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NOTE ON POLY-SUPERTEMPERATURES ON A STRIP DOMAIN

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0. Introduction

Let m be a positive integer and let

$$D = \{(X, t); X = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n, 0 < t < T\}$$

be a strip domain in the $(n + 1)$ -dimensional Euclidean space \mathbf{R}^{n+1} . We consider supersolutions of the m -th iterates of the heat operator

$$H = \Delta_X - \frac{\partial}{\partial t}$$

on D . A lower semi-continuous and locally integrable function u on D is called a poly-supertemperature of degree m , if $(-H)^m u \geq 0$ on D in the sense of distributions. If u and $-u$ are both poly-supertemperatures of degree m , then u is called a poly-temperature of degree m .

In our previous paper [2] (see also [1]), we have shown the following super-mean-value property for poly-supertemperatures.

Theorem A ([2, Theorem 2]). *Let u be a C^{2m-2} -function on D satisfying the growth condition*

$$(1) \quad |H^k u(X, t)| \leq M e^{a|X|^2}, \quad k = 0, 1, \dots, m-1,$$

with some constants $M > 0$ and $a > 0$ (here $H^0 u$ means u). If u is a poly-supertemperature of degree m on D , then

$$(2) \quad u(X_0, t_0) \geq A[u, c_1, c_2, \dots, c_m](X_0, t_0)$$

whenever $(X_0, t_0) \in D$ and $0 < c_1 < c_2 < \dots < c_m < \min\{1/4a, t_0\}$. (For notation, see (5) below.)

In the present note, we first point out that the above mean $A[u, c_1, \dots, c_m]$ is a decreasing function of each c_1, \dots, c_m and converges to $u(X_0, t_0)$ as c_1, \dots, c_m tend to 0 under the condition $0 < c_1 < \dots < c_m$ (Theorem 1). Secondly, in section 2, we show that the lower-regularization \hat{v} of a Borel measurable function v having the super-mean-value property (2) is a poly-supertemperature (Theorem 2). In the final section, we derive a minimum principle for poly-supertemperatures, from the super-mean-value property (Theorem 3). As its corollary, we have some uniqueness results for poly-supertemperatures. Especially, we obtain the existence and uniqueness of poly-supertemperatures satisfying the boundary conditions.

1. Monotonicity of the mean

Let W denote the fundamental solution for the heat equation on \mathbf{R}^{n+1} , that is,

$$W(X, t) = \begin{cases} (4\pi t)^{-\frac{n}{2}} \exp(-\frac{|X|^2}{4t}) & \text{if } t > 0, \\ 0 & \text{if } t \leq 0. \end{cases}$$

We set $W^1 := W$ and $W^k := W^{k-1} * W$ for $k \geq 2$, inductively, where $*$ denotes the convolution in \mathbf{R}^{n+1} . Then

$$(3) \quad W^m(X, t) = \frac{t^{m-1}}{(m-1)!} W(X, t)$$

and this is the fundamental solution of the equation $(-H)^m u = 0$, that is,

$$(4) \quad (-H)^m (W^m * \phi) = W^m * ((-H)^m \phi) = \phi$$

for all $\phi \in C_0^\infty(D)$ (cf. [2, Proposition 2]).

Now we recall the definition of the mean values $A[u, c_1, c_2, \dots, c_m]$:

$$(5) \quad A[u, c_1, c_2, \dots, c_m](X_0, t_0) := \sum_{k=1}^m A_k W[u, c_k](X_0, t_0),$$

where

$$W[u, c_k](X_0, t_0) := \int_{\mathbf{R}^n} u(X - X_0, t_0 - c_k) W(X, c_k) dX$$

and the coefficients $A_k, k = 1, 2, \dots, m$, are given by

$$A_k = A_k^m(c_1, \dots, c_m) := \prod_{j=1, j \neq k}^m \frac{c_j}{c_j - c_k} \quad (A_1 = 1 \text{ when } m = 1).$$

