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On the Lattice Homomorphisms of Infinite Groups II

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1. **Introduction.** In the present note we shall study the structure of infinite groups which admit proper lattice homomorphisms. As the structure of infinite p-groups is not so clear, it is difficult to determine those groups even if we assume that they admit a proper lattice homomorphism to a cyclic group. We shall prove in this note some preliminary results in general case and determine the soluble groups that admit proper lattice homomorphisms to cyclic groups of square free order under the condition that the lower kernels are mapped to the unit element group of the cyclic groups.

NOTATIONS. $L(G)$ is the lattice formed of all subgroups of a group G. $\{M\}$ is the subgroup generated by the elements of M, where M is a subset of a group. $Z(M)$ is the centralizer of a subset M in the whole group. 0 and I are the least element and the greatest element of a considering lattice respectively. Definitions of the lattice homomorphism and others are the same as in the previous paper[1].

2. **Lower kernel and upper kernel.** Let G be a group and ϕ a lattice homomorphism from G onto a lattice L. All the elements of $L(G)$, which are mapped to the least (greatest) element of L, form a (dual) ideal of $L(G)$. The least upper (greatest lower) bound of this (dual) ideal is called "*lower* (*upper*) *kernel*", or shortly "*l*— (*u*—) *kernel*" of ϕ in G. Of coruse, in the case of infinite groups the l-kernel or u-kernel may be no longer mapped to 0 or I of L. But we have next two theorems, which are the generalizations of the theorems in the case of finite groups[2].

Theorem 1. *Let G be a (finite or infinite) group, and ϕ a lattice homomorphism from G onto a lattice L. Then the l-kernel of ϕ is a normal subgroup of G.*

PROOF. Let N be the l-kernel. If $N=G$, then there is nothing

1) S. SATO [1].
2) M. SUZUKI [2].

to prove. Suppose $N \neq G$, and take an element $a \in G$ which is not contained in N. For any element b such that $\{a\} \cup N \supseteq \{b\} \nsubseteq N$, we can find a subgroup N' of N which is generated by a finite number of elements of N and satisfies $\{a\} \cup N' \supseteq \{b\}$. All cyclic subgroups in N and hence any join of finite number of them are mapped to 0 of L by ϕ. So $\phi N' = 0$. Hence

$$\begin{aligned}\phi(\{a\} \cap \{b\}) = \phi\{a\} \cap \phi\{b\} &= (\phi\{a\} \cup \Phi N') \cap \phi\{b\} \\ &= \phi(\{a\} \cup N') \cap \phi\{b\} = \phi\,\{b\} \neq 0\,.\end{aligned}$$

This means the cyclic group $\{a\} \cap \{b\}$ is not mapped to 0, i.e., $\{a\} \cap \{b\} \nsubseteq N$. Now assume that $aNa^{-1} \nsubseteq N$. Then for some $x \in N$ holds $axa^{-1} \notin N$. But above consideration shows $\{a\} \cap \{axa^{-1}\} \ni a^i = ax^ja^{-1} \notin N$ for some integers i and j, so we have $x^j = a^i \notin N$. This is a contradiction. Hence $aNa^{-1} \subseteq N$ holds for any element a of G, q. e. d.

Theorem 2. *Let G be a (finite or infinite) group, and ϕ a lattice homomorphism from G onto a lattice L. Then the u-kernel is a normal subgroup of G.*

PROOF. Let $G_\alpha \subseteq G$ and $\phi G_\alpha = I$. For any element $a \in G$

$$\phi(G_\alpha \cap \{a\}) = \phi G_\alpha \cap \phi\{a\} = I \cap \phi\{a\} = \phi\{a\}\,,$$

and we have

$$\phi a G_\alpha a^{-1} \geqq \phi(aG_\alpha a^{-1} \cap \{a\}) = \phi(G_\alpha \cap \{a\}) = \phi\{a\}\,.$$

But $I = \phi G_\alpha \leqq \phi(aG_\alpha a^{-1} \cup \{a\})$. Hence

$$I = \phi(aG_\alpha a^{-1} \cup \{a\}) = \phi a G_\alpha a^{-1} \cup \phi\{a\} = \phi a G_\alpha a^{-1}\,.$$

This means the u-kernel $\bigcap_{\phi G_\alpha = I} G_\alpha$ is invariant in G, q. e. d.

