Nagayasu, S.
Osaka J. Math.
42 (2005), 613–632

AN INVERSE PROBLEM FOR THE WAVE EQUATION IN PLANE-STRATIFIED MEDIA

SEI NAGAYASU

(Received January 6, 2004)

Abstract

Assume that two media are laying in a half-space and the interface wall is parallel to the boundary of the half-space. We can directly observe the data near the boundary of the half-space, but we cannot directly observe inside the half-space. In this situation, we try to identify these unknown things by creating an artificial explosion and observing on the boundary the waves generated by the explosion. In the previous works related to this problem, only the speeds of the waves were treated, but we also take into account the impedances of the media in our setting.

1. Introduction

Our problem originates from a simplified model of an experiment conducted by geophysicists. We cannot directly observe the structure inside the earth. Then, for example, we perform the following experiment in order to guess it: We create an artificial explosion at a certain point near the earth's surface. Waves generated by the explosion travel in the earth. We observe the waves on the earth's surface, and determine the structure inside the earth from the observation data.

We consider this problem, in particular, in the case when the earth consists of some layers. This problem has been studied by Bartoloni-Lodovici-Zirilli [1], Fatone-Maponi-Pignotti-Zirilli [2], and Hansen [3], for instance. However, from the experimental point of view, these results have some problem since we have to know in advance, some information on the interface or transmission conditions.

In [1] and [2], the authors consider the earth as a half-space, and assume that each interface wall of layers is parallel to the boundary of the half-space, and a property of a medium of each layer is uniform, that is, the speeds of the waves through the half-space are described by a piecewise constant function. In particular, they deal with

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \operatorname{div}(c(x)\nabla_x u)(t,x)$$

in order to express behavior of the waves inside the half-space, where $c(x)$ is a piecewise constant function. In this case, the interface or transmission conditions are characterized once we know the speeds of the waves.

On the other hand, Hansen [3] considers the case of a two-layer domain, whose

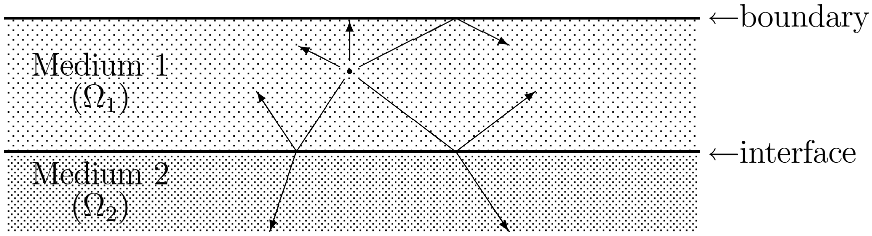


Fig. 1.

boundary may be not flat, and a property of a medium of each layer may be not uniform. The author deals with

$$\begin{cases} \Delta u - \alpha \dfrac{\partial^2 u}{\partial t^2} = 0 & \text{in each layer,} \\ u_+ = u_- & \text{on the interface wall,} \\ (\partial_{n_+} u)_+ + (\partial_{n_-} u)_- = 0 & \text{on the interface wall} \end{cases}$$

in order to describe behavior of the waves in the domain, where α is smooth in each layer, and has gaps on the interface wall. In this case, the author assumes that the interface or transmission conditions are independent of the speeds of the waves and are known in advance.

However, it is not natural to treat the interface or transmission conditions as the known data or the data directly given by the speeds of the waves since they actually depend on not only the speeds of the waves but also the impedances of media.

We discuss the following problem. Assume that two media, Medium 1 and Medium 2, are laying in a half-space, and the interface wall is parallel to the boundary of the half-space (see Fig. 1). We assume that the speed of the waves in Medium 1 and the way of the reflection by the boundary are known, but the width of Medium 1, the speed of the waves in Medium 2, and the interface or transmission conditions are unknown. In this situation, we try to identify these unknown things by using the known data or the data which can be observed near the boundary.

Now, we introduce the notations and formulate the problem above. Suppose $n \geq 2$. Let us write $x' = (x_1, \dots, x_{n-1})$, and $x'' = (x_2, \dots, x_n)$ for the coordinate $x = (x_1, \dots, x_n)$ in $\mathbb{R}^n$. The variable x_1 plays the role of the time and x'' the physical space. We introduce x' for short notation when we apply the Fourier-Laplace transformation with respect to $(x_1, \dots, x_{n-1})$.

Let $h > 0$ and $\Omega_1 := \{x'' \in \mathbb{R}^{n-1} : 0 < x_n < h\}$, $\Omega_2 := \{x'' \in \mathbb{R}^{n-1} : x_n > h\}$. We set $D_{x_j} := (1/i)(\partial/\partial x_j)$, $\Delta_{x''} := D_{x_2}^2 + \cdots + D_{x_n}^2$, and $D_{\xi_j} := (1/i)(\partial/\partial \xi_j)$. Let a_k be a positive real number and set $P_k(D_x) := a_k^2 \Delta_{x''} - D_{x_1}^2$ for $k = 1, 2$. The positive number a_k describes the speed of the waves in Ω_k. Let $Q(D_x)$ be a partial differential operator with constant coefficients of first order, and write $Q(D_x) = q_1 D_{x_1} + \cdots + q_n D_{x_n} + q_0$.

Furthermore we assume the coefficient q_n is not zero. Let b_1, b_2, c_1, c_2 be constants. Suppose $0 < y_n < h$. Set $y'' := (0, \dots, 0, y_n) \in \mathbb{R}^{n-1}$ and $y := (0, y'') \in \mathbb{R}^n$.

We discuss the following equations:

$$P_1(D_x)u(x) = \delta(x - y), \quad x_1 \in \mathbb{R}, \ x'' \in \Omega_1, \tag{1}$$

$$P_2(D_x)u(x) = 0, \quad x_1 \in \mathbb{R}, \ x'' \in \Omega_2, \tag{2}$$

$$Q(D_x)u(x)|_{x_n=0} = 0, \quad x' \in \mathbb{R}^{n-1}, \tag{3}$$

$$b_1 u(x)|_{x_n=h_-} = c_1 u(x)|_{x_n=h_+}, \quad x' \in \mathbb{R}^{n-1}, \tag{4}$$

$$b_2 D_{x_n} u(x)|_{x_n=h_-} = c_2 D_{x_n} u(x)|_{x_n=h_+}, \quad x' \in \mathbb{R}^{n-1}. \tag{5}$$

These equations describe the situation that the initial data is the delta function at a point y'' in Ω_1 at time $x_1 = 0$ with the boundary condition (3) and the interface or transmission conditions (4) and (5). In particular, if $Q(D_x) = D_{x_n}$ then the boundary condition is Neumann condition. If $b_1 = c_1$ then the equation (4) expresses the continuity of the displacement of the waves on the interface wall. The equation (5) expresses the continuity of the stress on the interface wall. We assume that the mixed problem for the operator system $\{P_1(D_x), P_2(D_x); Q(D_x); b_1, c_1; b_2 D_{x_n}, c_2 D_{x_n}\}$ is $\mathcal{E}$ well-posed (see Section 2).

The following main result says that except some special cases we can determine the data which we cannot directly observe, that is, the width h of Ω_1, the speed a_2 of the waves in Ω_2 and the interface or transmission conditions to a certain degree from the observation data $u(x)|_{x_n=0}$ when the speed a_1 of the waves in Ω_1 and the boundary condition are known. In particular, we remark that we can determine the interface or transmission conditions which other authors treated as the data known by some method or the data given in advance. For example, in [1] and [2] it is assumed that the interface or transmission conditions can be derived from the given data of the speeds of the waves. In [3], the interface or transmission conditions are known in advance. In our setting, the interface or transmission conditions are independent of the speeds of the waves, and moreover their coefficients are unknown data, together with the speed a_2 of the waves in Ω_2. In these senses, the restriction of our setting is weaker than the other authors'.

Main result. Let $a_1, Q(D_x), y_n$ be given. Assume that the observation data $u(x)|_{x_n=0}$ are given, where $u(x)$ denotes the solution of the equations (1)–(5). Then the constants h, a_2 and the ratio of $b_1 c_2$ to $b_2 c_1$ are determined in the following sense:

- The constant h is expressed with the known data $a_1, Q(D_x), y_n$ and the observation data $u(x)|_{x_n=0}$ unless $u(x)|_{x_n=0} \equiv \widetilde{u}(x)|_{x_n=0}$. Here $\widetilde{u}(x)$ is the waves in the situation that only one medium Medium 1 is laying in the half-space, that is, the solution of

$$P_1(D_x)\widetilde{u}(x) = \delta(x - y), \quad x \in \mathbb{R}^n,$$
$$Q(D_x)\widetilde{u}(x)|_{x_n=0} = 0, \quad x' \in \mathbb{R}^{n-1}.$$

- Suppose $n \geq 3$. Then the ratio of b_1c_2 to b_2c_1 is expressed with a_1, $Q(D_x)$, y_n and $u(x)|_{x_n=0}$. If $b_1b_2c_1c_2 \neq 0$ then a_2 is also determined.
- Suppose $n = 2$. Then the ratio of b_1c_2 to $a_2b_2c_1$ is expressed with a_1, $Q(D_x)$, y_n and $u(x)|_{x_n=0}$.

