_n \geq m_{m}$.

Since $\epsilon$, $\delta$ and $n$ are arbitrary, and the boundary of $DF_{2n}^\epsilon \cup \mathcal{P}_{r_{m}}$ is rectifiable $\lim \omega_{m}(p) = \lim m_{m+\epsilon}(p) \geq \lim \omega_{m}(p) \geq \lim \omega_{m+\epsilon}(p) = \tilde{H}_n^R(p)$ and the converse is true, then we have

$$\tilde{H}_n^R(p) = \omega(z, F).$$

Let $\Omega_n = \mathcal{A} - \mathcal{F}_n$ and $\omega_{m+\epsilon}$ be super-harmonic in $R$ such that
$0 \leq \omega_{m+m+\lambda}(p) \leq 1$ and 1 when $p \in \Omega_n \cap R-R_{m}$ and $\gamma_m \cap \Omega_n$ and vanishes on $\Gamma_{m+\lambda} \cap \Omega_n$, and harmonic in $R_{m+\lambda} \cap \Omega_n$. \lim \omega_{m+m+\lambda}(p) \darrow 0$ when $p \in \Omega_n \cap CO_n$, where $O_n$ is the image of $\Omega_n$. Denote by $J_n$ the set in $\Omega_n$ where $\bar{H}_\lambda^\infty(p) \geq \lambda$ and part $T^{\lambda,\infty}_m$ = $(R-R_{m}) \cap J_n$ and $\omega_{m+m+\lambda}(p)$ being super-harmonic in $R$ and harmonic in $R_{m+\lambda} \cap T^{\lambda,\infty}_m$ and attains 1 when $p \in T^{\lambda,\infty}_m$ and vanishes if $p \in \Gamma_{m} \cap T^{\lambda,\infty}_m$, clearly we have

$$\lim (\lim \omega_{m+m+\lambda}(p)) = \omega_{m+m+\lambda}(p).$$

Put $\omega_{m,m+\lambda}(p) = \min \left( \omega_{m+m+\lambda}, \frac{1}{\lambda} \bar{H}_\lambda^\infty(p) \right)$ this is contained in the upper class for $T^{\lambda,\infty}_m \cap \Omega_n$, but $\lim \omega_{m,m+\lambda}(p)$ and its lower envelope has angular limits 0 almost everywhere on $F_2$ and on $\partial \Omega \cap O_1$. Let $n \to \infty$, then this envelope has zero almost everywhere on $z = 1$. This follows that $\lim T^{\lambda,\infty}_n \cap \Omega_n$ has outer harmonic measure zero. Therefore there exists a non negative continuous super-harmonic function $U(p)$ such that $U(p) = \infty$ when $p$ tends to $T^{\lambda,\infty}_n$ for every $\lambda > 0$ and $n$. Since $\max (\bar{H}_\lambda^\infty(p)-\varepsilon U(p), 0)$ has limits zero when $p$ tends to $\partial \Omega \cap O_1$ this is contained in $\mathcal{B}_\lambda$ class, thus

$$\omega(R^{\lambda,\infty}_n, F) = \bar{H}_\lambda^\infty(p) \leq H_{\lambda,\infty}(p).$$

Similarly for open set $\Omega$ of $\Omega$, we have $\bar{H}^\infty_\lambda(p) = \bar{H}^\infty_\lambda(p)$.

Let $\varphi(p) : p \in \Omega$ be a real valued function on $\Omega$ (admitting $\pm \infty$). Define the upper class $U^\varphi(p)$ consisting of all the lower bounded continuous super-harmonic function such that $\lim u(p) \geq \varphi(p)$ and its lower envelope $\bar{H}_\varphi^\infty(p)$, and the lower class and $H_\varphi^\infty(p)$ for sub-harmonic function similarly. If $\bar{H}_\varphi^\infty(p) = H_\varphi^\infty(p)$ holds then $\varphi$ will be called a resolutive boundary function and common envelope will be denoted by $H_\varphi^\infty(p)$.

**Theorem 2.10.** Let $R$ be F-type and $\varphi$ be semi-continuous function, then $\varphi$ is resolutive.

In fact, let $\varphi$ be bounded upper semi-continuous function on $\Omega$ and $M \geq \varphi \geq m$, and divides $[m, M]$ such that $m = c_0 < c, \ldots, c_n = M$, $c_{i+1} - c_i = \frac{M-m}{n}$ and the set $E[\varphi \geq c_i]$ is denoted by $A_i$ being closed and $E_i = E[\varphi_i \geq \varphi]$. Let $U_{A_i}$ be upper $\mathcal{B}_{A_i}$ and lower class of characteristic function of $A_i$. Put $U_{E_i} = U_{A_i} - V_{A_{i+1}}(V_{E_i} = V_{A_i} - U_{A_{i+1}})$, where $U_{A_i}(V_{A_i})$ is a function contained in $U_{A_i}(\mathcal{B}_{A_i})$, then $U_{E_i}(p)(V_{E_i}(p))$
is supper (sub)-harmonic and \( \lim_{P \to E_i} U(P) \geq 1 \) (\( \lim_{P \to C E_i \cap \mathfrak{A}} V(P) \leq 0 \)) thus \( U_{E_i}(p) \) \( (V_{E_i}(p)) \) is contained in \( U_{A_i}(\mathfrak{B}_{A_i}) \) and \( U^\circ(p) = \sum_{i=0}^{n-1} c_i U_{E_i}(p) \) \( (V^\circ(p) = \sum_{i=1}^{n} c_i V_{E_i}(p)) \) is contained in \( U^\circ(\mathfrak{B}^\circ) \) but the lower envelope \( \bar{H}_{E_i}(p) \) \( (H_{E_i}(p)) \) of \( U^\circ(V^\circ) \) is equal to the harmonic measure of \( E_i \) in \( R^\circ : |z| < 1 \) then we have,
\[
\lim_{n} (U^\circ(p) - V^\circ(p)) \leq \frac{1}{n} \sum_{i=0}^{n-1} \omega_{E_i}(p) \leq \frac{1}{n}, \quad \text{let} \ n \to \infty
\]
\[
H^u_\psi(p) = H^u_\psi(p) = \int_\mathfrak{A} \psi \, d\mu
\]
where \( \mu \) is harmonic measure.

From general theory of Dirichlet problem, we have next

**Lemma.** \( H^u_\psi(p) \) is the upper envelope of \( H^u_\phi(p) \), where \( \psi < \phi \) and \( \psi \) is upper bounded and semi-continuous on \( \mathfrak{A} \), the similar fact holds for \( H^u_\phi(p) \).

In the same manner used by M. Brelot, we have

**Theorem 2.11.** In order that \( \phi \) is resolutive it is necessary and sufficient that \( \phi \) is integrable in the narrow sense.

**Remark 1.** We easily see \( H^u_\psi(p) \) is the same as the solution of R. Nevanlinna

**Remark 2.** This method is also applicable for \( \phi(p) \) on \( \mathfrak{A} \) even if \( \mu(\tilde{R}^\circ, \mathfrak{A}) = 1 \), if we restrict \( \phi \geq 0 \) and \( H^u_\phi \) is equal to \( \int_\mathfrak{A} \phi \, d\mu \)

**Remark 3.** We have seen in the proof of Lemma \( H^u_\phi(p) \) takes value zero or 1 almost everywhere on \( |z| = 1 \) according to the point is contained in the image of \( \mathfrak{B} \) or not more generally the solution \( H^u_\psi \) for continuous boundary function \( \phi \) takes value \( \phi \) almost everywhere on the image of \( \mathfrak{A} \). If we consider \( H^u_\phi(p) \) in \( R \), this fact means that \( H^u_\phi(p) \) takes the given \( \phi \) "almost" paths determining A.B.P.s so that the image of these paths form measure \( 2\pi \) set on \( |z| = 1 \). This follows the resolutivity of \( \phi \). But we dont know if \( H^u_\phi(p) \) tends to \( \phi \) along every curve ending at any point \( \mathfrak{A} - \mathfrak{A}_0 \), where \( \mathfrak{A}_0 \) is a subset of \( \mathfrak{A} \) being so small that its harmonic measure zero, this fact has stronger sense than the former.

On the other hand it is well known, for Dirichlet problem of the domain in the \( z \)-plane for continuous boundary value \( \phi \), there exists a subset \( I \) being \( F_\phi \) of the boundary of capacity zero called irregular set determined uniquely for the domain such that the solution takes the given value except \( I \). In general case when \( R \) is \( F \)-type Riemann
surface the problem whether there exists or not the negligible subset with the property determined by the domain only and not depending on the boundary function \( \varphi \) is open.

5.2. **Dirichlet problem on a covering surface over a positive boundary Riemann surface.**

Let \( R \) be a positive boundary Riemann surface and \( p_0 \) be an inner point of \( R \), and \( G(p, p_0) \) be Green's function of \( R \) and \( h(p, p_0) \) be its conjugate. Put \( \kappa(p) = \frac{e^{-\varphi}}{\varphi} \) and we define the length of a curve \( L \) as \[ \int_L dk(p) \] and distance between two points \( q_1, q_2 \in R \) be the lower limit of the length of all curves \( L \) connecting \( q_1, q_2 \) in \( R \) and by completion we have metric space \( R^* \), and the distance on a covering surface of two points \( q_1, q_2 \) is defined by the diameter of the projection on \( R^* \) of all curves connecting \( q_1, q_2 \) in \( R \). If a curve \( l \) on \( R \) tends to the boundary \( R \) and its projection converges to a point on \( R^* \) we say, \( l \) determines an A. B. P. this means that the mapping function \( f(z) \rightarrow R \rightarrow -k(p) = w, f(z) \) has limit \( w_0 \) on the \( w \)-plane along the image \( l \) on the \( z \)-plane with the image \( l \), since \( |\kappa(p)| < 1 \) the images \( l \) tends to equivalent points of Fuchsian group on \( |z| = 1 \) and at these points \( k(p) = k(z) \) has an angular limit \( w_0 \) i.e., \( l \) determines the same A. B. P. lying on \( R^* \). As \( f(z) \) has angular limits almost everywhere on \( |z| = 1 \) this follows that \( R \) is \( F \)-type.