Note that

$$(-1)^{k-1}A_k > 0, \quad k = 1, 2, \dots, m.$$

For integers m, p with $0 \leq p \leq m$ and real numbers c_1, \dots, c_m with $0 = c_0 < c_1 < \dots < c_m < c_{m+1} := \infty$, we consider the following functions:

$$(6) \quad \psi_p^m(t) = \psi_p^m(c_1, \dots, c_m; t) := t^{m-1} - \sum_{k=1}^p A_k^m(c_1, \dots, c_m)(t - c_k)^{m-1}$$

and

$$(7) \quad \Psi_m(t) = \Psi_m(c_1, \dots, c_m; t) := \frac{1}{(m-1)!} \sum_{p=0}^{m-1} \psi_p^m(t) \chi_{(c_p, c_{p+1}]}(t),$$

where $\psi_0^m(t) = t^{m-1}$ and $\chi_{(c_p, c_{p+1}]}$ denotes the characteristic function of the interval $(c_p, c_{p+1}]$. We remark that the above functions were already introduced in our previous paper [2] as $\phi_p(t) = \psi_p^m(t_0 - t)$ and $\psi_m(t) = (m-1)! \Psi_m(t)$. We have already obtained the following ([2, Lemma 1], for the proof see [1, Lemma 8]): for all integers p with $0 \leq p \leq m-1$,

$$(8) \quad \psi_p^m(t) = \sum_{k=p+1}^m A_k(t - c_k)^{m-1},$$

$$(9) \quad \psi_p^m(t) \geq 0 \quad \text{for } c_p \leq t \leq c_{p+1},$$

and

$$(10) \quad \psi_m^m(t) \equiv 0.$$

The function Ψ_m has the following properties.

Lemma 1. (A) $\Psi_m(c_1, \dots, c_m; t)$ is a continuous (for $m \geq 2$) and nonnegative function of $t \geq 0$. Moreover $\Psi_m(c_1, \dots, c_m; t) > 0$ if $0 < t < c_m$.

(B) $\Psi_m(c_1, \dots, c_m; t)$ is an increasing function of each variable c_j , $j = 1, 2, \dots, m$, and

$$\lim_{0 < c_1 < \dots < c_m \rightarrow 0} \Psi_m(c_1, \dots, c_m; t) = 0.$$

$$(C) \quad \int_0^{c_m} \Psi_m(c_1, \dots, c_m; t) dt = \frac{c_1 \cdots c_m}{m!}.$$

Proof. (A) The continuity of Ψ_m follows from the facts $\psi_p^m(c_p) = \psi_{p-1}^m(c_p)$, $p = 1, 2, \dots, m$. Inequalities (9) show the nonnegativity of Ψ_m . The positivity of Ψ_m is obtained immediately in the case of $c_{m-1} < t < c_m$ because

$$\Psi_m(c_1, \dots, c_m; t) = \frac{1}{(m-1)!} (-1)^{m-1} A_m(c_m - t)^{m-1}$$

and $(-1)^{m-1} A_m > 0$. As will be seen in the below, the proof of (B) is independent of the positivity of Ψ_m . Therefore the general case follows from the case of $c_{m-1} < t < c_m$ because of (B).

(B) For the proof, we use the following fact: Let $m \geq 2$. We define

$$\rho_p^m(t) = \rho_p^m(c_1, \dots, c_m; t) := \sum_{k=1}^p c_k A_k^m(c_1, \dots, c_m)(t - c_k)^{m-2}$$

where $\rho_0^m := 0$. Then we have

$$(11) \quad \rho_p^m(c_1, \dots, c_m; t) \geq 0 \quad \text{for } c_p \leq t \leq c_{p+1}$$

and

$$(12) \quad \rho_m^m(c_1, \dots, c_m; t) \equiv 0.$$

This can be proved by the quite same manner as in [1, Lemma 8], so we omit the proof.

Now we consider the first part of (B). Though the method of the proof is also similar to that of [1, lemma 8], we give the proof, because it is a little more complicated.

