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On Covering Surfaces

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CONTENTS

Introduction

The purpose of the present paper is to investigate the behaviour of the boundary of a covering surface. In chapter I we shall consider the correspondence of the boundary points when the universal covering surface is mapped onto the unit-circle, and extend Fatou's theorem in such a case. Chapter II is devoted to the study of accessible boundary points from measure theoretic views. Thus our study is a continuation of that due to M. Ohtsuka[1]. R. Nevanlinna discussed the Dirichlet

1) M. Ohtsuka: 1) Dirichlet problems on Riemann surface and conformal mapping. Nagoya Math. Journal, 3, 1951, pp. 91-135.
2) On a covering surface over an abstract Riemann surface. ibid., 4, 1952, pp. 109-118.
3) Note on the harmonic measure of the accessible boundary of acovering Riemann surface. ibid., 5, 1953, pp. 35-38.

problem in weak sense on a subsurface of an abstract Riemann surface. M. Ohtsuka proved that the Dirichlet problem for continuous boundary value is *resolutive* in Perron-Brelot's sense[2] which is stronger than that of R. Nevanlinna under the following three assumptions: A) the projection of accessible boundary points of the surface R is compact in the basic surface $\underline{R}$, B) the universal covering surface R^{∞} of R is F-type[3], C) the connectivity of R is finite. We shall show that it is possible to remove conditions A) and C). Thus we see that the resolutivity of the problem has not so much relation with the measure of irregular points. If we suppose that R covers $\underline{R}$ a finite number of times, then we can prove two theorems concerning the Dirichlet problem: i) the irregular accessible boundary points are so rare that the projection of them may be contained in the F_{σ} set of capacity zero, ii) the regularity for Green's function is equivalent to that for Dirichlet problem. Chapter III is related to R. S. Martin's topology concerning harmonic functions. We shall study *activity*[4] of points and *minimality*. R. S. Martin conjectured that the set of non minimal points would be non dense in the set of ideal boundary points. However we shall show by an example that his conjecture does not hold good and at the end of this chapter we shall prove miscellaneous theorems on the subsurfaces of abstract Riemann surfaces. In chapter IV we consider analytic functions on Riemann surfaces and the behaviour of covering surfaces. We shall give an example of Riemann surface belonging to the class O_{HP} and without Gross's property. From this we see that the validity of the Gross's property for every covering surface of a given Riemann surface does not depend upon the complexity of the boundary. It depends rather upon the "*force*" of the boundary, i.e., roughly speaking upon the size of the boundary.

Chapter I. Correspondence of boundaries

1. Image of boundary

Let $\underline{R}$ be an open abstract Riemann surface, and $\underline{R}_n$ $(n=1,2,\ldots)$ be an exhaustion with relative boundary Γ_n, then $R-R_n$ is composed of a finite number of disjoint non compact subsurfaces G_n, let G_n be a sequence of non compact subsurfaces such that $G_i \supset G_{i+1} \cdots$, $\bigcap^{\infty} G_n = 0$.

2) M. Brelot: Familles de Perron et Problème de Dirichlet. Acta Szeged **9**, 1939, pp. 133-153.

3) See Chap. II, 4.

4) M. Brelot: Remarques sur la variation des fonctions sousharmoniques et la masses associées. Application. Ann. Institut Fourier **2**, 1950, pp. 101-111.

Two sequences $\{G_n\}$ and $\{G_m'\}$ are called equivalent if and only if, for a given number m, there exists a number n such that $G_m' \supset G_n$ and vice versa. We correspond an ideal boundary point (component) to an equivalent sequence, and denote the set of all ideal boundary point by B_I. A-topology will be introduced on $\underline{R}+B_I$ by the completion of $\underline{R}$. It is clear that $R+B_I$ is closed, compact and B_I is totally disconnected. For simplicity we denote $\underline{R}+B_I$ by $\underline{R}^*$ which has the topology being isomorphic in $\underline{R}$ to the original topology defined by the local parameter.

When R is a covering surface over an abstract Riemann surface $\underline{R}^*$, $\underline{R}^*$ is a metric space, we introduce as O. Teichmüller and Mazurkiewicz a metric on R by $\inf \delta(p_{\_}\cdot p_2)$, where $\delta(p_{\_}\cdot p_2)$ is the diameter of the projection into $\underline{R}$ of $\widetilde{p\, p_2}$ which connects $p_{\_}$ with p_2 on R, and define accessible boundary points of R by the completion with respect to this metric, it is well known that this definition is equivalent to the other definition by deciding the equivalency of two curves on R which determine an accessible boundary point (we abbreviate by A. B. P). In this and next chapter we suppose $\underline{R}$ to be a null-boundary Riemann surface and discuss another Riemann surface R defined as the covering surface over $\underline{R}$.

Lemma. *Let R be a covering surface over $\underline{R}$, and does not cover a set of positive coapacity inner points of $\underline{R}$, and the universal covering surface R^∞ of R mapped conformally onto the unit-circle $U: |z|<1$. If a curve L_z on L^* contained in R and determining an A. B. P. ending at a point $z_0: |z_0|=1$, determines an A. B. P. having its projection on B_I, then the mapping function $w=f(z): w \in \underline{R}^*$ has the same limit when z_0 tends to z_0 along Stolz's path.*

Proof. Let p_0 be the projection of the A. B. P. on $\underline{R}^*$ and $V_n(p_0)$ be Γ_n. the sequence of neighbourhoods of p_0 with a compact relative boundary. We define a super-harmonic function $\omega_n(p)$ such as, $\omega_n(p)$ is harmonic in $(\underline{R}-\underline{R}_0-V_n(\rho_0) \bigcup (\text{proj } R \bigwedge R_0)$, $\omega_n(p)=0$ if $p \in$ boundary of proj R in $\underline{R}_0$, where R_0 is a compact part of in which the boundary of the projection of has positive capacity, and $\omega_n(p)=M_n$ if $p \in \Gamma_n+V_n(p_0)$, and is normalized by $\frac{1}{2\pi}\int\limits_{\Gamma_n} \frac{\partial \omega_n}{\partial n} ds = 1$. Since $\underline{R}$ is a null-boundary Riemann surface, this function is uniquely determined and $\lim\limits_n M_n = \infty$, we denote the domain $\arg|z-z_0|<\frac{\pi}{2}-\delta$, $|z-z_0|<1-r$ by $\Delta_{r,\delta}$ and the part of L outside of $|z|=r_0$ by Lr_0, then

$$\omega_n(f(z)) \geqq M_n \lambda\delta, \quad \text{if } z \in \Delta_{r\cdot\delta};\ n > i_0 \cdots \tag{1}$$

where $\lambda\delta > 0$ and i_0 is the minimal number such as $f(Lr_0) \in V_{i_0}(p)$. If $f(z)$ had not limit p_0 in $\Delta_{r\cdot\delta}$, then there would exist a sequence z_i such as $\lim_i z_i = z_0$: $z_i \in \Delta_{r\cdot\delta}$ and a number n_0 and a subsequence of $i_{n_{k_i}}$ such as $f(z_{n_{k_i}}) \bar{\in} V_{n_0}(p)$: $i_{n_{k_i}} \geqq i_0$, therefore there exists a number N such that

$$\omega_n(f(z_{i_k})) \leqq N : i_k \geqq i_{k_0} \quad n \geqq n_0 \cdots \tag{2}$$

from (1) and (2) we have $N \geqq \omega_n(f(z_n)) \geqq M_n\lambda\delta$: $\lim M_n = \infty$, this is a contradiction.

Remark. When the A. B. P. lies on $\underline{R}$, our assertion is trivial. From this lemma we can easily deduce the next proposition.

Proposition. *Under the same condition as the lemma, the set E on $|z| = 1$ where at least a curve ends along which $f(z)$ has cluster set of capacity zero is a linear measure zero set on $|z| = 1$.*

Proof. We denote a closed set $E' \subset E$ where $f(z)$ tends uniformly in A-topology, then $f(E')$ is closed and capacity zero, then there exists a sequence V_n with compact relative boundary γ_n such as $\bar{V}_n \supset f(E')$, where $\bar{V}_n$ is the closure of V_n. We denote continuous super-harmonic function such as $\omega_n(p)$ is harmonic in $(\underline{R} - \underline{R}_0 - V_n) \cup (R_0 \cap \text{proj } R)$ and $\omega_n(p) = 0$ if $p \in$ boundary of proj $R \cap \underline{R}_0$, and $\omega_n(p) = M_n$ if $p \in \gamma_n + V_n$, $\frac{1}{2\pi}\int \frac{\partial \omega_n}{\partial n} ds = 1$ then $\lim_n M_n = \infty$. We consider $\omega_n(p)$ in $|z| < 1$,

$$\omega_n(z) \geqq \frac{1}{2\pi}\int_{E'} M_n \frac{r_n^2 - \rho^2}{r_n^2 - 2r_n\rho\cos(\theta - \varphi) + \rho^2} d\theta, \text{ and let } r_n \to 1 \text{ then } \omega_n(z) \to \infty.$$

This is a contradiction.

Let R be a null-boundary Riemann surface, and R^∞ be mapped onto $|z| < 1$ conformally and D_0 be the normal polygon being a fundamental domain containing $z = 0$. We denote by $D_0'(\rho)$ the part of D_0, which lies in $1 - \rho < |z| < 1$: $0 < \rho < 1$; then $D_0'(\rho)$ consists of a finite number of simply connected closed domain, we consider only such domains and $D_n(\rho)$ be its equivalent ones and put

$$\Delta(\rho) = \sum_{n=0}^{\infty} D_n(\rho),$$

then $\Delta(\rho)$ consists of a enumerable number of disjoint simply connected continua: $\Delta_n(\rho)$ such that $\Delta(\rho) = \sum \Delta_n(\rho)$. Since in a normal polygon, equivalent points on the boundary of D_0 are equidistant from $z = 0$, $\Delta_n(\rho)$ is bounded by Jordan arcs λ_n^k $(k = 0, 1, 2, \ldots)$ and closed sets E_n

on $|z|=1$. We put $E(\rho)=\sum E_n$, λ_n^k ends at two points ξ_n^k, η_n^k on $|z|=1$ which are fixed points of some substitution, if $\xi_n^k=\eta_n^k$ for one k, then $\Delta_n(\rho)$ is bounded by a single Jordan curve which touches $|z|=1$ at $\xi_n^k=\eta_n^k$. It is easily seen that if $\xi_n^k \neq \eta_n^k$ then λ_n^k is contained between two circular arcs C_n^k, $C_n^{k\prime}$ through ξ_n^k, η_n^k which meets $|z|=1$ with an angle α_n^k, β_n^k respectively. Since λ_n^k $(n=0,1,2,\ldots)$ can be grouped into a finite number of quivalent classes, there exists α, β $(0<\alpha, \beta, <\pi)$ such that for any λ_n^k for which

$$\alpha \leqq \alpha_n^k \leqq \beta, \quad \alpha \leqq \beta_n^k \leqq \beta \quad (n=0,1,2,3,\ldots)$$

Since λ_n^k is a $\sum_n^\infty S^{\pm n}(l)$, where l is the part of $|z|=\rho$ in D_0. and multiplier of the substitution $k \neq 1$, then λ_n^k is rectifiable.

Lemma. *For any substitution T_i $(i=1,2,\ldots i_0)$ of Fuchsian group*

$$\underline{\delta}_0 \leqq \frac{\text{length } \lambda_n^k}{\text{length } T_i(\lambda_n^k)} \leqq \bar{\delta}_0 .$$

We denote by ds the line-element of λ_i, and by $T(ds)$ its image, since λ_n^k has a tangent almost everywhere on λ_n^k, let C^*ds be the circular arc ending at two points ξ and η passing ds at P and $C\,ds$ the circular arc passing ds at P and ending at two fix points ζ, ζ' of the substitution T, then

$$\frac{ds}{T(ds)}=\frac{\xi P}{T(\xi P)} \quad \text{or} \quad \frac{ds}{T(ds)}=\frac{\zeta P}{T(\zeta P)},$$

but $\dfrac{\xi P}{T(\xi P)}$ has extremal values when $C\,ds$ coincides with C' or C and the intersecting angle of λ_i and $|z|=1$ is contained between α and β, therefore we have

$$\underline{\delta}\frac{\xi\eta}{T(\xi\eta)}<\frac{ds}{T(ds)}<\bar{\delta}\frac{\xi\eta}{T(\xi\eta)}$$

and if we denote by $\widehat{\xi\eta}$ the length of $|z|=1$ between ξ, η then

$$\underline{\delta}_0\frac{\widehat{\xi\eta}}{T(\widehat{\xi\eta})}<\frac{\lambda_n^k}{T(\lambda_n^k)}<\delta_0\frac{\widehat{\xi\eta}}{T(\widehat{\xi\eta})}.$$

But $\Delta_n(\rho)$ are bounded by (ξ_n, η_n) and its image transformed, one of them is an inner side of $\Delta_n(\rho)$ with respect to $|z|=1$, and the others are outer sides and have non common points on $|z|=1$, therefore the length of $\Delta_n(\rho)$ is finite. Since Δ_n does not cover a set of positive capacity on $\underline{R}$, then we map Δ_n onto $|y|<1$ then from the proposition we see that E_n is linear measure zero set, and also $\sum E_n$ is too, thus

Theorem 1. 1.[5] *Let R be a null-boundary Riemann surface. When universal covering surface mapped onto the unit-circle,* then *the image of the boundary points is a set of linear measure zero.*

Remark. This theorem is proved by M. Tsuji, our proof is slightly simpler.

2. Generalization of Fatou's theorem

Lemma. *Let R be a null-boundary Riemann surface, we denote two points a and q of R and denote a harmonic function $U_a(p)$ such that*

$$U_a(p)+\log p = 0\,, \qquad \textit{in the neighbourhood of } q$$
$$U_a(p)-\log p = \text{harmonic} \quad \textit{in the neighbourhood of } a$$

then $U_a(p)$ is uniquely determined and let $U_a(p)^+ = 0$ if $U_a(p) \leq 0$, and $U_a(p) = U_a(p)$, if $U_a(p) > 0$, then for any points a and p

$$U_a^+(p)-d-U_a^+(0) \leqq U_0^+(p)\,, \quad U_0^+(p) \leqq U_a^+(p)+U_a^+(o)+d$$

where d depends on 0 and p only.

Proof. We denote by D_0 the domain such that $0 \geqq U_0(p)$ and C_0 the set such that $U_0(p)=0$ and D_a, C_a respectively.

Case 1. $a \in D_0$. $V(p)=U_a(p)-U_a(0)$ is harmonic in $R-D_0$ for fixed a, then it takes its maxium in $R-D_0$, when p lies on C_0, where $U_p(a)=U_p(a)-U_0(a)$, for varying a. Put $U_{a_s}(p_0)=U_{p_0}(a_s)=d=\max$: $a_s \in C_0, p_0 \in C_0$. We denote by D_a^+ the domain such as $U_a(p)-d-U_a<0$ and by C_a^+ its boundary, then it is clear $D_a^+ \supset D_0$, and $U_a^+(p)-d-U_a^+(0) \leqq 0$, $U_0^+(p) \geqq 0$ when $p \in R$.

Case 2. $a \in D_0$, If $p \in D_0$, then Max $U_a(p)-U_a(0)$ attains its maximum d when $p, a \in C_0$ then

$$U_a(p)-d = U_a^+(p)-d-U_a^+(0) \leq U_0^+(p)\,.$$

In the case when $p \in D_0$, we can prove similarly, the latter part of the lemma can be proved in the same way.

Let R be a covering surface over $\underline{R}$ and of positive boundary Riemann surface, and $f(z)$: $p \in \underline{R}$; $z \in R$ be the mapping function and $G_n(z,0)$ be the Green's function with its pole at 0 with respect to R_n and $\bigvee_n R_n = R$, and by $h_n(z,0)$ be its conjugate. Put $x_n = e^{-G_n-ih_n} = re^{i\theta}$, $\lim_n x_n = x$.

5) M. Tsuji: Some metrical theorems on Fuchsian groups. Kōdai Math. Sem Nos. 4-5, 1950, pp. 27-44.

In denoting by a_i, b_i the point where $f(a_i)=0$, $f(b_i)=q$ respectively then

$$U_0(f(z))=\frac{1}{2\pi}\int_0^{2\pi} U_0(f(re^{i\theta})d^{\theta}+\sum_{\nu} G_n(z, b_\nu)-\sum_{\nu} G_n(z, a_\nu)+\log|c_k|,$$

where c_k is the first non vanishing coefficient of the expansion of $f(z)$ with respect to the local parameter defined in the neighbourhood of 0. Put

$$m(r, f-a)=\int_0^{2\pi} U_a^+(f(re^{i\theta})\,d\theta$$

$$N(r, f-a)=\int_0^r \frac{n(q, a)-n(0, a)}{t}dt$$

from the lemma $|m(r, f-a)-m(r, q)|\leqq U_0^+(a)+d$

$$m(r, a)+N(r, a)=m(r, q)+N(r, q)+\varphi(r)$$

where $|\varphi(r)|\leqq U_0(a)+d+|\log|c_k||$, if

$$T(r, q)=\lim_{R_n\to R}(N(r, p)+m(r, q))<\infty$$

we call this mapping bounded type covering.

Theorem 1.2. *If $p=f(z)$ is bounded type, we map R^∞ onto $|\zeta|<1$ conformally, $\zeta=\varphi^{-1}(z)$ $0=\varphi(0)$, $z=\varphi(\zeta)$, then $p=f(\varphi(\zeta))=F(\zeta)$ has angular limits almost everwhere on $|\zeta|=1$.*

Proof. We can discuss the potential theory on $\underline{R}$, if μ is a unit mass distribution on the positive capacity set E of $\underline{R}$, then we have $T(r)=\int_E N(r, a)d\mu(a)+0(1)$, we easily see that $T(r)$ is finite only and if only when $N(r, a)$ is finite everywhere. A system of equivalent points a_i^j of ζ-plane ($j=1, 2, \ldots$) corresponds to a point $a_i\in R$ and $G(z, a_i)\geqq \sum_j G_r(\zeta, a_i^j)$ where $G_r(\zeta, a_i^j)$ is the Green's function of $|\zeta|<r$ with the poles at a_i^j, we have $A\geqq N(r, a)\geqq G_r(z, a_i^j)$ and a small circle in the neighbourhood of $x=0$ in the x-plane and the neighbourhood of 0 in $\underline{R}$ corresponds conformally each other, therefore in ring domain $\rho<|x|<\rho_2$, $N(r, a_i)\leqq A$ for any point a_i in the ring. Hence $F(\zeta)\to x$ is bounded type on $|\zeta|<1$, by Fatou's theorem $F(\zeta)$ has angular limits almost everywhere on $|\zeta|=1$, on the x-plane, let l be an asymptotic path ending at ζ_0; $|\zeta_0|=1$ along which $F(\zeta)$ converges to x_0 and the Green's function of R tends to zero, then it is easy to see that a curve L on R corresponds to l, determines an A. B. P. of R relative to $\underline{R}$.

Corollary.[6] *Let R be a covering surface over an abstract Riemann surface $\underline{R}$ and there exists a positive capacity set $E \subset \underline{R}$ such that, every point of E is covered by R a finite number of times, then on R^∞: $|\zeta| < 1$ the mapping function $F(\zeta)$ has angular limits almost everywhere on $|\zeta| = 1$, this is proved by M. Ohtsuka by another way.*

Corollary. *Let $\hat{R}$ be a covering surface over $\underline{R}$ and R is a bounded type covering over $\underline{R}$, then $\hat{R}$ is a bounded covering over $\underline{R}$, too.*

Proof. A system a_i^j corresponds to a point a_i, which lie on a_i, and $\sum_j G_{\hat{R}}(p, a_i^j) \leq G_R(a_i)$, then we easily have $T(\hat{R}) \leq T(R)$, where $G_{\hat{R}}$, G_R are Green's function of $\hat{R}$ and R respectively.

Chapter II. Accessible boundary point

1. Measures of accessible boundary points

Let $\underline{R}$ be a null-boundary Riemann surface with A-topology and R be a covering surface over $\underline{R}$. When a curve l converging to the boundary of R and its projection on $\underline{R}$ converges to a point $p \in \underline{R}^*$, then we say that l determines an accessible boundary point (A. B. P.) relative to $\underline{R}^*$. We consider non negative continuous super-harmonic functions such that $0 \leq v(p) \leq 1$, and $v(p) = 1$ when p tends to an A.B.P. along every curve, we denote by $\mu(R, \mathfrak{A})$ the lower envelope of $\{v(p)\}$ being harmonic on account of Perron-Brelot's theorem, and define $\mu(R^\infty, \mathfrak{A})$ similarly. Denoting by $\underline{R}'$ the projection of R over $\underline{R}^*$, if the universal covering surface of $\underline{R}'$ is parabolic type, i.e., it cannot be mapped conformally onto the unit-circle, we take off a finite number of points $p_1 \cdot p_2, \dots, p_n$ (when $\underline{R}'$ is closed and its genus is zero, three points, and when $\underline{R}'$ is closed and its genus is one, one point) and remove from R all the points p_{ij}; $j = 1, 2, \dots$, lying over them and denote the remaining surface by $\tilde{R}$ so that the universal covering surface of $(\underline{R}' - \sum^n p_i)$ may be hyperbolic. As the preceding we define $\mu(\tilde{R}, \mathfrak{A})$, and $\mu(\tilde{R}^\infty, \mathfrak{A})$. In the following we assume that R has at least one accessible point.