Let \( \mathcal{U}_z \) be the image of A. B. P.s on \( |z| = 1 \) and \( F_z \) be the image of closed subset \( \mathcal{F} \) of \( \mathcal{U} \), then we have

\[
\mathcal{U}_z = E_{\theta} \left[ \prod_{i} \prod_{k} \sum_{n} \prod_{m} \left[ \delta(f(\Delta_{m,n})\theta) \leq \frac{1}{k} \right] \right],
\]

\[
F_z = E_{\theta} \left[ \prod_{i} \prod_{k} \sum_{n} \prod_{m} \left[ d(f(\Delta_{m,n}), \mathcal{F}) \leq \frac{1}{k} \right] \right],
\]

where \( \Delta_{m,n} \) the set \( 0 \leq |\arg(1-e^{-i\theta}z)| \leq \frac{\pi}{2} - \frac{1}{l}, \frac{1}{n} \leq |z-\theta| < \frac{1}{m} \), \( \delta(f(\Delta)) \) is the diameter of \( f(\Delta) \) with respect to the topology of \( R^* \) and \( d(\Delta, \mathcal{F}) \) is the distance from \( \mathcal{F} \) with respect to the topology of \( R \), then we have the same result about Dirichlet problem as in the case when \( R \) is \( F \)-type Riemann surface over a null-boundary Riemann surface \( R \).

6. **Finitely sheeted Riemann surface**

Let \( R \) be a null-boundary Riemann surface with \( A \)-topology and \( R \) be a covering surface over \( R \). When a Jordan curve \( L \) on \( R \) converging to the boundary of \( R \) and its projection tends to a point \( p_0 \) of \( R^* \), we
say, $L$ determines an A.B.P. Let $V_n$ be the neighbourhood of $p_0$ of the projection of the A.B.P. with diameter $\frac{1}{n}$ and $\mathcal{B}_n$ be the set of $R$ lying on $V_n$, this composed of at most enumerable number of domains.

Associated domain $D_n$ of $L$ will be a domain containing an end-part of $L$ in $\mathcal{B}_n$. Two arcs $L_1, L_2$ determine the same A.B.P if and only if for any number $n$ two associated domains are the same. This definition of A.B.P. is equivalent with that of O. Teichmüller.

We denote by $n(p) : p \in R$ the number of times when $p$ is covered by $R$, then it is clear that $n(p)$ is lower semi-continuous. We call $R$ boundedly sheeted covering surface if $n(p) \leq M : p \in R$. In this paragraph we consider chiefly such a Riemann surface. When $\sup_{p \in R} n(p) = 1$, non accessible point of $R$ have no sense in itself, hence we consider only accessible boundary points.

**Barrier.** We call $B(p) : p \in R$, Barrier of $p$ the function such that $B(p)$ is non negative bounded continuous super-harmonic function $\lim_{q \to p} B(q) = 0$ and for every associated domain $D_m$, there exists a number such as, $\underline{\lim_{q \to p}} B(q) > \delta_m$; if $p \in D_m$ it is well known that $p$ is regular for Dirichlet problem of $R$ if and only if at $p$ Barrier exists.

**Lemma.** Let $R$ be a F-type Riemann surface over $R$ and $p$ be an A.B.P and $D(p)$ be an associated domain of $p$, we denote by $\text{proj } D(p)$ the projection of $D(p)$, if $p$ is lacunary of $D$ and further $\text{proj } p$ is regular for the domain $\text{proj } D(p)$, then $p$ is regular for Dirichlet problem on $R$.

In fact, let $T(\text{proj } p)$ be Barrier of $p$ with respect to $\text{proj } D$, then there exists a number $\delta_m$ such that $\lim T(\text{proj } p) \geq \delta_m$, if $\text{proj } p \in \text{proj } D_m$. Put $B(p) = \min(\delta_m, T(\text{proj } p))$, this is clearly Barrier of $p$ with respect to $R$. We have at once

**Lemma.** $\mathcal{X}$ be lacunary set being clearly closed on $R$, the all A.B.P.s on $\mathcal{X}$ are regular for Dirichlet problem except the set having the projection of capacity zero which is $F_\infty$.

In the sequel, let $R$ be boundedly sheeted covering surface over $R$ such as $n(p) \leq M$. It is known that $R$ is a null- or positive boundary Riemann surface according to the set $E[n(p) \leq N-1]$ where $N = \lim n(p): p \in R$ being clearly closed, is a null-capacity set or not. We suppose $R$ be a positive boundary Riemann surface on account of theorem 1.2, $R$ is of $F$-type. In this class of Riemann surface the following propositions hold.

1) Any A.B.P is a direct singular point.
Let \( p \) be an A.B.P. Case 1. \( p \) lies on \( R \), if \( p \) is not direct singular point, then there exists connected pieces \( v_1, \ldots, v_k; k \leq N \) with common points with the associated domain \( D_m \), such that these connected pieces have inner points \( p_1, \ldots, p_k \) with the same projection as \( p \) for any large number \( m \). Take \( c_1 \cdots c_k \) discs contained in \( v_1 \cdots v_k \), then any one of \( c_i \) has no branch points converging to \( p_i \), hence there exists a number \( m_0 \) such that \( D_m \cap c_i : m \geq m_0 \) has a finite number of branch points, then we can find \( m_0 \geq m_0' \) such that any one of \( c_i \cap D_m (m \geq m_0) \) has no common points with the other \( c_j \cap D_m (j \neq i) \). For any \( m, D_m \cap c_i \) has non common points, this implies that \( D_m \cap c_i = c_i \cap D_m = c_i \) and \( p \) is an inner point of \( R \). This is a contradiction.

Case 2. If \( p \) lies on the boundary of \( R \) for this case our assertion in trivial.

The order of an A.B.P. and of the associated domain.

The number \( \lim_{n \to \infty} [\sup_{p \in \text{proj } D \cap D_m} n(p) : p \in D \cap D_m] \) is called the order of an A.B.P. and \( \lim_{n \to \infty} [\sup_{p \in \text{proj } D \cap D_m} n(p) : p \in D \cap D_m] \) is the order of a associated domain \( D_m \) respectively, we denote by \( \mathcal{F}_n \) the set of 21 of order \( n \), and by \( \mathcal{F}_n \) the subset of \( \mathcal{F}_n \) such that \( [\sup n(p) : p \in D \cap D_m] = n \) and its projection by \( \mathcal{F}_n \), then we have \( \mathcal{F}_n = \bigcup_{m \geq n} \mathcal{F}_m \), \( A_n = \bigcup_{m \geq n} \mathcal{F}_m \).

2) We can prove that \( \mathcal{F}_n \) is closed relative to \( \text{proj } \mathcal{F}_n \), if we denote by \( \mathcal{F}_n = \sum_{i \geq n} A_i \).

3) We easily see that if \( [\sup n(p) : p \in \text{proj } D_m] \leq n, \) and \( D_m \) contains A.B.P.s of \( \mathcal{F}_n \) then the projection of \( \mathcal{F}_n \) is lacunary.

Put \( \mathcal{F}_n = \sum_{i \geq n} A_n \) then \( \mathcal{F}_n \subset \mathcal{F}_{n-1} \cdots \subset \mathcal{F}_1 \).

Theorem 2.12. Let \( R \) be a boundedly sheeted covering surface over a null-boundary Riemann surface, then all accessible points are regular for Dirichlet problem except at most the subset \( \mathcal{N}_r \) of \( \mathcal{N} \) having its projection contained in the \( F_\sigma \) set of capacity zero.

a) Any point of \( \mathcal{N}_x \) is regular except at most \( \mathcal{N}_x \), of which the projection is capacity zero and \( F_\sigma \). Because \( \mathcal{N}_x \) is lacunary, if \( \text{Cap} (\text{proj } \mathcal{N}_x) > 0, \) \( \mathcal{N}_x \) is lacunary of \( R_x \), where \( R_x \) is the points of \( R \) covered by \( R, N \) times being open set, then \( A_x \) is regular for the domain \( R_x \), except at most capacity zero \( F_\sigma \) denoted by \( A_x' \), hence let \( p \) be of \( \mathcal{N}_x \) with its projection in \( A_x - A_x' \) is, clearly regular for \( R \), because \( R \) has more boundary than \( \mathcal{N}_x \).

b) We suppose all points of \( \sum_{i \geq n} \mathcal{N}_i \) on \( \mathcal{N}_n - \mathcal{N}_{n-1} \), where \( \text{Cap} (\mathcal{N}_{n-1}) \)
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\[ N = 0 \text{ and } F_{\sigma}, \] on which at least an A. B. P. of \( \Sigma \) being irregular point
lies. We shall prove that \( \sum_{t=n-p}^{N} \mathcal{A}_t \) on \( \mathcal{B}_{n-p-1} \) is regular except the set
of which the projection is capacity zero and \( F_{\sigma} \), we suppose \( \sum_{t=n-p}^{N} \mathcal{A}_t \) is
regular except the set on \( \mathcal{B}_{n-p} \), then we have only to prove almost all
points of \( \mathcal{A}_{n-p-1} \) is regular, since \( \mathcal{B}_{n-p} = \bigcup_{i} \mathcal{B}_{n-p}^i, \mathcal{B}_{n-p-1} = \bigcup_{i} \mathcal{B}_{n-p-1}^i \),
there exists a number \( m \) such that \( \text{Cap}(\mathcal{B}_{n-p-1}^i) > 0 \), we cover by at most
enumerable discs \( c_1, c_2, \ldots \) of diameter \( \leq \frac{1}{2m} \), then there is at least a
\( c_t \) such that \( \text{Cap}(c_t \cap \mathcal{B}_{n-p-1}^i) > 0 \). Let \( \nu_1, \nu_2 \ldots \) be associated domains
of points of \( \mathcal{A}_{n-p-1} \), we have only to consider next three cases.

Case 1. \( \text{ord}(\nu_t) \geq n-p \) and \( \text{ord}(\nu_t) = \sup n(p): p \in \text{proj} \nu_t \)
Case 2. \( \text{ord}(\nu_t) < n-p-1 \).
Case 3. \( \text{ord}(\nu_t) = n-p-1 \).