3. Groups admitting lattice homomorphisms onto chains.

Definition. A lattice homomorphism ϕ of G with the l-kernel N is called regular if $\phi N = 0$.

Proposition. *Let ϕ be a lattice homomorphism from a group G onto a chain with the l-kernel N. If ϕ is not regular, holds $N = G$.*

PROOF. Suppose $G \neq N$. N is the set theoretical sum of the cyclic subgroups of G that are mapped to 0 by ϕ. So, for any element $a \in G$ which is not contained in N, we have $0 = \phi(\{a\} \cap N)$.

But $\phi\{a\} \neq 0$ and, from the assumption, $\phi N \neq 0$. Hence $\phi(\{a\} \frown N) = \Phi\{a\} \frown \phi N \neq 0$. This is a contradiction, q. e. d.

An example of the groups that admit above type of lattice homomorphism onto a chain is the locally cyclic infinite p-group. But generally in the case when the lattice homomorphism is not regular it is quite difficult to investigate the structure of groups, and from now on we mainly deal with regular lattice homomorphisms.

Lemma 1. *A group G admits a regular lattice homomorphism onto a two element chain with the l-kernel N if and only if it has one of the following three types:*

1) *G contains no element of infinite order, and there exists a proper normal subgroup N of G such that G/N is a p-group which has only one subgroup of order p and N contains either no element of order p or only one cyclic p-group in its center. Moreover, if $p \neq 2$, every finite p-subgroup of G is cyclic, and if $p=2$, every finite p-subgroup is a cyclic group or a generalized quaternion group (g. q. group). There is a cyclic p-group which is not contained in N and contained in the center of G, and further every p-subgroup of G either contains it or is contained in it.*

2) *All the elements of finite order form a characteristic subgroup N. For any element a of infinite order and for any element n of N holds $\{a\} \frown Z(n) \neq 1$. Moreover, for any pair of elements A and B of G/N holds $\{A\} \frown \{B\} \neq E$, where E is the unit group of G/N.*

3) *There exists a normal subgroup N that contains all the elements of finite order in G and G/N is a p-group that has a unique subgroup of order p. G contains no element of order p. For any pair of elements a and b of infinite order and for any element n of N, holds $\{a\} \frown \{b\} \neq 1$ and if a is not contained in N, holds $\{a\} \frown Z(n) \not\subseteq N$.*

PROOF. I) To prove "only if"-part, let G admit a regular lattice homomorphism ϕ onto a chain with the l-kernel N.

a) If z and y are not contained in N, then $\phi(\{x\} \frown \{y\}) = \phi\{x\} \frown \phi\{y\} \neq 0$. Hence $\{x\} \frown \{y\} \not\subseteq N$. Now for any $c \in N$ there is an integer i and $\{x\} \frown \{cx\} \ni x^i \not\ni N$. This shows that x^i is commutative with cx and hence with c itself. Hence first of all we have $\{x\} \frown Z(c) \not\subseteq N$.

b) Suppose there is an element a of finite order which is not contained in N. Then from above discussion we can find a minimal subgroup $\{b\} (\subseteq \{a\})$ that satisfies $\{b\} \not\subseteq N$ and $\{b\} \subseteq \{x\}$ for every element $x \notin N$. Every maximal subgroup of $\{b\}$ is contained in N, hence $\{b\}$ cannot have more than one maximal subgroup, which implies that $\{b\}$ is a cyclic p-group. Thus we see that any element of G not contained in N is of finite order and that, according to a), holds $b \in Z(N)$. Moreover G/N is a p-group that has only one subgroup of order p.

Now if an element $c \in N$ is of infinite order, then $bc \notin N$ is also of infinite order. But this is impossible. Thus G itself has no element of infinite order.