1. R is a null-boundary Riemann surface, then $\tilde{R}$ is also too, since no bounded harmonic function exists on a null-boundary Riemann surface, $\mu(R, \mathfrak{A}) = \mu(\tilde{R}, \mathfrak{A}) = 1$. On the other hand if R^∞ is parabolic, we see that $\mu(R^\infty, \mathfrak{A}) = 1$. If R^∞ is hyperbolic, we also on account of theorem 1.1. that any curve determining an A. B. P. converges to the set on the periphery of the unit-circle, of linear measure zero, thus we have

6) See M. Ohtsuka, 1).

$\mu(R^{\infty}, \mathfrak{A}) = \mu(\tilde{R}^{\infty}, \mathfrak{A}) = 0$.

2. R is a positive boundary Riemann surface, put $R - \tilde{R} = \sum_j^{\infty} \sum_i^{n} p_{ij}$ and let $G(p, p_{ij})$ be the Green's function on R with its pole at p_{ij}. For arbitrary point $p_0 \in R$, the function $g(p) = \sum_{ij} \frac{G(p \cdot p_{ij})}{(ij+i)^2 G(p_0, p_{ij})}$ represents a harmonic function tending to ∞ at any points of p_{ij}, accordingly for any positive number $\varepsilon > 0$ and $v(p)$, one of the class defining $\mu(R, \mathfrak{A})$, Min $(1, v(p) + \varepsilon g(p))$ is one of super-harmonic function defining $\mu(\tilde{R}, \mathfrak{A})$, this follows $\mu(R, \mathfrak{A}) \geqq \mu(\tilde{R}, \mathfrak{A})$. Conversely any $v(p)$ on $\tilde{R}$ can be considered as one of $v(p)$ of R if the value 1 is supplement to $\sum p_{ij}$. Hence $\mu(\tilde{R}, \mathfrak{A}) \geqq \mu(R, \mathfrak{A})$ and the equality follows. Further there holds $\mu(R, \mathfrak{A}) \geqq \mu(R^{\infty}, \mathfrak{A})$, because any $v(p)$ of R is considered on R^{∞}. Since for any $v(p)$ of R^{∞} and $\varepsilon > 0$, Min $(1, v(p) + \varepsilon g(p))$ can be considered on $\tilde{R}^{\infty}$, then we have $\mu(R^{\infty}, \mathfrak{A}) \geqq \mu(\tilde{R}^{\infty}, \mathfrak{A})$.

Theorem 2. 1.

$$\mu(R, \mathfrak{A}) = \mu(\tilde{R}, \mathfrak{A}) \geqq \mu(R^{\infty}, \mathfrak{A}) \geqq \mu(\tilde{R}^{\infty}, \mathfrak{A})^{7)}.$$

As in the case when R is a null-boundary Riemann surface, these relations are settled, in the sequel we suppose that R is a positive boundary Riemann surface and investigate when equality or inequality hold among these quantities.

We suppose $\underline{R}'$ the projection of R on $\underline{R}^*$ is a null-boundary Riemann surface and $\underline{R}^{\infty\prime}$ is hyperbolic type, i.e., it can be mapped conformally onto the unit-circle. We map $\underline{R}^{\infty\prime}$ onto $|\zeta| < 1$, then from Theorem 1. 1. on $|\zeta| = 1$ the set of points at which at least a curve converging to the boundary of $\underline{R}'$ ends, is a zero linear measure set and if we map R^{∞} onto $|z| < 1$. On $|z| = 1$, the set E_I of points at which at least a curve determining an A. B. P. lying on the ideal boundary of $\underline{R}'$ is also zero measure.

Let z_0 be a point at which a curve l (the image of L on R) determining an A. B. P. having its projection on $\underline{R}'$, then we can take a certain branch of $f(z)$ which converges to a point ζ_0, $|\zeta_0| < 1$ along l, then $f(z)$ has an angular limit ζ_0 also, this implies that any branch of $f(z)$ has angular limit relative to $\underline{R}'$. Let us fix a certain branch $f(z)$ mapping corresponding to $z \to \underline{R}'$ and denote by E the set at which a curve ends along which $f(z)$ determines an A. B. P. with projection on $\underline{R}'$.

Let $\underline{R}_\lambda$ be an exhaustion of $\underline{R} : \bigcup \underline{R}_\lambda = R$ and denote by $\Delta_{m \cdot n \cdot l}(\theta)$,

7) See M. Ohtsuka, 2).

the set $\frac{1}{n} \leqq |z-\theta| < \frac{1}{m}$, $0 < |\arg(z-\theta)| < \frac{\pi}{2} - \frac{1}{l}$ and by $\delta(f(z))$: $z \in \Delta_{m\cdot n\cdot l}(\theta)$ the diameter of the set $(f(z): z \in \Delta_{m\cdot n\cdot l}(\theta))$ with respect to A-topology of $\underline{R}^*$, then

$$E = E_\theta\left[\prod_l \sum_\lambda \prod_k \sum_n \prod_m (\delta f(\Delta_{m\cdot n\cdot l}(\theta)) \leqq \frac{1}{k}, \ f(\Delta_{m\cdot n\cdot l}) \subset \underline{R}_\lambda\right],$$

because $\delta(f(\Delta_{m\cdot n\cdot l}(\theta))$ is continuous with respect to θ, for fixed λ, l, m, n and κ, E is a Borel set on $|z|=1$.

Let $\mu(R^\infty, \mathfrak{A})$ be the lower envelope of the super-harmonicfunction $\{v(z)\}$ such as $0 \leq v(z) \leq 1$, andl im $v(z)=1$ along every path determining an A. B. P. relative to $\underline{R}$ and $\mu(R^\infty, \mathfrak{A}(E_I))$, $\mu(R^\infty, \mathfrak{A}(E))$ be the lower envelope of the super-harmonic functions such that $\lim v(p)=1$ along every path determining an A. B. P. lying on the boundary of $\underline{R}$, and on $\underline{R}$ respectively. We have $\mu(R^\infty, \mathfrak{A}(E_I)) \leqq \mu(\underline{R}^{\infty\prime}, \mathfrak{A})$ by Löwner's lemma and moreover $\mu(R^{\infty\prime}, \mathfrak{A})$ is zero as the solution of the Dirichlet problem on the ζ-plane, thus we have

$$\mu(R^\infty, \mathfrak{A}(E)) \leqq \mu(R^\infty, \mathfrak{A}) \leq \mu(R^\infty, \mathfrak{A}(E_I)) + \mu(R^\infty, \mathfrak{A}(E)) = \mu(R^\infty, \mathfrak{A}(E)).$$

On the other hand we denote by $\omega(E)$ the harmonic measure of measurable set E then, since $v(R^\infty, \mathfrak{A})=1$, when z converges radially to E, then we have easily the next equality $\mu(R^\infty, \mathfrak{A}) = \omega(z, E)$.

Theorem 2. 2. *When R is a covering surface over a null-boundary Riemann surface $\underline{R}$ if the universal covering surface of R is hyperbolic type then $\mu(R^\infty, \mathfrak{A}) = \omega(z, E)$.*

Theorem 2. 3.[8] *Let $\underline{R}$ be a null-boundary Riemann surface with A-topology and R be a positive boundary Riemann surface given as a covering surface over $\underline{R}$, if the universal covering surface of the projection of R is parabolic type, i. e. connot be mapped onto the unit-circle, take a finite number of points (p_i) $(i=1,2,\dots,n)$ and remove from R all the points (p_{ij}) $(j=1,2,3,\dots)$ lying over p_i so that the projection of the remaining surface $\tilde{R}$ may has an universal covering surface of hyperbolic type. Let $G(p, p_{ij})$ be the Green's function of R with the pole at p_{ij}. In the case when R covers p_i $(i=1,2,\dots,n)$ so rarely that $\sum_j^\infty \sum_i^n G(p, p_{ij}) < \infty$, if we map $\tilde{R}^\infty$ onto $|\eta|<1$ conformally, then we have*

$$\mu(R^\infty, \mathfrak{A}) = \mu(\tilde{R}^\infty, \mathfrak{A}).$$

8) This theorem is proved by M. Ohtsuka 2) under the condition that R covers p_i a finite number of times.

Proof. We map R^{∞} onto $|z|<1$, then a system z_{ijk} $(k=1, 2, \ldots)$ corresponds to a point p_{ij} and $\sum_{ji} G(p, p_{ij})=\sum_{k}\sum_{j}\sum_{i} \log \frac{|1-\bar{z}_{ijk}z|}{|z_{ijk}-z|}<\infty$ is equivalent to $\sum_{ijk}(1-|z_{ijk}|)<\infty$.

a) *A curve l in $|z|<1$ being the image of the curve L on R not passing through p_{ij} and determining an A.B.P. not lying on $p_i (i=1, 2, \ldots, n)$ converges to a point on $|z|=1$.*

We map $(\underline{R}-\sum^{n} p_i)^{\infty}$ onto $|\zeta|<1$ if l does not converge to a point, then converges to an arc γ; where mes $\gamma>0$. Since the mapping $f(z) \rightarrow \underline{R}$ is one-valued function, there exists a branch $\zeta=f(z)$ tending to a point ζ_0: $|\zeta_0|<1$ corresponding to p_0 along l. Let us take off radial segments S_{ijk} from z_{ijk} to the periphery $|z|=1$ and denote it by $(U_z-\sum_{ijk} S_{ijk})$ being simply connected domain, we map it onto $|\xi|<1$: $z=\varphi(\xi)$. Since the boundary of $(U_z-\sum_{ijk} S_{ijk})$ is rectifiable, γ is transformed onto a positive measure set γ' on $|\xi|=1$, $\zeta=f(\varphi(\xi))$ has angular limits on the ζ-plane almost everywhere on γ', therefore there exists a positive measure set γ'' on which $\zeta=f(\varphi(\xi))$ has angular limit ζ_0: $|\zeta_0|<1$, because the radius ending at γ'' is transformed to a curve in $|z|<1$, intersecting l infinitely many times on which $f(z)$ tends to ζ_0, this is a contradiction.

b) *The set E_z on $|z|=1$ at which at least an image of the curve not passing through p_i and determining A. B. P.s not lying on p_i ends is linearly measurable*[9].

In fact we map $\tilde{R}^{\infty}$ onto $|\eta|<1$, and denote by E_{η} the set at which at least a curve determining an A. B. P. ends. Let L be the above mentioned curve on R, since L does not pass through p_{ij}, L is contained in $\tilde{R}$ and the image l_{η} in η-plane of L converges. Since the projection of $\tilde{R}$ is $(\underline{R}-\sum^{n} p_i)$, $\zeta=\zeta(\eta)$ has angular limits relative to $\underline{R}$ at every point of E_{η}, then E_{η} is a Borel set. As $\tilde{R}^{\infty}$ can be considered as the universal covering surface of the domain (Unit-circle$-\sum_{ijk} z_{ijk}$) and l_z corresponds one-to-one manner to a system l_{η}, then E_z can be thought to be the image of E_{η} of the function $z=z(\eta)$, then E_z is an analytic set. Next, let l_z be a curve in R^{∞} determining an A. B. P. lying on p_i $(i=1, 2, \ldots, n)$ and E_{lz} be the set at least such a l_z converges, then it is easy to see that l_z converges and its outer measure of the set where at least a l_z converges is zero in regarding that the mapping $z \rightarrow \zeta$ and Löwner's lemma.

9) See M. Ohtsuka, 2).

c) *Let E_z be the measurable set of positive measure on $|z|=1$, and E_η be the set on $|\eta|=1$, where $z=z(\eta)$ has angular limits contained in E_z, then E_η is a set of inner measure positive.*

Let E_z' be a closed set of positive measure of E_z where $\eta=\eta(x)$ tends uniformly, then the set on $|\eta|=1$ where $z=z(\eta)$ has angular limits contained in E_z', is also measurable denoted by E_η'. On the other hand $(U-\sum_{ijk} S_{ijk})$ has a rectifiable curve as its boundary then E_ξ' on $|\xi|=1$ of positive measure and consider the composed function E_z' is transformed onto the set $\eta=\eta(z(\xi))$ in $U_z-\sum_{ijk} S_{ijk}$ and let E_η'' be the image of E_ξ' then $E_\eta''\subset E_\eta$, in mapping $z=z(\eta)$: $|\eta|<1$. We denote any super-harmonic function $v(\eta)$ such as $0\leqq v(\eta)\leqq 1$, $\lim\limits_{\eta\to E_\eta''} v(\eta)=1$, we consider this function on $|\xi|<1$ then clearly $v(\eta)\geqq\omega(E_\xi')$, where $\omega(E_\xi')$ is the harmonic measure of E_ξ' with respect to $|\xi|<1$, this follows that mes $E_\eta>0$.

Proof of the theorem. $\mu(R^\infty,\mathfrak{A})\leqq\mu(R^\infty,E_z)$ implies that $\mu(R^\infty,\mathfrak{A})$ has angular limits zero almost everywhere on the complementary set of E_z, on the other hand $\mu(\tilde{R}^\infty,\mathfrak{A})=\mu(\tilde{R}^\infty,E_\eta)=\omega_\eta(E_\eta,\eta)$ and $\mu(\tilde{R}^\infty,E_\eta)$ is one-valued harmonic function in $\tilde{R}$, this is also harmonic in R, there $\mu(\tilde{R}^\infty,E_\eta)$ is one-valued in $|z|<1$ and has angular limits almost everywhere on $|z|=1$. In denoting by $E_{1\cdot z}$ the subset of E_z being measurable where $\mu(\tilde{R}^\infty,E_\eta)$ has angular limits less than 1 and denoting by $E_{1\cdot\eta}$ the image of $E_{1\cdot z}$, then $\mu(\tilde{R}^\infty,E_{1\cdot\eta})=0$ follows in connection c) that $\mu(R^\infty,E_{1\cdot z})=0$. Thus we have $\mu(\tilde{R}^\infty,E_\eta)\geqq\mu(R^\infty,E_z)\geqq\mu(R,\mathfrak{A})$, because $\mu(R^\infty,E_z)$ is the lower envelope of super-harmonic functions having limits 1 along any curve ending at E_z. Since any super-harmonic determining $\mu(R^\infty,\mathfrak{A})$ can be considered as the function on $\tilde{R}^\infty$ if we supplement the value 1 on z_{ijk} then we have $\mu(\tilde{R}^\infty,\mathfrak{A})\leqq\mu(R^\infty,\mathfrak{A})$. Finally we have

$$\mu(\tilde{R}^\infty,E_\eta)\geqq\mu(R^\infty,E_z)\geqq\mu(R^\infty,\mathfrak{A})\geqq\mu(\tilde{R}^\infty,\mathfrak{A})\geqq\mu(\tilde{R}^\infty,E_\eta).$$

2. On the behaviour of Green's function in the neighbourhood of the ideal boundary

Theorem 2.4. *Let $G(p,p_0)$ be the Green's function of R with its pole at p_0, D_λ be the closed domain such as $G(p,p_0)\geqq\lambda$ where $\lambda>0$ and R_m be an exhaustion of R. We denote by B_λ the boundary elements defined by $\lim\limits_n (R-R_n)\cap D_\lambda$, then B_λ is outer harmonic measure zero.*

Proof. Let v_n^λ be continuous super-harmonic function in R such as $0\leqq v_n^\lambda(p)\leqq 1$, and $v(p)=1$ if $p\in D_\lambda\cap(R-R_n)$ and its lower envelope of $v_n^\lambda(p)$ be denoted by $\mu_n^\lambda(p)$, since $\frac{1}{\lambda}\,\text{Min}\,[\lambda, G(p, p_0)]$ is one of such functions, and $\mu_n^\lambda(p) \geqq \mu_{n+1}^\lambda(p)$ and $\lim\limits_n \mu_n^\lambda(p) = \mu^\lambda(p)$ is harmonic in R. $G(p, p_0) - \lambda\mu_n^\lambda(p)$ is harmonic and positive in R_n and has a logarithmic singularity p_0 then

$$G(p, p_0) - \lambda\mu_n^\lambda(p) \geqq G(p, p_0)_{R_n}$$

where $G(p, p_0)_{R_n}$ is the Green's function of the surface R_n, let $n \to \infty$

$$G(p, p_0) - \lambda\mu_n^\lambda(p) \geqq G(p, p_0),$$

because $G(p, p_0)$ is the lower envelope of positive harmonic function with the same singularity as $G(p, p_0) - \lambda\mu_n^\lambda(p)$, this follows that $\lim \mu_n^\lambda(p) = 0$.

We extract a sebsequence $v_{n_1}^\lambda(p), v_{n_2}^\lambda(p) \cdots$ of which the lower envelope is $\mu^\lambda(p)$, thus we have the theorem.

Corollary. *Let $\mu_n^{*\lambda}(p)$ be the lower envelope of non negative super-harmonic function $v_n^\lambda(p)$ in D_λ such as $0 \leqq v_n^\lambda(p) \leqq 1$, and $\lim\limits_{P \to B_\lambda} v^*(p) = 1$, then $\mu^{*\lambda}(p) = 0$.*

In fact, we can consider $v_n^\lambda(p)$ is one of $v_n^{*\lambda}(p)$, then we have the conclusion.

Corollary. *There exists a continuous positive super-harmonic function satisfying the following conditions* 1) $0 \leqq U(p) < \infty$ *in* R. 2) $\lim\limits_{P \to B} U(p) = \infty$.

Proof. Let $B = \sum\limits_{n=2}^{\infty} B_{\frac{1}{n}}$, since for every λ, B_λ is outer harmonic measure zero, then we can extract a sequence of continuous super-harmonic function $U^{\frac{1}{n}}(p)$ such that $\text{Max}\,(U^{\frac{1}{n}}(p)) \leqq \frac{1}{2^n}$, if $p \in R_n$, $U^{\frac{1}{n}}(p) = 1$ if $p \overline{\in} R_{n'}$ and $p \in D_{\frac{1}{n}}$, where $n' = n'(n)$. Put $U(z) = \sum U^{\frac{1}{n}}(p)$, then $U(p) \leqq n + \frac{1}{2^n}$ if $p \in R_n$ and uniformly convergent in every R_n then $U(p)$ is continuous and super-harmonic in R and $\lim\limits_{P \to B} U(p) = \infty$.

We consider the case when $\mu(R, \mathfrak{A})$ is equal to $\mu(\tilde{R}^\infty, \mathfrak{A})$.

Theorem 2. 5. *Let R be a positive boundary Riemann surface and $\underline{R}'$ be the projection of R over $\underline{R}$, we take off points p_i $(i = 1, 2, \ldots, n)$ so that the universal covering surface of $(\underline{R} - \sum\limits^{n} p_i)$ is hyperbolic type, and denote by $G(p, p_{ij})$ the Green's function with its pole at p_{ij} lying over p_i. When R^∞ is mapped conformally on $|z| < 1$, we denote by D_0 the normal polygon of Fuchsian Group containing $z = 0$, with arcs α_j $(j = 1, 2, \ldots)$ on $|z| = 1$ and by $\{T_i\}$ $(i = 1, 2, \ldots)$ the substitutions of Fuchsian group,* if

$$\sum_j \sum_i G(p, p_{ij}) < \infty, \quad \text{and}$$
$$\text{mes}\ (\sum_i^{\infty} T_i\ (\sum_j^{\infty} \alpha))_j = 2\pi, \quad \text{then}$$
$$\mu(R, \mathfrak{A}) = \mu(\tilde{R}, \mathfrak{A}) = \mu(\tilde{R}^{\infty}, \mathfrak{A}) = \mu(R^{\infty}, \mathfrak{A}).$$

Proof. We denote the set by E where at least a curve determining an A. B. P ends, we have seen that E is measurable and $\mu(R^{\infty}, \mathfrak{A}) = \mu(R^{\infty}, E) = \omega(z, E)$. For any given positive number $\varepsilon > 0$, there exists a number p and k such that

$$\text{mes}\ (\sum_j^{p} T_i\ (\sum \alpha_j)) > 2\pi - \varepsilon,\ \frac{1}{k} \leqq \frac{\text{mes}\ T_i(l)}{\text{mes}\ l} < k:\ i = 1, 2, \ldots, p$$

for any line element l on α_j, we can extract subarcs denoted by α_j^* having no common ends with α_j such that $\text{mes}\ (\sum_j^{\infty} \alpha_j - \sum_j^{\infty} \alpha_j^*) < \frac{\varepsilon}{2kp}$, then we have $\text{mes}\ (\sum_i^{\infty} T_i\ (\sum_j^{\infty} \alpha_j) - \sum_i^{p} T_i\ (\sum_j^{q} \alpha_j)) < 2\varepsilon$, if $\mu(R, R) \gneqq \mu(R^{\infty}, \mathfrak{A})$ we have $\mu(\tilde{R}, 0) > \mu(\tilde{R}^{\infty}, 0) + \delta : \delta > 0$, since $\mu(\tilde{R}, \mathfrak{A})$ is one-valued harmonic function in $|z| < 1$, denoting by $H_{\frac{1}{n}}$ the set where $\mu(\tilde{R}, \mathfrak{A})$ has angular limits larger than $\frac{1}{n}$, then $\text{mes}\ ((2\pi -)E \cap H_{\frac{1}{n}} \cap \sum^{j} T_i(\sum^{q} \alpha_j^*)) > \delta$ and $\text{mes}\ ((\pi - E) \cap \sum^{p} T_i\ (H_{\frac{1}{n}} \cap \sum^{q} \alpha_j^*) > \delta - \varepsilon$.