Case 1. \( \nu_t \) has no point of \( A_{n-p-1}^m \), case 2. \( \nu_t \) has no point of \( A_{n-p-1}^m \),
then we may regard the case 3 in this case \( \text{proj}(\nu_t \cap A_{n-p-1}) \) is lacunary
of the domain of \( \text{proj} \nu_t \), but in the case except at most capacity zero
of \( F_{\sigma} \) set all point of \( A_{n-p-1} \) is regular with respect to \( \text{proj}(\nu_t) \), hence
by lemma all points of \( \mathcal{A}_{n-p-1} \) is regular for \( R \) except the set with the
projection being \( F_{\sigma} \) and of capacity zero. Thus we have the conclusion.

**Theorem 2.13.** Let \( q \) be an A. B. P. with its projection on \( R \) and
\( G(p, p_0) \) be Green's function of \( R \), if \( \lim_{p \to q} G(p, p_0) = 0 \), then \( q \) is regular
for Dirichlet problem.

Proof. As we have proved, we can take \( V_m \) so that an A. B. P. may
be lacunary of \( D_m \). Without loss of generality we may suppose the projection
of the A. B. P. is \( q_0 \), let \( C_{p_0} \) be the circle of radius \( \rho_0 \) with
centre \( q_0 \) with respect to the local parameter defined in the neighbour-
hood of the projection of the A. B. P such as \( C_{p_0} \subset V_m \) and \( C_{p_0}^* \) be the
periphery of \( C_{p_0} \) and \( D_{p_0} \) be the associated domain of the A. B. P. lying
on \( C_{p_0} \). Denote by \( F_t \) the point of \( R \) covered \( i \) times by \( R \), then
\( F_{n} \subset F_{n-1} \subset \ldots \subset F_1 \). From the semi-continuity of \( n(p) \), \( F_t \) is open and
\( F_t \cap C_{p_0}^* \) is composed of at most enumerable number of intervals \( ^*I_j^*(i = 1, 2, \ldots N) \) \( (j = 1, 2, \ldots) \). Let \( s \) be a point of \( ^*I_j^* \) then there exists \( i \) points
\( p_1, p_2, \ldots, p_i \), where some of them may coincide. Put \( g(s_i) = G(s_i, p_0) \)
where \( G(s_1) \geq G(s_2) \geq \ldots \geq G(s_i) \). We can easily prove that \( g(s_i) \) is con-
tinuous in \( ^*I_j^*(j = 1, 2, \ldots) \) for fixed \( i \).

We choose a closed sub-set \( ^*e_j^* \) of \( ^*I_j^* \) such that \( \frac{1}{2\pi \rho_0} \) \( \min \left( \sum_{j}^{\infty} \rho_j e_j^* \right) \)
\( < 2\pi \rho_0 \) and denote by \( \kappa \) the minimum of \( g(s) \) in \( \sum_{j}^{\infty} e_j^*; \kappa > 0 \).
Introduce Poisson’s integral for $C_p$ with value $\rho_0$ on $\sum_{j}^\gamma \rho_j I_j - \sum_{j}^\gamma e_j I_j$ with $0$ on $C_p - \sum_{j}^\gamma e_j I_j + \sum_{j}^\gamma \rho_j I_j$

$$\rho(p) = \int \frac{1}{2\pi} \frac{\rho^2 - r^2}{\rho^2 - 2\rho r \cos(\theta - \varphi) + r^2} d\varphi,$$ then $\rho(q_0) \leq 2\pi \rho.$

Let $U(p)$ be the upper envelope of sub-harmonic function in $D_{p_0}$ such that $\lim V(p) \leq \overline{q_0 \rho_0}$ where $\overline{q_0 \rho_0}$ means euclidean distance between $q$ and $q_0$. Put $\alpha(p) = U(p) - \rho - \frac{\rho_0}{k} G(p, \rho_0) - \sum_{j}^\gamma \rho_j I_j(p) \leq 0$ because, at A. B. P. on $C_p$, $U(p) \leq \rho$ except negligible set, because $D_{p_0}$ is $F$-type covering, if $p \in R$ and $\text{proj} \ p \in \sum_{j}^\gamma e_j I_j$, $U(p) - \frac{\rho_0}{k} G(p) \leq 0$, if $p \in R$ and $\text{proj} \ p \in \sum_{j}^\gamma e_j I_j$, $U(p) - \frac{\rho_0}{k} G(p) \leq 0$, thus $U(p) \leq (2N + 1)\rho$ on the other hand since $s(p) = \text{proj} \ \overline{q_0 \rho_0}$ is sub-harmonic in $R$, $U(p) \geq \rho_0$, if $\text{proj} \ p \in C_p^\ast$. Put $\beta(p) = \text{Min} (\rho_0, U(p))$, then $\beta(p)$ is Barrier at the A. B. P. $q_0$.

In the case when the basic surfaces a positive boundary Riemann surface, if we introduce $C$-topology\textsuperscript{12} on $R$, the distance is defined as O. Teichmüller then in the same way all points are regular except the set having its projection being $F_\ast$ and negligible with respect to $R$.

\textbf{Remark 1.} When $R$ is boundedly sheeted covering surface the answer of the problem at the end of the preceding paragraph is affirmative.

\textbf{Remark 2.} There exists a harmonic function on $R$ which is $= \infty$ at all irregular accessible points lying on $R$ on account of G. C. Evans and M. Breiot, hence on such class of Riemann surface the resolutivity of continuous function is proved by the ordinary method as in the case when the domain is the subset of the $z$-plane i.e. without use of conformal mapping of the universal covering surface.

\textbf{Chapter III.} Martin’s\textsuperscript{13} topology and type of Riemann surface

\textbf{1. Singular point}

On a positive boundary Riemann surface $R$, the most precise topology for harmonic function is given by R. S. Martin. Let $a$ be a fixed point of $R$, and $K(p, q)$ be normalized Green’s function of its pole at $q$ defined for $q = a$, and $K(a, a) = 1$ when $\{r_t\}$ is a sequence of points tending

to the boundary, if \( K(r_i, a) \) determines a harmonic function on \( R \), then \( \{r_i\} \) is called a fundamental sequence of ideal points \( r_\infty \) corresponding to the limit function \( K(p, r_\infty) \). We denote by \( \Gamma \) all ideal points, then \( K(p, q) \) is defined for all points of \( R + \Gamma = \overline{R} \), \( K(p, q) \) is harmonic for \( p \) for fixed \( q \), and harmonic for \( q \) for fixed \( p \). The topology introduced by the distance

\[
\delta(r_1, r_2) \text{ of } r, r_2 \text{ by } \delta(r, r_2) = \sup_{r \in \overline{R}} \left| \frac{K(p, r_1)}{1 + K(p, r_1)} - \frac{K(p, r_2)}{1 + K(p, r_2)} \right|
\]

is called Martin's topology, the potential will be defined for the mass-distribution on \( \overline{R} \) is given by \( \int K(p, r) d\mu(r) \).

Activity. In the neighbourhood \( \nu(p) \) in Martin’s topology, if there exists an unit-mass-distribution \( \mu \) such that the potential by \( \mu \) is not bounded in the neighbourhood of \( \nu(p_2) \), then \( p_1 \) is called an active point to \( p_2 \).

**Theorem 3.1.** \( p_1 = \lim i p_i', p_2 = \lim j p_j' \) and \( \lim i G(p_i', a) > \delta_1 > 0 \), and \( \lim j G(p_j', a) > \delta_2 > 0 \), and if \( p_1 \) is active to \( p_2 \), then \( p_2 \) is active to \( p_1 \).

Proof. From the hypothesis there exists a sequence \( \{p_i'\} \) such that \( \lim K(p_i', p_j') = \infty \), then we can extract a subsequence such that

\[
K(p_i'^n, p_j'^n) = G(p_i'^n, p_j'^n) \geq n^2 \delta; \quad n = 1, 2, \ldots ; \quad \delta > 0
\]

then \( \sum \frac{1}{n^2} K(p_i'^n, p_j'^n) \geq \sum n^2 \delta \) is the potential required of the mass-distribution in \( \nu(p_2) \).

**Corollary.** If \( \lim i G(p_i', a) \geq \delta > 0 \), \( \lim j G(p_j', a) \geq \delta > 0 \), then the activity has symmetricity property.

If \( K(p, p_\infty) \) is bounded in \( R \), we shall call \( p_\infty \) a singular point.

Let \( K_{v_n}(p, p_\infty) \) be the lower envelope of super-harmonic functions \( u(p) \) such that \( u(p) \geq K(p, p_\infty) \) if \( p \in \overline{v_n(p_\infty)} \) then \( K_{v_n}(p, p_\infty) \geq K_{v_{n+1}}(p, p_\infty) \). Martin proved that \( \lim_{n} K_{v_n}(p, p_\infty) = K(p, p_\infty) \) or 0 and is called \( p_\infty \) a minimal point or not according to \( \lim_{n} K_{v_n}(p, p_\infty) = 0 \) or 0 respectively, and non minimal points set at most negligible (outer harmonic measure zero), and \( K(p, p_\infty) \) is minimal function when \( p_\infty \) is a minimal point.

**Theorem 3.2.** If \( p_\infty \) is inactive to almost all points of \( \Gamma - p_\infty \) (except

14) See 4).
at most negligible set) and moreover if \( K(p, p_\infty) \) is bounded, then \( p_\infty \) is singular minimal point.

Proof. For any given \( \varepsilon > 0 \), there exists a finite number of neighbourhoods \( v_i: \sum_{i=1}^{n_0} v_i \subset \Gamma - p_\infty \) such that the outer harmonic measure of \( (\sum v_i \cap \Gamma) < \varepsilon \) and in \( \sum v_i \), \( K(p, p_i) \leq M, p \in \sum v_i, p_i \in v_i \), then on almost all points of \( \sum v_i \cap \Gamma \) has limits value zero, but \( K(p, p_i) \leq M \) follows that \( K(a, a) = 0 \) if \( p_\infty \) is of harmonic measure zero since \( K(a, a) = 1 \), then \( p_\infty \) is positive harmonic measure thus \( p_\infty \) is a minimal point.