Let $c \in N$ be of order a power of p. $bc \notin N$ implies $\{bc\} \ni b$ and we find an integer i such that $b = (bc)^i = b^i c^i$. $c^i = b^{1-i}$. So if $p \not\mid (1-i)$, then $\{c\} \supseteq b$ in contradiction with $c \in N$. So $p \mid (1-i)$ and $\{c\} \subset \{b\}$. Thus we see that every element of order a power of p in N is contained in $\{b\}$. Hence every finite p-subgroup of G is a cyclic group or a g. q. group. This is the case 1) in the lemma.

c) Suppose there is an element a of infinite order such that $N \frown \{a\} = 1$. In this case too N contains no element of infinite order. For, take an arbitrary element $c \in N$. As we noticed in a), for some integer i a^i is commutative with c and $\{a^i c\} \frown \{a^i\} \neq 1$, i.e., for some integers $j \neq 0$ and k holds $(a^i c)^j = a^{ik}$. Hence $a^{ik-ij} = c^j$. But $\{a\} \frown \{c\} = 1$, then we obtain $c^j = 1$. Form 1) we see also that G has no element of finite order which is not contained in N. This is the case 2) in the lemma. Other conditions are easily verified from a).

d) Suppose there is an element a of infinite order such that $a \notin N$ and $\{a\} \frown N \neq 1$. From 1) and 2) we see that every element of G not contained in N is of infinite order and of finite order mod N. A power of a is of order p mod N for some prime number p. We put this power b. Take $c \in N$ and let $b^i \notin N$ be commutative with c. Then for some integers $j \neq 0$ and k holds $(b^i c)^j = b^{ik} \notin N$. Hence $b^{i(k-j)} = c^j$. So if the order of c is finite, then $c^j = 1$ and $k = j$. And $b^{ij} \notin N$ implies $p \not\mid j$, i.e., the order of c is prime to p. If the order of c is infinite, then above equation implies $\{b\} \frown \{c\} \neq 1$. From a) and this we see that for any pair of elements x and y of infinite order in G holds $\{x\} \frown \{y\} \neq 1$.

$L(G/N)$ cannot have two atoms, hence every subgroup of G/N contains $\{b\}N$. Thus we know that G/N is a p-group. This is the case 3) in the lemma.

II) Conversely we prove next that in any case of the lemma G admits a proper lattice homomorphism ϕ onto a two element chain $(I, 0)$.

Case 1) We define ϕ as follows:

$$\begin{aligned} \phi V &= I \quad \text{if} \quad V \nsubseteq N, \\ &= 0 \quad \text{if} \quad V \subseteq N. \end{aligned}$$

It is trivial that this ϕ is a lattice homomorphism.

Case 2) Put

$$\phi V = I \quad \text{if } V \text{ contains an element of infinite order,}$$
$$\quad = 0 \quad \text{if } V \text{ contains no element of infinite order.}$$

To prove this mapping ϕ is a lattice homomorphism it is sufficient to show $\{a\} \frown \{b\} \neq 1$ for every pair of elements a and b of infinite order. As $\{\bar{a}\} \frown \{\bar{b}\} \neq E$, for some integers i and $j \neq 0$, and for some elements $n \in N$ holds $a^i = b^j n$. On the other hand for some integer $k \neq 0$ b^{jk} is commutative with n. Then $a^{ik} = b^{jk} n'$, where n' is in N and commutative with b^{jk}. If the order of n' is s, then we have $a^{iks} = b^{iks} \neq 1$. This completes the proof.

Case 3) Put

$$\phi V = I \quad \text{if} \quad V \not\supseteq N,$$
$$\quad = 0 \quad \text{if} \quad V \supseteq N.$$

To prove this ϕ is a lattice homomorphism it is sufficient to show that for any pair of elements a and $b \notin N$ holds $\{a\} \frown \{b\} \not\supseteq N$. Let a^i and b^j be of order p mod N, and $a^i = b^j n$ for some $n \in N$. If $b^{jk} \notin N$ is commutative with n, then $a^{ik} = b^{jk} n'$ and $n' \in N$ is commutative with b^{jk}. If the order s of n' is finite, then $a^{iks} = b^{jks}$ and $a^{iks} \notin N$ since $p \not| s$. If the order of n' is infinite, then take the minimal positive integer s such that $(n')^s = b^{jkt}$ $(t \neq 0)$. Then $a^{iks} = b^{jk(t+s)}$. $p | t$ is obvious. But $p \not| s$, for otherwise $(n')^{s/p}(b^{jk})^{-t/p}$ is an element of order p in contradiction with the condition. Hence $a^{iks} \notin N$ and $\{a\} \frown \{b\} \not\supseteq N$, q. e. d.