In the case $m = 1$, assertion (B) is clear, because $\Psi_1(c_1; t) = \chi_{(0, c_1]}(t)$. Since for $m \geq 2$, $\Psi_m(c_1, \dots, c_m; t)$ is a continuous function of t , it is sufficient to show that for $p = 0, 1, \dots, m$ and $j = 1, \dots, m$,

$$(13) \quad \frac{\partial \psi_p^m}{\partial c_j}(c_1, \dots, c_m; t) \geq 0 \quad \text{if } c_p \leq t \leq c_{p+1}.$$

In the sequel, for $m \geq 1$, $0 \leq p \leq m$ and $1 \leq j \leq m$, we say that the assertion (m, p, j) holds if we have (13) for all real numbers $0 < c_1 < c_2 < \dots < c_m$. We shall prove the assertions (m, p, j) for all m, p, j by the induction on m , and at each step we consider the induction with respect to p . First remark that assertions $(m, 0, j)$ and (m, m, j) hold for all m and j , because $\partial \psi_0^m / \partial c_j = \partial \psi_m^m / \partial c_j = 0$. In particular, the assertions $(1, 0, 1)$ and $(1, 1, 1)$ hold, and hence the step $m = 1$ is obtained. Let $m \geq 2$ and assume that the assertions at the step $m - 1$ is valid. Since

$$\frac{\partial}{\partial c_j} A_k^m(c_1, \dots, c_m) = \frac{-c_k}{c_j(c_j - c_k)} A_k^m(c_1, \dots, c_m)$$

for $k \neq j$, it follows from (6) and (8) that

$$\begin{aligned} & \frac{\partial \psi_p^m}{\partial c_j}(c_1, \dots, c_m; t) \\ = & \begin{cases} \sum_{k=1}^p \frac{c_k}{c_j(c_j - c_k)} A_k^m(c_1, \dots, c_m)(t - c_k)^{m-1} & \text{for } p = 0, \dots, j-1, \\ \sum_{k=p+1}^m \frac{-c_k}{c_j(c_j - c_k)} A_k^m(c_1, \dots, c_m)(t - c_k)^{m-1} & \text{for } p = j, \dots, m. \end{cases} \end{aligned}$$

Now let j be fixed. First we deal with the case of $0 \leq p \leq j-1$. The assertion $(m, 0, j)$ has been obtained in the above. Assume $1 \leq p \leq j-1$. We shall show that the assertion (m, p, j) follows from induction assumptions $(m, p-1, j)$, $(m-1, p, j-1)$ and $(m-1, p-1, j-1)$. Assume that the function $f(t) := (\partial \psi_p^m / \partial c_j)(c_1, \dots, c_m; t)$ attains its minimum on $[c_p, c_{p+1}]$ at τ_0 . It is sufficient to show $f(\tau_0) \geq 0$. If $\tau_0 = c_p$, then

$$\begin{aligned} f(\tau_0) &= \frac{\partial \psi_p^m}{\partial c_j}(c_1, \dots, c_m; c_p) \\ &= \sum_{k=1}^p \frac{c_k}{c_j(c_j - c_k)} A_k^m(c_1, \dots, c_m)(c_p - c_k)^{m-1} \\ &= \frac{\partial \psi_{p-1}^m}{\partial c_j}(c_1, \dots, c_m; c_p) \geq 0 \end{aligned}$$