We shall show by an example the condition that \( p_\infty \) is inactive to almost all points of \( \Gamma - p_\infty \) is necessary.

\( \text{(Fig. 2)} \)

Example. Let \( R_n, R'_n, R''_n \) be rings in the unit-circle \(|z|<1\) such as
\[
1 - \frac{1}{2n} < |z| < 1 - \frac{1}{2n + 1}, \quad 1 - \frac{1}{2n + 1} < |z| < 1 - \frac{1}{2(n+1)},
\]
\[
1 - \frac{1}{2n + 1} < |z| < 1 - \frac{1}{2(n+1)}: \quad n = 1, 2, \ldots, \text{respectively}.
\]

Put \( p_n = 32e^\pi \frac{1}{M_n''} \) where \( M_n'' \) is the module of \( R''_n \) i.e.
\[
M'' = \log \left( 1 - \frac{1}{2(n+1)} \right)
\]
we map \( R_n \) onto the ring \( e^{-M_n} \leq |\zeta| \leq e^{M_n} \) where \( 2M_n \) is the module of \( R_n \), and \( H(\zeta) \) be a harmonic function such as \(|H(\zeta)|<p_n^{1+\delta_0}: e^{-M_n} \leq |\zeta| \leq e^{M_n}: \delta_0 > 0 \), then there exist a number \( N_n \) such that \(|\operatorname{Max} H(\zeta) - \operatorname{Min} H(\zeta)| \leq \frac{1}{n} \) where Maximum and Minimum mean the Max. and Min. on \(|\zeta| = 1 \) and \(|\arg \zeta| \leq \frac{2\pi}{2N_n}: N_n > N_n' \).

Let \( R_n \) be mapped onto the ring \( e^{-M_n} \leq |\zeta| \leq e^{M_n} \), and put \( m_n = \frac{M_n}{2(2N_n^2+1)} \), in ring \( R_n \) we denote by \( r_n^i, r_n'^i \) the rings such as
\[
e^{-M_n-m(i+2)} \leq |\zeta| \leq e^{-M_n-m(i+1)}, \quad e^{-M_n+m(i+1)} \leq |\zeta| \leq e^{-M_n+m(i+2)}
\]
In \( r_n^i, r_n'^i \) make systems of slits \( \{I_{ij}^i\}, \{I_{ij}'^i\}: i = 0, 2, 4, \ldots, 2N_n^2-2; \)
\( j = 1, 2, 3 \ldots 2N_n. \)
\[
I_{ij}^i: e^{-M_n-m(i+2)} \leq |\zeta| \leq e^{-M_n-m(i+1)}, \quad \theta = \frac{2\pi}{2N_n} j \quad j = 1, 2, 3, \ldots, 2N_n
\]
\[
I_{ij}'^i: e^{-M_n-m(i+2)} \leq |\zeta| \leq e^{-M_n-m(i+1)}, \quad \theta = \frac{2\pi}{2N_n} j
\]

15) See Chap. IV.
Where $N_n$ is\(^{16}\) the number to be defined in the following manner.

Let $T^1$ be the indirect conformal mapping $p \sim \tilde{p}$ in itself, where $\tilde{p}$ is the symmetric $\bar{p}$ with respect to arg $\xi = 0$, and $T^2$ is the indirect mapping $p \sim \tilde{p}$, where $\tilde{p}$ is the symmetric point of $p$ with respect to arg $\xi = \frac{\pi}{2}$, $-\frac{\pi}{2}$ and $T^3$, corresponds to $\frac{3}{4}$, $\frac{5}{4}$, $\frac{7}{4}$, and $T^4$ so on.

We cut $R_n$ by a cut lying on the real axis and identify the new boundaries lying symmetrically with respect to the imaginary axis and obtain two new surface $R^1_n$, $R^2_n$.

If $|U(p)| \leq p^{n+1}b''(b'' > 0)$, let $N_n \to \infty$, then $|U(p) - U(T_1(p))| \to 0$ and $|U(p) - U(T_2(p))| \to 0$ and we have $\frac{\partial U}{\partial n} \to 0$ on the real axis, because $U(p)$ is harmonic on real axis, thus by the\(^{17}\) boundary problem $U(p)$ converges to a harmonic function on the part of the new surface $R^1_n(R^2_n)$ lying on $e^{-M_n+\frac{1}{2}M_n} \leq |\xi| \leq e^{M_n-\frac{1}{2}m_n}$ and $|U(p) - U(T_2(p))| \to 0$ in this part, consequently there exists a number $N_n$ such that $U(p)$ is symmetric and periodic except at most $\frac{1}{n}$ with respect to $2N_n$ directions in $e^{-\frac{M_n}{2}} \leq |\xi| \leq e^{\frac{M_n}{2}}$.

We make slits $\{S^0\}$ in $R^1_n$, $S^n = z = re^{\theta}$ such as $\theta = 0$

\[ 1 - \frac{1}{2(n+1)} \geq r \geq -1 - \frac{1}{2n+1} - \frac{1}{n^2}. \]

We denote the image in $R_n$ of $\{I^i\}$ and $\{I^{i'}\}$ of $R_n$ by the same letter and $F_1$ and $F_2$ two unit-circles $|z| < 1$ with $\{I^i\}$, $\{I^{i'}\}$ and $\{S^0\}$. We shall get a Riemann surface $F$ composed of $F_1$ and $F_2$ with identifying in the following manner.

Identity two symmetric edges of $\{I^i\}$, $\{I^{i'}\}$: $(n = 1, 2, \ldots)$, $i, j = 1, 2, \ldots, 2N_n$, lying symmetrically with respect to the axis: arg $z = 0$, or $-\frac{\pi}{2}$ of $F_i (i = 1, 2)$ respectively. In general for $i$ in the sector $\frac{2\pi}{2k-1} k \leq \arg z$


\[ 17) \quad \text{Let } ABCD \text{ be an rectangle with sides } AB, BC, CD, DA \text{ and } U(z) \text{ be a harmonic function such that } U(z) = 0; z \in AB+CD \text{ and } \frac{\partial U}{\partial n} \text{ is given when } z \in BC+AD. \]

On the other hand we denote by $N(z, z_0)$ the harmonic function with logarithmic singularity at $z_0$ and $N(z) = 0$: $z \in AB+CD$ and $\frac{\partial N}{\partial n} = 0$, $z \in BC+DA$, then we have $U(z) = \int_{BC+DA} N \frac{\partial U}{\partial n} \cdot ds$.

From this we see that if $U(z) \to 0; z \in AB+CD$ and $\frac{\partial U}{\partial n} \to 0; z \in BC+AD$, then $U(z) \to 0$. 
17) In general $|U(z) - U(T_i(z))| \to 0$ and $|U(z) - U(T'_i(z))| \to 0$ follow that $|U(z) - UT_i(z)| \to 0$ on

$$e^{- M_a + m_i (2i + 2 + \frac{1}{2})} \leq |z| \leq e^{M_a - m_i (2i + 2 + \frac{1}{2})},$$
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\[ \leq \frac{2\pi(k+1)}{2^{i-1}} \quad (k = 0, 1, 2, 3, \ldots, 2^{-i-1}) \quad (n = 1, 2, \ldots) \quad (i, j = 1, 2, 3, \ldots). \]

We identify \( \{I_n\} \) and \( \{I_n'\} \) with edges lying symmetrically on the axis \( \arg z = \frac{2\pi}{2^{i-1}}(k + \frac{1}{2}) \). We identify \( \{S^n\} \) of \( F_1 \) and \( F_2 \) with the edges.

On the surface let \( u(p) : p \in F \) be a positive harmonic function and denote by Max \( u(p) \) the maximum of \( u(p) \) when \( p \) lies on \( |z| = r \), then

\[ \lim_{r_n} \max_{|z| = r} u(p) \leq P_n^{1+\delta_0} \quad \text{where} \quad r = -\frac{1}{2n+1}. \]

If \( \lim_{r_n} \max_{|z| = r} u(p) \geq P_n^{\delta+\epsilon} \), then there exists at least one sequence of \( R_n \) of \( F_1 \) or \( F_2 \) on which \( \min u(p), p \in R_n' \geq P_0^{\delta'} \), because in \( R_n'' \) there exists a curve connecting two boundaries of \( R_n'' \) on which \( u(p) \geq P_0^{\delta'} \) where \( \delta \) and \( \delta' \) are positive numbers and \( \Min u(p) \geq P_0^{\delta'} ; p \in \nu' \) follows that \( u(0) \) tends to infinity. Thus we may suppose without loss of generality \( \lim_{r_n} u(p) \leq P_n^{1+\delta_0} ; p \in R_n \). Next by the property of \( N_n \), and \( N_n', \) we have (Max \( u(p) - \Min u(p) \)) ; \( p \) lies on \( |\xi| = 1 \) \( \leq \frac{1}{n} + (\Max \( u(p) - \Min u(p) \)) \) \( \text{when} \ \arg \xi \leq \frac{\pi}{2}\nu' \) \( \text{and} \ |\xi| = 1 \)

Thus on \( \lambda_n = |z| = \sqrt{(1-\frac{1}{2n})(1-\frac{1}{2n+1}-\frac{1}{n^2})} \), \( \Max u(z) - \Min u(z) \leq \frac{2}{n} \).

This follows that on \( F_\iota \) with projection on \( |z| = \lambda_n \), \( u(p) \) tends to constant \( a_\iota (i = 1, 2) \). It is clear that \( a_\iota < \infty \). Finally on \( F_\iota \) there is no non-constant unbounded positive harmonic function, and the dimension of the class of bounded harmonic functions at most two. In \( F_\iota \) there exists a non constant bounded function vanishing on \( \{S^\iota\} \), therefore the dimension of the class is at least two.

On the surface \( K(p, p_0) \leq M \) for every ideal point of Martin's topology and is spanned by two bounded harmonic functions \( H_\iota(p) \) and \( H_\iota(p) \) and since \( F \) has only one boundary components \( \Gamma \) has infinitely\(^{18}\) many points (density of continuum) singular points and only two of them are minimal. R. S. Martin proved the set of non minimal points is \( F_\sigma \) and conjectured\(^{19}\) that this set was non dense in \( \Gamma \), but our example shows that his conjecture is not always true on an abstract Riemann surface, because the set of ideal points of this example is isomorphic to the closed interval \([0, 1]\) and only two ends correspond to minimal points.