We remark that the constants b_1, b_2, c_1, c_2 themselves cannot be identified. Namely, in some cases of $n \geq 3$, the observation data are the same even if the coefficients b_1, b_2, c_1, c_2 of the condition (4) and (5) are different.

We mention the impedances in the case $n = 2$, that is, the physical space dimension is one. If we assume the continuity of the displacement of the waves, namely $b_1 = c_1$, then we determine the ratio of b_2/a_1 to c_2/a_2 by our main result because a_1 is known. Here, b_2/a_1 and c_2/a_2 are the impedance of Medium 1 and 2, respectively. Then, if the impedance of Medium 1 is known, we can determine the impedance of Medium 2, however we cannot identify the speed of the waves in Medium 2. When viewed from a different angle, we obtain the same observation data if the impedance of Medium 2 is same even if the speed of the waves in Medium 2 is different. This result is not obtained by [1], in which they deal with the case when the physical space dimension is one.

In Section 4, we state our main results more precisely, and prove them by using the solution formula of the problem (1)–(5). The solution formula is given by Matsumura [4] as the fundamental solution of the mixed problem for the operator system $\{P_1(D_x),\ P_2(D_x);\ Q(D_x);\ b_1,\ c_1;\ b_2D_{x_n},\ c_2D_{x_n}\}$. Theoretically, this formula must describe the dependence between behavior of the solution and information of the media. However this dependence is rather intricate and is not expressed straightforwardly. In this section, we give a process of reduction to clear the dependence. This is the main part of our methods.

Finally, we explain the plan of this paper. In Section 2, we explain necessary and sufficient conditions for the $\mathcal{E}$ well-posedness for the mixed problem for the operator system $\{P_1(D_x),\ P_2(D_x);\ Q(D_x);\ b_1,\ c_1;\ b_2D_{x_n},\ c_2D_{x_n}\}$. In Section 3, we rewrite the equations (1)–(5), construct the solution, and discuss some properties of the solution. In Section 4, we state the main theorems in this paper and give the proofs.

2. The $\mathcal{E}$ Well-posedness

In Section 1, we assume that the mixed problem for the operator system $\{P_1(D_x),$ $P_2(D_x);\ Q(D_x);\ b_1,\ c_1;\ b_2D_{x_n},\ c_2D_{x_n}\}$:

(6) $P_1(D_x)u(x) = f(x), \quad x_1 > 0,\ x'' \in \Omega_1,$

(7) $P_2(D_x)u(x) = f(x), \quad x_1 > 0,\ x'' \in \Omega_2,$

(8) $u(0, x'') = g_0(x''), \quad x'' \in \mathbb{R}^{n-2} \times (0, \infty),$

(9) $D_{x_1}u(0, x'') = g_1(x''), \quad x'' \in \mathbb{R}^{n-2} \times (0, \infty),$

(10) $\quad Q(D_x)u(x)|_{x_n=0} = k_0(x'), \quad x' \in (0,\infty)\times\mathbb{R}^{n-2},$

(11) $\quad b_1u(x)|_{x_n=h_-} = c_1u(x)|_{x_n=h_+} + k_1(x'), \quad x' \in (0,\infty)\times\mathbb{R}^{n-2},$

(12) $\quad b_2D_{x_n}u(x)|_{x_n=h_-} = c_2D_{x_n}u(x)|_{x_n=h_+} + k_2(x'), \quad x' \in (0,\infty)\times\mathbb{R}^{n-2}$

is $\mathcal{E}$ well-posed. In this section, we explain necessary and sufficient conditions for the well-posedness. We mainly refer to Matsumura [4] and Sakamoto [5].

We first introduce the notations in order to refer to the results of [4]. We set $\Gamma_k := \{\eta \in \mathbb{R}^n : \eta_1 > 0,\ \eta_1^2 > a_k^2|\eta''|^2\}$ and $\Gamma_k' := \{\eta' \in \mathbb{R}^{n-1} : (\eta', 0) \in \Gamma_k\}$ for $k = 1, 2$. We denote by $\lambda_k^+(\xi' + i\eta')$ the root which has positive imaginary part of the equation in λ: $P_k(\xi' + i\eta', \lambda) = 0$ for $\xi' \in \mathbb{R}^{n-1}$, $\eta' \in -\Gamma_k'$, and $k = 1, 2$. Also we set

$$R_0(\zeta') := Q\big(\zeta', \lambda_1^+(\zeta')\big)\begin{vmatrix} b_1 & c_1 \\ -b_2\lambda_1^+(\zeta') & c_2\lambda_2^+(\zeta') \end{vmatrix},$$

$$R_1(\zeta') := Q\big(\zeta', -\lambda_1^+(\zeta')\big)\begin{vmatrix} b_1 & c_1 \\ b_2\lambda_1^+(\zeta') & c_2\lambda_2^+(\zeta') \end{vmatrix}$$

for $\zeta' = \xi' + i\eta'$ with $\xi' \in \mathbb{R}^{n-1}$ and $\eta' \in -(\Gamma_1' \cap \Gamma_2')$. Then the Lopatinski's determinant of the operator system $\{P_1(D_x), P_2(D_x);\ Q(D_x);\ b_1, c_1;\ b_2D_{x_n}, c_2D_{x_n}\}$ for the mixed problem (6)–(12) is defined by

$$R(\zeta') := R_0(\zeta')e^{-i\lambda_1^+(\zeta')h} - R_1(\zeta')e^{i\lambda_1^+(\zeta')h}.$$

Furthermore we have the following theorem:

Theorem 1 ([4]). *The mixed problem* (6)–(12) *is $\mathcal{E}$ well-posed if and only if the following conditions are satisfied*:
(i) $R_0^0(\theta') \neq 0$.
(ii) *There exists γ_1 such that $R_0(\xi' - i\gamma\theta') \neq 0$ for any $\xi' \in \mathbb{R}^{n-1}$ and $\gamma \geq \gamma_1$.*
Here $\theta' := (1, 0, \dots, 0) \in \mathbb{R}^{n-1}$ and R_0^0 is the principal part of R_0. Moreover, under this condition, $R(\xi' - im\log(2 + |\xi'|)\theta') \neq 0$ for all $\xi' \in \mathbb{R}^{n-1}$, where m is a positive real large enough.

Now, we have

$$R_0^0(\theta') = -\left(q_1 - \frac{q_n}{a_1}\right)\left(\frac{b_1c_2}{a_2} + \frac{b_2c_1}{a_1}\right)$$

by calculating it. Then the condition (i) holds if and only if $q_1 - q_n/a_1 \neq 0$ and $b_1c_2/a_2 + b_2c_1/a_1 \neq 0$ hold. Hence the mixed problem (6)–(12) is $\mathcal{E}$ well-posed if and only if the following conditions hold:
(I) $q_1 - q_n/a_1 \neq 0$ and there exists γ_1 such that

$$Q\big(\xi' - i\gamma\theta', \lambda_1^+(\xi' - i\gamma\theta')\big) \neq 0$$

for any $\xi' \in \mathbb{R}^{n-1}$ and $\gamma \geq \gamma_1$.
(II) $b_1c_2/a_2 + b_2c_1/a_1 \neq 0$ and there exists γ_1 such that

$$b_1c_2\lambda_2^+(\xi' - i\gamma\theta') + b_2c_1\lambda_1^+(\xi' - i\gamma\theta') \neq 0$$

for any $\xi' \in \mathbb{R}^{n-1}$ and $\gamma \geq \gamma_1$.
Here the condition (I) is a condition for $Q(D_x)$ and (II) the constants b_1, b_2, c_1, c_2, and a_2.

We state necessary and sufficient conditions for (II).

Proposition 2. *If* $n = 2$ *then* (II) *is equivalent to* $b_1c_2/a_2 + b_2c_1/a_1 \neq 0$. *If* $n \geq 3$ *then* (II) *is equivalent to* $b_1c_2/a_2 + b_2c_1/a_1 \neq 0$ *and the following*:
(III) $(b_1^2c_2^2 - b_2^2c_1^2)\left(\dfrac{\overline{b_1}^2\overline{c_2}^2}{a_2^2} - \dfrac{\overline{b_2}^2\overline{c_1}^2}{a_1^2}\right) \geq 0,$
or
(IV) $a_2 > a_1$ *and* $a_1^2b_1^2c_2^2\overline{b_2}^2\overline{c_1}^2 - a_2^2|b_2c_1|^4 \in \mathbb{C}$ *is not a nonnegative number and* $\mathrm{Re}(b_1c_2\overline{b_2}\overline{c_1}) > 0$,
or
(V) $a_1 > a_2$ *and* $a_2^2\overline{b_1}^2\overline{c_2}^2b_2^2c_1^2 - a_1^2|b_1c_2|^4 \in \mathbb{C}$ *is not a nonnegative number and* $\mathrm{Re}(b_1c_2\overline{b_2}\overline{c_1}) > 0$.