by the assumption $(m, p-1, j)$. Next, if $\tau_0 = c_{p+1}$, then

$$\begin{aligned} f(\tau_0) &= \frac{\partial \psi_p^m}{\partial c_j}(c_1, \dots, c_m; c_{p+1}) \\ &= \sum_{k=1}^p \frac{c_k}{c_j(c_j - c_k)} A_k^m(c_1, \dots, c_m)(c_{p+1} - c_k)(c_{p+1} - c_k)^{m-2} \\ &= \sum_{k=1}^p \frac{c_k \cdot c_{p+1}}{c_j(c_j - c_k)} A_k^{m-1}(c_1, \dots, \check{c}_{p+1}, \dots, c_m)(c_{p+1} - c_k)^{m-2} \\ &= \begin{cases} c_{p+1} \frac{\partial \psi_p^{m-1}}{\partial c_{j-1}}(c_1, \dots, \check{c}_{p+1}, \dots, c_m; c_{p+1}) & \text{if } p < j-1 \\ c_{p+1}^{-1} \rho_p^m(c_1, \dots, c_m; c_{p+1}) & \text{if } p = j-1 \end{cases} \\ &\geq 0 \end{aligned}$$

by the assumption $(m-1, p, j-1)$ and (11); here by \check{c}_{p+1} we indicate that the factor

c_{p+1} is missing. Finally, if $\tau_0 \in (c_p, c_{p+1})$, then $f'(\tau_0) = 0$, that is,

$$\frac{\partial^2 \psi_p^m}{\partial t \partial c_j}(c_1, \dots, c_m; \tau_0) = (m-1) \sum_{k=1}^p \frac{c_k}{c_j(c_j - c_k)} A_k^m(c_1, \dots, c_m) (\tau_0 - c_k)^{m-2} = 0.$$

Hence we have

$$\begin{aligned} f(\tau_0) &= \frac{\partial \psi_p^m}{\partial c_j}(c_1, \dots, c_m; \tau_0) \\ &= \sum_{k=1}^{p-1} \frac{c_k}{c_j(c_j - c_k)} A_k^m(c_1, \dots, c_m) (\tau_0 - c_k)^{m-2} (\tau_0 - c_k) \\ &\quad + \frac{c_p}{c_j(c_j - c_p)} A_p^m(c_1, \dots, c_m) (\tau_0 - c_p)^{m-2} (\tau_0 - c_p) \\ &= \sum_{k=1}^{p-1} \frac{c_k}{c_j(c_j - c_k)} A_k^m(c_1, \dots, c_m) (\tau_0 - c_k)^{m-2} (c_p - c_k) \\ &= \sum_{k=1}^{p-1} \frac{c_k \cdot c_p}{c_j(c_j - c_k)} A_k^{m-1}(c_1, \dots, \check{c}_p, \dots, c_m) (\tau_0 - c_k)^{m-2} \\ &= c_p \frac{\partial \psi_{p-1}^{m-1}}{\partial c_{j-1}}(c_1, \dots, \check{c}_p, \dots, c_m; \tau_0) \geq 0 \end{aligned}$$

by the assumption $(m-1, p-1, j-1)$. Therefore we have the assertions (m, p, j) for $0 \leq p \leq j-1$ by the induction with respect to p . Next we deal with the case of $j \leq p \leq m$. In this case, likewise remarking that for $j \leq p \leq m-1$,

$$\begin{aligned} \frac{\partial \psi_p^m}{\partial c_j}(c_1, \dots, c_m; c_{p+1}) &= \frac{\partial \psi_{p+1}^m}{\partial c_j}(c_1, \dots, c_m; c_{p+1}), \\ \frac{\partial \psi_p^m}{\partial c_j}(c_1, \dots, c_m; c_p) &= \begin{cases} c_p \frac{\partial \psi_{p-1}^{m-1}}{\partial c_j}(c_1, \dots, \check{c}_p, \dots, c_m; c_p) & \text{if } p > j \\ c_p^{-1} \rho_p^m(c_1, \dots, c_m; c_p) & \text{if } p = j \end{cases} \end{aligned}$$

and that if

$$\frac{\partial^2 \psi_p^m}{\partial t \partial c_j}(c_1, \dots, c_m; \tau_0) = (m-1) \sum_{k=p+1}^m \frac{-c_k}{c_j(c_j - c_k)} A_k^m(c_1, \dots, c_m) (\tau_0 - c_k)^{m-2} = 0,$$

then

$$\frac{\partial \psi_p^m}{\partial c_j}(c_1, \dots, c_m; \tau_0) = c_{p+1} \frac{\partial \psi_p^{m-1}}{\partial c_j}(c_1, \dots, \check{c}_{p+1}, \dots, c_m; \tau_0),$$

we can obtain the assertion (m, p, j) from induction assumptions $(m, p + 1, j)$, $(m - 1, p - 1, j)$ and $(m - 1, p, j)$. Here we note that the induction on p goes downward from (m, m, j) .