Theorem 3.3. Let \( R \) be a \( F \)-type covering surface, over a null-boundary Riemann surface then on \( R \) there is no singular minimal function.


\(^{19}\) See 13).
Proof. Let $u(p)$ be a singular minimal function, it is easy to see that $u(p)$ takes zero or maximum $M$ on $\Gamma$ except outer harmonic measure. Consider $u(p)$ in $R^\omega$: $|z| < 1$ then $u(z)$ has angular limits $M$ on a positive measure set $G$, we divide $R$ into a system of enumerable neighbourhoods non overlapping and without lacunary, then there exists at least one $\nu_1$ such that the image $E_1$ of A.B.P.s lying on $\nu_1$ has common positive measure set $E_1$ with $G$. If $M\mu(R^\omega, E_1) = u(z)$, where $\mu(R^\omega, E_1)$ harmonic measure of $E_1$, then we divide $\nu_1$, $\nu_2$... non overlapping and on lacunary and consider $M\mu(R^\omega, E_1^?)$, it is clear that $\mu(R^\omega, E_1^?) \leq \mu(R^\omega, E_1)$ and $\lim M\mu(R^\omega, E_1^?) = 0$ by adequate subdivision, consequently there exists a harmonic function such as $u(p) \equiv u^*(p) \equiv 0$. This contradicts the minimality of $u(p)$.

2. On subsurfaces

Theorem. 3.4. Let $R \in O_\sigma$ and $S$ be a subsurface with at most enumerable infinity of analytic curves $\{C_i\}$ as its relative boundaries which are converging to the boundary of $R$. When we denote by $\hat{S}$ the doubled surface of $S$ being symmetric with respect to $\{C_i\}$, then $\hat{S}$ is contained in $O_\sigma$ too.

Proof. Denote by $R_m$ the exhaustion of $R$ with relative boundary $\Gamma_m$, and by $\omega_m(p)$ the harmonic function in $R_m - F_0$ such that $\omega_m(p) = M_m$, if $p \in \Gamma_m$, $\omega_m(p) = 0$ if $p$ is contained in the boundary of $F_0$, where $F_0$ is a compact disc contained in $S \cap R$ and $\int_{\partial F_0} \frac{\partial \omega_m}{\partial n} ds = 2\pi$, then it is clear $\lim M_m = \infty$. Put $z_m = e^{\omega_m + ih_m} = re^\theta$, where $h_m(p)$ is the conjugate of $\omega_m(p)$. Denoting by $F_0^*$ the symmetric disc is $\hat{S}$ of $F_0$.

If $\hat{S}$ is a positive boundary Riemann surface, then there exists a non negative harmonic function $w(p)$ such that $0 \leq w(p) \leq 1$, $w(p) = 0$ if $p \in$ boundary of $F_0$ and $F_0^*$ and $D(w) \leq M'$, $\frac{\partial w}{\partial n} = 0$ if $p \in \sum C_i$, and $\int_{\partial F_0} \frac{\partial w}{\partial n} ds = \pi$.

Regarding $w(p)$ in $S \cap R_m - F_0$, and denoting by $\theta_m$ the curve on which $|z_m| = r$. Put $L(r_m) = \int_{\theta_m} \left| \frac{\partial w}{\partial r_m} \right| r_m d\theta (\geq \pi)$, then $L^2(r_m) \leq 2\pi r_m \int_{\theta_m} \left| \frac{\partial w}{\partial r_m} \right|^2 r_m d\theta$.
Let $R_m \to R$ then $e^{M_n} \to \infty$, we have $D(w) \to \infty$, this is a contradiction. We denote by $S_{OB}$ and $S_{oB}$ the class of sub-Riemann surface on which any bounded harmonic function vanishing or with vanishing normal derivative on the relative boundary of $S$ must reduce to a constant.

**Theorem 3.5.** If there is no singular minimal point on the boundary of $S$, then

$S_{oB} \subset S_{OB}$.

**Proof.** Let $u_i(p) : p \in S$ be a non-constant harmonic function such as $0 \leq u_i(p) \leq 1$ and vanishes on the relative boundary of $S$, since there is no singular points there exists $u_2(p) \leq u_1(p)$: \( \lim u_2(p) = \lim u_1(p) = 1 \). Put $u_3(p) = u_1(p) - u_2(p)$, we define new functions $u_i(p^*) : i = 1, 2, 3$ such that $u_i^*(p) = u_i(p)$ on $S \cap \Gamma_m$ and symmetric with respect to the relative boundary of $S$ where $\Gamma_m$ is the relative boundary of $S_m$ being an exhaustion of $S$, and we extract \{ $u_i^*(p)$\} converging uniformly to $u_i^*(p)$, then $1 \geq u_i^*(p) = u_i^*(p) + u_i^*(p)$, \( \lim u_i^*(p) \geq 0 \), hence there is at least a non-constant bounded harmonic function with vanishing normal derivative.

As the special case if the genus of $S$ is finite then this theorem is affirmed. Because $S$ can be mapped onto the subsurface of a null-boundary Riemann surface, then by theorem 1.2 the universal covering surface is $F$-type.

**Remark.** The condition that there is no singular point is necessary is easily shown by examples.

Extension of L. Myrberg's theorem\(^2\).\

**Theorem 3.6.** Let $w(p) : p \in R$ and $D(w(p)) < \infty$, then $w(p)$ is expressed by Poisson's integral in $R^n$; $|z| < 1$.

**Proof.** Let $V(z)$ be sub-harmonic function such as \( \int_{\|z\|} |V(z)|d\theta \leq M \) and $G$ be a simply connected domain in $|z| < 1$ with boundary $\Gamma$, then

\[ \int_{\|z\|} |V|d\omega \leq M, \]
where \( d_\omega \) is harmonic measure with respect to \( G \). In fact denote by \( V^*(z) \) the upper envelope of sub-harmonic function such as \( V^*(p) \leq |V(p)| \) if \( p \in \{ |z| < 1 \} - G \), then \( V^*(0) \geq |V(0)| \), hence

\[
M \geq \int_{|z|=1} |V| d\theta \geq V^*(0) = \int_{r} |V(p)| d_\omega.
\]

Let \( G(p, p_0) \) be the Green’s function of \( R \) with its pole at \( p_0 \), when \( z = 0 \) is the of \( p_0 \), and put \( e^{-\varphi} = \rho \), then \( M \geq D_F(u(p)) \)

\[
= \int g \frac{2\pi}{2} \frac{\partial \rho \theta (e^\rho)}{\partial \rho} d\varphi \geq \int u^2 d\varphi, \quad \text{where} \quad F_r = \log e^{G/\log r} R \setminus E[G > \log r], \quad \text{since} \quad F_r^* \text{ is a subsurface of } R \text{ and has the boundary in } |z| > 1 \text{ except measure zero set.}
\]

We map \( F_r^* \) onto \(|\xi| < 1\), then the \(|z| = r \) in \( F_r^* \) is transformed onto a closed curve in \(|\xi| < 1\), by the preceding we have

\[
M \geq \int u^2 r^2 d\varphi = \int u^2 d\theta \quad \text{implies} \quad \int_{|\xi|=1} |u|^2 d\theta \leq M : \lim_{\imath \to \infty} r_\imath = 1
\]

thus, this can be expressed by Poisson’s integral.

**Corollary.** \( O_{\mathcal{H}B} \subset O_{\mathcal{H}BD} = O_{\mathcal{H}B} \).

Assuming that \( D_\mathcal{H}(u(p)) \leq M \), we map \( R^\imath \) onto \(|z| < 1\), then there exists a constant \( M_0 \) such that any one of the set where \( u(z) = u(p) \) has angular limits \( M_0 + \delta_0 \) between \( M_0 + \delta_0 \) and \( M_0 - \delta_0 \) and \( \leq M_0 - \delta_0 \) has positive measure. We denote by \( G_{M_0 + \delta_0} = E[u(p) \geq M_0 + \delta_0] \setminus R, \quad G_{M_0 - \delta_0} = E[u(p) \geq M_0 - \delta_0] \setminus R, \quad G_{M_0 - \delta_0} = E[u(p) \leq M_0 - \delta_0] \setminus R \) and \( R_\imath \) be an exhaustion with relative boundary \( \Gamma_\imath \), let \( u_\imath(p) = M_0 + \delta_0 \) when \( p \in G_{M_0 + \delta_0} \setminus \Gamma_\imath, \quad u_\imath(p) = u(p) \), when \( p \in G_{M_0 - \delta_0} \setminus \Gamma_\imath, \quad u_\imath(p) = M_0 - \delta_0 \) when \( p \in G_{M_0 - \delta_0} \setminus \Gamma_\imath \), respectively, we define a harmonic function \( u^*(p) \) by extracting from uniformly convergent sequence of \( u_\imath(p) \). It is clear by Dirichlet principle \( D_\mathcal{H}(u^*(p)) \leq D(u(p)) \)

\[\leq M \]. On the other hand, we map \( G_{M_0 + \delta_0}(G_{M_0 - \delta_0}) \) onto \(|\xi| < 1\), then the boundary of \( G_{M_0 + \delta_0}(G_{M_0 - \delta_0}) \) except relative boundary where \( u(p) = M_0 + \delta_0(M_0 - \delta_0) \) has positive measure, consequently \( u^*(p) \) is non constant.

**Chapter IV. Behaviour of analytic functions**

1. **Analytic functions on** \( O_{\mathcal{H}B} \)

We consider the behaviour of analytic functions defined on an
Theorem 5.1. Let $R'$ be the remaining surface after a compact set of $R$ is removed, if $R \in O_{\sigma}$ and $\in O_{HB}$ then there exists no analytic bounded functions exist.

In fact on $R'$, let $A(p)$ be a bounded analytic function, then by theorem 3.3 on $R'$ no singular minimal function exists, but on the ideal boundary has positive measure, hence there is a non-constant bounded harmonic function with the value 0 on the relative boundary of $R'$, it follows $R \in O_{HB}$.