Proof. For all $\zeta_1 = \xi_1 - i\gamma$ with $\xi_1 \in \mathbb{R}$ and $\gamma > 0$, we have

$$b_1c_2\lambda_2^+(\zeta_1\theta') + b_2c_1\lambda_1^+(\zeta_1\theta') = -\left(\frac{b_1c_2}{a_2} + \frac{b_2c_1}{a_1}\right)\zeta_1$$

since $\lambda_k^+(\zeta_1\theta') = -\zeta_1/a_k$. This completes the proof in the case of $n = 2$.

Hereafter we assume $n \geq 3$. By noting the imaginary part of λ_k^+, (II) holds if
- $b_1c_2/a_2 = b_2c_1/a_1 \neq 0$,
- $b_1c_2 = 0$ and $b_2c_1 \neq 0$,
- $b_2c_1 = 0$ and $b_1c_2 \neq 0$,
- $a_1 = a_2$.

Accordingly we suppose

$$b_1b_2c_1c_2 \neq 0, \quad a_1 \neq a_2, \quad \frac{b_1^2c_2^2}{a_2^2} \neq \frac{b_2^2c_1^2}{a_1^2}$$

hereafter. We denote by $\nu_k^+(z)$ the root with positive imaginary part of the equation in ν: $z - a_k^2(1 + \nu^2) = 0$ for $z \in \mathbb{C}$, $k = 1, 2$. Then (II) holds if and only if the following condition (VI) is satisfied since we have

$$\lambda_k^+(\xi' - i\gamma\theta') = w\nu_k^+\left(\left(\frac{\xi_1}{w} - i\frac{\gamma}{w}\right)^2\right)$$

for $\xi' \in \mathbb{R}^{n-1}$ with $w \neq 0$ and $\gamma > 0$, where $w = \sqrt{\xi_2^2 + \cdots + \xi_{n-1}^2}$:
(VI) for all $X \in \mathbb{R}$ and $Y > 0$,

$$b_1 c_2 \nu_2^+\big((X - iY)^2\big) + b_2 c_1 \nu_1^+\big((X - iY)^2\big) \neq 0.$$

On the other hand, solving

$$\big\{b_1 c_2 \nu_2^+\big((X - iY)^2\big)\big\}^2 = \big\{b_2 c_1 \nu_1^+\big((X - iY)^2\big)\big\}^2$$

for $(X - iY)^2$, we obtain $(X - iY)^2 = M$, where

$$M = \frac{b_1^2 c_2^2 - b_2^2 c_1^2}{(b_1^2 c_2^2 / a_2^2) - (b_2^2 c_1^2 / a_1^2)}.$$

Then (VI) holds if and only if $M \geq 0$ or
(VII) $b_1 c_2 \nu_2^+(M) + b_2 c_1 \nu_1^+(M) \neq 0$. ($M \in \mathbb{C}$ is not a nonnegative number.)
Here, $M \geq 0$ holds if and only if the condition (III) holds. Finally, we consider necessary and sufficient conditions for (VII). Put

$$A = \frac{a_2^2 - a_1^2}{a_1^2 b_1^2 c_2^2 - a_2^2 b_2^2 c_1^2}, \quad B = b_1 c_2, \quad C = b_2 c_1$$

for short notation. Then $\nu_1^+(M)^2 = AB^2$ and $\nu_2^+(M)^2 = AC^2$. Let Z be $Z^2 = A$ and $\operatorname{Im} Z \geq 0$. Then we have

$$b_1 c_2 \nu_2^+(M) + b_2 c_1 \nu_1^+(M) = \big\{\operatorname{sgn}\big(\operatorname{Im}(ZB)\big) + \operatorname{sgn}\big(\operatorname{Im}(ZC)\big)\big\} ZBC.$$

Hence (VII) is equivalent to $\operatorname{Im}(ZB)\operatorname{Im}(ZC) > 0$. Put $\beta = B/a_2$ and $\alpha = C/a_1$. Then

$$\begin{aligned}&\operatorname{Im}(ZB)\operatorname{Im}(ZC)\\ &= \frac{1}{2}\frac{a_1 a_2}{|\beta^2 - \alpha^2|^2}\operatorname{Re}(\alpha\overline{\beta})\left\{\left|\frac{1}{a_1^2} - \frac{1}{a_2^2}\right||\beta^2 - \alpha^2| - \left(\frac{1}{a_1^2} - \frac{1}{a_2^2}\right)\big(|\beta|^2 - |\alpha|^2\big)\right\}.\end{aligned}$$

Therefore $\operatorname{Im}(ZB)\operatorname{Im}(ZC) > 0$ holds if and only if (IV) or (V) because

$$\frac{a_1 a_2}{|\beta^2 - \alpha^2|^2}\left\{\left|\frac{1}{a_1^2} - \frac{1}{a_2^2}\right||\beta^2 - \alpha^2| - \left(\frac{1}{a_1^2} - \frac{1}{a_2^2}\right)(|\beta|^2 - |\alpha|^2)\right\} \geq 0, \tag{13}$$

and we have equality in (13) if and only if

$$a_2 > a_1 \text{ and } (\beta^2 - \alpha^2)\overline{\alpha}^2 \geq 0,$$

or

$$a_1 > a_2 \text{ and } (\alpha^2 - \beta^2)\overline{\beta}^2 \geq 0.$$

The proof is completed. $\square$

3. The Solution Formula and some Properties of the Solution

In this section, we construct the solution to the equations (1)–(5), and prove a lemma needed later. We mainly refer to Matsumura [4] and Sakamoto [5] about the construction of the solution.

We first rewrite these equations. The fundamental solution of the forward Cauchy problem for $P_1(D_x)$ in the whole physical space $\mathbb{R}^{n-1}_{x''}$ is defined as the inverse Fourier-Laplace transform of $1/P_1(\xi + i\eta)$ in the sense of the distribution:

$$E_1(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_\xi} \frac{e^{ix\cdot(\xi+i\eta)}}{P_1(\xi + i\eta)}\, d\xi,$$

where $\eta \in -s\theta - \Gamma_1$ and $\theta := (1, 0, \dots, 0) \in \mathbb{R}^n$ with a positive real s large enough. Moreover we define $F_1(x)$ and $F_2(x)$ by

$$F_1(x) := E_1(x - y) - u(x), \quad x'' \in \Omega_1, \tag{14}$$

$$F_2(x) := u(x), \quad x'' \in \Omega_2, \tag{15}$$

respectively. Since the distribution $E_1(x-y)$ describes the incident or primary propagation of the waves due to a point source $\delta(x'' - y'')$, $F_1(x)$ in (14) describes the propagation in Ω_1 of secondary waves caused by the primary waves, the boundary wall $\{x_n = 0\}$ and the interface wall $\{x_n = h\}$. By (14) and (15), the equations (1)–(5) are equivalent to the following equations:

$$P_1(D_x)F_1(x) = 0, \quad x_1 \in \mathbb{R},\ x'' \in \Omega_1, \tag{16}$$

$$P_2(D_x)F_2(x) = 0, \quad x_1 \in \mathbb{R},\ x'' \in \Omega_2, \tag{17}$$

$$Q(D_x)F_1(x)|_{x_n=0} = Q(D_x)E_1(x - y)|_{x_n=0}, \quad x' \in \mathbb{R}^{n-1}, \tag{18}$$

$$b_1\big(E_1(x - y) - F_1(x)\big)|_{x_n=h_-} = c_1 F_2(x)|_{x_n=h_+}, \quad x' \in \mathbb{R}^{n-1}, \tag{19}$$

$$b_2 D_{x_n}\big(E_1(x - y) - F_1(x)\big)|_{x_n=h_-} = c_2 D_{x_n} F_2(x)|_{x_n=h_+}, \quad x' \in \mathbb{R}^{n-1}. \tag{20}$$