From this lemma we can easily derive some results about the structure of the group that admits lattice homomorphisms onto a finite or infinite cyclic group if the lower kernel is not the whole group. But we omit this and proceed to the case of soluble groups.

REMARK. It seems to be a difficult problem to see in this lemma whether G has in the case 1) the normal subgroup N_0 formed of all elements of order prime to p or not, except the obvious case when $p = 2$ and G contains a quaternion group. This concerns with a generalization of Schur's lemma, i. e.: Let G be an infinite group that contains no element of infinite order, P cyclic p-group contained in the center of G and G/P have no element of order p, then does G split over P? A trivial case is the locally finite group, in which any finite number of elements of order prime to p generate a subgroup that contains no element of order p, because this subgroup and P

generate a finite group and P becomes a direct factor of it. Another case we can prove that is the soluble group, as we see in the following paragraph.

4. Soluble groups.

Theorem 3. *A soluble group G admits a regular lattice homomorphism onto a two element chain $(I, 0)$ with l-kernel N if and only if G has one of the following two types:*

1) *G contains no element of infinite order and splits into two subgroups P and N_0: $G = PN_0$ with $P \frown N_0 = 1$, where P is a p-group which is locally cyclic or a generalized quaternion group and N_0 is normal in G and contains no element of order p. Moreover P contains a normal subgroup P_0 which is contained in the center of G.*

2) *All the elements of finite order of G form a normal subgroup M and G/M is a torsion free locally cyclic group. Moreover, for any pair of elements $n \in M$ and $a \notin M$ holds $\{a\} \frown Z(n) \neq 1$.*

PROOF.

Lemma A. *If a group G contains in its center a torsion free locally cyclic subgroup N and if G/N is abelian and has no element of infinite order, then G is itself an abelian group.*

PROOF OF LEMMA A. Take a pair of elements a and b in G. Let $a^i \in N$ and $bab^{-1} = az$ for $z \in N$. Then $a^i = (bab^{-1})^i = (az)^i = a^i z^i$. Hence $z^i = 1$ and so $z = 1$. This shows $ba = ab$.

Lemma B. *A torsion free soluble group is locally cyclic if for any pair of elements a and b holds $\{a\} \frown \{b\} \neq 1$.*

PROOF OF LEMMA B. Under the condition of the lemma a locally cyclic normal subgroup of this group is always contained in the center. Hence the lemma follows immediately in virtue of lemma A.

Lemma C. *Let N be an infinite soluble group that contains no element of infinite order, P a cyclic p-subgroup in the center of G and let G/P have no element of order p, then N is a direct product of P and a normal subgroup N_0: $N = P \times N_0$.*

PROOF OF LEMMA C. Let $N/P = N_1/P \supset N_2/P \supset \cdots \supset N_s/P = E$ be the derived series of N/P. Suppose that P is a direct factor of N_{i+1} but not of N_i. Put $N_{i+1} = P \times N'_{i+1}$. As N_i/N_{i+1} is abelian and N_{i+1}/N'_{i+1} is finite, N_i/N'_{i+1} is a locally finite group. Hence, according to the remark of the preceding paragraph, N_i/N'_{i+1} decomposes into

N_{i+1}/N'_{i+1} and a normal subgroup N'_i/N'_{i+1}. Obviously $P \frown N'_i = 1$, $P \smile N'_i = N_i$, and so $P \times N'_i = N_i$ in contradiction with our assumption. That P is a direct factor of N_{s-1} is trivial, so N itself contains P as its direct factor.

I) "Only if"-part of the theorem.

i) Let G contains an elements of infinite order and let $N = N_1 \supset N_2 \supset \cdots \supset N_r = 1$ be the derived series of N.

a) Suppose that N_{i+1} does not contain an element of infinite order but N_i does. Let M/N_{i+1} be the subgroup of N_i/N_{i+1} formed of all elements of finite order in N_i/N_{i+1}. Then M contains no element of infinite order. According to lemma 1, for any pair of elements $a \in G$ and $n \in N$ holds $\{a\} \frown Z(n) \neq 1$, where a is of infinite order. Hence under the transformation by any element of N, there must be a corresponding invariant element in N_i/M. As N_i/M is a torsion free locally cyclic group, N_i/M is contained in the center of N/M, whence, according to lemma A, N/M is itself an abelian group. Thus we see that M is the characteristic subgroup formed of all elements of finite order in N and hence in G, (according to lemma 1). Now, according to lemma B, we conclude that G/M is a torsion free locally cyclic group. Other conditions in the theorem are obtained immediately from lemma 1.