Let (complement of E) $\cap\ \alpha_j^* = \alpha_j^{**}$, and we can take a closed subset denoted by $F^*_{\frac{1}{n}}$ in $\sum \alpha_j^{**}$ such that $\sum_i^{p} T_i\ (F^*_{\frac{1}{n}}) > \frac{\delta}{2}$. On the other hand let $\omega_{F^*_{\frac{1}{n}}}(z)$ be the harmonic measure of $\sum^{\infty} T_i\ (F^*_{\frac{1}{n}})$, then $\omega_{F^*_{\frac{1}{n}}}(0) \geqq \frac{\delta}{2}$ and one-valued on R. We consider non negative super-harmonic function $v^*(p)$ in $\tilde{R}$ such that $\lim v^*(p) = 1$ when p tends to $F^*_{\frac{1}{n}}$ along every curve and denote by $\mu^*(\tilde{R}, p)$ their lower envelope, then we have $\mu^*(\tilde{R}, \mathfrak{A}) \geqq \omega_{F^*_{\frac{1}{n}}}(F^*_{\frac{1}{n}})$. Since $F^*_{\frac{1}{n}}$ is a set on arcs $\alpha_1 \cdots \alpha_2$, the behaviour of $\mu^*(\tilde{R}, \mathfrak{A})$ is the same as that of Green's function i.e., if $\lim G(p, p_0) = 0$ when p tends along every curve converging out of $F^*_{\frac{1}{n}}$ then $\lim \mu^*(R, p) = 0$ along these curves. But we have seen in corollary of theorem 2. 4 that there exists a non negative continuous super-harmonic function $V(p)$ such that $\lim_{P \to B} V(p) = \infty$, where $B = \sum_{\lambda} B_{\lambda}:\ G(p, p_0) > \lambda$. Let $v(p)$ be of the class defining $\mu(\tilde{R}, \mathfrak{A})$ and consider next function

$$S(p) = \text{Min}\ (\varepsilon V(p) + v(p) - \frac{1}{n}\omega(p).\ 1).$$

case 1. p tends to an A.B.P. then this curve ends out side of $\sum T(F^*_{\frac{1}{n}})$, we have to consider in D_0, since $F^*_{\frac{1}{n}}$ is closed then $\omega_{F^*_{\frac{1}{n}}}(p)=0$ thus $S(p)=1$.

case 2. p tends to $F^*_{\frac{1}{n}}$ on the z-plane, since $v(p)\geqq\frac{1}{n}$ along Stolz's path converging to $F^*_{\frac{1}{n}}$, $S(p)\geqq 0$.

case 3. p tends to the boundary of R, except $F^*_{\frac{1}{n}}$, then $\varepsilon V(p)-\frac{1}{n}\omega_{F^*_{\frac{1}{n}}}(p)\geqq 0$. Hence $S(p)$ has angular limits $\geqq 0$ almost everywhere on $|z|=1$ and bounded, therefore $S(p)\geqq 0$ in R, finally $S(p)$ is a one of super-harmonic function but ε is arbitrary we have, the lower envelope of $S(p)=\mu(R,\mathfrak{A})-\frac{1}{n}\omega_{F^*_{\frac{1}{n}}}(p)\lneqq\mu(R,\mathfrak{A})$. This is a contradiction, then we have $\mu(\tilde{R},\mathfrak{A})=\mu(\tilde{R}^\infty,\mathfrak{A})$ any by the preceding theorems we have the conclusion.

Corollary. *The connectivity of R is finite and universal covering surface of the projection of R on $\underline{R}$ is hyperbolic then we have $\mu(R,\mathfrak{A})=\mu(R^\infty,\mathfrak{A})$.*

Theorem 2. 6. *Let R be a positive boundary Riemann surface and the universal covering surface of the projection of R over $\underline{R}$ be hyperbolic type. If $\mu(R,\mathfrak{A})=\mu(R^\infty,\mathfrak{A})$ then we have $\mu(R,\mathfrak{F})=\mu(R^\infty,\mathfrak{F})$ for every closed subset $\mathfrak{F}$ of $\mathfrak{A}$.*

We denote by $\mathfrak{F}_n$ the set of of $R+\mathfrak{A}$ such as all points of R with distance $\leqq\frac{1}{n}$ from $\mathfrak{F}$, then $\mathfrak{F}=\bigwedge_n\mathfrak{F}_n$ and denote by F_n the image of $\mathfrak{A}\bigwedge\mathfrak{F}_n$ on $|z|=1$: $(|z|<1)\equiv R^\infty$, then F_n is measurable. Since the image $\mathfrak{A}$ on $|z|=1$ with its projection lying on the boundary of $\underline{R}$ has outer zero measure where the mapping function has not necessarily angular limits, we can suppose the projectionof $\mathfrak{A}$ lies on $\underline{R}$. Let R_m be the exhaustion of R, Γ_m be its relative boundary, and α_n be the relative boundary of $\mathfrak{F}_n$, we define

$\omega^n_{m\cdot m+i}(p)$: $(i=1,2,\ldots)$ satisfying the following conditions,

$\omega^n_{m\cdot m+i}(p)$ is non negative continuous super-harmonic function in R_{m+i}

$\omega^n_{m\cdot m+i}(p)=1$, if $p\in\mathfrak{F}_n\bigwedge(R_{m+i}-R_m)$, $\omega^n_{m\cdot m+i}(p)$ is harmonic in $R_{m+i}-(\mathfrak{F}_n\bigwedge(R_{m+i}-R_m))$, $\omega^n_{m\cdot m+i}(p)=0$ if $p\in\Gamma_{m+i}-(\Gamma_{m+i}\bigwedge\mathfrak{F}_n)$.

$\omega^n_{m\cdot m+i}(p)\leqq\omega^n_{m\cdot m+i+j}(p)$. Let $\omega^n_m(p)$ be the lower envelope of $\omega^n_{m\cdot m+i}$: $i=1,2,\ldots$ $\omega^n_m(p)$ is super-harmonic therefore $\mu(R,\mathfrak{F}_n)\leqq\omega^n_m(p)$, $\mu(R,\mathfrak{F})<\lim_n\omega^n_m(p)$ for every n.

If $\mu(R,\mathfrak{F})\neq\mu(R^\infty,\mathfrak{F})$, we would have mes $(\mathfrak{A}_z\bigwedge C\mathfrak{F}_z)>0$, where $\mathfrak{A}_z$ means the image of $\mathfrak{A}$ and $C\mathfrak{F}_z$ means the complement of the image $\mathfrak{F}$ with respect to $|z|=1$.

Let $f(z)$ be the mapping function from the z-plane onto $R+\mathfrak{A}$, $D_{\delta.F_l}$ the domain in $|z|<1$ such that it contains the endpart of the angular domain: $|\arg(1-e^{-i\theta}z)|<\frac{\pi}{2}-\delta$: $(\delta>0)$ at every point $e^{i\theta}$ of closed set F_l and $C_{\delta'}$ be the ring domain $1-\delta'<|z|<1$.

Since A. B. P corresponding to $\mathfrak{A}\frown C\mathfrak{F}$ which has positive distance from ones corresponding to $\mathfrak{F}$, then for given positive number $\varepsilon>0$ and δ, there exists a closed subset F_l of $\mathfrak{A}_z\frown C\mathfrak{F}_n$ such that mes $(C\mathfrak{A}_z-F_l)$ $<\varepsilon$, and dist $(f(z),\mathfrak{F}_n)\geqq\frac{1}{2n}$ if $z\in D_{\delta.F_l}\frown C_{\delta'}$. The boundary Γ_n of R_m is transformed onto a simple closed Jordan curve γ_m in $|z|<1$. Denote by r_m the distance γ_m from $z=0$, and let m_0 be the smallest number such as $r_{m_0}>1-\delta$, then for $\Gamma_m:(m\geqq m_0)$, the image of Γ_m is contained in $C_{\delta'}$ and the part of the image contained in $D_{\delta.F_l}$ are the arcs where $\omega^n_{m.m+i}$: $i=1,2,\ldots$ takes value 0 except a set of linear measure zero.

Let C'_{r_m} be the circle $|z|<r_m$, and $D^*_{\delta.F_l}$ be the set $D_{\delta.F_l}\smile C'_{r_m}$, and $U_m(z)$ be the harmonic function in $D^*_{\delta.F_l}$ such that $0\leqq U_m(z)\leqq 1$, $U_m(z)=1$ when z lies on the boundary of $D^*_{\delta.F_l}$ except on the boundary on $|z|=1$ and vanishes on the boundary lying on $|z|=1$. Since the boundary of $D^*_{\delta.F_l}$ is rectifiable $U_m(z)\leqq 1$ and $U_m(z)=0$ when z tends to F_l almost everywhere. It is clear that $U_m(z)\geqq\omega^{2n}_{m.m+i}(z)\geqq\mu(R,\mathfrak{F})$ if $z\in D_{\delta.F_l}\frown R^\infty_m$ for every i and n, but ε and δ are arbitrary, therefore $\mu(R,\mathfrak{F})$ has angular limits 0 almost everywhere on $\mathfrak{A}_z\frown C\mathfrak{F}_z$. On the other hand $\mu(R,\mathfrak{A})>\mu(R,\mathfrak{F})$, it follows $\mu(R,\mathfrak{F})$ has angular limits zero almost everywhere on $C\mathfrak{F}_z$, thus $\mu(R,\mathfrak{F})$ has angular limit zero almost everywhere on $C\mathfrak{F}_z$ and so $\mu(R,\mathfrak{F})=\mu(R^\infty,\mathfrak{F})$.

Theorem 2. 7. *Suppose that R'^∞ is hyperbolic type. Take a finite number of $\{p_i\}$ from $\underline{R}$ and remove from R all the points lying over them and denote the remaining surface by $\tilde{R}$, then there holds*

$$\mu(R^\infty,\mathfrak{A})=\mu(\tilde{R}^\infty,\mathfrak{A}).$$

This is proved by M. Ohtsuka in regarding that the mapping function from R^∞ onto $\underline{R}$ has angular limits and the set where at least a curve determining A.B.P. lying on $\{p_i\}$ is outer harmonic measure zero.

Till now we have investigated the case when harmonic measures have equal values, in the following we show by example there are cases when inequalities hold.

3. Counter examples of the preceding theorems

Example. Let B_{2n}, B_{2n+1}: $n=1,2,\dots$ be the system closed domains in $|z|<1$ such as

$$B_{2n}:\ 1-\frac{1}{4n+3}\leqq r\leqq 1-\frac{1}{4n+4}:\ \frac{3\pi}{4}\leqq\theta\leqq\frac{\pi}{4}\ \left(\text{containing } -\frac{\pi}{2}\right),$$

$$B_{2n+1}:\ 1-\frac{1}{4n}\leqq r\leqq 1-\frac{1}{4n+1}:\ -\frac{3\pi}{4}\geqq\theta\geqq-\frac{5\pi}{4}\ \left(\text{containing } \frac{\pi}{2}\right).$$

We can construct the holomorphic function[10] $f(z)$: $|z|<1$ by Runge's theorem such that

$$|f(z)-1|<\frac{1}{n}\ \text{ if }\ z\in B_{2n},\ \ |f(z)|<\frac{1}{n}\ \text{ if }\ z\in B_{2n+1}.$$

It is clear that $f(z)$ is not bounded in $|z|<1$ and $w=f(z)=\infty$ is an asymptotic value, therefore there exists an asymptotic path l along which $f(z)$ tends to infinity, this path determines an A. B. P. lying on $w=\infty$, let p_0 be the starting point on l where $|f(z)|=M_0$ and this path lies in the unit-circle except $\sum\limits_{n_0}^{\infty}(B_{2n}+B_{2n+1})$ and denote by p_r the first point when p lies on $|z|=r$ in coverging to the boundary $|z|=1$ from p_r and by l_r the endpart from p_r.

Let $\{v(z)\}$ be non negative continuous super-harmonic function in $|z|<1$ such as $0\leqq v(z)\leqq 1$ and $\lim v(z)=1$ when z tends to $|z|=1$ along l, take a $v(z)$, then there is r_0 such that $v(z)\geqq 1-\delta$, if $z\in l_{r_0}$ for every δ where l_{r_0} has a branch passing left half plane and let n_0 be the number such as $1-\frac{1}{4n_0+3}\geqq r_0$, and A and E be the point $\left(r=1-\frac{1}{4n_0+1},\ \theta=\frac{5\pi}{4}\right)$. $B=\left(r=\frac{1}{4n_0+3},\ \theta=\frac{5\pi}{4}\right)=E$. $C=\left(r=1-\frac{1}{4n_0+3},\ \theta=\frac{3\pi}{4}\right)$ and $D=\left(r=1-\frac{1}{4n_0+3},\ \theta=0\right)$ and $U(z)$ be a harmonic function such as $U(z)=1$, if $z\in\overline{AB}$ and on the circular arc $\widehat{BC}$, and vanishes on the circular arc $\widehat{CDE}$, or on the segment $\overline{EF}$ then $V(0)\geqq U(0)=\delta_0>0$, for every $v(z)$[11]. Hence $\mu(R, w=\infty)$ being the lower envelope of $\{v(z)\}$ is $\geqq\delta_0>0$.

We remove all points where $f(z)=0, 1$ and 2 from the unit-circle, then the universal covering surface of the projection of the remaining surface R' is hyperbolic type then it is clear $\mu('R,\infty)>0$ and $\mu(R'^{\infty},\infty)=\mu(\tilde{R}^{\infty},\infty)=0$. Hence by theorem 2.6. $\mu(R',\mathfrak{A})\gneqq\mu(R'^{\infty},\mathfrak{A})$.

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10) L. Bieberbach: Funktionentheorie II.

11) When l_{r_0} has right half plane, similar fact occurs.

1) If we consider the R' to be given surface R, then $\mu(R, \mathfrak{A}) \gneqq \mu(R^\infty, \mathfrak{A})$.

2) If we consider R given surface $R^\infty = R$, then $\mu(R^\infty, \mathfrak{A}) \gneqq \mu(\tilde{R}^\infty, \mathfrak{A})$.

We easily see that these surfaces do not satisfy the conditions of preceding theorems.

4. Type of covering surfaces

1) If there holds $u(p) \leqq 0$ on R whenever $u(p)$ is upper bounded continuous sub-harmonic function on R such that $\lim_{P \to \mathfrak{A}} u(p) \leqq 0$, we call R a surface of D-type.

2) Let R^∞ be hyperbolic hyperbolic type, when the function cores-ponding to the mapping onto the unit circle $R^\infty \to R \to \underline{R}^*$ has limits along Stolz's path almost everywhere on $|z| = 1$, where $R^\infty \equiv (|z| < 1)$.

We call in this case R F-type covering.

3) If $T(r)$ of the mapping function from R to $\underline{R}$ is bounded, we say R is bounded covering, we have easily

$$D\text{-type} \rightleftarrows \mu(R, \mathfrak{A}) = 1\,. \quad B\text{-type} \to F\text{-type} \to D\text{-type}\,,$$

and if the universal covering surface of the projection of R is hyper-bolic, we have F-type $\leftrightarrows \mu(R^\infty, \mathfrak{A}) = 1$.

In the sequal we investigate type of covering surface among covering surfaces over the same basic surface. We have at once from theorem 1.2. the next

Theorem 2.8. *When $\hat{R}$ is a covering surface over R and R is of bounded type over $\underline{R}$, then $\hat{R}$ is of bounded type over $\underline{R}$, too.*

Theorem 2.9. *When the universal covering surface of the projection of R over $\underline{R}$ is hyperbolic type, if R^∞ is F-type and if $\hat{R}$ is a covering surface over R, then $\hat{R}$ is also F-type relative to $\underline{R}$.*

Proof. We map $\underline{R}^\infty, R^\infty, \hat{R}^\infty$, onto $|\xi| < 1$, $|z| < 1$ and $|\zeta| < 1$ and denote by $E_{I\xi}$, E_{Iz} and $E_{I\zeta}$ the set where at least a curve determining an A. B. P. lying on the boundary of $\underline{R}$, since $E_{I\xi}, E_{Iz}, E_{I\zeta}$ are of outer harmonic measure zero, we may suppose without loss of generality that A. B. P.s lie on $\underline{R}$. Let $\mathfrak{A}_\xi, \mathfrak{A}_z, \mathfrak{A}_\zeta$ the set being measurable where the mapping function onto $\underline{R}$ has angular limits on $\underline{R}$, then $\mu(R^\infty, \mathfrak{A}_z) = 1$. If mes $(C\mathfrak{A}_\zeta) > 0$, where $C\mathfrak{A}_\zeta$ means the complementary set of $\mathfrak{A}_\zeta$ and let $z = z(\zeta)$, $0 = z(0)$, $\xi(z(\zeta)) = \xi(\zeta)$ be the mapping function onto the z-plane and ξ-plane respectively, they have angular limits almost everywhere on $|z|<1$ and ξ-plane. Denote by l_ζ the radial path ending at $C\mathfrak{A}_\zeta$ along which $z=z(\zeta)$. $\xi=\xi(\zeta)$ has angular limits on z, ξ-planes.

$z=z(\zeta)$ does not converges to $z_0: |z_0|<1$, because if it were so, l_ζ determines an A.B.P. on $\underline{R}$, therefore $z=z(\zeta)$ converges to $z_0: |z_0|=1$ and further the image l_z of l_ζ does not intersect infinitely many times with the angular endpart $\arg|z-e^{i\theta}|<\frac{\pi}{2}-\delta: \delta>0$, $e^{i\theta}\in\mathfrak{A}_z$, because if l_z intersects it infinitely, since $\xi=\xi(z)$ tends in this angular domain to a point $\xi_0: |\xi_0|<1$ with regarding that $\xi=\xi(\zeta)=\xi(z(\zeta))$ has limit $\xi_0: |\xi_0|<1$ on the ξ-plane, it follows that $\xi=\xi(\zeta)$ determines an A.B.P along l_z, this is a contradiction, thus $z=z(\zeta)$ has no angular limit or converges to $z: z\in C\mathfrak{A}_z$ or converges tangentially to $z: z\in\mathfrak{A}_z$.

For given numbers ε, δ there exist numbers ε', δ' such that $z=z(\zeta)$ is not contained in $D_{z.\delta.F_\varepsilon}\bigwedge C_{\delta^*}$ if $\zeta\in D_{\zeta.\varepsilon'.\delta'.F_{\varepsilon'}}\bigwedge C_{\delta^*}$, where F_ε, $F_{\varepsilon'}$ are closed sets of $\mathfrak{A}_z$, $C\mathfrak{A}_\zeta$ and mes $|\mathfrak{A}_z-F_\zeta|<\varepsilon$, mes $|C\mathfrak{A}_\zeta-F_{\varepsilon'}|<\varepsilon'$, $D_{z.\delta.F_\varepsilon}$, $D_{\zeta.\delta'.F_{\varepsilon'}}$ are angular domains on z, ζ-plane $\arg|1-ze^{-i\theta}|<\frac{\pi}{2}-\delta$ for any point of $e^{i\theta}\in F_\varepsilon$, $\arg|1-\zeta e^{-i\theta}|<\frac{\pi}{2}-\delta': e^{i\theta}\in F_{\varepsilon'}$, respectively, $C_{z.\delta^*}$, $C_{\zeta.\delta'^*}$, $\bar{C}_\delta$ are the rings $1-\delta<|z|<1$, $1-\delta'<|\zeta|<1$ and a circle $1-\delta>|z|$, $\lim\limits_{\substack{\varepsilon'\to0\\ \delta'\to0}}\varepsilon=0$, $\lim\limits_{\substack{\varepsilon'\to0\\ \delta'\to0}}\delta=0$.

Let $U_{\delta.\varepsilon}(z)$ be the harmonic function in $D_{z.\delta.F_\varepsilon}\bigvee\bar{C}_\delta$ such as $U_{\delta.\varepsilon}(z)=0$ on the boundary of $D_{z.\delta.F_\varepsilon}\bigvee\bar{C}_\delta$ except its boundary on $|z|=1$ and 1 on the boundary lying on $|z|=1$, and $\omega(\zeta)$ be the harmonic measuse of $F_{\varepsilon'}$, then we have $1-\omega(\zeta)\geqq U_{\delta.\varepsilon}(z)$ when $U_{\delta.\varepsilon}(z)$ is considered on ζ-plane, since $\mathfrak{A}_z$, $C\mathfrak{A}_\zeta$ are measurable and ε, δ are arbitrary and $\lim\varepsilon'=0$, $\lim\delta'=0$ then we have

$$\mu(\hat{R}, \mathfrak{A})=\lim(1-\omega_{F_{\varepsilon'}}(\zeta))\geqq\lim_{\substack{\delta\to0\\ \varepsilon\to0}}U_{\delta.\varepsilon}(z)=\mu(R^\infty, \mathfrak{A})$$

This implies the conclusion.

Corollary. *We can weaken the conditions of the preceding theorem in such a way the condition that the universal covering surface of the projection R is hyperbolic is changed by the condition that $\sum\limits_{ij}G(p, p_{ij})<\infty$, where p_{ij} are points of R lying on p_i of $\underline{R}$, so that the universal covering surface of $(\underline{R}-\sum p_i)$ is hyperbolic.*

Corollary. *Let R be a positive boundary Riemann surface and covers p_i so rarely as $\sum G(p, p_{ij})<\infty$, then $\mu(R^\infty, \mathfrak{A})\leqq\mu(R^\infty, \hat{\mathfrak{A}})$.*

Proof. We denote by p_{ijk}; $(k=1, 2, \ldots)$ the points lying on p_{ij} then we have

$$\sum_k \sum_j \sum_i G(p, p_{ijk}) \leq \sum_j \sum_i G(p, p_{ij}),$$

where $G(p, p_{ijk})$, $G(p, p_{ij})$ are Green's function of R and $\hat{R}$. Let $\tilde{\hat{R}}$ and $\tilde{R}$ be the remaining surface after removing p_{ijk} and p_{ij} from $\hat{R}$ or R then we have from theorem 4. 4.

$$\mu(\hat{R}^\infty, \mathfrak{A}) = \mu(\tilde{\hat{R}}^\infty, \mathfrak{A}) \geq \mu(\tilde{R}^\infty, \mathfrak{A}) = \mu(R^\infty, \mathfrak{A}).$$

Example. *A covering surface over a D-type covering surface is not always D-type.* In the preceding theorem we take as B_{2n} and B_{2n+1} the domains shown in the figure 1.