Put $Z'_m(\xi') := \xi' - im\log(2 + |\xi'|)\theta'$ and $S'_m := \{Z'_m(\xi') : \xi' \in \mathbb{R}^{n-1}\}$. We can solve the problem (16)–(20) by the Fourier-Laplace transformation along S'_m, Theorem 2 and so on, where m is a positive real large enough (see Matsumura [4]). In particular, the Fourier-Laplace transforms of $F_1(x)$ and $F_2(x)$ with respect to $\zeta' = Z'_m(\xi') \in S'_m$ can be expressed in the forms of

$$(\mathcal{L}_{x'}F_1)(\zeta'; x_n) = \Phi_1(\zeta')e^{i\lambda_1^+(\zeta')x_n} + \Psi_1(\zeta')e^{-i\lambda_1^+(\zeta')x_n}, \tag{21}$$

$$(\mathcal{L}_{x'}F_2)(\zeta'; x_n) = \Phi_2(\zeta')e^{i\lambda_2^+(\zeta')x_n}, \tag{22}$$

where $\Phi_1(\zeta')$, $\Psi_1(\zeta')$ and $\Phi_2(\zeta')$ are of class C^∞ with respect to $\xi' \neq 0$ and continuous with respect to ξ'. We remark that the Fourier-Laplace transform of E_1 can be expressed in the following form by the inversion formula and the residue theorem and so on:

$$\begin{aligned} Q(\zeta', D_{x_n})(\mathcal{L}_{x'}E_1)(\zeta'; x_n - y_n)|_{x_n=0} &= \frac{i}{2a_1^2}\frac{Q(\zeta', -\lambda_1^+(\zeta'))}{\lambda_1^+(\zeta')}e^{i\lambda_1^+(\zeta')y_n}, \\ (\mathcal{L}_{x'}E_1)(\zeta'; x_n - y_n)|_{x_n=h_-} &= \frac{i}{2a_1^2}\frac{1}{\lambda_1^+(\zeta')}e^{i\lambda_1^+(\zeta')(h-y_n)}, \\ D_{x_n}(\mathcal{L}_{x'}E_1)(\zeta'; x_n - y_n)|_{x_n=h_-} &= \frac{i}{2a_1^2}e^{i\lambda_1^+(\zeta')(h-y_n)}. \end{aligned}$$

Set $\chi_1(\xi') := \xi_1 - im\log(2 + |\xi'|)$ for $\xi' \in \mathbb{R}^{n-1}$. We remark that $S'_m = \{(\chi_1(\xi'), \xi_2, \dots, \xi_{n-1}) : \xi' \in \mathbb{R}^{n-1}\}$. Set $\chi(\xi_1) := \chi_1(\xi_1\theta')$ for $\xi_1 \in \mathbb{R}$ for short notation.

Finally, we prove a lemma needed later.

Lemma 3. *Let* $\{F_1(x),\ F_2(x)\}$ *be the solution of the problem* (16)–(20). *If* $\Psi_1(\chi(\xi_1)\theta') \equiv 0$ *for all* $\xi_1 \in \mathbb{R}$, *then for some* $\xi_1 \in \mathbb{R}$

$$\Phi_1\big(\chi(\xi_1)\theta'\big) \not\equiv -\frac{i}{2a_1\chi(\xi_1)}e^{i\chi(\xi_1)y_n/a_1}$$

holds, where $\Phi_1(\zeta')$ *and* $\Psi_1(\zeta')$ *are defined by* (21).

REMARK 4. We remark that

$$\frac{i}{2a_1^2}\frac{1}{\lambda_1^+(\zeta')}e^{-i\lambda_1^+(\zeta')y_n}\bigg|_{\zeta'=\chi(\xi_1)\theta'} = -\frac{i}{2a_1\chi(\xi_1)}e^{i\chi(\xi_1)y_n/a_1}.$$

Indeed, we have $\lambda_1^+(\chi(\xi_1)\theta') = -\chi(\xi_1)/a_1$ since $\lambda_1^+(\chi(\xi_1)\theta')^2 = \{\chi(\xi_1)/a_1\}^2$ and $\operatorname{Im}\chi(\xi_1) < 0$.

Proof of Lemma 3. We use reduction to absurdity. Assume

$$\Phi_1\big(\chi(\xi_1)\theta'\big) \equiv -\frac{i}{2a_1\chi(\xi_1)}e^{i\chi(\xi_1)y_n/a_1} \quad \text{and} \quad \Psi_1\big(\chi(\xi_1)\theta'\big) \equiv 0 \tag{23}$$

for all $\xi_1 \in \mathbb{R}$. We have

$$Q\big(\zeta', \lambda_1^+(\zeta')\big)\Phi_1(\zeta') + Q\big(\zeta', -\lambda_1^+(\zeta')\big)\Psi_1(\zeta') = \frac{i}{2a_1^2}\frac{Q\big(\zeta', -\lambda_1^+(\zeta')\big)}{\lambda_1^+(\zeta')}e^{i\lambda_1^+(\zeta')y_n} \tag{24}$$

by (21) and the Fourier-Laplace transform of (18). By substituting $\zeta' = \chi(\xi_1)\theta'$ into (24) and using the assumption (23), we have

$$\left\{\left(q_1 - \frac{q_n}{a_1}\right)\chi(\xi_1) + q_0\right\}e^{i\chi(\xi_1)y_n/a_1} = \left\{\left(q_1 + \frac{q_n}{a_1}\right)\chi(\xi_1) + q_0\right\}e^{-i\chi(\xi_1)y_n/a_1}. \tag{25}$$

Now, we deduce a contradiction by observing the behavior of the equality (25) as $|\xi_1| \to \infty$. We have

$$|(\text{the left-hand side of (25)})| = \left| \left(q_1 - \frac{q_n}{a_1} \right) \chi(\xi_1) + q_0 \right| (2 + |\xi_1|)^{y_n m/a_1} \to +\infty$$

as $|\xi_1| \to \infty$ because $q_1 - q_n/a_1 \neq 0$ (see Section 2 (I)). On the other hand,

$$|(\text{the right-hand side of (25)})| = \left| \left(q_1 + \frac{q_n}{a_1} \right) \chi(\xi_1) + q_0 \right| (2 + |\xi_1|)^{-y_n m/a_1} \to 0$$

as $|\xi_1| \to \infty$ since m is a positive real large enough. Hence we have a contradiction. □

4. The Main Theorem and its Proof

In this section, we prove the main result. For the purpose, first we show that the behavior of the waves in Ω_1 is determined by the known data and the observation data (cf. Lemma 5). Next, we classify cases by the observation data, and we determine the unknown constants for each case (cf. Cases (a), (b), (c)).

In Section 3, we rewrite the equations (1)–(5) to the equations (16)–(20) by using (14) and (15). Since $E_1(x-y)|_{x_n=0}$ can be expressed concretely, we remark that the following are equivalent:

- $u(x)|_{x_n=0}$ is given.
- $F_1(x)|_{x_n=0}$ is given.

We first prove that we obtain the solution $F_1(x)$ in Ω_1 when the observation data $F_1(x)|_{x_n=0}$ are given, that is, the following lemma:

Lemma 5. *Let a_1, $Q(D_x)$, y_n be given. Assume that the observation data $N(x') = F_1(x)|_{x_n=0}$ are given. Then the Fourier-Laplace transform of $F_1(x)$ with respect to $\zeta' = Z'_m(\xi') \in S'_m$ is expressed in the form of*

$$(\mathcal{L}_{x'} F_1)(\zeta'; x_n) = \Phi_1^N(\zeta') e^{i\lambda_1^+(\zeta') x_n} + \Psi_1^N(\zeta') e^{-i\lambda_1^+(\zeta') x_n}, \tag{26}$$

where $\Phi_1^N(\zeta')$ and $\Psi_1^N(\zeta')$ are defined by

$$\begin{aligned}\Phi_1^N(\zeta') := &- \frac{1}{Q(\zeta', \lambda_1^+(\zeta')) - Q(\zeta', -\lambda_1^+(\zeta'))} \\ &\times \left\{ Q(\zeta', -\lambda_1^+(\zeta'))(\mathcal{L}_{x'} N)(\zeta') - \frac{i}{2a_1^2} \frac{Q(\zeta', -\lambda_1^+(\zeta'))}{\lambda_1^+(\zeta')} e^{i\lambda_1^+(\zeta') y_n} \right\},\end{aligned} \tag{27}$$

$$\begin{aligned}\Psi_1^N(\zeta') := &\frac{1}{Q(\zeta', \lambda_1^+(\zeta')) - Q(\zeta', -\lambda_1^+(\zeta'))} \\ &\times \left\{ Q(\zeta', \lambda_1^+(\zeta'))(\mathcal{L}_{x'} N)(\zeta') - \frac{i}{2a_1^2} \frac{Q(\zeta', -\lambda_1^+(\zeta'))}{\lambda_1^+(\zeta')} e^{i\lambda_1^+(\zeta') y_n} \right\}.\end{aligned} \tag{28}$$

Proof. By the discussion in Section 3, $(\mathcal{L}_{x'}F_1)(\zeta'; x_n)$ can be expressed in the form of (26). In particular, we have

$$(\mathcal{L}_{x'}N)(\zeta') = \Phi_1^N(\zeta') + \Psi_1^N(\zeta') \tag{29}$$

by (26) with $x_n = 0$. On the other hand, we have

$$Q\big(\zeta', \lambda_1^+(\zeta')\big)\Phi_1^N(\zeta') + Q\big(\zeta', -\lambda_1^+(\zeta')\big)\Psi_1^N(\zeta') = \frac{i}{2a_1^2}\frac{Q\big(\zeta', -\lambda_1^+(\zeta')\big)}{\lambda_1^+(\zeta')}e^{i\lambda_1^+(\zeta')y_n} \tag{30}$$

by (26) and the Fourier-Laplace transform of (18). Since $q_n\lambda_1^+(\zeta') \neq 0$, we can solve the simultaneous linear equation (29) and (30), and we obtain (27) and (28). □