We investigate the behaviour of the boundary of Riemann surface when it is given as a covering surface over the a $z$-plane.

Theorem 5.2. (W. Gross). Let $z = z(p) : p \in R$, be meromorphic and $R$ be a Riemann surface of $O_{\sigma}$, denoting by $p = p(z)$ its inversed function, if $p = p(z)$ is regular at $z_0$, then we can continue $z(p)$ analytically on half lines: $z = z_0 + re^{i\theta} (0 \leq r \leq +\infty)$ except for $\theta$ of angular measure zero.

2. Gross's property

Let $R$ be a Riemann surface of $O_{HB}$, then any connected piece has no lacunary of positive capacity projection, this implies $R$ has Iversen's property. In regarding above result we ask if the Gross's property holds in a surface of $O_{HB}$.

Theorem 5.3. A Riemann surface of $O_{HP}$ has not alivays Gross's property. This fact tells us the essential condition for $R$ to have the Gross's property is the "force" and not complexity of the ideal boundary and character of $O_{\sigma}$.

Example.

1) Let $G$ be a curvilinear rectangle with sides $C_1 -a \leq \Re(z) \leq a$, $\Im(z) = 0$. $C_2 : x+a = \phi(y), a = \phi(0), 0 \leq y \leq b$. $C_3 : -a \leq \Re(z) \leq a$, $\Im(z) = b$. $C_4 : x-a = \phi(y), -a = \phi(0), b \geq y \geq 0$, and $U(z) \geq 0$ be a positive harmonic function such that $U(z) \geq M$, if $z \epsilon C_2 + C_4 : U(z) \geq 0$, if $z \epsilon C_1 + C_3$, then there exist a curve $l$ connecting $C_2$ and $C_4$ on which $\text{Min} U(z) \geq \omega(0, \frac{ib}{2})$ where $\omega(x)$ is a harmonic function in a rectangle with

---

vertices \((-a, 0), (a, 0)\) and \((a, ib), (-a, ib)\) and \(\omega(z) = 1\) if \(z \in \text{segment } (a, 0)(a, ib), (-a, 0)(-a, b)\) and \(\omega(z) = 0\) if \(z \in (-a, 0)(a, 0), (-a, ib)(a, ib)\).

Because, let \(f(z)\) be a function mapping the curvilinear rectangle onto the rectangle \((-a', 0)(a', 0)(a, ib)(-a', ib)\) then \(2a' \leq \int_{-a' \mp \eta}^{a' \mp \eta} |f'| \, dx\) and \(4a'^2b \leq 2a \int_0^b \int_0^\eta |f'|^2 \, dx \, dy = 4ax'b\), it follows \(\frac{b}{a} \leq \frac{b'}{a'}\). From this fact we easily have our assertion.

2) The number \(P_{n-1}\).

Put \(r_n = \frac{1}{4} \times 2^{1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}}}, s_n = t_{n-1}\)

\[= \frac{1}{4} (r_n - r_{n-1}) = \frac{2^{1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}}} - 1}{10},\]

and \(R_n\) be a ring \(r_{n-1} + t_{n-1}\) \(\leq |z| \leq r_n - s_n\), and \(M_n\) be the module of \(R_n, M_n = \log \frac{3 \cdot 2^{1} + 1}{3 \cdot 2^{n-1}}\). The transformations "ring \(R_n(z\text{-plane}) \rightarrow \text{the rectangle } (-\pi, 0), (\pi, 0)(\pi, iM_n) (-\pi, iM_n) \rightarrow \text{upper half } \eta\text{-plane} (A = -\frac{1}{\kappa}, B = -1, D = 1, E = \frac{1}{\kappa}) \rightarrow \text{the unit circle of the } \xi\text{-plane}," are carried by

\[\xi = \frac{1}{\eta} \int_0^\eta \frac{d\eta}{\sqrt{(1-\eta^2)(1-\kappa^2 \eta^2)}}, \quad \xi = \frac{(1+i) \eta + \sqrt{\frac{1}{\kappa}} (1-i)}{(1-i) \eta + \sqrt{\frac{1}{\kappa}} (1+i)}\]

respectively, through some calculation we have \(\omega(iM_n) = e^{-\pi^2 (\frac{1}{M_n})}\).

Put \(P_{n-1} = 32e^{\pi^2 (\frac{1}{M_n})}\). (Fig. 3)

3) The number \(\mu_n\). Let \(I_n\) be slits such as; \(r_n - s_n \leq r \leq r_n + t_n\).

\(\theta = \frac{2\pi \nu}{2^\nu \theta}, \nu = 1, 2, 3, \ldots, 2^\nu\) and \(R_n' > R_n\) be rings such as \(R_n' = r_n + t_{n-1} \geq r \geq r_n - s_{n-1}. \quad R_n''; r_{n-1} + \frac{11}{10} t_{n-1} \leq r \leq r_n - \frac{11}{10} s_n\), and \(U(z)\) be harmonic in \(R_n' - \sum_I I_n - \sum_{n-1} I_n', 0 \leq U(z) \leq 2P_{n+1}^{1+\delta_0}: \delta_0 > 0, \) and vanishes on \(\sum_I I_n + \sum_{n-1} I_n',\) then there exists \(\mu_n''\) such that \(\max U(z), (\text{when } z \text{ is }\hfill\text{ contained in } R_n'' \leq \frac{1}{2\nu} \text{ for } \mu_n > \mu_n'\). \ We define \(\mu_n' = \max (\mu_n', \mu_n'')\).

4) The number \(\mu_n'''\) and \(\mu_n\). Let \(U(z)\) be a harmonic function such as \(|U(z)| \leq 2P_{n+1}^{1+\delta_0}\) if \(z \in R_n''\) then there exists \(\mu_n'''\) such that \(\max U(z) - \min U(z)\), (when \( z \) lies on \( r_n^* = \sqrt{\frac{11}{10} t_{n-1}} \text{ and } \frac{11}{10} s_n\) and
On Covering Surfaces

5) The number \( N_n \). Let \( U(z) \) be a harmonic function such as 

\[
0 \leq U(z) \leq 2P_{n+1}^{\frac{1}{2n}} + \sum_{n=1}^{\infty} \mu_n^2 \frac{1}{n!}
\]

for \( \mu_n > 0 \), then there exists a number \( N_n \) such as if 

\[
U(z) \leq \frac{3}{n}
\]

on \( I_n, I_n' \): \( \nu = 1, 2, \ldots, 2n^2, \nu' = 1, 2, \ldots, 2n^2+1 \), 

except any measurable set of having measure \( \frac{\text{length } I_n}{N_n} \), \( \frac{\text{length } I_n'}{N_n} \) respectively, \( U(z) \) must satisfy the condition that \( \text{Max } U(z) \leq \frac{6}{n} \), when 

\( z \in R_n'' \). We denote such number by \( N_n \).

6) Let \( \varphi(t) : [0 \leq t \leq 1] \) be absolutely continuous, almost every where derivable and integrable and \( \text{mes } E[\varphi(t)] = 0 = \text{mes } E[\varphi(t) = \infty] = 0 \), then there exist \( \kappa_1 \) and \( \kappa_2 \) depending only on \( \varphi \) but on \( E \) such that 

\[
\kappa_2 \geq \int_0^1 |\varphi(t)| dt > \kappa_1, \quad \text{where } E \text{ is any measurable set with measure } \delta_0.
\]

7) Let \( G \) be a domain in the \( z \)-plane with boundaries composed of analytic curves \( \gamma_1, \gamma_2 \ldots \gamma_2n-1, \gamma_n \) if we map \( G \) onto the ring \( 1 \leq |\xi| \leq e^{2\pi} \) so that \( \gamma_1, \gamma_2, \ldots, \gamma_2n-1 \) may correspond to \( |\xi| = 1, |\xi| = e^{2\pi} \) and radial slits in this ring \( R \). Let \( U(z) \) be harmonic in \( G \) and \( U(z) = \varphi \xi(z) \) if \( z \in \gamma_1, U(z) = \varphi \xi(z) \) if \( z \in \gamma_n \) respectively, then 

\[
D(U(z)) = D(U)^2(U, \varphi) + (U, \varphi) = 0.
\]

but clearly \( D(\tau) \geq \frac{1}{2\pi |\xi|} \int_0^{2\pi} |\varphi \xi(\xi) - \varphi \xi(\xi')| d\theta \).

(Fig. 4)

8) In the \( z \)-plane with slits \( I_n, \nu = 1, 2, \ldots, 2n^2 \), we enclose \( I_n \) by a simple closed curve \( J_n \) such that
and let $L_n^\prime$ and $L_n^\prime\prime$ half lines such that

$|z| = \frac{r_n + r_{n-1}}{2}, \quad \frac{2\pi \nu}{2^n a} - \frac{\pi}{2^n a} \leq \arg z \leq \frac{2\pi \nu}{2^n a} + \frac{\pi}{2^n a},$

$\frac{r_n + r_{n-1}}{2} \leq |z| \leq \frac{r_{n+1} + r_n}{2}, \quad \arg z = \frac{2\pi \nu}{2^n a} - \frac{\pi}{2^n a},$

$|z| = \frac{r_{n+1} + r_n}{2}, \quad \frac{2\pi \nu}{2^n a} - \frac{\pi}{2^n a} \leq \arg z \leq \frac{2\pi \nu}{2^n a} + \frac{\pi}{2^n a},$

$\frac{r_n + r_{n-1}}{2} \leq |z| \leq \frac{r_{n+1} + r_n}{2}, \quad \arg z = \frac{2\pi \nu}{2^n a} + \frac{\pi}{2^n a},$

and let $L_n^\prime$ and $L_n^\prime\prime$ half lines such that

$L_n^\prime: 0 \leq |z| \leq \infty, \quad \arg z = \frac{2\pi \nu}{2^n a} + \pi - \frac{\pi}{2^n a},$

$L_n^\prime\prime: 0 \leq |z| \leq \infty, \quad \arg z = \frac{2\pi \nu}{2^n a} + \pi + \frac{\pi}{2^n a},$