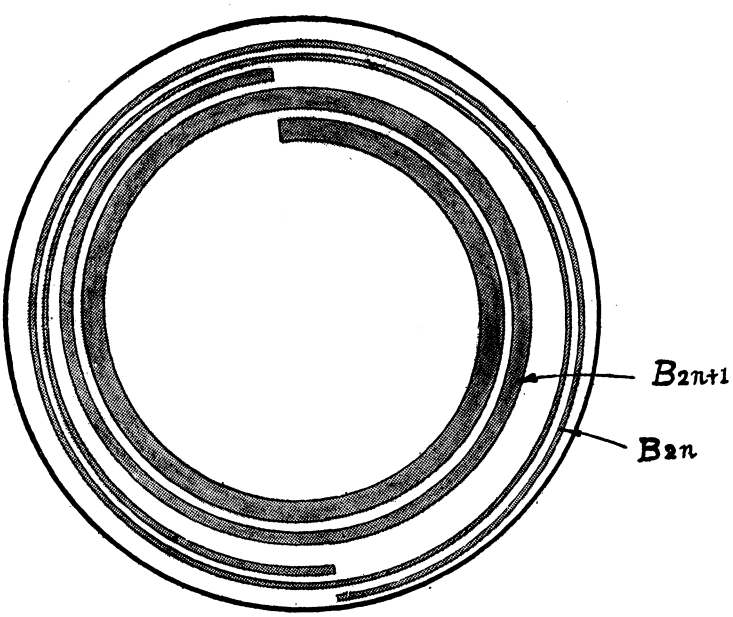

Fig. 1

5. Dirichlet problem

5. 1. Dirichlet problem on a covering surface over a null-boundary Riemann surface

Lemma. *Let $\underline{R}$ be a null-boundary Riemann surface and R be a F-type covering surface over $\underline{R}$ and $\mathfrak{F}$ be a closed set of $\mathfrak{A}$, the upper class $U^R_{\mathfrak{F}}$ is defined by all non negative continuous super-harmonic functions such that $\underline{\lim}_{P\to\mathfrak{F}} u\ (p) = 1$, then the lower envelope $\bar{H}^R_{\mathfrak{F}}(p)$ is harmonic. Similarly the lower class $\mathfrak{B}_{\mathfrak{F}}$ and $H^R_{\mathfrak{F}}(p)$ are defined for upper bounded continuous sub-harmonic function such that $\overline{\lim}_{P\to C\mathfrak{F}\cap\mathfrak{A}} v(p) \leq 0$. Further it is clear that*

$\bar{H}^{R}_{\mathfrak{F}}(p) \geqq \underline{H}^{R}_{\mathfrak{F}}(p)$ *on D-type Riemann surface, and if they coincide at a point, they are identical in R.*

Lemma. *Let $\mathfrak{F}$ be a closed subset of $\mathfrak{A}$ of F-type covering surface then there holds*

$$\bar{H}^{R}_{\mathfrak{F}}(p) = \underline{H}^{R}_{\mathfrak{F}}(p).$$

Proof. Mapping R^{∞} onto $|z|<1$, since the set E_I the image of the A. B. P.s lying on the boundary of $\underline{R}$ has outer harmonic measure zero i. e. $\mu(R, \mathfrak{A} \cap E_I) = 0$. We may suppose that A. B. P.s lie on $\underline{R}$. Let $\mathfrak{F}_n$ be the set of $R+\mathfrak{A}$ which has distance from $\mathfrak{F} \leqq \frac{1}{n}$ and F_n be the image of $\mathfrak{F}_n \cap \mathfrak{A}$ on $|z|=1$, then $\mathfrak{F} = \bigcap_n \mathfrak{F}_n$ and $F = \bigcap_n F_n$ and they are measurable.

Let R_m be an exhaustion of R with relative boundary Γ_m and γ_n be the boundary of $\mathfrak{F}_n$ and $\omega^n_{m\cdot m+i}(p)$ $(i=1, 2, \ldots)$ be non negative continuous super-harmonic function in R such that $\omega^n_{m\cdot m+i} \leqq 1$ and harmonic in $(R_{m+i} - (\mathfrak{F}_n \cap R_{m+1} - R_m))$ and 1, if $p \in \mathfrak{F}_n \cap (R - R_m)$ and vanishes on $C\mathfrak{F}_n \cap \Gamma_{m+i}$, $\omega^n_{m\cdot m+i}(p)$ is increasing with respect to i, and $\omega^n_m = \lim_{i=\infty} \omega^n_{m\cdot m+i}$ is contained in $U_{\mathfrak{F}}$ class and harmonic in R_m and ω^n_m is decreasing with respect to m. $\lim_m \omega^n_m(p)$ is harmonic in $R - F_n$, then $\bar{H}^{R}_{\mathfrak{F}}(p) \leqq \omega^n$ for every n. On the other hand at any point of $\mathfrak{A}_z \cap CF_n$, $f(z)$ does not tend to F_{2n} in angular domain, because dist $[(\mathfrak{A} \cap C\mathfrak{F}_n). \mathfrak{F}_n] \geqq \frac{1}{2n}$.

Let $D_{\kappa\delta}$ be the domain with arg $|-ze^{-i\theta}| < \frac{\pi}{2} - \delta$, $e^{i\theta} \in \kappa$, $r_{m+i} =$ dist (image of γ_{m+i} in $|z| < z=0$, $C_{r_{m'}}$ be the ring $r'_m < |z| < 1$, and $C^*_{r_m}$ be open circle $|z| < r'_m$. For any given number $\varepsilon > 0$, and $\delta > 0$ we can find a closed subset F^*_{2n} of $\mathfrak{A}_z$ and $\mathfrak{A} \cap CF_{2n}$, such that, mes$|\mathfrak{A}_z - F_{2n}| < \varepsilon$, if $z \in D_{F^*_{2n}\cdot\delta} \cap C_{r_{m'}}$, then $f(z)$ is contained in the complement of F_n. Let $U^n_m(z)$ be harmonic in $D_{F^*_{2n}\cdot\delta} \cup C^*_{r_m}$ and $\leqq 1$ and 1 on this domain not lying on $|z|=1$ and vanishes on the boundary on $|z|=1$, then

$$U^n_m \geqq \omega^n_{m\cdot m+i} \text{ and harmonic in } D_{F^*_{2n}\cdot\delta} \cup C^*_{r_m},$$

$$U^n_m \geqq \omega^n_{m\cdot m+i} \text{ for every } i, \text{ there holds } U^n_m \geqq \omega^n_m.$$

Since ε, δ and n are arbitrary, and the boundary of $D_{F^*_{2n}\cdot\delta} \cup C^*_{r_m}$ is rectifiable $\lim_n (\lim_m U^n_m) \geqq \lim_n \omega_{F_n}(z) \geqq \lim_n \lim_m \omega^n_m(p) \geqq \lim_n \omega^n(p) \geqq \bar{H}^{R}_{\mathfrak{F}}(p)$ and the converse is true, then we have

$$\bar{H}^{R}_{\mathfrak{F}}(p) = \omega(z, F).$$

Let $\Omega_n = \mathfrak{A} - \mathfrak{F}_n$ and $\acute{\omega}^n_{m\cdot m+i}$ be super-harmonic in R such that

$0 \leqq \acute{\omega}^n_{m\cdot m+i}(p) \leqq 1$ and 1 when $p \in \Omega_n \cap R - R_m$ and $\gamma_m \cap \Omega_n$ and vanishes on $\Gamma_{m+i} - \Gamma_{m+i} \cap \Omega_n$, and harmonic in $R_{m+i} - \Omega_n$. $\lim\limits_i \acute{\omega}^n_{m\cdot m+i}(p) \uparrow = \acute{\omega}^n_m(p)$, $\lim\limits_m \acute{\omega}^n_m(p) \downarrow = \acute{\omega}^n(p)$ is harmonic and has angular limits zero almost everywhere on $\mathfrak{A}_z \cap CO_{\frac{n}{2}}$, where $O_{\frac{n}{2}}$ is the image of $\Omega_{\frac{n}{2}}$. Denote by J^λ_n the set in Ω_n where $\bar{H}^R_{\mathfrak{F}}(p) \geqq \lambda$ and part $T^{\lambda\cdot n}_m = (R - R_m) \cap J^\lambda_n$ and $\overset{\prime\prime}{\omega}{}^{n\cdot\lambda}_{m\cdot m+i}(p)$ being super-harmonic in R and harmonic in $R_{m+i} - T^{\lambda\cdot n}_m$ and attains 1 when $p \in T^{\lambda\cdot n}_m$ and vanishes if $p \in \Gamma_{m+i} - T^{\lambda\cdot n}_m$, clearly we have

$$\lim_m (\lim_i \overset{\prime\prime}{\omega}{}^{n\cdot\lambda}_{m\cdot m+i}(p)) = \overset{\prime\prime}{\omega}{}^{n\cdot\lambda}(p).$$

Put $\overset{*}{\omega}{}^n_{m\cdot m+i}(p) = \mathrm{Min}\left(\omega^n_{m\cdot m+i}, \overset{\prime\prime}{\omega}{}^{\lambda\cdot n}_{+\cdot m+i}, \frac{1}{\lambda}\bar{H}^R(p)\right)$ this is contained in the upper class for $T^{\lambda\cdot n}_{m\cdot m+i}$, but $\lim\limits_m \lim\limits_n \overset{*}{\omega}{}^n_{m\cdot m+i}(p)$ and its lower envelope has angular limits 0 almost everywhere on F_{2n} and on $\mathfrak{A} \cap O_{\frac{1}{2n}}$. Let $n \to \infty$, then this envelope has zero almost everywhere on $z = 1$. This follows that $\lim\limits_n T^{\lambda n} \cap \mathfrak{A}$ has outer harmonic measure zero. Therefore there exists a non negative continuous super-harmonic function $U(p)$ such that $U(p) = \infty$ when p tends to $T^{\lambda\cdot n}$ for every $\lambda > 0$ and n. Since $\mathrm{Max}(\bar{H}^R(p) - \varepsilon U(p), 0)$ has limits zero when p tends to $\mathfrak{A} \cap C\mathfrak{F}$ this is contained in $\mathfrak{B}_{\mathfrak{F}}$ class, thus

$$\omega(R^\infty, F) = \bar{H}^R_{\mathfrak{F}}(p) \leqq \underline{H}^R_{\mathfrak{F}}(p).$$

Similarly for open set Ω of $\mathfrak{A}$, we have $\bar{H}^R_\Omega(p) = \underline{H}^R_\Omega(p)$.

Let $\varphi(p)$: $p \in \mathfrak{A}$ be a real valued function on $\mathfrak{A}$ (admitting $\pm\infty$). Define the upper class U^R_φ consisting of all the lower bounded continuous super-harmonic function such that $\varliminf\limits_{R \to \mathfrak{F}} u(p) \geqq \varphi(p)$ and its lower envelope $\bar{H}^R_\varphi(p)$, and the lower class and $\underline{H}^R_\varphi(p)$ for sub-harmonic function similarly. If $\bar{H}^R_\varphi(p) = \underline{H}^R_\varphi(p)$ holds then φ will be called a *resolutive* boundary function and common envelope will be denoted by $H^R_\varphi(p)$.

Theorem 2. 10. *Let R be F-type and φ be semi-continuous function, then φ is resolutive.*

In fact, let φ be bounded upper semi-continuous function on $\mathfrak{A}$ and $M \geqq \varphi \geqq m$, and divides $[m, M]$ such that $m = c_0 < c, \cdots c_n = M$, $c_{i+1} - c_i = \dfrac{M-m}{n}$ and the set $E[\varphi \geqq c_i]$ is denoted by A_i being closed and $E_i = E[c_{i+1} \geqq \varphi > c_i]$. Let U_{A_i} be upper $\mathfrak{B}_{A_i}$ and lower class of characteristic function of A_i. Put $U_{E_i} = U_{A_i} - V_{A_{i+1}} (V_{E_i} = V_{A_i} - U_{A_{i+1}})$, where $U_{A_i}(V_{A_i})$ is a function contained in $U_{A_i}(\mathfrak{B}_{A_i})$, then $U_{E_i}(p)(V_{E_i}(p))$

is supper (sub)-harmonic and $\underline{\lim}_{P\to E_i} U(p)\geqq 1$ ($\overline{\lim}_{P\to CE_i\cap\mathfrak{A}} V(p)\leqq 0$) thus $U_{E_i}(p)$ $(V_{E_i}(p))$ is contained in $U_{A_i}(\mathfrak{B}_{A_i})$. and $U^n_\varphi(p)=\sum_{i=0}^{n-1} c_{i+1}U_{E_i}(p)\,(V^n_\varphi(p)$ $=\sum_{i=1}^{n} c_i V_{E_i}(p))$ is contained in $U^R_\varphi(\mathfrak{B}^R_\varphi)$ but the lower envelope $\bar{H}_{E_i}(p)\,(\underline{H}_{E_i}(p))$ of $U^n_{E_i}(V^n_{E_i})$ is equal to the harmonic measure of E_i in R^∞: $|z|<1$ then we have,

$$\overline{\lim_n}\,(U^n_\varphi(p)-V^n_\varphi(p))\leqq \frac{1}{n}\sum_{i=0}^{n-1}\omega_{E_i}(p)\leqq\frac{1}{n},\quad \text{let } n\to\infty$$

$$H^R_\varphi(p)=\underline{H}^R_\varphi(p)=\int\varphi\,d\mu$$

where μ is harmonic measure.

From general theory of Dirichlet problem, we have next

Lemma. *$\underline{H}^R_\varphi(p)$ is the upper envelope of $H^R_\psi(p)$, where $\psi<\varphi$ and ψ is upper bounded and semi-continuous on $\mathfrak{A}$, the similar fact holds for $\bar{H}^R_\varphi(p)$.*

In the same manner used by M. Brelot, we have

Theorem 2. 11. *In order that φ is resolutive it is necessary and sufficient that φ is integrable in the narrow sense.*

Remark 1. *We easily see $H^R_\varphi(p)$ is the same as the solution of R. Nevanlinna*

Remark 2. *This method is also applicable for $\varphi(p)$ on $\mathfrak{A}$ even if $\mu(\tilde{R}_\infty,\mathfrak{A})\neq 1$, if we restrict $\varphi\geqq 0$ and $\bar{H}^R_\varphi$ is equal to $\int_{\mathfrak{A}}\varphi\,d\mu$*

Remark 3. We have seen in the proof of Lemma $H^R_{\mathfrak{F}}(p)$ takes value zero or 1 almost everywhere on $|z|=1$ according to the point is contained in the image of $\mathfrak{F}$ or not more generally the solution H^R_φ for continuous boundary function φ takes value φ almost everywhere on the image of $\mathfrak{A}$. If we consider $H^R_\varphi(p)$ in R, this fact means that $H^R_\varphi(p)$ takes the given φ "*almost*" paths determining A. B. P.s so that the image of these paths form measure 2π set on $|z|=1$. This follows the resolutivity of φ. But we dont know if $H^R_\varphi(p)$ tends to φ along every curve ending at any point $\mathfrak{A}-\mathfrak{A}_0$, where $\mathfrak{A}_0$ is a subset of $\mathfrak{A}$ being so small that its harmonic measure zero, this fact has stronger sense than the former.

On the other hand it is well known, for Dirichlet problem of the domain in the z-plane for continuous boundary value φ, there exists a subset I being F_σ of the boundary of capacity zero called irregular set determined uniquely for the domain such that the solution takes the given value except I. In general case when R is F-type Riemann surface the problem whether there exists or not the negligible subset with the property determined by the domain only and not depending on the boundary function φ is open.

5.2. Dirichlet problem on a covering surfaceover a positive boundary Riemann surface.

Let $\underline{R}$ be a positive boundary Riemann surface and p_0 be an inner point of R, and $G(p, p_0)$ be Green's function of R and $h(p, p_0)$ be its conjugate. Put $k(p)=e^{-G-ih}$ and we define the length of a curve L as $\int_L dk(p)$ and distance between two points $\underline{q}_1$, and $\underline{q}_2 \in \underline{R}$ be the lower limit of the length of all curves L connecting $\underline{q}_1, \underline{q}_2$ in $\underline{R}$ and by completion we have metric space $\underline{R}^*$, and the distance on a covering surface of two points q_1, q_2 is defined by the diameter of the projection on $\underline{R}^*$ of all curves connecting q_1, q_2 in R. If a curve l on R tends to the boundary R and its projection converges to a point on $\underline{R}^*$ we say, l determines an A. B. P. this means that the mapping function $f(z)\rightarrow R\rightarrow \underline{R}\rightarrow k(p)=w$, $f(z)$ has limit w_0 on the w-plane along the image l_z on the z-plane of l, since $|k(p)|<1$ the images l_z tends to equivalent points of Fuchsian group on $|z|=1$ and at these points $k(p)=k(z)$ has an angular limit w_0 i.e., l_z determines the same A. B. P. lying on $\underline{R}^*$. As $f(z)$ has angular limits almost everywhere on $|z|=1$ this follows that R is F-type.

Let $\mathfrak{A}_z$ be the image of A. B. P.s on $|z|=1$ and F_z be the image of closed subset $\mathfrak{F}$ of $\mathfrak{A}$, then we have

$$\mathfrak{A}_z=E_\theta\left[\prod_l \prod_k \sum_n \prod_m \left[\delta(f(\Delta_{m\cdot n\cdot l}(\theta)\leqq \frac{1}{k}\right]\right],$$

$$F_z=E_\theta\left[\prod_l \prod_k \sum_n \prod_m \left[df(\Delta_{m\cdot n\cdot l}(\theta), \mathfrak{F})\leqq \frac{1}{k}\right]\right].$$

where $\Delta_{m\cdot n\cdot l}$ the set $0\leqq|\arg(1-e^{-i\theta}z)|\leqq\frac{\pi}{2}-\frac{1}{l}$, $\frac{1}{n}\leqq|z-\theta|<\frac{1}{m}$, $\delta(f(\Delta))$ is the diameter of $f(\Delta)$ with respect to the topology of $\underline{R}^*$ and $d(\Delta, \mathfrak{F})$ is the distance from $\mathfrak{F}$ with respect to the topology of R, then we have the same result about Dirichlet problem as in the case when R is F-type Riemann surface over a null-boundary Riemann surface $\underline{R}$.

6. Finitely sheeted Riemann surface

Let $\underline{R}$ be a null-boundary Riemann surface with A-topology and R be a covering surface over $\underline{R}$. When a Jordan curve L on R converging to the boundary of R and its projection tends to a point p_0 of $\underline{R}^*$, we say, L determines an A. B. P. Let V_n be the neighbourhood of p_0 of the projection of the A. B. P. with diameter $\frac{1}{n}$ and $\mathfrak{V}_n$ be the set of R lying on V_n, this composed of at most enumerable number of domains.

Associated domain D_n of L will be a domain containing an end-part of L in $\mathfrak{V}_n$. Two arcs L_1, L_2 determine the same A. B. P if and only if for any number n two associated domains are the same. This definition of A. B. P. is equivalent with that of O. Teichmüller.

We denote by $n(p): p \in \underline{R}$ the number of times when p is covered by R, then it is clear that $n(p)$ is lower semi-continuous. We call R boundedly sheeted covering surface if $n(p) \leqq M : p \in \underline{R}$. In this paragraph we consider chiefly such a Riemann surface. When $\sup_{p \in \underline{R}} n(p) \gneqq 1$, non accessible point of R have no sense in itself, hence we consider only accessible boundary points.

Barrier. We call $B(p): p \in R$, Barrier of p the function such that $B(p)$ is non negative bounded continuous super-harmonic function $\lim_{q \leftarrow p} B(q) = 0$ and for every associated domain D_m, there exists a number such as, $\underline{\lim}\, B(q) > \delta_m$; if $p \overline{\in} D_m$ it is well known that p is regular for Dirichlet problem of R if and only if at p Barrier exists.

Lemma. *Let R be a F-type Riemann surface over R and p be an A.B.P and $D(p)$ be an associated domain of p, we denote by proj $D(p)$ the projection of $D(p)$, if p is lacunary of D and further proj p is regular for the domain proj $D(p)$, then p is regular for Dirichlet problem on R.*

In fact, let $T(\text{proj}\, p)$ be Barrier of proj p with respect to proj D, then there exists a number δ_m such that $\lim T(\text{proj}\, p) \geqq \delta_m$, if $\text{proj}\, p \overline{\in} \text{proj}\, D_m$. Put $B(p) = \text{Min}\,(\delta_m, T(\text{proj}\, p))$, this is clearly Barrier of p with respect to R. We have at once

Lemma. *χ be lacunary set being clearly closed on R, the all A.B.P.s on χ are regular for Dirichlet problem except the set having the projection of capacity zero which is F_σ.*

In the sequal, let R be boundedly sheeted covering surface over R such as $n(p) \leqq M$. It is known that R is a null- or positive boundary Riemann surface according to the set $E[n(p) \leqq N-1)]$ where $N = \overline{\lim}\, n(p): p \in \underline{R}$ being clearly closed, is a null-capacity set or not. We suppose R be a positiveboundary Riemann surface on account of theorem 1.2, R is of F-type. In this class of Riemann surface the following propositions hold.