Next, we remark that there is a possibility that the same observation data can be obtained even if the unknown constants are different. That it, we have the following:

Proposition 6. *Let a_1, $Q(D_x)$, y_n be given. If the ratio of b_1c_2 to b_2c_1 is the same, the same solution $F_1(x)$ is obtained.*

Proof. The Fourier-Laplace transform of $F_1(x)$ can be expressed by (21), and $\Phi_1(\zeta')$ and $\Psi_1(\zeta')$ are expressed in the following forms:

$$\begin{aligned}
\Phi_1(\zeta') &= i\,Q\big(\zeta', -\lambda_1^+(\zeta')\big) \\
&\quad\times \Big[\kappa\lambda_2^+(\zeta')\Big\{e^{-i\lambda_1^+(\zeta')(h-y_n)} - e^{i\lambda_1^+(\zeta')(h-y_n)}\Big\} \\
&\qquad\quad + \lambda_1^+(\zeta')\Big\{e^{-i\lambda_1^+(\zeta')(h-y_n)} + e^{i\lambda_1^+(\zeta')(h-y_n)}\Big\}\Big] \\
&\quad\times \Big[2a_1^2\Big\{Q\big(\zeta', \lambda_1^+(\zeta')\big)\big(\kappa\lambda_2^+(\zeta') + \lambda_1^+(\zeta')\big)e^{-i\lambda_1^+(\zeta')h} \\
&\qquad\qquad - Q\big(\zeta', -\lambda_1^+(\zeta')\big)\big(\kappa\lambda_2^+(\zeta') - \lambda_1^+(\zeta')\big)e^{i\lambda_1^+(\zeta')h}\Big\}\lambda_1^+(\zeta')\Big]^{-1}, \\
\Psi_1(\zeta') &= i e^{i\lambda_1^+(\zeta')h}\big(\kappa\lambda_2^+(\zeta') - \lambda_1^+(\zeta')\big) \\
&\quad\times \Big\{Q\big(\zeta', \lambda_1^+(\zeta')\big)e^{-i\lambda_1^+(\zeta')y_n} - Q\big(\zeta', -\lambda_1^+(\zeta')\big)e^{i\lambda_1^+(\zeta')y_n}\Big\} \\
&\quad\times \Big[2a_1^2\Big\{Q\big(\zeta', \lambda_1^+(\zeta')\big)\big(\kappa\lambda_2^+(\zeta') + \lambda_1^+(\zeta')\big)e^{-i\lambda_1^+(\zeta')h} \\
&\qquad\qquad - Q\big(\zeta', -\lambda_1^+(\zeta')\big)\big(\kappa\lambda_2^+(\zeta') - \lambda_1^+(\zeta')\big)e^{i\lambda_1^+(\zeta')h}\Big\}\lambda_1^+(\zeta')\Big]^{-1},
\end{aligned}$$

where $\kappa := b_1c_2/b_2c_1$ if $b_2c_1 \neq 0$. We remark that these forms express the dependence of $\Phi_1(\zeta')$ and $\Psi_1(\zeta')$ on κ, rather than that on each constant b_1, b_2, c_1, c_2. Therefore, we obtain this proposition except the case when $b_2c_1 = 0$. If $b_2c_1 = 0$ then we can prove it by the concrete forms of $\Phi_1(\zeta')$ and $\Psi_1(\zeta')$ in the same way. □

From Proposition 6, we see that the constants b_1, b_2, c_1, c_2 themselves are not identified even if the observation data $F_1(x)|_{x_n=0}$ is given. However we can expect that the ratio b_1c_2 to b_2c_1 can be determined, and it is actually true to some extent (see Proposition 11 and Theorem 15).

In the following remark, we state particular cases. We can prove these in the same way as Proposition 6.

REMARK 7. Let a_1, $Q(D_x)$, y_n, and $F_1(x)$ be given.
(i) If $b_1c_2 = 0$ then the constants b_2, c_1, a_2 are not identified.
(ii) If $b_2c_1 = 0$ then the constants b_1, c_2, a_2 are not identified.
(iii) If $a_2 = a_1$ and $b_1c_2 = b_2c_1$ then the constant h is not identified.
(iv) If $n = 2$ then the constants b_1, b_2, c_1, c_2, and a_2 are not identified except the ratio of b_1c_2 to $a_2b_2c_1$.
(v) If $n = 2$ and $a_1b_1c_2 = a_2b_2c_1$ then the constant h is not identified.

Hereafter, we define $\Phi_1^N(\zeta')$ and $\Psi_1^N(\zeta')$ by (27) and (28) for the observation data $N(x') = F_1(x)|_{x_n=0}$. Before we determine unknown things, we prove a lemma needed later.

Lemma 8. *Let a_1, $Q(D_x)$, y_n be given. Assume that the observation data $N(x') = F_1(x)|_{x_n=0}$ are given. Then we have*

$$\tag{31} \begin{aligned} &b_1c_2\lambda_2^+(\zeta')\left[\frac{i}{2a_1^2}\frac{1}{\lambda_1^+(\zeta')}e^{i\lambda_1^+(\zeta')(h-y_n)} - \left\{\Phi_1^N(\zeta')e^{i\lambda_1^+(\zeta')h} + \Psi_1^N(\zeta')e^{-i\lambda_1^+(\zeta')h}\right\}\right] \\ &= b_2c_1\left[\frac{i}{2a_1^2}e^{i\lambda_1^+(\zeta')(h-y_n)} - \left\{\Phi_1^N(\zeta')\lambda_1^+(\zeta')e^{i\lambda_1^+(\zeta')h} - \Psi_1^N(\zeta')\lambda_1^+(\zeta')e^{-i\lambda_1^+(\zeta')h}\right\}\right]. \end{aligned}$$

REMARK 9. The equality (31) is equal to the following equality:

$$\tag{32} \begin{aligned} &\left(b_1c_2\lambda_2^+(\zeta') - b_2c_1\lambda_1^+(\zeta')\right)\left\{\frac{i}{2a_1^2}\frac{1}{\lambda_1^+(\zeta')}e^{-i\lambda_1^+(\zeta')y_n} - \Phi_1^N(\zeta')\right\} \\ &\qquad = \left(b_1c_2\lambda_2^+(\zeta') + b_2c_1\lambda_1^+(\zeta')\right)\Psi_1^N(\zeta')e^{-2i\lambda_1^+(\zeta')h}. \end{aligned}$$

Proof of Lemma 8. By Lemma 5, $(\mathcal{L}_{x'}F_1)(\zeta'; x_n)$ can be expressed in the form of (26). On the other hand, $(\mathcal{L}_{x'}F_2)(\zeta'; x_n)$ can be expressed in the form of (22) by the discussion in Section 3. So, we multiply $c_2\lambda_2^+(\zeta')$ and the Fourier-Laplace transform of (19). In the same way, we multiply c_1 and the Fourier-Laplace transform of (20). We subtract the latter from the former, and substitute the equations (26) and (22) into it. Then we have the equality (31). $\square$

Hereafter, we determine unknown things case by case. We first consider the following three cases:

(a) $\Psi_1^N(\zeta') \equiv 0$ for all $\zeta' \in S'_m$.
(b) $\Psi_1^N(\zeta')$ is not identically zero with respect to $\zeta' \in S'_m$, but $\Psi_1^N(\chi(\xi_1)\theta') \equiv 0$ for all $\xi_1 \in \mathbb{R}$.
(c) $\Psi_1^N(\chi(\xi_1)\theta') \not\equiv 0$ for some $\xi_1 \in \mathbb{R}$.

We remark that the solution $\widetilde{F}_1(x)$ of the equations

$$P_1(D_x)\widetilde{F}_1(x) = 0, \quad x_n > 0, \tag{33}$$

$$Q(D_x)[E_1(x-y) - \widetilde{F}_1(x)]|_{x_n=0} = 0 \tag{34}$$

satisfies

$$(\mathcal{L}_{x'}\widetilde{F}_1)(\zeta'; x_n)|_{x_n=0} = \frac{i\,Q(\zeta', -\lambda_1^+(\zeta'))e^{i\lambda_1^+(\zeta')y_n}}{2a_1^2 Q(\zeta', \lambda_1^+(\zeta'))\lambda_1^+(\zeta')}$$

when $(\mathcal{L}_{x'}\widetilde{F}_1)(\zeta'; x_n) \to 0$ as $x_n \to +\infty$. On the other hand, the condition (a) holds if and only if

$$(\mathcal{L}_{x'}N)(\zeta') = \frac{i\,Q(\zeta', -\lambda_1^+(\zeta'))e^{i\lambda_1^+(\zeta')y_n}}{2a_1^2 Q(\zeta', \lambda_1^+(\zeta'))\lambda_1^+(\zeta')}.$$

Then the condition (a) means the observation data are equal to the behavior of the waves on the boundary of the half-space under the condition that Medium 1 is laying in the half-space.