In the end, we have the assertion (m, p, j) for all $m \geq 1, 0 \leq p \leq m, 1 \leq j \leq m$. Clearly $\lim_{0 < c_1 < \dots < c_m \rightarrow 0} \Psi_m(t) = 0$ for $t > 0$, and thus we achieve the proof of (B).

(C) By a direct calculation, we have

$$\begin{aligned} & \int_0^{c_m} \Psi_m(c_1, \dots, c_m; t) dt \\ &= \frac{1}{(m-1)!} \left(\int_0^{c_m} t^{m-1} dt - \sum_{p=1}^{m-1} \int_{c_p}^{c_{p+1}} \sum_{k=1}^p A_k^m(c_1, \dots, c_m) (t - c_k)^{m-1} dt \right) \\ &= \frac{1}{(m-1)!} \left(\frac{c_m^m}{m} - \sum_{k=1}^{m-1} A_k^m(c_1, \dots, c_m) \sum_{p=k}^{m-1} \int_{c_p}^{c_{p+1}} (t - c_k)^{m-1} dt \right) \\ &= \frac{1}{m!} \left(c_m^m - \sum_{k=1}^{m-1} A_k^m(c_1, \dots, c_m) (c_m - c_k)^m \right). \end{aligned}$$

Since for $1 \leq k \leq m - 1, A_k^m(c_1, \dots, c_m) (c_m - c_k) = c_m A_k^{m-1}(c_1, \dots, c_{m-1})$, and since $\psi_{m-1}^{m-1}(c_1, \dots, c_{m-1}; t) \equiv 0$ by (10), we have

$$\begin{aligned} \int_0^{c_m} \Psi_m(c_1, \dots, c_m; t) dt &= \frac{c_m}{m!} \left(c_m^{m-1} - \sum_{k=1}^{m-1} A_k^{m-1}(c_1, \dots, c_{m-1}) (c_m - c_k)^{m-1} \right) \\ &= \frac{c_m}{m!} \left((m-1) \int_{c_{m-1}}^{c_m} \psi_{m-1}^{m-1}(c_1, \dots, c_{m-1}; t) dt \right. \\ &\quad \left. + c_{m-1}^{m-1} - \sum_{k=1}^{m-1} A_k^{m-1}(c_1, \dots, c_{m-1}) (c_{m-1} - c_k)^{m-1} \right) \\ &= \frac{c_m}{m} \int_0^{c_{m-1}} \Psi_{m-1}(c_1, \dots, c_{m-1}; t) dt, \end{aligned}$$

which shows (C), because $\int_0^{c_1} \Psi_1(c_1; t) dt = c_1$. This completes the proof of Lemma 1.

Theorem 1. *Let u be the same as in Theorem A and let $(X_0, t_0) \in D$. Suppose that u is a poly-supertemperature of degree m on D and $0 < c_1 < \dots < c_m < \min\{1/4a, t_0\}$. Then the mean value $A[u, c_1, c_2, \dots, c_m](X_0, t_0)$ is a decreasing function of each c_j ($1 \leq j \leq m$) and converges to $u(X_0, t_0)$ as c_m tends to 0.*

Proof. Put $\mu := (-H)^m u$. Then by [2, Theorems 1 and 2] and their proofs, we

have

$$\begin{aligned} & u(X_0, t_0) - A[u, c_1, c_2, \dots, c_m](X_0, t_0) \\ = & W^m * \mu(X_0, t_0) - A[W^m * \mu, c_1, c_2, \dots, c_m](X_0, t_0) \\ = & \iint_{\mathbf{R}^{n+1}} \Psi_m(c_1, \dots, c_m; t_0 - t) W(X_0 - X, t_0 - t) d\mu(X, t). \end{aligned}$$

Hence Theorem 1 follows from Lemma 1 (B).