1) *Any A.B.P is a direct singular point.*

Let p be an A. B. P. Case 1. p lies on $\underline{R}$, if p is not direct singular point, then there exists connected pieces $\mathfrak{v}_1, \dots, \mathfrak{v}_k$; $k \leqq N$ with common points with the associated domain D_m, such that these connected pieces have inner points $p_1 \cdots p_k$ with the same projection as p for any large number m. Take $c_1 \cdots c_k$ discs contained in $\mathfrak{v}_1 \cdots \mathfrak{v}_k$, then any one of c_i has no branch points converging to p_i, hence there exists a number m_0' such that $\mathfrak{V}_m \bigwedge c_i$: $m \geqq m_0'$ $(i = 1, 2, \dots, k)$ has a finite number of branch points, then we can find $m_0 \geqq m_0'$ such that any one of $c_i \bigwedge \mathfrak{V}_m$ $(m \geqq m_0)$ has no common points with the other $c_j \bigwedge \mathfrak{V}_m$ $(j \neq i)$. For any m, $D_m \bigwedge \mathfrak{V}_m$ has non common points, this implies that $D_m \bigwedge \mathfrak{V}_m = c_i \bigwedge \mathfrak{V}_m = c_i$ and p is an inner point of R. This is a contradiction

Case 2. If p lies on the boundary of $\underline{R}$ for this case our assestion in trivial.

The order of an A.B.P. and of the associated domain.

The number $\lim_{m\to\infty} [\sup n(p) : p \in \text{proj}\, D \bigwedge \mathfrak{V}_m]$ is called the *order* of an A. B. P. and $[\lim \sup n(p) : p \in D \bigwedge \mathfrak{V}_m]$ is the order of a associated domain D_m respectively, we denote by $\mathfrak{A}_n$ the set of $\mathfrak{A}$ of order n, and by $\mathfrak{A}_n^m$ the subset of $\mathfrak{A}_n$ such that $[\sup n(p) : p \in \text{proj}\, D \bigwedge \mathfrak{V}_m] = n$ and its projection by $\mathfrak{A}_n^m$, then we have $\mathfrak{A}_n = \bigvee_m \mathfrak{A}_n^m$, $A_n = \bigvee_m A_n^m$.

2) *We can prove that $\mathfrak{F}_n$ is closed relative to proj $\mathfrak{A}$, if we denote by $\mathfrak{F}_n = \sum_{i \geqq n} A_i$.*

3) *We easily see that if $[\sup n(p) : p \in \text{proj}\, D_m] \leqq n$, and D_m contains A.B.P.s of $\mathfrak{A}_n$ then the projection of $\mathfrak{A}_n$ is lacunary.*

Put $\mathfrak{F}_n = \sum_{i \geqq n}^{N} A_n$ then $\mathfrak{F}_N \subset \mathfrak{F}_{N-1} \cdots\cdots \subset \mathfrak{F}_1$.

Theorem 2. 12. *Let R be a boundedly sheeted covering surface over a null-boundary Riemann surface, then all accessible points are regular for Dirichlet problem except at most the subset $\mathfrak{A}^I$ of $\mathfrak{A}$ having its projection contained in the F_σ set of capacity zero.*

a) Any point of $\mathfrak{A}_N$ is regular except at most $\mathfrak{A}_N^I$, of which the projection is capacity zero and F_σ. Because $\mathfrak{A}_N$ is lacunary, if Cap (proj $\mathfrak{A}_N$) > 0, $\mathfrak{A}_N$ is lacunary of R_N, where R_N is the points of $\underline{R}$ covered by R, N times being open set, then A_N is regular for the domain R_N, except at most capacity zero F_σ denoted by A_N^I, hence let p be of $\mathfrak{A}_N$ with its projection in $A_N - A_N^I$ is, clearly regular for R, because R has more boundary than $\mathfrak{A}_N$.

b) We suppose all points of $\sum_{i=n-p}^{n} \mathfrak{A}_i$ on $\mathfrak{F}_{n-p} - \mathfrak{F}_{n-p}^I$, where Cap $(\mathfrak{F}_{n-p}^I)$

$=0$ and F_σ on which at least an A. B. P. of $\sum_{i=n-p}^{N} \mathfrak{A}_i$ being irregular point lies. We shall prove that $\sum_{i=n-p-1}^{N} \mathfrak{A}_i$ on $\mathfrak{F}_{n-p-1}$ is regular except the set of which the projection is capacity zero and F_σ, we suppose $\sum_{i=n-p}^{N} \mathfrak{A}_i$ is regular except the set on $\mathfrak{F}_{n-p}^{I}$, then we have only to prove almost all points of $\mathfrak{A}_{n-p-1}$ is regular, since $\mathfrak{F}_{n-p}=\bigcup_m \mathfrak{F}_{n-p}^m$, $\mathfrak{F}_{n-p-1}^{I}=\bigcup_m \mathfrak{F}_{n-p-1}^{I\cdot n}$, there exists a number m such that $\mathrm{Cap}(\mathfrak{F}_{n-p-1}^{I\cdot m})>0$, we cover by at most enumerable discs $c_1, c_2, \ldots$ of diameter $\leqq \frac{1}{2m}$, then there is at least a c_i such that $\mathrm{Cap}(c_i \frown \mathfrak{F}_{n-p-:}^{I\cdot n})>0$. Let $\mathfrak{d}_1, \mathfrak{d}_2 \ldots$ be associated domains of points of $\mathfrak{A}_{n-p-1}^{I\cdot m}$, we have only to consider next three cases.

Case 1. $\mathrm{ord}(\mathfrak{d}_i) \geqq n-p$ and $\mathrm{ord}(\mathfrak{d}_i)=\sup n(p): p \in \mathrm{proj}\, \mathfrak{d}_i)$

Case 2. $\mathrm{ord}(\mathfrak{d}_i) < n-p-1$.

Case 3. $\mathrm{ord}(\mathfrak{d}_i) = n-p-1$.

Case 1. $\mathfrak{d}_i$ has no point of A_{n-p-1}^m, case 2. $\mathfrak{d}_i$ has no point of A_{n-p-1}^n, then we may regard the case 3 in this case $\mathrm{proj}(\mathfrak{d}_i \frown A_{n-p-1})$ is lacunary of the domain of $\mathrm{proj}\, \mathfrak{d}_i$, but in the case except at most capacity zero of F_σ set all point of A_{n-p-1} is regular with respect to $\mathrm{proj}(\mathfrak{d}_i)$, hence by lemma all points of $\mathfrak{A}_{n-p-1}$ is regular for R except the set with the projection being F_σ and of capacity zero. Thus we have the conclusion.

Theorem 2. 13. *Let q be an A.B.P. with its projection on $\underline{R}$ and $G(p, p_0)$ be Green's function of R, if $\lim_{p \to q} G(p, p_0)=0$, then q is regular for Dirichlet problem.*

Proof. As we have proved, we can take V_m so that an A. B. P. may be lacunary of D_m. Without loss of generality we may suppose the projection of the A. B. P. is q_0, let C_{ρ_0} be the circle of radius ρ_0 with centre q_0 with respect to the local parameter defined in the neighbourhood of the projection of the A. B. P such as $C_{\rho_0} \subset V_m$ and C_ρ^* be the periphery of C_{ρ_0} and D_{ρ_0} be the associated domain of the A. B. P. lying on C_{ρ_0}. Denote by F_i the point of $\underline{R}$ covered i times by R, then $F_N \subset F_{N-1} \cdots \subset F_1$. From the semi-continuity of $n(p)$, F_i is open and $F_i \frown C_\rho^*$ is composed of at most enumerable number of intervals ${}^\rho I_j^i (i= 1, 2, \ldots N)$ $(j=1, 2, \ldots)$. Let s be a point of ${}^\rho I_j^i$ then there exists i points $p_1, p_2, \ldots, p_i$, where some of them may coincide. Put $g(s_i)=G(s_i, p_0)$ where $G(s_1) \geqq G(s_2) \cdots \geqq G(s_i)$. We can easily prove that $g(s_i)$ is continuous in ${}^\rho I_j^i$ $(j=1, 2, \ldots)$ for fixed i.

We choose a closed sub-set ${}^\rho e_j^i$ of ${}^\rho I_j^i$ such that $\frac{1}{2\pi\rho}\,\mathrm{mes}\,(\sum_j^{\infty} {}^\rho I_j^i - \sum_j^{N_j} {}^\rho e_j^i) < 2\pi \frac{\rho}{\rho_0}$ and denote by κ the minimum of $g(s)$ in $\sum_j^{N_i} \sum_i^{N} e_j^i$; $k>0$.

Introduce Poisson's integral for C_ρ with value ρ_0 on $\sum_j^\infty {}^\rho I_j^i - \sum {}^\rho e_j^i$ with 0 on $C_\rho - \sum {}^\rho I_j^i + \sum {}^\rho e_j^i$

$$\rho^i(p) = \int\limits_{\sum^\rho I_j^i - {}^\rho\sum e_j^i} \frac{1}{2\pi} \frac{\rho^2 - r^2}{\rho^2 - 2\rho r \cos(\theta - \varphi) + r^2} d\varphi\,, \quad \text{then } p^i(q_0) \leqq 2\pi\rho\,.$$

Let $U(p)$ be the upper envelope of sub-harmonic function in D_{ρ_0} such that $\overline{\lim}\, V(p) \leqq \overline{q\, q_0}$ where $\overline{q\, q_0}$ means euclidean distance between q and q_0. Put $\alpha(p) = U(p) - \rho - \frac{\rho_0}{k} G(p, p_0) - \sum_i^N {}^\rho\rho^i(p) \leqq 0$ because, at A. B. P. on C_ρ^*, $U(p) \leqq \rho$ except negligible set, because D_{ρ_0} is F-type covering, if $p \in R$ and $\operatorname{proj} p \in \sum {}^\rho e_j^i$, $U(p) - \frac{\rho_0}{k} G(p) < 0$, if $p \in R$ and $\operatorname{proj} p \overline{\in} \sum {}^\rho e_j^i$, $U(p) - \sum \frac{\rho_0}{k} G(p) \leqq 0$, thus $\underset{p \to q_0}{U}(p) \leqq (2N+1)\rho$ on the other hand since $s(p) = \operatorname{proj} \overline{p\, q_0}$ is sub-harmonic in R, $U(p) \geqq \rho_0$, if $\operatorname{proj} p \in C_{\rho_0}^*$. Put $\beta(p) = \operatorname{Min}(\rho_0, U(p))$, then $\beta(p)$ is Barrier at the A. B. P. q_0.

In the case when the basic surfaces a positive boundary Riemann surface, if we introduce C-topology[12] on $\underline{R}$, the distance is defined as O. Teichmüller then in the same way all points are regular except the set having its projection being F_σ and negligible with respect to R.

Remark 1. *When R is boundedly sheeeted covering surface the answer of the problem at the end of the preceding paragraph is affirmative.*

Remark 2. *There exists a harmonic function on R which is $= \infty$ at all irregular accessible points lying on $\underline{R}$ on account of G. C. Evans and M. Breiot, hence on such class of Riemann surface the resolutivity of continuous function is proved by the ordinary method as in the case when the domain is the subset of the z-plane i.e. without use of conformal mapping of the universal covering surface.*

Chapter III. Martin's[13] topology and type of Riemann surface

1. Singular point

On a positive boundary Riemann surface R, the most precise topology for harmonic function is given by R. S. Martin. Let a be a fixed point of R, and $K(p, q)$ be normalized Green's function of its pole at q defined for $q \neq a$, and $K(a, a) = 1$ when $\{r_i\}$ is a sequence of points tending

12) Z. Kuramochi: Potential theory and its applications, II. Osaka Math. Journ., 3, 1952, pp. 87-99.

13) R. S. Martin: Minimal positive harmonic functions. Trans. Amer. Math. Soc., 19, 1941, pp. 137-172.

to the boundary, if $K(r_i, a)$ determines a harmonic function on R, then $\{r_i\}$ is called a fundamental sequence of ideal points r_∞ corresponding to the limit function $K(p, r_\infty)$. We denote by Γ all ideal points, then $K(p, q)$ is defined for all points of $R+\Gamma=\bar{R}$, $K(p, q)$ is harmonic for p for fixed q, and harmonic for q for fixed p. The topology introduced by the distance

$$\delta(r_1, r_2) \text{ of } r, r_2 \text{ by } \delta(r, r_2)=\sup_{p\in\bar{R}}\left|\frac{K(p, r_1)}{1+K(p, r_1)}-\frac{K(p, r_2)}{1+K(p, r_2)}\right|$$

is called Martin's topology, the potential will be defined for the mass-distribution on $\bar{R}$ is given by $\int K(p, r)\, d\mu(r)$.

Activity.[14] In the neighbourhood $\mathfrak{v}(p_1)$ in Martin's topology, if there exists an unit-mass-distribution μ such that the potential by μ is not bounded in the neighbourhood of $\mathfrak{v}(p_2)$, then p_1 is called an *active point* to p_2.

Theorem 3. 1. $p_1=\lim_i p_1^i$, $p_2=\lim_j p_2^j$ *and* $\varliminf_i G(p_1^i, a)>\delta_1>0$, *and* $\varliminf G(p_2^j, a)>\delta_2\geqq 0$, *and if* p_1 *is active to* p_2, *then* p_2 *is active to* p_1.

Proof. From the hypothesis there exists a sequence $\{p_1^i\}$, $\{p_2^j\}$ such that $\lim K(p_1^i, p_2^j)=\infty$, then we can extract a subsequence such that

$$K(p_2^{j_n}, p_1^{i_n})=\frac{G(p_1^{i_n}, p_2^{j_n})}{G(p_1^{i_n}, a)}\geqq n^3\delta\,;\; n=1, 2, \ldots : \delta>0$$

then $\sum\frac{1}{n^3}K(p_1^{i_n}, p_2^{j_n})>\sum n\delta$ is the potential required of the mass-distribution in $\mathfrak{v}(p_2)$.

Corollary. *If* $\varliminf_i G(p_1^i, a)\geqq\delta>0$, $\varliminf_j G(p_2^j, a)\geqq\delta>0$, *then the activity has symmetricity property.*

If $K(p, p_\infty)$ is bounded in R, we shall call p_∞ *a singular point.*

Let $K_{\mathfrak{v}_n}(p, p_\infty)$ be the lower envelope of super-harmonic functions $u(p)$ such that $u(p)\geqq K(p, p_\infty)$ if $p\in\bar{\mathfrak{v}}_n(p_\infty)$ then $K_{\mathfrak{v}_n}(p, p_\infty)\geqq K_{\mathfrak{v}_{n+1}}(p, p_\infty)$. Martin proved that $\lim_n K_{\mathfrak{v}_n}(p, p_\infty)=K(p, p_\infty)$ or 0 and is called p_∞ a minimal point or not according to $\lim_n K_{\mathfrak{v}_n}(p, p_\infty)\neq 0$ or 0 respectively, and non minimal points set at most negligible (outer harmonic measure zero), and $K(p, p_\infty)$ is minimal function when p_∞ is a minimal point.

Theorem 3. 2. *If* p_∞ *is inactive to almost all points of* $\Gamma-p_\infty$ *(except*

14) See 4).

at most negligible set) and moreover if $K(p, p_\infty)$ is bounded, then p_∞ is singular minimal point.

Proof. For any given $\varepsilon > 0$, there exists a finite number of neighbourhoods $\mathfrak{v}_i : \sum_i^{n_0} \mathfrak{v}_i \subset \Gamma - p_\infty$ such that the outer harmonic measure of $(\sum \mathfrak{v}_i \wedge \Gamma) < \varepsilon$ and in $\sum \mathfrak{v}_i$, $K(p, p_i) \leq M$, $p \in \sum^{n_0} \mathfrak{v}_i$, $p_i \in \mathfrak{v}_0$, then on almost all points of $\sum \mathfrak{v}_i \wedge \Gamma$ has limits value zero, but $K(p, p_i) \leq M$ follows that $K(a, a) = 0$ if p_∞ is of harmonic measure zero since $K(a, a) = 1$, then p_∞ is positive harmonic measure thus p_∞ is a minimal point.

We shall show by an example the condition that p_∞ is inactive to almost all points of $\Gamma - p_\infty$ is necessary.

(Fig. 2)

Example. Let R_n, R'_n, R''_n be rings in the unit-circle $|z| < 1$ such as

$$1 - \frac{1}{2n} < |z| < 1 - \frac{1}{2n+1} - \frac{1}{n}, \quad 1 - \frac{1}{2n+1} < |z| < 1 - \frac{1}{2(n+1)},$$

$$1 - \frac{1}{2n+1} < |z| < 1 - \frac{1}{2(n+1)} : \quad n = 1, 2, \ldots, \text{ respectively.}$$

Put $p_n \doteqdot 32\, e^{\pi^2 \frac{1}{M''_n}}$,[15] where M''_n is the module of R''_n i.e. $M'' = \log \dfrac{1 - \frac{1}{2(n+1)}}{1 - \frac{1}{2n+1}}$ we map R_n onto the ring $e^{-M_n} \leqq |\zeta| \leqq e^{M_n}$ where $2M_n$ is the module of R_n, and $H(\zeta)$ be a harmonic function such as $|H(\zeta)| < p_n^{1+\delta_0}$: $e^{-M_n} \leqq |\zeta| \leqq e^{M_n}$: $\delta_0 > 0$, then there exist a number N'_n such that $|\text{Max}\, H(\zeta) - \text{Min}\, H(\zeta)| \leqq \frac{1}{n}$ where Maximum and Minimum mean the Max. and Min. on $|\zeta| = 1$ and $|\arg \zeta| \leqq \frac{2\pi}{2^{N_n}}$; $N_n > N'_n$.

Let R_n be mapped onto the ring $e^{-M_n} \leqq |\zeta| \leqq e^{M_n}$, and put $m_n = \dfrac{M_n}{2(2^{N'_n}+1)}$, in ring R_n we denote by $r_i^n, r_i^{n\prime}$ the rings such as

$$e^{M_n - m(i+2)} \leqq |\zeta| \leqq e^{M_n - m(i+1)}. \quad e^{-M_n + m(i+1)} \leqq |\zeta| \leqq e^{-M_n + m(i+2)}$$

In $r_i^n, r_i^{n\prime}$ make systems of slits $\{I_{ij}^n\}$, $\{I_{ij}^{n\prime}\}$: $i = 0, 2, 4, 6, \ldots, 2^{N_n} - 2$; $j = 1, 2, 3 \ldots 2^{N_n}$.

$$\left.\begin{array}{ll} I_{ij}^n : e^{M_n - m(i+2)} \leqq |\zeta| \leqq e^{M_n - m(i+1)}, & \theta = \left(\dfrac{2\pi}{2^{N_n}}\right) j \\ I_{ij}^{n\prime} : e^{-M_n - m(i+2)} \leqq |\zeta| \leqq e^{-M_n - m(i+1)}, & \theta = \left(\dfrac{2\pi}{2^{N_n}}\right) j \end{array}\right\} : j = 1, 2, 3, \ldots, 2^{N_n}$$

15) See Chap. IV.

Where N_n is[16] the number to be defined in the following manner.

Let T^1 be the indirect conformal mapping $p \sim \tilde{p}$ in itself, where $\tilde{p}$ is the symmetric $\tilde{p}$ is the symmetric point of p with respect to $\arg\zeta = 0$, and T^2 is the indirect mapping $p \sim \tilde{p}$, where $\tilde{p}$ is the symmetric point of p with respect to $\arg\zeta = \frac{\pi}{2}, -\frac{\pi}{2}$ and T^3, corresponds to $\frac{\pi}{4}, \frac{3}{4}\pi, \frac{5\pi}{4}, \frac{7\pi}{4}$, and T^j so on.

We cut R_n by a cut lying on the real axis and identify the new boundaries lying symmetrically with respect to the imaginary axis and obtain two new surface $R_n^1, R_n^{1\prime}$.

If $|U(p)| \leqq p_n^{1+\delta''}$ $(\delta'' > 0)$, let $N_n \to \infty$, then $|U(p) - U(T_1(p))| \to 0$ and $|U(p) - U(T_2(p))| \to 0$ and we have $\frac{\partial U}{\partial n} \to 0$ on the real axis, because $U(p)$ is harmonic on real axis, thus by the[17] boundary problem $U(p)$ converges to a harmonic function on the part of the new surface $R_n^1(R_n^{1\prime})$ lying on $e^{-M_n + (4+\frac{1}{2})m_n} \leqq |\zeta| \leqq e^{M_n - (4-\frac{1}{2})m_n}$ and $|U(p) - U(T_3(p)| \to 0$ in this part, consequently there exists a number N_n such that $U(p)$ is symmetric and periodic except at most $\frac{1}{n}$ with respect to $2^{N'_n}$ directions in $e^{\frac{M_n}{2}} \leqq |\zeta| \leqq e^{\frac{M_n}{2}}$.

We make slits $\{S^n\}$ in R'_n, $S^n = z = re^{i\theta}$ such as $\theta = 0$, $1 - \frac{1}{2(n+1)} \geqq r \geqq -\frac{1}{2n+1} - \frac{1}{n^2}$.