$L_n: 0 \leq |z| \leq \infty, \quad \arg z = \frac{2\pi \nu}{2^n a} + \pi.$

We denote by $G_n$ the domain with boundaries $L_n^\prime$, $L_n^\prime\prime$, $\sum_{i=\nu} J_n^i$, $I_n$ and map it onto the ring $1 \leq |w| \leq e^{\frac{2\pi (1)}{2^n a}}$, so that $L_n^\prime + L_n^\prime\prime$, $I_n$ and $\sum_{i=\nu} J_n^i$ may be transformed onto $|w| = e^{\frac{2\pi (1)}{2^n a}}$, $|w| = 1$ and radial slits $J_n^\nu (\omega)$, $\sum_{i=\nu} J_n^i (\omega)$ respectively. In this mapping any measurable set of positive measure $\geq \frac{1}{N_n}$ on $I_n$ is transformed onto a measurable set of positive angular measure than $-\frac{1}{k_n}$ and smaller than $k_n$ from (6), and the doubly connected domain bounded by $J_n^\nu$ and $I_n$ of module $\mathfrak{M}_n (\gamma)$ is transformed onto a domain bounded by their images denoted by $J_n^\nu (\omega)$ and $I_n (\omega) = (|w| = 1)$. Let $\rho_n^\nu$ be the distance of $J_n^\nu (\omega)$ from $w = 0$ and define the number by $\rho_n^\nu = \frac{\mathfrak{M}_n^2 (\gamma)}{2\pi \cdot k_n \cdot n^2 \cdot P_{n+1}^2 + 2\pi n a}$. We choose $\alpha_n$ such that $e^{\alpha_n} \leq e^{\rho_n^\nu}$, $e^{2\alpha_n} \leq e^{\rho_n^\nu}$.

9) a) Let $R$ be a ring $1 \leq |\varsigma| \leq e^\theta$ and $S_n, S_n^\prime$ be slits such that

$S_k: e^\frac{\beta}{l} \leq |\varsigma| \leq e^{2\frac{\beta}{l}}, \quad \arg \varsigma = \frac{2k\pi}{l} \quad (k = 1, 2, \ldots, l),$

$S_k^\prime: e^\frac{\beta}{l} \leq |\varsigma| \leq e^{\frac{5\beta}{l}}, \quad \arg \varsigma = \frac{2\pi l}{l} \quad (k = 1, 2, \ldots, l),$

we make two same rings with the same slits $(R - \sum S_k - \sum S_k^\prime)$ and connect crossweise two surfaces with $S_k$ and $S_k^\prime$ with same projection then we have two-sheeted Riemann surface $\hat{R}$. Denote by $\tilde{\varsigma}$ the point having the same projection as $\varsigma$ and $S(\varsigma) = |U(\varsigma) - U(\tilde{\varsigma})|$ be a subharmonic function on $R$ such that $0 \leq U(\varsigma) \leq P$ and vanishing on all branch points (end points of $S_k$ and $S_k^\prime$) then there exists a constant $\lambda$,
such that $|S(\varsigma)| < \lambda\rho$: $\lambda < 1$, when $|\varsigma| = e^{\frac{\gamma}{l}}$ where $\lambda$ depends continuously on only the ratio $\frac{\beta}{l}$. In the following we fix two bounds $M \geq \frac{\beta}{l} \geq N$ so that $\lambda$ may be always smaller than $\lambda_0$.

b) The number $s_n$ and $q_n$. We choose $s_n$ and $q_n$ such as

$$\frac{s_n}{2\pi n^2 R_n(\nu_1,v_1) - \alpha_n} \geq \frac{P_n^{2+2\alpha_n} 2^{2\alpha_n}}{\rho_{n+1}}, \quad 2P_n^{1+\delta_n} q_n - s_n \leq \frac{1}{n}$$

In the ring $R_n^0$: $1 \leq |w| \leq 2^{2\alpha_n}$, denote by $C^\alpha_{ij}$, $\tilde{C}^\alpha_{ij}$ ($n, \nu$ fixed $i, j = 1, \ldots, q_n)$ rings and $H_n^\nu$ circle such as

$$C^\alpha_{ij}: 2\alpha_n - \gamma(i(i-1)+2j-1) \leq \log |w| \leq 2\alpha_n - \gamma(i(i-1)+2j-1+1)$$

$$\tilde{C}^\alpha_{ij}: \gamma(i(i-1)+2j-1) \leq \log |w| \leq \gamma(i(i-1)+2j-1+1), \quad \gamma = \frac{\alpha_n}{2(q_n^2 + q_n + 1)}$$

$$H_n^\nu: \log |w| = \alpha_n, \quad i \geq j, \quad \kappa = 1, 2, 3, \ldots, l_n$$

We make slits $S^\nu_{ij}$, $\tilde{S}^\nu_{ij}$, $\tilde{S}^\nu_{ij}$ such as $(n, \nu$, fixed $i, j = 1, \ldots, q_n$, $i \geq j) = 0$.

$$S^\nu_{ij}: 2\alpha_n - \gamma(i(i-1)+2j-1-\frac{4}{6}) \leq \log |w| \leq 2\alpha_n - \gamma(i(i-1)+2j-1+\frac{5}{6})$$

$$S^\nu_{ij}: 2\alpha_n - \gamma(i(i-1)+2j-1-\frac{1}{6}) \leq \log |w| \leq 2\alpha_n - \gamma(i(i-1)+2j-1+\frac{2}{6})$$

$$\tilde{S}^\nu_{ij}: \gamma(i(i-1)+2j-1+\frac{4}{6}) \leq \log |w| \leq \gamma(i(i-1)+2j-1+\frac{5}{6})$$

$$\tilde{S}^\nu_{ij}: \gamma(i(i-1)+2j-1+\frac{1}{6}) \leq \log |w| \leq \gamma(i(i-1)+2j-1+\frac{2}{6})$$

$$\arg w = \frac{2\pi \kappa}{l_n} \quad \kappa = 1, 2, 3, \ldots, l_n$$

where $M \geq \frac{\gamma}{l_n} \geq N$.

Let $F(1), F(2) \ldots F(2^{\alpha_n-1}), \hat{F}(1), \hat{F}(2) \ldots \hat{F}(2^{\alpha_n-1})$ be $2^{\alpha_n}$ equal examplars of ring $R$ with slits $S^\nu_{ij}$, $\tilde{S}^\nu_{ij}$, $\tilde{S}^\nu_{ij}$.

We connect $F(i)$, $\hat{F}(i)$ crosswise on $S^\nu_{ij}$, $\tilde{S}^\nu_{ij}$, $\tilde{S}^\nu_{ij}$ such as $i = 1, 2, 3, \ldots, q_n$, $\kappa = 1, 2, \ldots, l_n$.

---

22) We map $R$ with slits $S, \tilde{S}, \tilde{S}, \tilde{S}$ onto a band by $\eta = \log \zeta$. Put $V(\eta) = U(\zeta) - U(\bar{\zeta})$, then $V(\eta)$ is single-valued and has same absolute values on two edges of each $S$ or $\tilde{S}$ and opposite signature. Let $V^*(\zeta)$ be a harmonic function with value Max $(0, V(\eta))$ on the boundary and $S', S, \tilde{S}, \tilde{S}$, then $|V(\zeta)| = V^*(\zeta) = \int V^*(\zeta) \frac{\partial g}{\partial n}(\zeta, \xi) d\xi$, from this we see easily $\lambda$ depends on the ratio $\frac{\beta}{l}$.
Fig. 6

\[ F_n = F_{n-1} = \ldots = F_{n-2} \]

\[ = \ldots = F_{n-3} = \ldots = F_{n-2} \]

\[ = \hat{F}_n = \hat{F}_{n-2} = \ldots = \hat{F}_{n-2} \]
Let $T_1$ be transformation $z \leftrightarrow \bar{z}$, where $z$ and $\bar{z}$ are points of $F(i)$ and $\hat{F}(i)$ having the same projection.

We connect

$$
\begin{align*}
F(1) & \leftrightarrow \hat{F}(2) & F(3) & \leftrightarrow \hat{F}(4) & \ldots & \text{on } S_{i,2,1}, \text{ etc. } i = 2, \ldots, q_n
\end{align*}
$$

and by $T_2$ corresponding transformation.

In general connecting and corresponding transformation are

$$
T_j \begin{align*}
F(2^m+1) & \leftrightarrow \hat{F}(2^m+2^j-2+1) \\
F(2^m+2) & \leftrightarrow \hat{F}(2^m+2^j-2+2) \\
\vdots & \vdots \\
F(2^m+2^j-2) & \leftrightarrow \hat{F}(2^m+2^j-1) \\
F(2^m+2^j-2+1) & \leftrightarrow \hat{F}(2^m+1) \\
F(2^m+2^j-2+2) & \leftrightarrow \hat{F}(2^m+2) \\
\vdots & \vdots \\
F(2^m+2^j-1) & \leftrightarrow \hat{F}(2^m+2^j-2)
\end{align*}
$$

$m = 0, 1, 2, \ldots, q_n-j$

We denote by $|F(i)-\hat{F}(i)|$ the maximum $|U(w_i)-U(w_j)|$, where $w_i$, $w_j$ have the same projection lying $|w|=e^{\alpha_n}$, then by a) we have

$$
\begin{align*}
|F(1)-\hat{F}(1)| & \leq \frac{2\beta_1}{n+1} \\
|F(1)-F(2^m+1)| & \leq \frac{2\beta_1}{n+1} \\
\vdots & \vdots
\end{align*}
$$

Taking account of the property of $s_n$ and $q_n$ we see that there is at least $s_n$ examplars on which $U(w)$ has equal value except at most $\frac{1}{n}$ when $w$ lies on $|w|=e^{\alpha_n}: H^{\nu_1}$.