Proposition 10. *Let a_1, $Q(D_x)$, y_n be given. Assume that the observation data $N(x') = F_1(x)|_{x_n=0}$ are given. Suppose that* (a) *is satisfied. If $n \geq 3$ then $a_2 = a_1$ and $b_1c_2 = b_2c_1$. If $n = 2$ then $b_1c_2/a_2 = b_2c_1/a_1$.*

Proof. Set

$$V_1 := \left\{\xi_1 \in \mathbb{R} : \Phi_1^N(\chi(\xi_1)\theta') \neq -\frac{i}{2a_1\chi(\xi_1)}e^{i\chi(\xi_1)y_n/a_1}\right\},$$
$$V_2 := \left\{\xi' \in \mathbb{R}^{n-1} : \Phi_1^N(\zeta') \neq \frac{i}{2a_1^2}\frac{1}{\lambda_1^+(\zeta')}e^{-i\lambda_1^+(\zeta')y_n}, \ \ \zeta' = Z'_m(\xi').\right\}.$$

Note $V_1 \times \{0\} \subset V_2$. The sets V_1 and V_2 are open since $\Phi_1^N(\zeta')$ is continuous. They are not empty by Lemma 3. Note the equality (32) holds by Lemma 8 and Remark 9. Then we have

$$b_1c_2\lambda_2^+(\zeta') = b_2c_1\lambda_1^+(\zeta') \tag{35}$$

for $\zeta' = Z'_m(\xi')$ with $\xi' \in V_2$ by the equality (32) and the assumption $\Psi_1^N(\zeta') \equiv 0$. Hence we have $b_1c_2/a_2 = b_2c_1/a_1$ by the equality (35) with $\zeta' = \chi(\xi_1)\theta'$ and $\xi_1 \in V_1$.

Hereafter, we assume $n \geq 3$. We prove $a_2 = a_1$ and $b_1c_2 = b_2c_1$. By squaring the equality (35) and substituting $b_1c_2/a_2 = b_2c_1/a_1$ into it, we have

$$(b_1c_2 - b_2c_1)(b_1c_2 + b_2c_1)(\xi_2^2 + \cdots + \xi_{n-1}^2) = 0$$

on V_2. There exists $\xi' \in V_2$ such that $(\xi_2, \dots, \xi_{n-1}) \neq (0, \dots, 0)$ because V_2 is open and not empty. Hence we have $b_1c_2 = \pm b_2c_1$. By this and $b_1c_2/a_2 = b_2c_1/a_1$, we have $a_2 = a_1$ and $b_1c_2 = b_2c_1$ since a_1 and a_2 are positive. □

Proposition 11. *Let $a_1, Q(D_x), y_n$ be given. Assume that the observation data $N(x') = F_1(x)|_{x_n=0}$ are given. If* (b) *is satisfied, then $b_1c_2/a_2 = b_2c_1/a_1$ and the constants a_2 and h are expressed as*

$$a_2^2 = \frac{\chi_1(\xi')}{\xi_2^2 + \cdots + \xi_{n-1}^2}\left\{\chi_1(\xi') - \frac{3(D_{\xi_1}\chi_1)(\xi')\varphi_2^N(\xi')}{(D_{\xi_1}\varphi_2^N)(\xi')}\right\}, \tag{36}$$

$$h = -ia_1\left\{\frac{1}{a_2\lambda_2^+(\zeta')} - \frac{a_1\lambda_1^+(\zeta')(D_{\xi_1}\varphi_1^N)(\xi')}{2\chi_1(\xi')(D_{\xi_1}\chi_1)(\xi')\varphi_1^N(\xi')}\right\}, \quad \zeta' = Z_m'(\xi') \tag{37}$$

on $V_4 := \{\xi' \in \mathbb{R}^{n-1} : \Psi_1^N(\zeta') \neq 0,\ \zeta' = Z_m'(\xi')\}$, where $\varphi_1^N(\xi')$ and $\varphi_2^N(\xi')$ are defined by

$$\varphi_1^N(\xi') := \frac{(i/2a_1^2\lambda_1^+(\zeta'))e^{-i\lambda_1^+(\zeta')y_n} - \Phi_1^N(\zeta')}{\Psi_1^N(\zeta')}, \quad \zeta' = Z_m'(\xi'), \tag{38}$$

$$\varphi_2^N(\xi') := -2\chi_1(\xi')(D_{\xi_1}\chi_1)(\xi') \times \left[a_1 D_{\xi_1}\left\{\frac{\lambda_1^+(Z_m'(\xi'))(D_{\xi_1}\varphi_1^N)(\xi')}{\chi_1(\xi')(D_{\xi_1}\chi_1)(\xi')\varphi_1^N(\xi')}\right\}\right]^{-1}. \tag{39}$$

Proof. We remark $n \geq 3$. Set

$$V_3 := \left\{\xi_1 \in \mathbb{R} : \Phi_1^N\big(\chi(\xi_1)\theta'\big) \neq -\frac{i}{2a_1\chi(\xi_1)}e^{i\chi(\xi_1)y_n/a_1}\right\}.$$

The set V_3 is open since $\Phi_1^N(\zeta')$ is continuous. It is not empty by Lemma 3. Note the equality (32) holds by Lemma 8 and Remark 9. Then we have $b_1c_2/a_2 = b_2c_1/a_1$ by the equality (32) with $\zeta' = \chi(\xi_1)\theta'$ and $\xi_1 \in V_3$ since we assume $\Psi_1^N(\chi(\xi_1)\theta') \equiv 0$. Note $b_1c_2 \neq 0$ holds by this and the condition (II) (see Section 2). Therefore we have

$$\begin{aligned}&\left\{\lambda_2^+(\zeta') - \frac{a_1}{a_2}\lambda_1^+(\zeta')\right\}\left\{\frac{i}{2a_1^2}\frac{1}{\lambda_1^+(\zeta')}e^{-i\lambda_1^+(\zeta')y_n} - \Phi_1^N(\zeta')\right\}\\ &\qquad = \left\{\lambda_2^+(\zeta') + \frac{a_1}{a_2}\lambda_1^+(\zeta')\right\}\Psi_1^N(\zeta')e^{-2i\lambda_1^+(\zeta')h}\end{aligned} \tag{40}$$

by substituting $b_2c_1 = (a_1/a_2)b_1c_2$ into (32). Note $\lambda_2^+(\zeta') + (a_1/a_2)\lambda_1^+(\zeta')$ is not zero because its imaginary part is positive. Set $V_4 := \{\xi' \in \mathbb{R}^{n-1} : \Psi_1^N(\zeta') \neq 0,\ \zeta' = Z_m'(\xi')\}$.

The set V_4 is open since $\Psi_1^N(\zeta')$ is continuous. It is not empty by the assumption. Then by (40), we have

$$\varphi_1^N(\xi') = \widetilde{\kappa}(\xi')e^{-2i\lambda_1^+(Z_m'(\xi'))h} \ (\neq 0) \tag{41}$$

on V_4, where $\varphi_1^N(\xi')$ is defined by (38), and $\widetilde{\kappa}(\xi')$ is defined by

$$\widetilde{\kappa}(\xi') = \frac{a_2\lambda_2^+(Z_m'(\xi')) + a_1\lambda_1^+(Z_m'(\xi'))}{a_2\lambda_2^+(Z_m'(\xi')) - a_1\lambda_1^+(Z_m'(\xi'))}.$$

Note $D_{\xi_1}(\lambda_k^+(Z_m'(\xi'))) = \chi_1(\xi')(D_{\xi_1}\chi_1)(\xi')/a_k^2\lambda_k^+(Z_m'(\xi'))$. So we have

$$(D_{\xi_1}\varphi_1^N)(\xi') = \left\{\frac{(D_{\xi_1}\widetilde{\kappa})(\xi')}{\widetilde{\kappa}(\xi')} - 2ih\frac{\chi_1(\xi')(D_{\xi_1}\chi_1)(\xi')}{a_1^2\lambda_1^+(Z_m'(\xi'))}\right\}\widetilde{\kappa}(\xi')e^{-2i\lambda_1^+(Z_m'(\xi'))h}$$

by applying D_{ξ_1} to the equality (41). Then we have

$$\frac{1}{a_2\lambda_2^+(Z_m'(\xi'))} - \frac{ih}{a_1} = \frac{a_1\lambda_1^+(Z_m'(\xi'))(D_{\xi_1}\varphi_1^N)(\xi')}{2\chi_1(\xi')(D_{\xi_1}\chi_1)(\xi')\varphi_1^N(\xi')} \tag{42}$$

by

$$\frac{(D_{\xi_1}\widetilde{\kappa})(\xi')}{\widetilde{\kappa}(\xi')} = \frac{2\chi_1(\xi')(D_{\xi_1}\chi_1)(\xi')}{a_1a_2\lambda_1^+(Z_m'(\xi'))\lambda_2^+(Z_m'(\xi'))}$$

and the equality (41). Applying D_{ξ_1} to the equality (42), we obtain

$$\left\{a_2\lambda_2^+(Z_m'(\xi'))\right\}^3 = \varphi_2^N(\xi'),$$

where $\varphi_2^N(\xi')$ is defined by (39). Squaring this equality, we have

$$\left\{\chi(\xi_1)^2 - a_2^2(\xi_2^2 + \cdots + \xi_{n-1}^2)\right\}^3 = \left\{\varphi_2^N(\xi')\right\}^2. \tag{43}$$