b) If N contains no element of infinite order, then the theorem follows immediately from lemma B and lemma 1.

ii) Suppose G contains no element of infinite order. From lemma 1 and lemma C we can find a normal subgroup N_0 in N such that G/N_0 is a p-group for some prime number p and N_0 contains no element of order p. Let $G/N_0 = G_1/N_0 \supset \cdots \supset G_r/N_0 = N_0/N_0$ be the derived series of G/N_0. As G/N_0 has only one subgroup of order p, G_{r-1}/N_0 is a locally cyclic p-group. Suppose that $G_{r-1} \neq G$.

a) Suppose $p \geqq 3$. Take an element $\bar{a}$ such that $G_{r-2}/N_0 \ni \bar{a} \notin G_{r-1}/N_0$. Then, as any finite subgroup S of G_{r-1}/N_0 is normal in G/N_0, $\{\bar{a}, S\}$ is a finite p-group that has only one subgroup of order p. Hence $\{\bar{a}, S\}$ is cyclic and $\{\bar{a}\} \supset S$, whence we have $\{\bar{a}\} \supset G_{r-1}/N_0$ and know G_{r-1}/N_0 is a finite group. Thus any finite set of elements in G_{r-2}/N_0 generate a finite cyclic group, whence G_{r-2}/N_0 is abelian in contradiction with $G_{r-1} \neq G$. Hence $G_{r-1} = G$ and G/N_0 is a locally cyclic p-group.

For any subgroup G' such that $G \supset G' \supset N_0$ G'/N_0 is a finite p-group. Then we have $G' = P'N_0$ with $P' \frown N = 1$ for a cyclic

p-group P', and all these P' is conjugate to each other[3]. Hence in any such G' all maximal p-subgroups are of same order and conjugate to each other. Now we conclude that, even if a pair of maximal p-subgroups of G may not be conjugate to each other, all maximal p-subgroups are locally cyclic and isomorphic to one another and G splits over N.

b) Suppose $p=2$.

If G/N_0 does not contain a quaternion group, then in the same way as above we can prove the theorem. So assume that G contains at least one quaternion group. G_{r-1}/N_0 is a locally cyclic 2-group and every finite number of elements of G_{r-2}/N_0 generate a finite group which is a cyclic group or a g. q. group. Hence if G_{r-1}/N_0 is finite, necessarily holds $[G_{r-2}/N_0 : G_{r-1}/N_0]=4$, and if G_{r-1}/N_0 is infinite, necessarily holds $[G_{r-2}/N_0 : G_{r-1}/N_0]=2$. In either case G_{r-2}/N_0 is a g. q. group. Then we can prove easily $G_{r-2}=G$.

II) "If"-part of the theorem follows immediately from lemma 1, q. e. d.

In the rest of this note we study those groups that are lattice homomorphic to a direct product of chains. Put $L=C_1\times C_2\times\cdots\times C_r$, where C_i are finite chains whose greatest and least elements are I_i and 0_i respectively. Then we have

Lemma 2. *If a group G admits a regular lattice homomorphism ϕ onto L, then $\phi^{-1}(C_i)$ are all principal ideals of $L(G)$.*

PROOF. Suppose that $\phi^{-1}(C_i)$ is not principal for some i. Let A be the least upper bound of the ideal $\phi^{-1}(C_i)$. Then for some $j\neq i$ $\phi A\frown I_j\neq 0$. As ϕ is regular, there must be a cyclic subgroup $\{a\}$ such that $a\notin N$ and $0\neq\phi\{a\}\leq\phi A\frown I_j$, where N is the l-kernel. But, as A is the set theoretical sum of all cyclic subgroups in $\phi^{-1}(C_i)$, we have $\phi(A\frown\{a\})\leqq I_i$. Hence $\phi\{a\}\leq I_i\frown I_j=0$, which is a contradiction, q. e. d.