We denote the image in R_n of $\{I_{ij}^n\}$ and $\{I_{ij}^{n\prime}\}$ of $R_{n\zeta}$ by the same letter and F_1 and F_2 two unit-circles $|z| < 1$ with $\{I_{ij}^n\}$, $\{I_{ij}^{n\prime}\}$ and $\{S^n\}$. We shall get a Riemann surface F composed of F_1 and F_2 with identifying in the following manner.

Identity two symmetric edges of $\{I_{1j}^n\}$, $\{I_{1j}^{n\prime}\}$: $(n = 1, 2, \ldots)$ $(i, j = 1, 2, \ldots, 2^{N'_n})$ lying symmetrically with respect to the axis: $\arg z = 0$, or $-\frac{\pi}{2}$ of F_i $(i = 1, 2)$ respectively. In general for i in the sector $\frac{2\pi}{2^{i-1}}k \leqq \arg z$

16) Y. Tôki: On the classification of open Riemann surfaces. Osaka Math., Journal, Vol. 4, 1952, pp. 191-201.

17) Let $ABCD$ be an rectangle with sides AB, BC, CD, DA and $U(z)$ be a harmonic function such that $U(z) = 0$; $z \in AB + CD$ and $\frac{\partial U}{\partial n}$ is given when $z \in BC + AD$.

On the other hand we denote by $N(z, z_0)$ the harmonic function with logarithmic singularity at z_0 and $N(z) = 0$: $z \in AB + CD$ and $\frac{\partial N}{\partial n} = 0$: $z \in BC + DA$, then we have

$$U(z_0) = \int_{BC+DA} N \frac{\partial U}{\partial x} ds\,.$$

From this we see that if $U(z) \to 0$: $z \in AB + CD$ and $\left|\frac{\partial U}{\partial n}\right| \to 0$: $z \in BC + AD$, then $U(z) \to 0$.

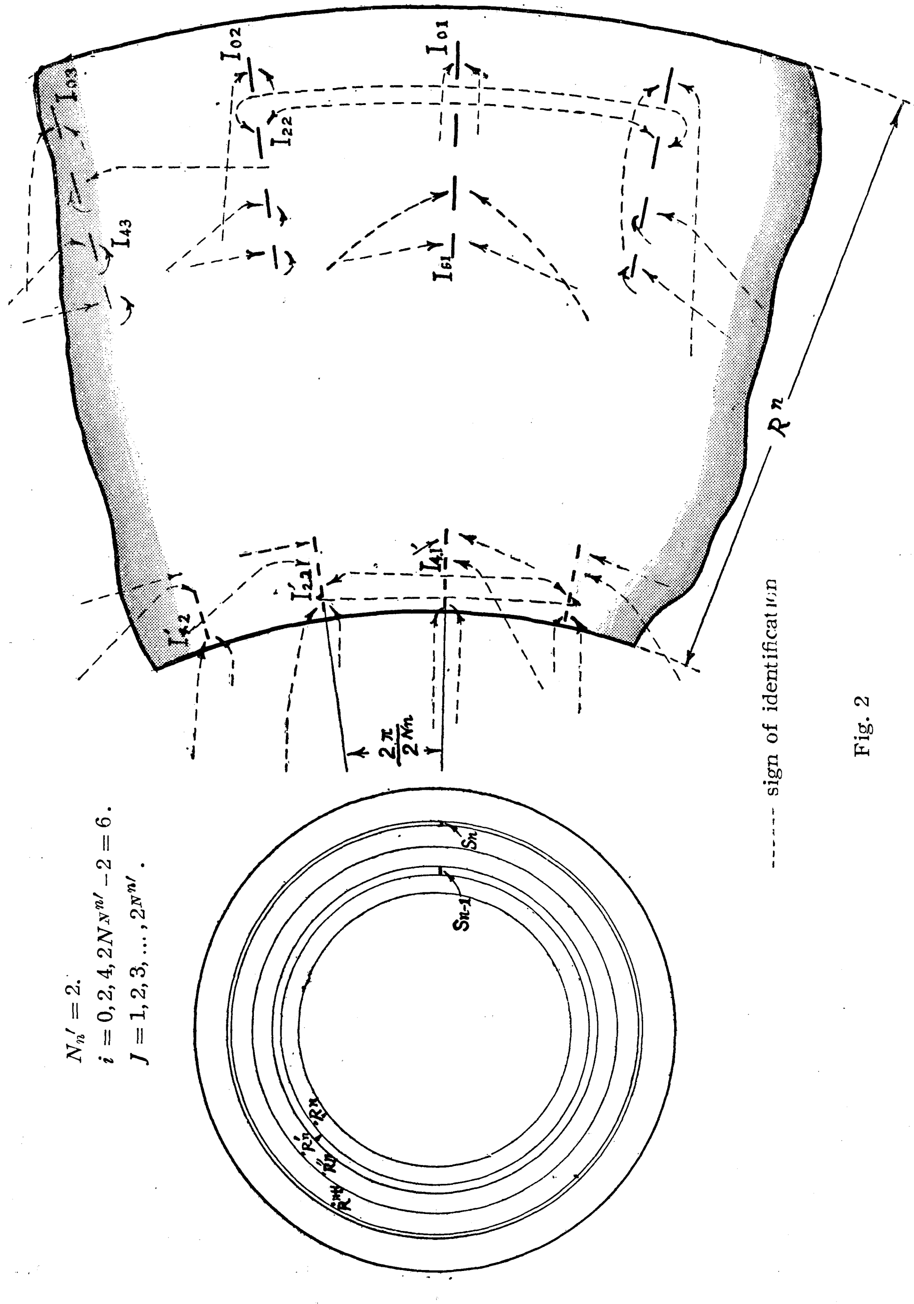

Fig. 2

17) In general $|U(z)-U(T_i(z))|\to 0$ and $|U(z)-U(T_i(z))|\to 0$ follow that $|U(z)-UT_{i+2}(z)|\to 0$ on

$$e^{-M_n+m\left(2i+2+\frac{1}{2}\right)} \leq |\zeta| \leq e^{M_n-m\left(2i+2+\frac{1}{2}\right)}.$$

$\leqq \frac{2\pi(k+)}{2^{i-1}}$: $(k=0, 1, 2, 3, \dots, 2^{i-1})$; $(n=1, 2, \dots)$ $(i, j=1, 2, 3, \dots)$. We identify $\{I^n_{ij}\}$ and $\{I'^n_{ij}\}$ with edges lying symetrically on the axis $\arg z = \frac{2\pi}{2^{i-1}}\left(k+\frac{1}{2}\right)$. We identify $\{S^n\}$ of F_1 and F_2 with the edges.

On the surface let $u(p)$: $p \in F$ be a positive harmonic function and denote by $\underset{|P|=r}{\mathrm{Max}}\, u(p)$ the maximum of $u(p)$ when p lies on $|z|=r$, then $\overline{\lim_{r_n}} \max u(p) \leqq P_n^{1+\delta_0}$ *where* $r = -\frac{1}{2n+1}$.

If $\overline{\lim_{r_n}} \max u(p) \geqq P_n^{1+\delta}$, then there exists at least one sequence of R_n of F_1 or F_2 on which $\min u(p)$, $p \subset R''_n \geqq P^{\delta'}$, because in R''_n there exists a curve connecting two boundaries of R''_n on which $u(p) \geqq P_n^{\delta'}$ where δ and δ' are positive numbers and $\mathrm{Min}\, u(p) \geqq P\delta'$; $p \in {}''_n$ follows that $u(0)$ tends to infinity. Thus we may suppose without loss of generality $\overline{\lim_{r_n}} u(p) \leqq P_n^{1+\delta_0}$; $p \in R_n$. Next by the property of N_n, and N'_n, we have $(\mathrm{Max}\, u(p) - \mathrm{Min}\, u(p))$; p lies on $|\zeta|=1) \leqq \frac{1}{n} + ((\mathrm{Max.}\, u(p) - \mathrm{Min.}\, u(p))$: when p lies in the sector $0 \leqq |\arg \zeta| \leqq \frac{\pi}{2N'_n}$ and $|\zeta|=1)$ Thus on $\lambda_n = |z| = \sqrt{\left(1-\frac{1}{2n}\right)\left(1-\frac{1}{2n+1}-\frac{1}{n^2}\right)}$, $|\mathrm{Max}\, u(z) - \mathrm{Min}\, u(z)| \leqq \frac{2}{n}$. This follows that on F_i with projection on $|z|=\lambda_n$, $u(p)$ tends to constant a_i $(i=1, 2)$. It is clear that $a_i < \infty$. Finally on F_i there is no non-constant unbounded positive harmonic function, and the dimention of the class of bounded harmonic functions at most two. In F_i there exists a non constant bounded function vanishing on $\{S^n\}$, therefore the dimension of the class is at least two.

On the surface $K(p, p_0) \leqq M$ for every ideal point of Martin's topology and is spanned by two bounded harmonic functions $H_1(p)$ and $H_2(p)$ and since F has only one boundary components Γ has infinitely[18] many points (density of continum) singular points and only two of them are minimal. R. S. Martin proved the set of non minimal points is F_σ and conjuctured[19] that this set was non dense in Γ, but our example shows that his conjucture is not always true on an abstract Riemann surface, because the set of ideal points of this example is isomorphic to the closed interval $[0, 1]$ and only two ends correspond to minimal points.

Theorem 3. 3. *Let R be a F-type covering surface, over a null-bouudary Riemann surface then on R there is no singular minimal function.*

18) M. Brelot: Sur le principe des singularités positive et la topologie de R. S. Martin. Annales Univ. Grenoble, sci. math. phys., 23, 1947, pp. 113-138.

19) See 13).

Proof. Let $u(p)$ be a singular minimal function, it is easy to see that $u(p)$ takes zero or maximum M on Γ except outer harmonic measure. Consider $u(p)$ in R^{∞}: $|z|<1$ then $u(z)$ has angular limits M on a positive measure set G, we divide $\underline{R}$ into a system of enumerable neighbourhoods non overlapping and without lacunary, then there exists at least one $\mathfrak{v}_i^1$ such that the image E_i of A. B. P.s lying on $\mathfrak{v}_i^1$ has common positive measure set E_1^i with G. If $M\mu(R^{\infty}, E_1^i)=u(z)$, where $\mu(R^{\infty}, E_1^i)$ harmonic measure of E_1^i, then we divide $\mathfrak{v}_i^1$ into finite number set $\mathfrak{v}_1^2, \mathfrak{v}_2^2 \ldots$ non overlapping and on lacunary and consider $M\mu(R^{\infty}, E_2^{i2})$, it is clear that $\mu(R^{\infty}, E_2^{i2}) \leqq \mu(R^{\infty}, E_1^i)$ and $\lim\limits_{n} M\mu(R^{\infty}, E_n^{in})=0$ by adequate subdivision, consequently there exists a harmonic function such as $u(p) \gneqq u^*(p) \gneqq 0$. This contradicts the minimality of $u(p)$.

2. On subsurfaces

Theorem. 3. 4. *Let $R \in O_G$ and S be a subsurface with at most enumerable infinity of analytic curves $\{C_i\}$ as its relative boundaries which are converging to the boundary of R. When we denote by $\hat{S}$ the doubled surface of S being symmetric with respect to $\{C_i\}$, then $\hat{S}$ is contained in O_G too.*

Proof. Denote by R_m the exhaustion of R with relative boundary Γ_m, and by $\omega_m(p)$ the harmonic function in $R_m - F_0$ such that $\omega_m(p)=M_m$, if $p \in \Gamma_m$, $\omega_m(p)=0$ if p is contained in the boundary of F_0, where F_0 is a compact disc contained in $S \frown R$ and $\displaystyle\int_{\text{boundary of } F_0} \frac{\partial \omega_n}{\partial n} ds = 2\pi$, then it is clear $\lim\limits_{m} M_n = \infty$. Put $z_m = e^{\omega_m + i h_m} = re^{\theta}$, where $h_m(p)$ is the conjugate of $\omega_m(p)$. Denoting by F_0^* the symmetric disc is $\hat{S}$ of F_0.

If $\hat{S}$ is a positive boundary Riemann surface, then there exists a non negative harmonic function $w(p)$ such that $0 \leqq w(p) \leqq 1$, $w(p)=0$ if $p \in$ boundary of F_0 and F_0^* and $\underset{\hat{S}-F_0-F_0^*}{D(w)} \leqq M'$, $\dfrac{\partial w}{\partial n}=0$ if $p \in \sum C_i$, and $\displaystyle\int_{\text{bounday of } F_0} \frac{\partial w}{\partial n} ds = \pi$.

Regarding $w(p)$ in $S \frown R_m - F_0$, and denoting by θ_r^m the curve on which $|z_m| = r$. Put $L(r_m) = \displaystyle\int_{\theta_{r_m}} \left|\frac{\partial w}{\partial r_m}\right| r_m d\theta \; (\geqq \pi)$,

then
$$L^2(r_m) \leqq 2\pi r_m \int_{\theta_r} \left|\frac{\partial w}{\partial r_m}\right|^2 r_m d\theta \,,$$

$$\underset{S\cap R_m - F_0}{D(w)} = \int_\theta \int_0^{M_n} \left\{\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)\right\} r dr d\theta , \int_1^{e^{Mn}} \frac{L^2(r)}{2\pi r} dr \leqq \int_0^{e^{Mn}} \frac{dD}{dr} dr \leqq D(w) ,$$

Let $R_m \rightarrow R$ then $e^{M_n} \rightarrow \infty$, we have $D(w) \rightarrow \infty$, this is a contradiction.

We denote by S_{OB} and S_{ON} the class of sub-Riemann surface on which any bounded harmonic function vanishing or with vanishing normal derivative on the relative boundary of S must reduce to a constant.

Theorem 3.5. *If there is no singular minimal point on the boundary of S, then*

$$S_{ON} \subset S_{OB} .$$

Proof. Let $u_1(p) : p \in S$ be a non-constant harmonic function such as $0 \leqq u_1(\mathrm{p}) \leqq 1$ and vanishes on the relative boundary of S, since there is no singular points there exists $u_2(p) \lneqq u_1(p) : \overline{\lim}\, u_1(p) = \overline{\lim}\, u_2(p) = 1$. Put $u_3(p) = u_1(p) - u_2(p)$, we denfine new functions $u_i(p)^* : i = 1, 2, 3$ such that $u_i^*(p) = u_i(p)$ on $S \wedge \Gamma_m$ and symmetric with respect to the relative boundary of S where Γ_m is the relative boundary of S_m being an exhaustion of S, and we extract $\{u_i^*(p)\}$ converging uniformly to $u_i^*(p)$, then $1 \geqq u_1^*(p) = u_2^*(p) + u_3^*(p)$. $\overline{\lim}\, u_i^*(p) > 0$, hence there is at least a non-constant bounded harmonic function with vanishing normal derivative.

As the special case if the genus of S is finite then this theorem is affirmed. Because S can be mapped onto the subsurface of a null-boundary Riemann surface, then by theorem 1.2 the universal covering surface is F-type.

Remark. The condition that there is no singular point is necessary is easily shown by examples.

Extension of L. Myrberg's theorem[20].

Theorem 3.6. *Let $u(p)$; $p \in R$ and $D(u(p)) < \infty$, then $u(p)$ is expressed by Poisson's integral in R^∞; $|z| < 1$.*

Proof. Let $V(z)$ be sub-harmonic function such as $\int_{|z|} |V(z)| d\theta \leqq M$ and G be a simply connected domain in $|z| < 1$ with boundary Γ, then

$$\int_\Gamma |V| d\omega \leqq M ,$$

20) L. Myrberg: Bemerkungen zur Theorie der harmonischen Funktionen. Annales Acad. Sci. Fenn., 107, 1952.

where $d\omega$ is harmonic measure with respect to G. In fact denote by $V^*(z)$ the upper envelope of sub-harmonic function such as $V^*(p) \leqq |V(p)|$ if $p \in \{(|z|<1)-G\}$, then $V^*(0) \geqq |V(0)|$, hence

$$M \geqq \int_{|z|=1} |V| d\theta \geqq V^*(0) = \int_{\Gamma} |V(p)| d\omega .$$

Let $G(p, p_0)$ be the Green's function of R with its pole at p_0, when $z=0$ is the of p_0, and put $e^{-G-ih} = re^{i\theta}$, then $M \geqq D_{F_r}(u(p)) = \iint_{G>\log r} \operatorname{grad}^2 u(p) r dr d\theta = \frac{1}{2} \int_0^{2\pi} \frac{\partial^2(re^{i\varphi})}{\partial r} d\varphi \geqq \int u^2 d\varphi$, where $F_r = R \cap E[G > \log r]$, since F_r^∞ is a subsurface of R and has the boundary in $|z|>1$ except measure zero set. We map F_r^* onto $|\xi|<1$, then the $|z|=r$ in F_z^∞ is transformed onto a closed curve in $|\xi|<1$, by the preceding we have

$$M \geqq \int u^2 r d\varphi = \int_{|\xi|=1} u^2 d\theta \ \text{ implies } \int_{|z|=r_i} |u|^2 d\theta \leqq M : \lim_{i=\infty} r_i = 1$$

thus, this can be expressed by Poisson's integral.

Corollary. $O_{HB} \subset O_{HBD} = O_{HD}$.

Assuming that $D_R(u(p)) < \infty$, we map R^∞ onto $|z|<1$, then there exists a constant M_0 such that any one of the set where $u(z)=u(p)$ has angular limits $\geqq M_0+\delta_0$ between $M_0+\delta_0$ and $M_0-\delta_0$ and $\leqq M_0-\delta_0$ has positive measure. We denote by $G_{M_0+\delta_0} = E[u(p) \geqq M_0+\delta_0] \cap R$, $G_{M_0-\delta_0}^{M_0+\delta_0} = E[M_0+\delta_0 > u(p) > M_0-\delta_0] \cap R$, $G_{M_0-\delta_0} = E[u(p) \leqq M_0-\delta_0] \cap R$, and R_m be an exhaustion with relative boundary Γ_m, let $u_m(p) = M_0+\delta_0$ when $p \in G_{M_0+\delta_0} \cap \Gamma_m$, $u_m(p) = u(p)$, when $p \in G_{M_0-\delta_0}^{M_0+\delta_0} \cap \Gamma_m$, $u(p) = M_0-\delta_0$ when $p \in G_{M_0-\delta_0} \cap \Gamma_m$, respectively, we define a harmonic function $u^*(p)$ by extracting from uniformly convergent sequence of $u_m(p)$. It is clear by Dirichlet principle $D_R(u^*(p)) \leqq D(u(p)) \leqq M$. On the other hand, we map $G_{M_0+\delta_0}(G_{M_0-\delta_0})$ onto $|\xi|<1$, then the boundary of $G_{M_0+\delta_0}(G_{M_0-\delta_0})$ except relative boundary where $u(p) = M_0+\delta_0(M_0-\delta_0)$ has positive measure, consequently $u^*(p)$ is non constant.

Chapter IV. Behaviour of analytic functions

1. Analytic functions on O_{HB}

We consider the behaviour of analytic functions defined on an abstract Riemann surface or a covering surface, at first we prove next simple theorem.

Theorem 5. 1. *Let R' be the remaining surface after a compact set of R is removed, if $R \overline{\in} O_G$ and $\in O_{HB}$ then there exists no analytic bounded functions exist.*

In fact on R', let $A(p)$ be a bounded analytic function, then by theorem 3. 3. on R' no singular minimal function exists, but on the ideal boundary has positive measure, hence there is a non-constant bounded harmonic function with the value 0 on the relative boundary of R', it follows $R \overline{\in} O_{HB}$.

We investigate the behaviour of the boundary of Riemann surface when it is given as a covering surface over the a z-plane.

Theorem 5. 2.[21] (*W. Gross*). *Let $z = z(p): p \in R$, be meromorphic and R be a Riemann surface of O_G, denoting by $p = p(z)$ its inversed function, if $p = p(z)$ is regular at z_0, then we can continuate $z(p)$ analytically on half lines: $z = z_0 + re^{i\theta}$ $(0 \leqq r \leqq +\infty)$ except for θ of angular measure zero.*

2. Gross's property

Let R be a Riemann surface of O_{HB}, then any connected piece has no lacunary of positive capacity projection, this implies R has Iversen's property. In regarding above result we ask if the Gross's property holds in a surface of O_{HB}.

Theorem 5. 3. *A Riemann surface of O_{HP} has not always Gross's property.* This fact tells us the essential condition for R to have the Gross's property is the "force" and not complexity of the ideal boundary and character of O_G.

Example.