Applying D_{ξ_1} to (43), we have

$$\left\{\chi(\xi_1)^2 - a_2^2(\xi_2^2 + \cdots + \xi_{n-1}^2)\right\}^2 = \frac{\varphi_2^N(\xi')(D_{\xi_1}\varphi_2^N)(\xi')}{3\chi_1(\xi')(D_{\xi_1}\chi_1)(\xi')}. \tag{44}$$

Substituting the equality (44) into the equality (43), we have the equality (36) because if $\xi_2 = \cdots = \xi_{n-1} = 0$ then $\xi' \notin V_4$. We determine a_2 because it is positive. The constant h is expressed as the equality (37) by (42). □

Theorem 12. *Let $a_1, Q(D_x), y_n$ be given. Assume that the observation data $N(x') = F_1(x)|_{x_n=0}$ are given. If* (c) *is satisfied, then the constant h is expressed as*

$$h = \frac{a_1(D_{\xi_1}\varphi_3^N)(\xi_1)}{2i(D_{\xi_1}\chi)(\xi_1)\varphi_3^N(\xi_1)} \tag{45}$$

on $V_5 := \{\xi_1 \in \mathbb{R} : \Psi_1^N(\chi(\xi_1)\theta') \neq 0\}$, *where*

$$\varphi_3^N(\xi_1) := \frac{-(i/2a_1\chi(\xi_1))e^{i\chi(\xi_1)y_n/a_1} - \Phi_1^N(\chi(\xi_1)\theta')}{\Psi_1^N(\chi(\xi_1)\theta')}. \tag{46}$$

Proof. Note the equality (32) holds by Lemma 8 and Remark 9. Then we have

$$\left(\frac{b_1c_2}{a_2} - \frac{b_2c_1}{a_1}\right)\left\{-\frac{i}{2a_1\chi(\xi_1)}e^{i\chi(\xi_1)y_n/a_1} - \Phi_1^N(\chi(\xi_1)\theta')\right\} = \left(\frac{b_1c_2}{a_2} + \frac{b_2c_1}{a_1}\right)\Psi_1^N(\chi(\xi_1)\theta')e^{2i\chi(\xi_1)h/a_1}$$

by (32) with $\zeta' = \chi(\xi_1)\theta'$. Set $V_5 := \{\xi_1 \in \mathbb{R} : \Psi_1^N(\chi(\xi_1)\theta') \neq 0\}$. By the condition (II) (see Section 2), we have

$$\varphi_3^N(\xi_1) = \widetilde{\kappa}_1 e^{2i\chi(\xi_1)h/a_1} \quad (\neq 0) \tag{47}$$

on V_5, where $\varphi_3^N(\xi_1)$ is defined by (46) and $\widetilde{\kappa}_1$ is defined by

$$\widetilde{\kappa}_1 := \frac{a_1b_1c_2 + a_2b_2c_1}{a_1b_1c_2 - a_2b_2c_1}.$$

Applying D_{ξ_1} to the equality (47) and mixing it with (47) multiplied by $(2ih/a_1)(D_{\xi_1}\chi)(\xi_1)$, we have the equality (45). □

Here, we explain whether the width h of Ω_1 can be determined with the given data or not. In the cases of (b) and (c), the constant h is determined by Proposition 11 and Theorem 12, respectively. However, in the case of (a), that is, in the case when the observation data are equal to the behavior of the waves on the boundary of the half-space under the condition that only Medium 1 is laying in the half-space, the constant h cannot be identified by Proposition 10 and Remark 7-(iii) (if $n \geq 3$), or by Proposition 10 and Remark 7-(v) (if $n = 2$). In other words, for example the case of $n \geq 3$, if $a_2 = a_1$ and $b_1c_2 = b_2c_1$ then we obtain the same observation data defined as the solution of (33) and (34) even if the constant h is different.

Hereafter, we set

$$K_1^N(\xi'; h) := \frac{i}{2a_1^2}\frac{1}{\lambda_1^+(\zeta')}e^{i\lambda_1^+(\zeta')(h-y_n)} - \left\{\Phi_1^N(\zeta')e^{i\lambda_1^+(\zeta')h} + \Psi_1^N(\zeta')e^{-i\lambda_1^+(\zeta')h}\right\}, \tag{48}$$

$$K_2^N(\xi'; h) := \frac{i}{2a_1^2}e^{i\lambda_1^+(\zeta')(h-y_n)} - \left\{\Phi_1^N(\zeta')\lambda_1^+(\zeta')e^{i\lambda_1^+(\zeta')h} - \Psi_1^N(\zeta')\lambda_1^+(\zeta')e^{-i\lambda_1^+(\zeta')h}\right\}, \tag{49}$$

where $\zeta' = Z'_m(\xi')$ for the observation data $N(x') = F_1(x)|_{x_n=0}$. Then we can rewrite the equality (31) as the following:

$$b_1c_2\lambda_2^+(\zeta')K_1^N(\xi'; h) = b_2c_1K_2^N(\xi'; h), \quad \zeta' = Z'_m(\xi'). \tag{50}$$

We remark that if $K_1^N(\xi'; h) \equiv 0$ and $K_2^N(\xi'; h) \equiv 0$ hold for all $\xi' \in \mathbb{R}^{n-1}$ then we have a contradiction by Lemma 3, because we have $\Phi_1^N(\zeta') \equiv -(i/2a_1^2\lambda_1^+(\zeta'))e^{-i\lambda_1^+(\zeta')y_n}$ and $\Psi_1^N(\zeta') = 0$ for all $\zeta' = Z_m'(\xi') \in S_m'$.

By Theorem 12, the constant h is determined when (c) is satisfied. Therefore we determine the constant a_2 and the ratio of b_1c_2 to b_2c_1 under the assumption that h is also given.

Proposition 13. *Let a_1, $Q(D_x)$, y_n and h be given. Assume that the observation data $N(x') = F_1(x)|_{x_n=0}$ are given. If $K_1^N(\xi'; h) \equiv 0$ [resp. $K_2^N(\xi'; h) \equiv 0$] for all $\xi' \in \mathbb{R}^{n-1}$, then $b_2c_1 = 0$ [resp. $b_1c_2 = 0$].*

Proof. By Lemma 8, we have the equality (50). We can prove easily this proposition by this equality and the remark above. □

Proposition 14. *Suppose $n = 2$. Let a_1, $Q(D_x)$, y_n and h be given. Assume that the observation data $N(x') = F_1(x)|_{x_n=0}$ are given. Suppose $K_1^N(\xi'; h)$ is not identically zero with respect to ξ' and $K_2^N(\xi'; h)$ is not identically zero with respect to ξ'. Then the constant $b_1c_2/a_2b_2c_1$ is expressed as*

$$\frac{b_1c_2}{a_2b_2c_1} = -\frac{K_2^N(\xi_1; h)}{\chi(\xi_1)K_1^N(\xi_1; h)}$$

on $\{\xi_1 \in \mathbb{R}\colon K_1^N(\xi_1; h) \neq 0\}$.

Proof. Note $\lambda_2^+(\chi(\xi_1)) = -\chi(\xi_1)/a_2$. Then we can prove this proposition in the same way as for Proposition 13. □

Theorem 15. *Suppose $n \geq 3$. Let a_1, $Q(D_x)$, y_n and h be given. Assume that the observation data $N(x') = F_1(x)|_{x_n=0}$ are given. Suppose $K_1^N(\xi'; h)$ is not identically zero with respect to ξ' and $K_2^N(\xi'; h)$ is not identically zero with respect to ξ'. Put $\kappa := b_1c_2/b_2c_1$. Then the constants a_2 and κ are expressed as*

$$a_2^2 = \frac{\chi_1(\xi')\{(D_{\xi_1}K^N)(\xi'; h)\chi_1(\xi') - K^N(\xi'; h)(D_{\xi_1}\chi_1)(\xi')\}}{(D_{\xi_1}K^N)(\xi'; h)(\xi_2^2 + \cdots + \xi_{n-1}^2)}, \tag{51}$$

$$\kappa = \frac{K^N(\xi; h)}{\lambda_2^+(\zeta')}, \quad \zeta' = Z_m'(\xi') \tag{52}$$

on $V_7 := \{\xi' \in \mathbb{R}^{n-1}\colon K_1^N(\xi'; h) \neq 0\} \setminus \{\xi_1\theta'\colon \xi_1 \in \mathbb{R}\}$, where $K^N(\xi'; h) := K_2^N(\xi'; h)/K_1^N(\xi'; h)$.