Lemma 3. *Let G be a soluble group that admits a lattice homomorphism ϕ onto L with the l-kernel 1. Then G contains no element of infinite order and is a direct product of p_i-group P_i $i=1, 2, \cdots, r$, where $p_i\neq p_j$ if $i\neq j$ and each P_i is a locally cyclic group or a g.q. grnup.*

PROOF. Denote the lattice homomorphisms $L\rightarrow C_i$ by ϕ_i. Then,

3) A trivial generalization of well known theorem in the case of finite groups. Cf. Zassenhaus [3] p. 126, Satz 27.

according to lemma 2, $\phi_i\phi$ are regular lattice homomorphisms from G onto G_i. So, in virtue of theorem 3, we find a set of normal subgroups N_i $i=1,2,\cdots,r$ such that $G=P_iN_i$ with $P_i\frown N_i=1$, where each P_i is a locally cyclic p_i-group or a g. q. group and N_i contains no element of order p_i, or otherwise G/N_i is torsion free locally cyclic group and N_i is the group formed of all elements of finite order in G. But the l-kernel of ϕ is 1, so $\bigwedge N_i=1$. Hence $G=P_1\times\cdots\times P_r$ and $\phi P_i=I_i$. Suppose that a P_i is torsion free and take a pair of elements $(1\neq)\, a\in P_i$ and $(1\neq)\, b\in P_j$ for $j\neq i$. Obviously $0\neq\phi\{a\}=A\leq I_i$ and $0\neq\phi\{b\}=B\leq I_j$. So $\phi(\{a\}\cup\{ab\})=\phi(\{ab\}\cup\{b\})=\phi(\{a\}\cup\{b\})=A\cup B$. Hence $\phi\{ab\}=A\cup B$. But, as a is of infinite order and b of finite order, we have $P_j\frown\{ab\}=1$. This is impossible. Hence G contains no element of infinite order, q. e. d.

Theorem 4. *Let $L=C_1\times C_2\times\cdots\times C_r$, where C_i are all two element chains $(I_i, 0_i)$ and $r\geqq 2$. A soluble group G admits a regular latticte homomorphism ϕ onto L if and only if G has one of the following two types:*

1) *G contains no element of infinite order. $G=HN$ with $H\frown N=1$, H is a direct product of p_i-group P_i $i=1,2,\cdots,r$, where $p_i\neq p_j$ if $i\neq j$ and each P_i is a locally cyclic group or a g.q. group. N is a normal subgroup that has no element of order p_i. Each P_i has a subgroup $(\neq 1)$ that is contained in the center of G.*

2) *G contains elements of infinite order. All elements of finite order form a normal subgroup M and G/M is locally cyclic. There is a normal subgroup $N\supset M$ such that G/N is a locally cyclic group that is a direct product of locally cyclic p_i-group P_i/N $i=1,2,\cdots,r$. Moreover, for any pair of elements $(N\not\ni)\, a\in P_i$ and $n\in N$ holds $\{a\}\frown Z(n)\not\subseteq N$. M contains no element of order p_i, $i=1,2,\cdots,r$.*

PROOF. 1) Suppose that G contains no element of infinite order. Then, in the similar way as in the proof of lemma 3, we obtain a set of normal subgroup N_i such that $G=P_iN_i$ with $P_i\frown N_i=1$ $i=1,2,\cdots,r$, where each P_i is a locally cyclic p_i-group or a g. q. group and N_i contains no element of order p_i. Then $N=\bigwedge N_i$ is the desired normal subgroup, for G splits over N[4]. Other conditions are obvious from theorem 3.

2) Suppose that G contains elements of infinite order. According to lemma 3, the l-kernel N cannot be the subgroup formed of all elements of finite order in G. Hence, applying theorem 3 to the greatest

4) See footnote 3.

subgroup P_i that is mapped to I_i (cf. lemma 2), wee see that all the elements of finite order actually from a subgroup M and P_i/M are all locally cyclic, because $P \supset N$. G/N is locally cyclic (lemma 3) and has no element of infinite order, whence G/M is a torsion free locally cyclic group[5]. Other conditions are all obtained from lemma 1.

II) "If"-part of the thorem is quite easy to see, q. e. d.

To investigate the group G that is lattice homomorphic to a chain of dimension $\geqq 2$ is of no interest when G contains no element of infinite order, because it is the same as in the case of finite groups. But if G contains elements of infinite order, it is closely related with the study of non-regular lattice homomorphisms.

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5) Cf. Lemma A and S. Sato [1], Lemma 6.