2. Lower-regularization

For a Borel measurable function v on D , its lower-regularization \hat{v} is defined by

$$\hat{v}(X, t) := \min \left\{ \liminf_{(Y,s) \rightarrow (X,t)} v(Y, s), v(X, t) \right\}.$$

Remark that \hat{v} is lower semi-continuous on D . Our result is the following

Theorem 2. *Let v be a Borel measurable function on D satisfying the growth condition*

$$(14) \quad |v(X, t)| \leq M e^{a|X|^2}, \quad \forall (X, t) \in D$$

with some constants $M > 0$ and $a > 0$. Suppose that v has the super-mean-value property, that is,

$$(15) \quad v(X, t) \geq A[v, c_1, \dots, c_m](X, t)$$

for all $(X, t) \in D$ and $0 < c_1 < \dots < c_m < \min\{1/4a, t\}$. Then \hat{v} is a poly-supertemperature of degree m and is equal to v a.e. on D .

We make some preparations for the proof of Theorem 2. The following assertion was noted in [2, Theorem 4] without proof. It can be shown by the similar manner to [1, Lemma 6], but we here give the proof for the sake of completeness.

Proposition 1. *Let v be a Borel measurable function on D satisfying the growth condition (14). Then*

$$\lim_{\substack{0 < c_1 < \dots < c_m \\ c_m \rightarrow 0}} \frac{m!}{c_1 \cdots c_m} \left(v - A[v, c_1, c_2, \dots, c_m] \right) = (-H)^m v$$

in the sense of distributions.

Proof. Let $\phi \in C_0^\infty(D)$ be fixed. Then for sufficiently small $c_m > 0$, we have

$$\begin{aligned} & \iint_D \{v(X, t) - A[v, c_1, \dots, c_m](X, t)\} \phi(X, t) \, dX \, dt \\ &= \iint_D v(X, t) \{ \phi(X, t) - A^*[\phi, c_1, \dots, c_m](X, t) \} \, dX \, dt, \end{aligned}$$

where

$$A^*[\phi, c_1, \dots, c_m](X, t) := \sum_{k=1}^m A_k W^*[\phi, c_k](X, t)$$

and

$$W^*[\phi, c](X, t) = \int_{\mathbf{R}^n} W(Y, c) \phi(X - Y, t + c) \, dY.$$

Put $\psi(X, t) = \phi(X, T - t)$. Then $\psi \in C_0^\infty(D)$ and hence $\psi = W^n * ((-H)^m \psi)$ by (4). Since $A^*[\phi, c_1, \dots, c_m](X, t) = A[\psi, c_1, \dots, c_m](X, T - t)$, an argument in [2, Proof of Theorem 2] gives

$$\begin{aligned} & \phi(X, t) - A^*[\phi, c_1, \dots, c_m](X, t) \\ &= \psi(X, T - t) - A[\psi, c_1, \dots, c_m](X, T - t) \\ &= W^m * ((-H)^m \psi)(X, T - t) - A[W^m * ((-H)^m \psi), c_1, \dots, c_m](X, T - t) \\ &= \int_{T-t-c_m}^{T-t} \left(\Psi_m(c_1, \dots, c_m; T - t - s) \right. \\ & \quad \left. \times \int_{\mathbf{R}^n} W(X - Y, T - t - s) ((-H)^m \psi)(Y, s) \, dY \right) \, ds \\ &= \int_{T-t-c_m}^{T-t} \left(\Psi_m(c_1, \dots, c_m; T - t - s) \right. \\ & \quad \left. \times \int_{\mathbf{R}^n} W(X - Y, T - t - s) ((-H^*)^m \phi)(Y, T - s) \, dY \right) \, ds \\ &= \int_0^{c_m} \Psi_m(c_1, \dots, c_m; \tau) \int_{\mathbf{R}^n} W(X - Y, \tau) ((-H^*)^m \phi)(Y, t + \tau) \, dY \, d\tau, \end{aligned}$$