Proof. Set $V_6 := \{\xi' \in \mathbb{R}^{n-1}\colon K_1^N(\xi';h) \neq 0\}$. The set V_6 is open since $\Phi_1^N(\zeta')$ and $\Psi_1^N(\zeta')$ are continuous. It is not empty by the assumption. Then we have

$$\kappa\lambda_2^+(\zeta') = K^N(\xi';h), \quad \zeta' = Z'_m(\xi') \tag{53}$$

on V_6 since we obtain the equality (50) from Lemma 8, where κ and $K^N(\xi';h)$ are defined as the statement above. Squaring the equality (53), we have

$$\frac{\kappa^2}{a_2^2}\chi_1(\xi')^2 - \kappa^2(\xi_2^2 + \cdots + \xi_{n-1}^2) = K^N(\xi';h)^2. \tag{54}$$

Moreover we have

$$\frac{\kappa^2}{a_2^2} = \frac{K^N(\xi';h)(D_{\xi_1}K^N)(\xi';h)}{\chi_1(\xi')(D_{\xi_1}\chi_1)(\xi')} \tag{55}$$

by applying D_{ξ_1} to the equality (54). Note $V_7 := V_6 \setminus \{\xi_1\theta'\colon \xi_1 \in \mathbb{R}\}$ is not empty because $n \geq 3$ and V_6 is open. We obtain

$$\kappa^2 = \frac{K^N(\xi;h)\{(D_{\xi_1}K^N)(\xi';h)\chi_1(\xi') - K^N(\xi';h)(D_{\xi_1}\chi_1)(\xi';h)\}}{(D_{\xi_1}\chi_1)(\xi')(\xi_2^2 + \cdots + \xi_{n-1}^2)} \tag{56}$$

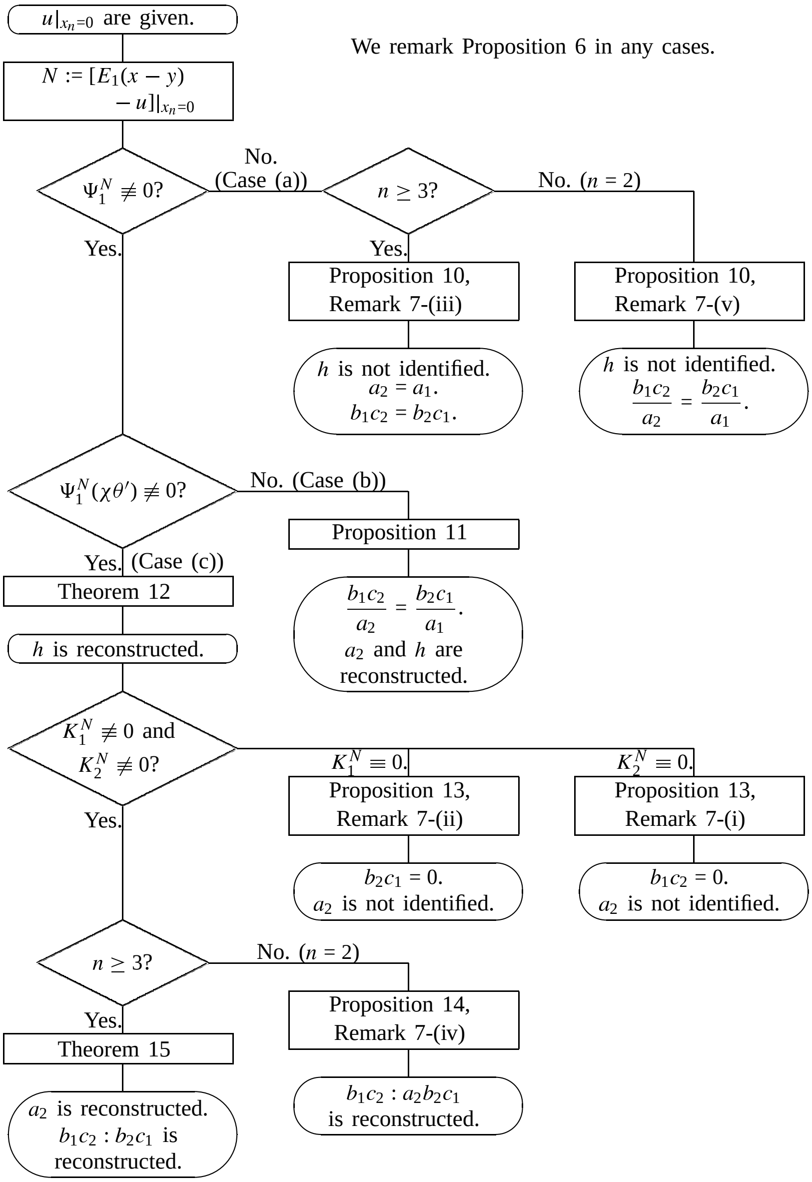
on V_7 from substituting (55) into (54). Then we have the equality (51) by substituting (56) into (55). Hence the constant a_2 is determined because it is positive. By (53), the constant κ is expressed as the equality (52). □

In short, we can determine unknown things from the given data as the flowchart in Fig. 2, where $\Phi_1^N(\zeta')$, $\Psi_1^N(\zeta')$, $K_1^N(\xi';h)$, $K_2^N(\xi';h)$ are defined by (27), (28), (48), (49), respectively.

Last, we mention a positive large real m in Remark 16.

REMARK 16. We deal with m as a fixed number. Indeed, this m depends on the unknown constants. However, we can check whether this m is so large that the Lopatinski's determinant does not vanish for the determined constants or not after we determine the unknown constants in the above way. If this m is large enough, then there is no problem. If this m is not large enough, then we take a larger number as new m instead of this m, and determine the unknown constants once again. This procedure is sure to conclude since there exists a large enough number m certainly by Theorem 1.

ACKNOWLEDGEMENTS. The author would like to express his sincere gratitude to Professor Mitsuru SUGIMOTO for his unfailing guidance and his useful advice. The author would like to express his thanks to Dr. Tetsuya SONOBE and Dr. Mitsuji TAMURA for their useful advice.



$u|_{x_n=0}$ are given.
We remark Proposition 6 in any cases.
$N := [E_1(x-y) - u]|_{x_n=0}$
$\Psi_1^N \not\equiv 0$?
No. (Case (a))
$n \geq 3$?
No. ($n = 2$)
Yes.
Yes.
Proposition 10, Remark 7-(iii)
Proposition 10, Remark 7-(v)
h is not identified. $a_2 = a_1$. $b_1c_2 = b_2c_1$.
h is not identified. $\frac{b_1c_2}{a_2} = \frac{b_2c_1}{a_1}$.
$\Psi_1^N(\chi\theta') \not\equiv 0$?
No. (Case (b))
Proposition 11
Yes. (Case (c))
Theorem 12
$\frac{b_1c_2}{a_2} = \frac{b_2c_1}{a_1}$. a_2 and h are reconstructed.
h is reconstructed.
$K_1^N \not\equiv 0$ and $K_2^N \not\equiv 0$?
$K_1^N \equiv 0$.
$K_2^N \equiv 0$.
Proposition 13, Remark 7-(ii)
Proposition 13, Remark 7-(i)
Yes.
$b_2c_1 = 0$. a_2 is not identified.
$b_1c_2 = 0$. a_2 is not identified.
$n \geq 3$?
No. ($n = 2$)
Proposition 14, Remark 7-(iv)
Yes.
Theorem 15
$b_1c_2 : a_2b_2c_1$ is reconstructed.
a_2 is reconstructed. $b_1c_2 : b_2c_1$ is reconstructed.

Fig. 2.

References

[1] A. Bartoloni, C. Lodovici and F. Zirilli: *Inverse problem for a class of one-dimensional wave equations with piecewise constant coefficients*, J. Optim. Theory Appl. **76** (1993), 13–32.

[2] L. Fatone, P. Maponi, C. Pignotti and F. Zirilli: *An inverse problem for the two-dimensional wave equation in a stratified medium*, Inverse problems of wave propagation and diffraction, 263–274, Lecture Notes in Phys., **486**, Springer, Berlin, 1997.

[3] S. Hansen: *Solution of a hyperbolic inverse problem by linearization*, Comm. Partial Differential Equations, **16** (1991), 291–309.

[4] M. Matsumura: *On the Singularities of the Riemann Functions of Mixed Problems for the Wave Equation in Plane-Stratified Media* I, Proc. Japan Acad., **52** (1976), 289–292.

[5] R. Sakamoto: Hyperbolic Boundary Value Problems, Cambridge University Press, Cambridge-New York, 1982.

Department of Mathematics
Graduate School of Science
Osaka University
1-16 Machikaneyama-cho
Toyonaka, Osaka 560-0043, JAPAN
e-mail: sei@cr.math.sci.osaka-u.ac.jp