1) Let G be a curvilinear rectangle with sides C_1 $-a \leqq \Re(z) \leqq a$, $\Im(z) = 0$. $C_2: x + a = \varphi(y)$, $a = \varphi(0)$, $0 \leqq y \leqq b$. $C_3: -a \leqq \Re(z) \leqq a$, $\Im(z) = b$. $C_4: x - a = \varphi(y)$, $-a = \varphi(0)$, $b \geqq y \geqq 0$, and $U(z) \geqq 0$ be a positive harmonic function such that $U(z) \geqq M$, if $z \in C_2 + C_4: U(z) \geqq 0$, if $z \in C_1 + C_3$, then there exist a curve l connecting C_2 and C_4 on which $\mathrm{Min}\, U(z) \geqq \omega(0, \frac{ib}{2})$ where $\omega(z)$ is a harmonic function in a rectangle with

21) K. Noshiro: Open Riemann surface with null boundary. Nagoya Math. Journ., 3, 1951, pp. 73-79.
Z. Kuramochi: Potential theory and its applications, I. Osaka Math. Journ., 2, 1951, pp. 123-175.

vertices $(-a,0)$, $(a,0)$ and (a,ib), $(-a,\mathrm{ib})$ and $\omega(z)=1$ if $z\in$ segment $\overleftrightarrow{(a,0)(a,\mathrm{ib})}$, $\overleftrightarrow{(-a,0)(-a,\mathrm{b})}$ and $\omega(z)=0$ if $z\in\overleftrightarrow{(-a,0)(a,0)}$, $\overleftrightarrow{(-a,\mathrm{ib})(a,\mathrm{ib})}$. Because, let $f(z)$ be a function mapping the curvilinear rectangle onto the rectangle $(-a',0)(a',0)(a,\mathrm{ib})(-a',\mathrm{ib})$ then $2a'\leqq\int_{-a+\varphi}^{a+\varphi}\left|\frac{\partial f}{\partial x}\right|dx$ and $4a'^2b\leqq 2a\int_0^b\int_{\varphi+a}^{\varphi-a}|f'|^2\,dxdy=4aa'b$, it follows $\frac{b}{a}\leqq\frac{b'}{a'}$. From this fact we easily have our assertion.

2) *The number* P_{n-1}. Put $r_n=\frac{1}{4}\times 2^{1+\frac{1}{2}+\cdots+\frac{1}{2^{n-1}}}$, $s_n=t_{n-1}$ $=\frac{1}{4}(r_n-r_{n-1})=\frac{2^{1+\frac{1}{2}+\cdots+\frac{1}{2^{n-2}}}}{10}\cdot(2^{\frac{1}{2^{n-1}}}-1)$, and R_n be a ring $r_{n-1}+t_{n-1}\leqq|z|\leqq r_n-s_n$, and M_n be the module of R_n, $M_n=\log\frac{3\cdot 2^{\frac{1}{2^{n-1}}}+1}{3\cdot 2^{\frac{1}{2^{n-1}}}}$. The transformations "ring R_n (z-plane) $\rightarrow$ the rectangle $(-\pi,0)$, $(\pi,0)$ (π,iM_n) $(-\pi,iM_n)\rightarrow$ upper half η-plane $\left(A=-\frac{1}{\kappa},\ B=-1,\ D=1,\ E=\frac{1}{\kappa}\right)$ $\rightarrow$ the unit circle of the ς-plane", are carried by

$$\xi=\frac{1}{h}\int_0^\eta\frac{d\eta}{\sqrt{(1-\eta^2)(1-\kappa^2\eta^2)}},\qquad \varsigma=\frac{(1+i)\,\eta+\sqrt{\frac{1}{\kappa}}\,(1-i)}{(1-i)\,\eta+\sqrt{\frac{1}{\kappa}}\,(1+i)}$$

respectively, through some calculation we have $\omega\left(\frac{iM_n}{2}\right)\doteqdot\frac{e^{-\pi^2\left(\frac{1}{M_n}\right)}}{32}$

Put $P_{n-1}=32e^{\pi^2\left(\frac{1}{M_n}\right)}$. (Fig. 3)

3) *The number* μ_n. Let I_n^ν be slits such as; $r_n-s_n\leqq r\leqq r_n+t_n$. $\theta=\frac{2\pi\nu}{2^{\mu_n}}$, $\nu=1,2,3,\ldots,2^{\mu_n}$ and $R_n'\supset R_n$ be rings such as R_n': $r_n+t_{n+1}\geqq r\geqq r_{n-1}-s_{n-1}$. R_n''; $r_{n-1}+\frac{11}{10}t_{n-1}\leqq r\leqq r_n-\frac{11}{10}s_n$, and $U(z)$ be harmonic in $R_n'-\sum_\nu I_n^\nu-\sum_\nu{}_{n-1}^\nu$, and $0\leqq U(z)\leqq 2P_{n+1}^{1+\delta_0}$: $\delta_0>0$, and vanishes on $\sum_\nu I_n^\nu+\sum_\nu I_{n-1}^\nu$, then there exists μ_n'' such that Max $U(z)$, (when z is contained in R_n'') $\leqq\frac{1}{2n}$ for $\mu_n>\mu_n'$. We define $\mu_n'=\mathrm{Max}\,(\mu_n'',\mu_{n-1}'')$.

4) *The number* μ_n''' *and* μ_n. Let $U(z)$ be a harmonic function such as $|U(z)|\leqq 2P_{n+1}^{1+\delta_0}$ if $z\in R_n''$ then there exists μ_n''' such that $|\mathrm{Max}\,U(z)$ $-\mathrm{Min}\,U(z)|$, (when z lies on $r_n^*=\sqrt{(r_{n-1}+\frac{11}{10}t_{n-1})(r_n-\frac{11}{10}s_n)}$ and

$0 \leqq |\arg z| \leqq \frac{2\pi}{2^{\mu\mu}}$, for $\mu_n \geqq \mu_n'''$. Put $\mu_n = \text{Max}\,[\mu_n', \mu_n''']$.

5) *The number* N_n. Let $U(z)$ be a harmonic function such as $0 \leqq U(z) \leqq 2P_{n+1}^{1+\delta_0}$ in $R_n' - \sum_{\nu}^{2^{\mu n}} I_n^\nu - \sum^{2^{\mu n+1}} I_{n+1}^\nu$, then there exists a number N_n such as if $U(z) \leqq \frac{3}{n}$ on I_n^ν, $I_{n+1}^{\nu'}$: $\nu = 1, 2, \dots, 2^{\mu_n}$, $\nu' = 1, 2, \dots, 2^{\mu_n+1}$, except any measurable set of having measure $< \frac{\text{length}\, I_\nu^n}{N_n}$, $< \frac{\text{length}\, I_\nu^{n+1}}{N_{n-1}}$ respectively, $U(z)$ must satisfy the condition that $\text{Max}\, U(z) \leqq \frac{6}{n}$, when $z \in R_n''$. We denote such number by N_n.

6) Let $\varphi(t)$: $[0 \leqq t \leqq 1]$ be absolutely continuous, almost every where derivable and integrable and $\text{mes}\, E[\varphi'(t)] = 0 = \text{mes}\, E[\varphi'(t) = \infty] = 0$, then there exist κ_1 and κ_2 depending only on φ but on E such that $\kappa_2 \geqq \int_E |\varphi'(t)|\, dt > \kappa_1$, where E is any measurable set with measure δ_0.

7) Let G be a domain in the z-plane with boundaries composed of analytic curves $\gamma_1, \gamma_2 \cdots \gamma_{n-1}, \gamma_n$ if we map G onto the ring $1 \leqq |\zeta| \leqq e^{\mathfrak{M}}$ so that $\gamma_1, \gamma_n, \gamma_2, \dots, \gamma_{n-1}$ may correspond to $|\zeta| = 1$, $|\zeta| = e^{\mathfrak{M}}$ and radial slits in this ring R. Let $U(z)$ be harmonic in G and $U(z) = \varphi_1(z)$ if $z \in \gamma_1$, $U(z) = \varphi_2(z)$ if $z \in \gamma_n$ respectively, then $D_G(U(z)) = D_R(U(\varsigma))$ $\geqq \frac{1}{2\pi\mathfrak{M}} \int_0^{2\pi} |\varphi_1(e^{i\theta}) - \varphi_2(e^{i\theta})|^2\, d\theta$ where $U(e^{me})$ and $\varphi_i(e^{i\theta})$ are the transformed functions from $U(z)$ and $\varphi_i(z)$.

Proof. Let $\tau(z)$ be harmonic such that $\tau(z) = \varphi_i(\varsigma)$ if $|\zeta| = 1$ or $|\varsigma| = e^{\mathfrak{M}}$ and $\frac{\partial \tau}{\partial n} = 0$ if $\varsigma \in$ radial slits $\gamma_{i\varsigma}$, $i = 2, 3, \dots, n-1.$, where $\gamma_{i\varsigma}$ are the images of γ_i, then

$$D(\tau, U-\tau) = \int_{\gamma_1 + \gamma_n + \sum_2^{n-1} \gamma_i} (U-\tau) \frac{\partial \tau}{\partial n}\, ds = 0\,,$$

$$D(U) - D(\tau) = D(U) - 2(U, \tau) + D(\tau) = D(U-\tau) \geqq 0\,.$$

but clearly $D(\tau) \geqq \frac{1}{2\pi\mathfrak{M}} \int_0^{2\pi} |\varphi_1(\varsigma) - \varphi_2(\varsigma)|\, d\theta$.

(Fig. 4)

8) In the z-plane with slits I_n^ν, $\nu = 1, 2, \dots, 2^{\mu_n}$, we enclose I_n^ν by a simple closed curve J_n^ν such that

$$|z| = \frac{r_n + r_{n-1}}{2}, \quad \frac{2\pi\nu}{2^{\mu_n}} - \frac{\pi}{2^2 \cdot 2^{\mu_\mu}} \leqq \arg z \leqq \frac{2\pi\nu}{2^{\mu_n}} + \frac{\pi}{2^2 \cdot 2^{\mu_n}},$$

$$\frac{r_n + r_{n-1}}{2} \leqq |z| \leqq \frac{r_{n+1} + r_n}{2}, \qquad \arg z = \frac{2\pi\nu}{2^{\mu_n}} - \frac{\pi}{2^2 \cdot 2^{\mu_n}},$$

$$|z| = \frac{r_{n+1} + r_n}{2}, \quad \frac{2\pi\nu}{2^{\mu_n}} - \frac{\pi}{2^2 \cdot 2^{\mu_n}} \leqq \arg z \leqq \frac{2\pi\nu}{2^{\mu_n}} + \frac{\pi}{2^2 \cdot 2^{\mu_n}},$$

$$\frac{r_n + r_{n-1}}{2} \leqq |z| \leqq \frac{r_{n+1} + r_n}{2}, \qquad \arg z = \frac{2\pi\nu}{2^{\mu_n}} + \frac{\pi}{2^2 \cdot 2^{\mu_n}},$$

and let L_n^ν and $L_n^{\nu\prime}$ half lines such that

$$L_n^\nu:\ 0 \leqq |z| \leqq \infty, \ \arg z = \frac{2\pi\nu}{2^{\mu_n}} + \pi - \frac{\pi}{2 \cdot 2^{\mu_n}},$$

$$L_n^{\nu\prime}:\ 0 \leqq |z| \leqq \infty, \ \arg z = \frac{2\pi\nu}{2^{\mu_n}} + \pi + \frac{\pi}{2 \cdot 2^{\mu_n}},$$

$$L_n^\nu:\ 0 \leqq |z| \leqq \infty, \ \arg z = \frac{2\pi\nu}{2^{\mu_n}} + \pi .$$

We denote by G_n^ν the domain with boundaries L_n^ν, L_n^ν, $\sum_{i \mp \nu} J_n^i$, I_n^ν and map it onto the ring $1 \leqq |w| \leqq e^{\mathfrak{M}_n^{(1)}}$, so that $L_n^\nu + L_n^{\nu\prime}$, I_n^ν and $\sum_{i \mp \nu} J_n^i$ may be transformed onto $|w| = e^{\mathfrak{M}_n^{(1)}}$, $|w| = 1$ and radial slits $I_{n(w)}^\nu$, $\sum_{i \mp \nu} J_{n(w)}^i$ respectively. In this mapping any measurable set of positive measure $\geqq \frac{1}{N_n}$ on I_n^ν is transformed onto a measurable set of positive angular meaure than $\frac{1}{k_n}$ and smaller than k_n' from (6), and the doubly connected domain bounded by J_n^ν and I_n^ν of module $\mathfrak{M}_n^{(2)}$ is transformed onto a domain bounded by their images denoted by $J_{n(w)}^\nu$ and $I_{n(w)}^\nu \equiv (|w| = 1)$. Let e^{ρ_n} be the distance of $J_{n(w)}^\nu$ from $w = 0$ and define the number by $\rho_n' = \dfrac{\mathfrak{M}_{(n)}^2}{2\pi \cdot k_n \cdot n^2 \cdot P_{n+1}^{2+2\delta_0 2^{\mu_n}}}$. We choose α_n such that $e^{\alpha_n} \leqq e^{\rho_n'}$, $e^{2\alpha_n} \leqq e^{\rho_n}$.

9) a) Let R be a ring $1 \leqq |\varsigma| \leqq e^\beta$ and S_k, S_k' be slits such that

$$S_k:\ e^{\frac{\beta}{6}} \leqq |\varsigma| \leqq e^{\frac{2\beta}{6}}, \quad \arg \varsigma = \frac{2\kappa\pi}{l} \qquad (k = 1, 2, \dots, l),$$

$$S_k':\ e^{\frac{4\beta}{6}} \leqq |\varsigma| \leqq e^{\frac{5\beta}{6}}, \quad \arg \varsigma = \frac{2\pi l}{l} \qquad (k = 1, 2, \dots, l),$$

we make two same rings with the same slits $(R - \sum S_k - \sum S_n')$ and connect crossweise two surfaces with S_k and S_k' with same projection then we have two-sheeted Riemann surface $\hat{R}$. Denote by $\tilde{\varsigma}$ the point having the same projection as ς and $S(\varsigma) = |U(\varsigma) - U(\tilde{\varsigma})|$ be a subharmonic function on R such that $0 \leqq U(\varsigma) \leqq P$ and vanishing on all branch points (end points of S_k and S_k') then there exists a constant λ,

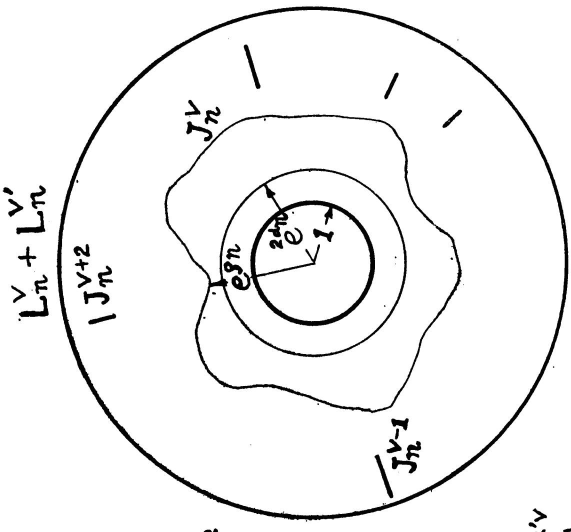

Fig. 3

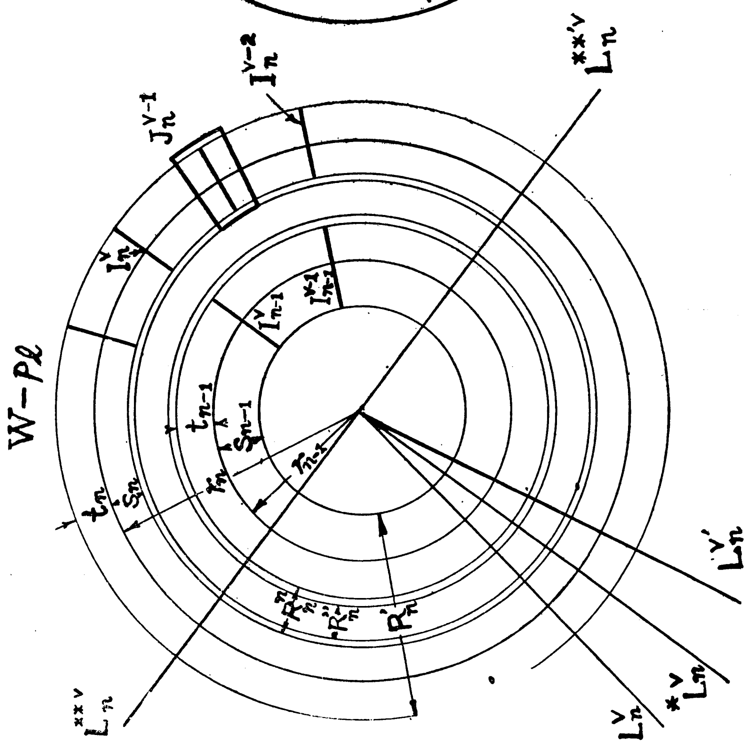

Fig. 4

such that $|S(\varsigma)|<\lambda P$: $\lambda<1$, when $|\varsigma|=e^{\frac{\beta}{2}}$ where λ depends continuously on only the ratio $\frac{\beta}{l}$.[22] In the following we fix two bounds $M\geqq\frac{\beta}{l}\geqq N$ so that λ may be always smaller than λ_0.

b) The number s_n and q_n. We choose s_n and q_n such as

$$\frac{s_n}{2\pi n^2 k_n(\mathfrak{M}_n^{(1)}-\alpha_n)}\geqq\frac{P_{n+1}^{2+2\delta_0}2^{2^{\mu n}}}{\mathfrak{M}_n^{(2)}}, \quad 2P_{n+1}^{1+\delta_0}\lambda_0^{q_n-s_n}\leqq\frac{1}{n}$$

(Fig. 5)

In the ring; R_n^ν; $1\leqq|w|\leqq e^{2\alpha_n}$, denote by $C_{ij}^{n\cdot\nu}$, $\tilde{C}_{ij}^{n\cdot\nu}$ (n,ν fixed $i,j=1,\dots,q_n$) rings and H_n^ν circle such as

$$C_{i\cdot j}^{n\cdot\nu}:\ 2\alpha_n-\gamma(i(i-1)+2j-1)\leqq\log|w|\leqq 2\alpha_n-\gamma i((i-1)+2j-1+1)$$

$$\tilde{C}_{ij}^{n\cdot\nu}:\ \gamma i(i-1)+2j-1\leqq\log|w|\leqq\gamma(i(i-1)+2j-1+1),\quad \gamma=\frac{\alpha_n}{2(q_n^2+q_n+1)}$$

$$H^{n\cdot\nu}:\qquad \log|w|=\alpha_n,\qquad i\geqq j,$$

We make slits $S_{ij\kappa}^{n\cdot\nu\prime}$, $S_{ij\kappa}^{n\cdot\nu}$, $\tilde{S}_{ij\kappa}^{n\cdot\nu}$, $\tilde{S}_{ij\kappa}^{n\cdot\nu}$ such as (n,ν, fixed $i,j=1,\dots,q_n$, $i\geqq j)=0$.

$$S_{\stackrel{|}{ij\kappa}}^{n\cdot\nu}:\ 2\alpha_n-\gamma\left(i(i-1)+2j-1-\frac{4}{6}\right)\leqq\log|w|\leqq 2\alpha_n-\gamma\left(i(i-1)+2j-1+\frac{5}{6}\right)$$

$$S_{ij\kappa}^{n\cdot\nu}:\ 2\alpha_n-\gamma\left(i(i-1)+2j-1-\frac{1}{6}\right)\leqq\log|w|\leqq 2\alpha_n-\gamma\left(i(i-1)+2j-1+\frac{2}{6}\right)$$

$$\tilde{S}_{ij\kappa}^{n\cdot\nu}:\ \gamma\left(i(i-1)+2j-1+\frac{4}{6}\right)\leqq\log|w|\leqq\gamma\left(i(i-1)+2j-1+\frac{5}{6}\right)$$

$$\tilde{S}_{ij\kappa}^{n\cdot\nu}:\ \gamma\left(i(i-1)+2j-1+\frac{1}{6}\right)\leqq\log|w|\leqq\gamma\left(i(i-1)+2j-1+\frac{2}{6}\right)$$

$$\arg w=\frac{2\pi\kappa}{l_n}\qquad \kappa=1,2,3,\dots,l_n$$

where $M\geqq\frac{\gamma}{l_n}\geqq N$.

Let $F(1), F(2)\dots F(2^{q_n-1})$, $\hat{F}(1)\,\hat{F}(2)\dots\hat{F}(2^{q_n-1})$ be 2^{q_n} equal examplars of ring R with slits $S_{\stackrel{|}{ij\nu}}^{n\nu}$, $S_{ij}^{n\nu}$, $\tilde{S}_{ij\kappa}^{n\nu\prime}$, $\tilde{S}_{ij\kappa}^{n\nu}$.

We connect $F(i)\,\hat{F}(i)$ crosswise on $S_{i\cdot 1\kappa}^{n\nu}$, $S_{i\cdot 1\kappa}^{n\nu\prime}$, $\tilde{S}_{i\cdot 1\kappa}^{n\nu}$, $\tilde{S}_{i\cdot 1\kappa}^{n\nu}$: $i=1,2,3,\dots,q_n$, $\kappa=1,2,\dots,l_n$.

22) We map R with slits $S, S, \tilde{\tilde{S}}, \tilde{S}'$ onto a band by $\eta=\log\zeta$. Put $V(\eta)=U(\zeta)-U(\bar{\zeta})$, then $V(\eta)$ is single-valued and has same absolute values on two edges of each S or $\tilde{S}$ and opposite signature. Let $V^*(\zeta)$ be a harmonic function with value $\text{Max}(0, V(\eta))$ on the boundary and $S', S, \tilde{S}, \tilde{S}'$, then $|V(\zeta)|\leqq V^*(\zeta)=\int V^*(\zeta)\frac{\partial g}{\partial n}(\zeta,\xi)ds_\xi$, from this we see easily λ depends on the ratio $\frac{\beta}{l}$.