where $H^* = \Delta_X + \partial/\partial t$ is the adjoint operator of H . Remarking the growth condition (14), Lemma 1 (C) and

$$\lim_{\tau \rightarrow 0} \int_{\mathbf{R}^n} W(X - Y, \tau) ((-H^*)^m \phi)(Y, t + \tau) \, dY = (-H^*)^m \phi(X, t),$$

we obtain

$$\begin{aligned} & \lim_{\substack{0 < c_1 < \dots < c_m \\ c_m \rightarrow 0}} \frac{m!}{c_1 \cdots c_m} \iint_D (v(X, t) - A[v, c_1, c_2, \dots, c_m](X, t)) \phi(X, t) dX dt \\ &= \iint_D v(X, t) ((-H^*)^m \phi)(X, t) dX dt \end{aligned}$$

by the Lebesgue dominated convergence theorem. This completes the proof.

The following lemma is the key in our argument.

Lemma 2. *Let v be a Borel measurable function on D satisfying (14) and (15).*

(A) *If $(X_0, t_0) \in D$ and $0 < c_0 < c_1 < \dots < c_m < \min\{1/4a, t_0\}$, then*

$$A[v, c_0, \dots, c_{m-1}](X_0, t_0) \geq A[v, c_1, \dots, c_m](X_0, t_0).$$

(B) *If $(X_0, t_0) \in D$ and $0 < d_1 < \dots < d_m < c_1 < \dots < c_m < \min\{1/4a, t_0\}$, then*

$$A[v, d_1, \dots, d_m](X_0, t_0) \geq A[v, c_1, \dots, c_m](X_0, t_0).$$

Proof. Before giving the proof, we remark that Theorem 1 is not applicable to this case directly, because we do not assume the condition (1) for v .

Integrating both sides of $v(Y, t_0 - c_0) \geq A[v, c_1 - c_0, \dots, c_m - c_0](Y, t_0 - c_0)$ with respect to $W(X_0 - Y, c_0)dY$, we have

$$W[v, c_0](X_0, t_0) \geq \sum_{k=1}^m A_k^m(c_1 - c_0, \dots, c_m - c_0) W[v, c_k](X_0, t_0).$$

Hence the fact $A_1^m(c_0, \dots, c_{m-1}) > 0$ implies

$$\begin{aligned} & A[v, c_0, \dots, c_{m-1}](X_0, t_0) \\ &= A_1^m(c_0, \dots, c_{m-1}) W[v, c_0](X_0, t_0) + \sum_{k=1}^{m-1} A_{k+1}^m(c_0, \dots, c_{m-1}) W[v, c_k](X_0, t_0) \\ &\geq A_1^m(c_0, \dots, c_{m-1}) \sum_{k=1}^m A_k^m(c_1 - c_0, \dots, c_m - c_0) W[v, c_k](X_0, t_0) \\ &\quad + \sum_{k=1}^{m-1} A_{k+1}^m(c_0, \dots, c_{m-1}) W[v, c_k](X_0, t_0) \\ &= \sum_{k=1}^m A_k^m(c_1, \dots, c_m) W[v, c_k](X_0, t_0), \end{aligned}$$

because $A_1^m(c_0, \dots, c_{m-1})A_m^m(c_1 - c_0, \dots, c_m - c_0) = A_m^m(c_1, \dots, c_m)$ and for $k = 1, \dots, m - 1$,

$$\begin{aligned} & A_1^m(c_0, \dots, c_{m-1})A_k^m(c_1 - c_0, \dots, c_m - c_0) + A_{k+1}^m(c_0, \dots, c_{m-1}) \\ = & \prod_{j=1}^{m-1} \frac{c_j}{c_j - c_0} \cdot \prod_{j=1, j \neq k}^m \frac{c_j - c_0}{c_j - c_k} + \frac{c_0}{c_0 - c_k} \cdot \prod_{j=1, j \neq k}^{m-1} \frac{c_j}{c_j - c_k} \\ = & \prod_{j=1, j \neq k}^{m-1} \frac{c_j}{c_j - c_k} \left(\frac{c_k}{c_k - c_0} \cdot \frac{c_m - c_0}{c_m - c_k} + \frac{c_0}{c_0 - c_k} \right) \\ = & A_k^m(c_1, \dots, c_m). \end{aligned}$$