**Structure of the surface**

(Fig. 6)

$F_0$ is the unit-circle $|z|<1$ with slits $I_\nu: n = 1, 2, \ldots, \nu = 1, 2, \ldots, 2^{\nu_n}$. $F_n (n = 1, 2, \ldots)$ be the $z$-plane with slits $I_n$. We map $G_n$ onto $1 \leq |w| \leq e^{\frac{\lambda n}{3}}$ and define the ring $1 \leq |w| \leq e^{\frac{\lambda n}{3}}$ and in 9) in $R_n$ we have made slits $S_{t, j, k}^n, \bar{S}_{t, j, k}^n, S_{t, j, k}^n, \bar{S}_{t, j, k}^n$, conversely in the $z$-plane i.e., we denote by the same letter $S_{t, j, k}^n$ the image $S_{t, j, k}^n, \bar{S}_{t, j, k}^n, S_{t, j, k}^n, \bar{S}_{t, j, k}^n$ in $F_n$ of the $w$-plane. Now by $F_1^n$ we denote the $z$-plane with slits $I_n, S_{t, j, k}^n, \bar{S}_{t, j, k}^n, S_{t, j, k}^n, \bar{S}_{t, j, k}^n, \bar{S}_{t, j, k}^n$ and by $F_2^n, F_3^n, \ldots, \hat{F}_n, \ldots, \hat{F}_n$ examplars of the $z$-plane with equal slits $S_{t, j, k}^n, S_{t, j, k}^n, \bar{S}_{t, j, k}^n, S_{t, j, k}^n$.

On the unit-circle $|z|<1$ we take a disc $|z| \leq r_m+s_m$ having slits $I_1, \ldots, I_m$, we connect $F_1^n (n = 1, \ldots, m)$ and $F_0 \cap (|z| \leq r_m+s_m)$, on
I_n (\nu = 1, \ldots, 2^{m_n}) crosswise. Connect F_n^1 with F_n^2 \ldots \hat{F}_n^{2q_{n-1}}, \hat{F}_n^1 \ldots \tilde{F}_n^{2q_{n-1}} on S_{\nu_{ek}}, \tilde{S}_{\nu_{ek}}, S_{\nu_{ek}}, \tilde{S}_{\nu_{ek}} (\nu = 1, 2, \ldots, \mu_{r}) in the manner mentioned in 9) b).

In such a way we have a Riemann surface R_m which covers the part \(|z| \leq r_m + s_m, 1 + 2^q_1 + 2^q_2 + \ldots + 2^q_m\) times and the part \(|z| > r_m + s_m, 2^q_1 + 2^q_2 + \ldots + 2^q_m\) times, and R_m has only one boundary component.

Put R = \bigcup R_m, then R is the required Riemann surface.

**Proof of the theorem.**

1) There exists no positive harmonic function on R.

Let U(z) be a positive harmonic function in R and Max U(z) be the maximum of U(z) on F_0 with projection on \(|z| = r_n + t_n\), then \(\lim \max U(z) \leq P_{n+\delta}; \delta > 0\).

If there exist a infinitely many \(r_n + t_n\) on which Max U(z) > \(P_{n+\delta}\), where \(\delta > 0\) then by 1) the ring \(r_n - t_n \leq |z| \leq r_n - s_n\) contained in \(F_0\), there is a closed curve which is a proper cut (dividing cut) on which Min U(z) > \(\delta P_n\) this follows that U(0) \(\to \infty\). Hence if U(z) \(\neq\) constant infinity, we may suppose \(\lim U(z) = P_{n+\delta}\). On \(R_n - R_{n-1}\) we denote by \(T_n'(z) (\nu = 1, 2, 3, \ldots, 2^{m_n})\) the conformal mapping in itself such that \(z \leftrightarrow \tilde{z}\) where \(\tilde{z}\) is the symmetric point of \(z\) with respect to the setraight \(L_n^*\): \(\arg z = \frac{2\pi (\nu + 1/2)}{2^{m_n}} \) or \(\pi + \frac{2\pi (\nu + 1/2)}{2^{m_n}}\), \(R_n - R_{n-1}\) is a covering surface over \(0 \leq |z| \leq \infty\), covers \(r_n - 1 + t_n \leq |z| \leq r_n + s_n + 2^{q_{n-1}} + 1\) times the other part of the z-plane \(2^{q_{n-1}}\) times, \(R_n - R_{n-1}\) has only two boundary components on which U(z) \(\leq P_{n+\delta}\) then also in \(R_n - R_{n-1}\). Put \(\tilde{U}(z) = |U(z) - U(T_n'(z))|\), then \(\tilde{U}(z) = 0\) on all points of \(R_n - R_{n-1}\) with its projection contained in \(L_n^*\); Denote by \(V^{(n)}(z)\) a harmonic function such that \(V^{(n)}(z) = 0\) if \(z \in R_n - R_{n-1}\) if \(z\) has projection on \(L_n^*, L_n', V^{(n)}(z) = P_{n+\delta}\) if \(z\) is contained in the boundary of \(R_n - R_{n-1}\) i.e. \(z \in F_0\) and \(|z| = \frac{r_n - 1 + t_n}{2}\) or \(|z| = \frac{r_n + s_n + 1}{2}\). Denote by \(R_n^*\) the surface composed of \(\hat{F}_n^i\), \(\hat{F}_n^i: i = 1, 2, 3, \ldots, 2^{q_{n-1}}\) which is a part of \(F_n^i\) bounded \(L_n^*\) and \(L_n'^*\) and the part of \(F_0\) contained in \(\sum J_n^*\) which is connected with \(F_n^i\) on \(\sum I_n^*\) and \(\sum S' + S + \tilde{S}' + \tilde{S}\) and \(V_n^{(n)}(z)\) be a harmonic function on \(R_n^*\) such that \(V_n^{(n)}(z) = 0\) if \(z\) lies on \(L_n^*\) or \(L_n'^*\) of \(F_n^i\) or \(\hat{F}_n^i\) and \(V_n^{(n)}(z) = P_{n+\delta}\) if \(z\) lies on \(\sum J_n^*\) of \(F_0\). We investigate the behaviour of \(V_n^{(n)}(z)\) on \(R_n^*\), since the part of \(F_0\) bounded by \(\sum J_n^*\) is composed of at most \(2^{m_n} - 1\) doubly connected domains with module \(M_{y}\), then the Dirichlet integral of \(V_n^{(n)}(z)\)
on $R_\alpha^2$ is smaller than the Dirichlet integral of the harmonic function being zero on $\sum J^\nu_\alpha$ and $= 2 P^{1+\delta_0}_n$ on $\sum I^\nu_\alpha$, over the part of $F_0$ bounded by $\sum J^\nu_\alpha$ and $\sum I^\nu_\alpha$; $\nu = 1, 2, 3, ..., 2^{\mu_\alpha-1}$, then we have

$$D_{R^2_\alpha}(V(z)) \leq \frac{1}{2\pi} \left(2^{\mu_\alpha-1}\right) \frac{P^{2+2\delta_0}_{n+1}}{\Im \Gamma(\nu)}.$$

If $V^\lambda(z) \geq \frac{2}{n}$ on the measurable set of measure larger than $\frac{1}{k_\alpha}$ of at $H^\alpha_\nu$ of $F'_{\alpha'}$, then on account of 9) there exist at least $s_\alpha$ examplars of $F'$ or $\hat{F}'$ such that $V^\alpha(z) \geq \frac{1}{n}$ on a measurable set of measure $> \frac{1}{k_\alpha}$ on $H^\alpha_\nu$ where measure of set means the measure of the image when $G_{\nu}^\alpha$ is mapped on $\exp(\frac{1}{n}) \geq |w| \geq 1$. Then

$$D(V^\alpha(z)) \geq 2\pi n^2 \frac{s_\alpha}{k_\alpha (\Im \Gamma(1)-\alpha_\alpha)} \geq \frac{P^{2+2\delta_0}_{n+1}}{\Im \Gamma(2)}.$$

This contradicts the property of $q_\alpha$ and $s_\alpha$.

If $|V^\alpha(z_1)-V^\alpha(z_2)| \geq \frac{1}{n}$ on the measurable set of measure $> \frac{1}{k_\alpha}$ of at $H^\alpha_\nu$, where $\arg V^\alpha(z_1) = \arg V^\alpha(z_2)$ and lie on $|w| = 1$ and $|w| = e^{\sigma_\alpha}$ of $F'_{\alpha'}$ respectively, then by 7) and 8), we have

$$D_{R^2_\alpha}(V(z)) \geq \frac{1}{2\pi n^2 k_\alpha \alpha_\alpha} \geq \frac{P^{2+2\delta_0}_{n+1}}{\Im \Gamma(2)}.$$

where right hand is the Dirichlet integral over the part of $F'_{\alpha'}$ bounded by $J^\nu_\alpha$ and $I^\nu_\alpha$, this contradicts to 3) therefore $V^\alpha(z) \leq \frac{3}{n}$ except $\frac{1}{k_\alpha}$ angular measure at any of $I^\nu_\alpha$. On the other hand $\bar{U}(z)$ is sub-harmonic and $\leq P^{1+\delta_0}_{n+1}$ and vanishes on $L^\nu_\alpha+L^{*\nu}_\alpha$, then $\bar{U}(z) \leq V^\lambda(z)$: $\lambda' = \lambda, \lambda+1, ... \lambda+2^{\mu_\alpha-1}$ where $|\arg L^\nu_\alpha - \arg L^{*\nu}_\alpha| = |\arg L^\nu_\alpha - \arg L^{*\nu}_\alpha| = \frac{\pi}{2}$. $|\bar{U}(z)| \leq \frac{3}{n}$ on every $I^\nu_\alpha$, except $\frac{\text{length } I^\nu_\alpha}{N^\nu_\alpha}$ measure set thus $|U(z)-U(T^\nu_\alpha(z))| \leq \frac{6}{n}$ in ring $R^\nu_\alpha$ but $\nu$ is arbitrary, accordingly $U(z)$ is symmetric and periodic with respect to $2^{\mu_\alpha}$ directions $L^\nu_\alpha$ except at most $\frac{6}{n}$ in step by 2) $|\text{Max } U(z)} - \text{Min } U(z)| < \frac{7}{n}$, when $|z| = r^\nu_\alpha$, where $r^\nu_\alpha = \sqrt{(r^{\nu-1}_n+\frac{11}{10}l_n-s^\nu_\alpha)}$. It follows that $U(z)$ must be a constant.

2) As $F_0$ is a subset of the unit-circle, it is clear that $R$ has not Gross's property and on account of theorem 5.2, $R$ is a positive boundary Riemann surface.

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