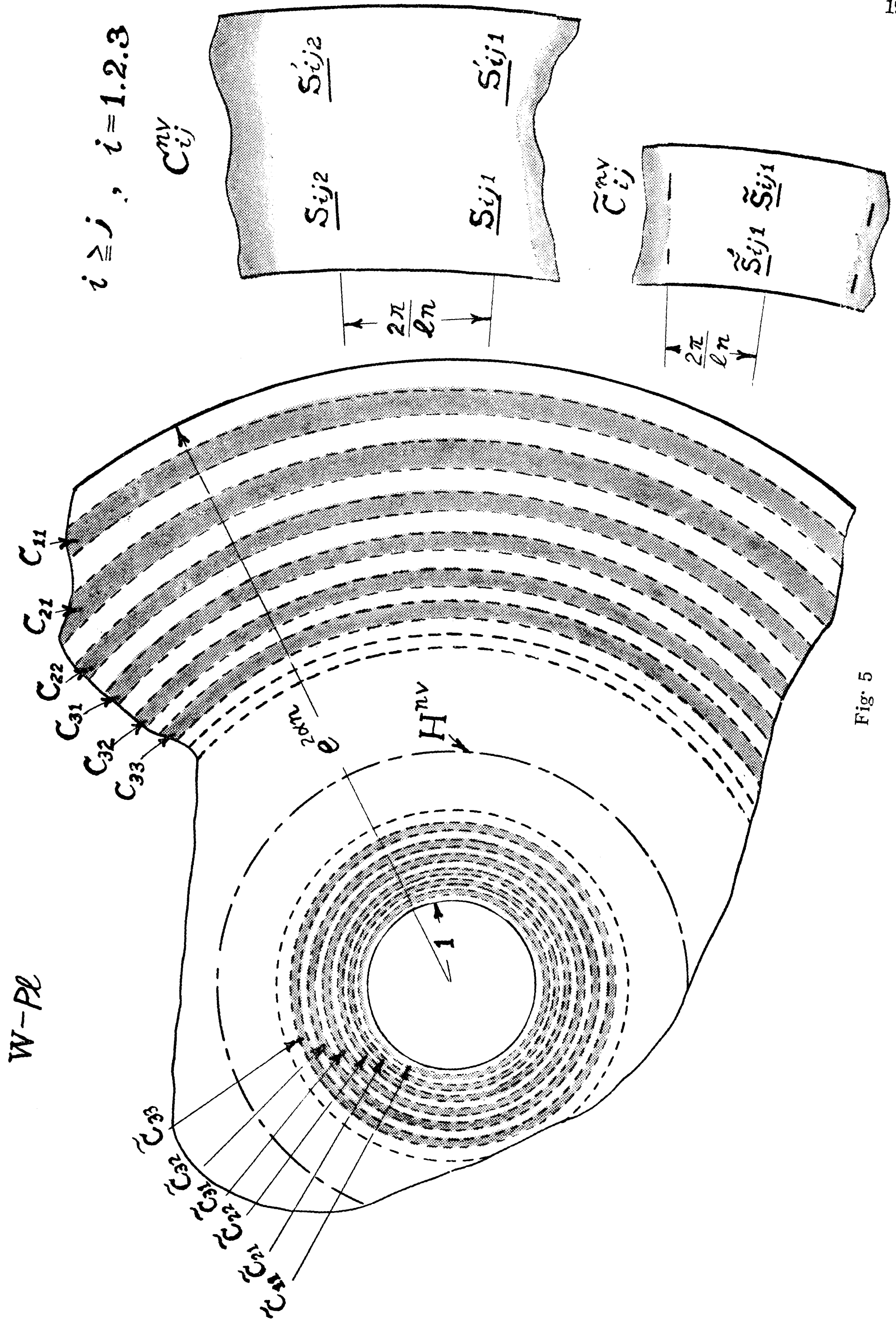
W-Pℓ
i ≥ j , i = 1.2.3
C_ij^nν
S_ij2
S'_ij2
S_ij1
S'_ij1
2π/ℓn
C̃_ij^nν
S̃_ij1
S̃'_ij1
C11
C21
C22
C31
C32
C33
e^2κn
H^nν
1
C̃33
C̃32
C̃31
C̃22
C̃21
C̃11

Fig. 5

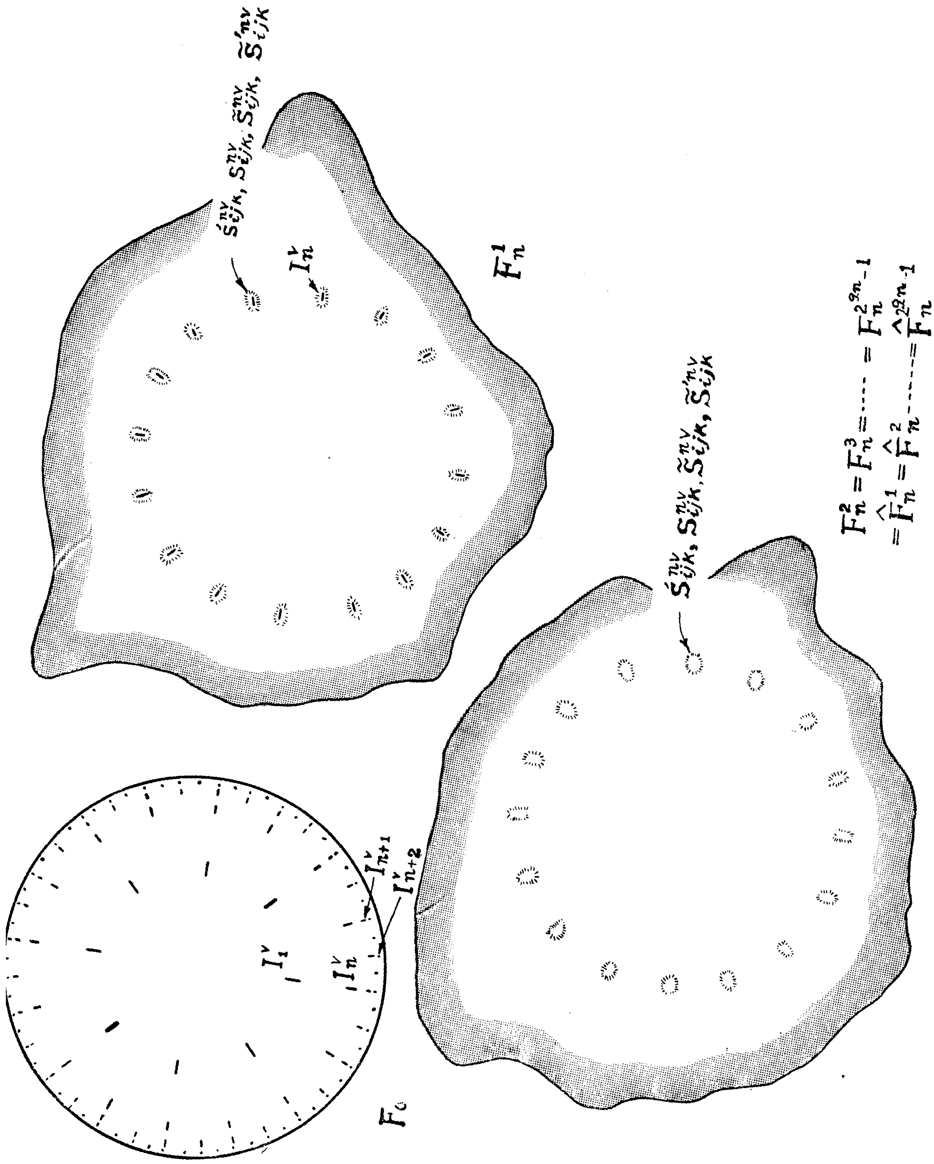
$\acute{S}^{n\nu}_{ijk}$, $S^{n\nu}_{ijk}$, $\tilde{S}^{n\nu}_{ijk}$, $\tilde{S}'^{n\nu}_{ijk}$
I^{ν}_{n}
F^{1}_{n}
$\acute{S}^{n\nu}_{ijk}$, $S^{n\nu}_{ijk}$, $\tilde{S}^{n\nu}_{ijk}$, $\tilde{S}'^{n\nu}_{ijk}$
$F^{2}_{n} = F^{3}_{n} = \cdots = F^{2^{2n}-1}_{n}$
$= \hat{F}^{1}_{n} = \hat{F}^{2}_{n} \cdots = \hat{F}^{2^{2n}-1}_{n}$
I^{ν}_{n+1}
I^{ν}_{n+2}
I^{ν}_{1}
I^{ν}_{n}
F_{0}

Fig. 6

Let T_1 be transformation $z \leftrightarrow \tilde{z}$, where z and $\tilde{z}$ are points of $F(i)$ and $\hat{F}(i)$ having the same projection.

We connect

$$\begin{array}{ccc} F(1) & & F(2) \\ & \times & \\ \hat{F}(1) & & \hat{F}(2) \end{array} \qquad \begin{array}{ccc} F(3) & & F(4) \\ & \times & \\ \hat{F}(3) & & \hat{F}(4) \end{array} \quad \begin{array}{c} \cdots \\ \\ \cdots \end{array} \quad \text{on } S_{i,2,\kappa}, \text{ etc. } i = 2. \dots q_n$$

and by T_2 corresponding transformation.

In general connecting and corresponding transformation are

$$\begin{array}{lll} T_j & F(2^m+1) & \leftrightarrow \hat{F}(2^m+2^{j-2}+1) \\ & F(2^m+2) & \leftrightarrow \hat{F}(2^m+2^{j-2}+2) \\ & \vdots & \quad \vdots \\ & F(2^m+2^{j-2}) & \leftrightarrow \hat{F}(2^m+2^{j-1}) \\ & F(2^m+2^{j-2}+1) & \leftrightarrow \hat{F}(2^m+1) \\ & F(2^m+2^{j-2}+2) & \leftrightarrow \hat{F}(2^m+2) \\ & \vdots & \quad \vdots \\ & F(2^m+2^{j-1}) & \leftrightarrow \hat{F}(2^m+2^{j-2}) \end{array} \qquad \begin{array}{l} m = 0, 1, 2, \cdots, q_n - j \\ \\ \text{on} \quad S_{ij\kappa}: \quad i = j, \; j+1 \dots q_n \end{array}$$

We denote by $|F(i)-\hat{F}(i)|$ the maximum $|U(w_i)-U(w_j)|$, where w_i, w_j have the same projection lying $|w| = e^{\alpha_n}$, then by a) we have

$$\begin{array}{l} |F(1)-\hat{F}(1)| < \lambda^{q_n} 2P_{n+1}^{1+\delta_0} \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ |F(1)-F(2^{j-1}+1)| < \lambda^{q-j+1} 2P_{n+1}^{1+\delta_0}. \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \end{array}$$

Taking account of the property of s_n and q_n we see that there is at least s_n examplars on which $U(w)$ has equal value except at most $\frac{1}{n}$ when w lies on $|w| = e^{\alpha_n} : H^{n\nu}$.

Structure of the surface (Fig. 6)

F_0 is the unit-circle $|z|<1$ with slits $I_n^\nu : n = 1, 2, \dots, \nu = 1, 2, \dots, 2^{\mu n}$ F_n $(n = 1, 2, \dots)$ be the z-plane with slits I_n^ν. We map G_n^κ onto $1 \leq |w| \leq e^{\mathfrak{M}_n^{(1)}}$ and define the ring $1 \leq |w| \leq e^{2\alpha_n}$ and in 9) in R_n^ν we have made slits $S_{ij\kappa}^{n\cdot\nu}$, $S_{ij\kappa}^{n\cdot\nu\prime}$, $\tilde{S}_{ij\kappa}^{n\cdot\nu}$, $\tilde{S}_{ij\kappa}^{n\cdot\nu}$, conversely in the z-plane i. e., we denote by the same letter S_{--}^{--} the image $S_{ij\kappa}^{n\nu}$, $S_{ij\kappa}^{n\cdot\nu\prime}$, $\tilde{S}_{ij\kappa}^{n\cdot\nu}$, $\tilde{S}_{ij\kappa}^{n\cdot\nu}$ in F_n of the w-plane. Now by F_n^1 we denote the z-plane with slits I_n^ν, $S_{ij\kappa}^{n\nu}$, $\tilde{S}_{ij\kappa}^{n\nu}$, $\tilde{S}_{ij\kappa}^{n\nu}$, $S_{ij\kappa}^{n\nu\prime}$ and by $F_n^2, F_n^3 \dots F_n^{2q_n-1}, \hat{F}_n^1 \dots \hat{F}_n^{2q_n-1}$ examplars of the z-plane with equal slits $S_{ij\kappa}^{n\nu}$, $S_{ij\kappa}^{n\nu\prime}$, $\tilde{S}_{ij\kappa}^{n\nu}$, $\tilde{S}_{ij\kappa}^{n\nu}$.

On the unit-circle $|z|<1$ we take a disc $|z| \leq r_m + s_m$ having slits $I_1^\nu, \dots, I_m^\nu$, we connect F_n^1 $(n = 1, \dots, m)$ and $F_0 \wedge (|z| \leq r_m + s_m)$, on

I_n^ν ($\nu = 1, \dots, 2^{\mu_n}$) crossweise. Connect F_n^1 with $F_n^2 \dots F_n^{2q_{n-1}}$, $\hat{F}_n^1 \dots \hat{F}_n^{2q_{n-1}}$ on $S_{ij\kappa}^{n\cdot\nu}$, $S_{ij\kappa}^{n\nu}$, $\tilde{S}_{ii\kappa}^{n\nu}$, $\tilde{S}_{ij\kappa}^{n\nu}$ ($\nu = 1, 2, \dots,$) in the manner mentioned in 9) b). In such a way we have a Riemann surface R_m which covers the part $|z| \leqq r_m + s_m$, $1 + 2^{q_1} + 2^{q_2} + \cdots + 2^{q_m}$ times and the part $|z| > r_m + s_m$, $2^{q_1} + 2^{q_2} + \cdots + 2^{q_m}$ times, and R_m has only one boundary component. Put $R = \bigcup_m R_m$, then R is the required Riemann surface.

Proof of the theorem.

1) *There exists no positive harmonic function on R.*

Let $U(z)$ be a positive harmonic function in R and Max $U(z)$ be the maximum of $U(z)$ on F_0 with projection on $|z| = r_n + t_n$, then $\overline{\lim_n} \operatorname{Max}_{r_n} U(z) \leqq P_n^{1+\delta}$; $\delta > 0$.

If there exist a infinitely many $r_n + t_n$ on which Max $U(z) > P_n^{1+\delta}$, where $\delta > 0$ then by 1) the ring $r_{n-1} + t_{n-1} \leqq |z| \leqq r_n - s_n$ contained in F_0, there is a closed curve which is a proper cut (dividing cut) on which Min $U(z) \geqq \delta' P_n$ this follows that $U(0) \to \infty$. Hence if $U(z) \neq$ constant infinity, we may suppose $\overline{\lim_n} \; U_{|z| = r_{n-1} + t_{n-1}} \leqq P_n^{1+\delta_0}$. On $R_n - R_{n-1}$ we denote by $T_n^\nu(z)$ ($\nu = 1, 2, 3, \dots, 2^{\mu_n}$) the conformal mapping in itself such that $z \leftrightarrow \tilde{z}$ where $\tilde{z}$ is the symmetric point of z with respect to the setraight $\overset{*}{L}{}_n^\nu$: $\arg z = \dfrac{2\pi\left(\nu + \frac{1}{2}\right)}{2^{\mu_n}}$ or $\pi + \dfrac{2\pi\left(\nu + \frac{1}{2}\right)}{2^{\mu_n}}$, $R_n - R_{n-1}$ is a covering surface over $0 \leqq |z| \leqq \infty$, covers $\frac{r_{n-1} + r_n}{2} \leqq |z| \leqq \frac{r_n + r_{n+1}}{2}$ $2^{q_{n-1}} + 1$ times the other part of the z-plane 2^{q_n} times, $R_n - R_{n-1}$ has only two boundary components on which $U(z) \leqq P_{n+1}^{1+\delta_0}$ then also in $R_n - R_{n-1}$. Put $\tilde{U}(z) = |U(z) - U(T^\nu(z))|$, then $\tilde{U}(z) = 0$ on all points of $R_n - R_{n-1}$ with its projection contained in $\overset{*}{L}{}_n^\nu$; Denote by $V'(z)$ a harmonic function such that $V'(z) = 0$, $z \in R_n - R_{n-1}$ if z has projection on L_n^ν, $L_n^{\nu'}$, $V'(z) = P_{n+1}^{1+\delta_0}$ if z is contained in the boundary of $R_n - R_{n-1}$ i. e. $z \in F_0$ and $|z| = \frac{r_{n-1} + r_n}{2}$ or $|z| = \frac{r_n + r_{n+1}}{2}$. Denote by R_n^ν the surface composed of $\hat{F}_n^i$, $\hat{F}_n^i$: $i = 1, 2, 3, \dots, 2^{q_n - 1}$ which is a part of F_n^i bounded L_n^ν and $L_n^{\nu'}$ and the part of F_0 contained in $\sum_\nu J_n^\nu$ which is connected with F_n^i on $\sum_\nu I_n^\nu$ and $\sum S' + S + \tilde{S}' + S$ and $V^\lambda(z)$ be a harmonic function on R_n^ν such that $V^\lambda(z) = 0$ if z lies on L_n^λ or $L_n^{\lambda'}$ of F_n^i or $\hat{F}_n^i$ and $V^\lambda(z) = P_{n+1}^{1+\delta_0}$ if z lies on $\sum J_n^\lambda$ of F_0. We investigate the behaviour of $V^\lambda(z)$ on R_n, since the part of F_0 bounded by $\sum_i J_n^i$ is composed of at most $2^{\mu_n} - 1$ doubly connected domains with module $\mathfrak{M}_2^{(n)}$, then the Dirichlet integral of $V^\lambda(z)$

on R_n^λ is smaller than the Dirichlet integral of the harmonic function being zero on $\sum_\nu J_n^\nu$ and $=2\ P_{n+1}^{1+\delta_0}$ on $\sum_\nu I_n^\nu$, over the part of F_0 bounded by $\sum_\nu J_n^\nu$ and $\sum_\nu I_n^\nu$; $\nu=1, 2, 3, \dots, 2^{\mu_n-1}$, then we have

$$D_{R_n^\nu}(V(z)) \leqq \frac{1}{2\pi}(2^{\mu_n}-1)\frac{P_{n+1}^{2+2\delta_0}}{\mathfrak{M}_{(n)}^2}.$$

If $V^\lambda(z) \geqq \frac{2}{n}$ on the measurable set of measure larger than $\frac{1}{k_n}$ of at H_n^λ of F_n', then on account of 9) there exist at least s_n examplars of F or $\hat{F}$ such that $V^\lambda(z) \geqq \frac{1}{n}$ on a measurable set of measure $>\frac{1}{k_n}$ on H_n^ν where measure of set means the measure of the image when G_n^ν is mapped on $e^{\mathfrak{M}_1^{(n)}} \geqq |w| \geqq 1$. then

$$D(V^\lambda(z)) \geqq \frac{s_n}{2\pi n^2 k_n(\mathfrak{M}_n^{(1)}-\alpha_n)} \geqq \frac{P_{n+1}^{2+2\delta_0}\, 2^{\mu_n}}{\mathfrak{M}_n^{(2)}}.$$

This contradicts the property of q_n and s_n.

If $|V^\lambda(z_1)-V^\lambda(z_2)| \geqq \frac{1}{n}$ on the measurable set of measuere $\geqq \frac{1}{k_n}$ of at H_n^λ, where arg $V^\nu(z_1)=$ arg $V^\lambda(z_2)$ and lie on $|w|=1$ and $|w|=e^{\alpha_n}$ of F_n' respectively, then by 7) and 8), we have

$$D_{R_n^\nu}(V^\lambda(z) \geqq \frac{1}{2\pi n^2 k_n \alpha_n} \geqq \frac{P_{n+1}^{2+2\delta_0} 2^{\mu_n}}{\mathfrak{M}_{(n)}^2},$$

where right hand is the Dirichlet integral over the part of F_n' bounded by J_ν^n and I_n^ν, this contradicts to 3) therefore $V^\lambda(z) \leqq \frac{3}{n}$ except $\frac{1}{k_n}$ angular measure at any of I_n^ν. On the other hand $\tilde{U}(z)$ is sub-harmonic and $\leqq P_{n+1}^{1+\delta_0}$ and vanishes on $\overset{**}{L_n^\nu}+\overset{**}{L_n'^\nu}$, then $\tilde{U}(z) \leqq V^{\lambda'}(z)$: $\lambda'=\lambda, \lambda+1, \dots$ $\lambda+2^{\mu_n-1}$ where $|\arg \overset{**}{L_n^\nu} - \arg L_n^\nu| = |\arg \overset{**\prime\nu}{L_n} - \arg L_n^\nu| = \frac{\pi}{2}$. $|\tilde{U}(z)| \leqq \frac{3}{n}$ on every I_n^ν except $\frac{\text{length } I_n^\nu}{N_n}$ measure set thus $|U(z)-U(T_n^\nu(z))| \leqq \frac{6}{n}$ in ring R_n' but ν is arbitrary, accordingly $U(z)$ is symmetric and periodic with respect to 2^{μ_n} directions $\overset{*}{L_n^\nu}$ except at most $\frac{6}{n}$ in step by 2) $|\text{Max}\, U(z)$ $-\text{Min}\, U(z)| < \frac{7}{n}$, when $|z|=r_n^*$, where $r_n^* = \sqrt{(r_{n-1}+\frac{11}{10}t_{n-1})(r_n-\frac{11}{10}s_n)}$. It follows that $U(z)$ must be a constant.

2) As F_0 is a subset of the unit-circle, it is clear that R has not Gross's property and on account of theorem 5.2, R is a positive boundary Riemann surface.

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