This shows the assertion (A). The assertion (B) follows from (A) immediately.

Now we shall prove Theorem 2.

Proof of Theorem 2. Let $1 \leq d_1 < d_2 < \dots < d_m \leq 2$ be fixed and $\rho \in C_0^\infty(0, \infty)$ satisfy $\rho \geq 0$, $\text{supp}[\rho] \subset [1, 2]$ and $\int_1^2 \rho(t) dt = 1$. For each integer $j \geq 1$, we put

$$A_j(X, t) := \sum_{k=1}^m \frac{4^j A_k}{d_k} \rho\left(\frac{4^j t}{d_k}\right) W(X, t).$$

Then for $t > 4^{1-j}$,

$$A_j * v(X, t) = \int A[v, 4^{-j} d_1 \tau, \dots, 4^{-j} d_m \tau](X, t) \rho(\tau) d\tau.$$

Next we consider the function Rv defined by

$$Rv(X, t) := \sup_{0 < c_1 < \dots < c_m} A[v, c_1, \dots, c_m](X, t).$$

Then Lemma 2 (B) shows

$$Rv(X, t) = \lim_{j \rightarrow \infty} A_j * v(X, t) \left(= \lim_{\substack{0 < c_1 < \dots < c_m \\ c_m \rightarrow 0}} A[v, c_1, \dots, c_m](X, t) \right)$$

Since $\{A_j * v\}$ is an increasing sequence of continuous functions, Rv is lower semi-continuous on D , so that

$$v(X, t) \geq \hat{v}(X, t) \geq Rv(X, t) \text{ on } D.$$

Moreover Proposition 1 gives $v = Rv$ a.e. and $(-H)^m v \geq 0$ in the sense of distributions. These mean that $v = \hat{v}$ a.e. and \hat{v} is poly-supertemperature of degree m , which completes the proof.

REMARK 1. In the theorem, if v is continuous, then we see $Rv \equiv v$ without difficulty. But unfortunately, in case that v is lower semi-continuous (that is, $v \equiv \hat{v}$), we do not know whether $Rv = v$ everywhere or not.

3. Minimum principle

From the super-mean-value property, we obtain the following minimum principle.

Theorem 3. *Let u be a C^{2m-2} -function on D satisfying the growth condition (1). We assume further*

$$(16) \quad a \leq \frac{1}{4T}.$$

Let p be an integer with $1 \leq p \leq m$ and $\{t_j\}_{j=1}^p$ be real numbers such that $T > t_1 > \dots > t_p > 0$. If u is a poly-supertemperature of degree m on D and if u satisfies

$$(17) \quad (-1)^{k-1} u(Y, t_k) \geq 0, \quad \forall k = 1, \dots, p \text{ and } \forall Y \in \mathbf{R}^n,$$

$$(18) \quad (-1)^{p-1} (-H)^k u(Y, t_p) \geq 0, \quad \forall k = 1, \dots, m - p \text{ and } \forall Y \in \mathbf{R}^n,$$

then $u(\Xi, \tau) \geq 0$ for $(\Xi, \tau) \in \mathbf{R}^n \times (t_1, T)$.

In addition, if $u(X_0, t_0) = 0$ for some $(X_0, t_0) \in \mathbf{R}^n \times (t_1, T)$, then $u = 0$ on $\mathbf{R}^n \times (t_p, t_0)$.

Corollary 1. *Let $T > t_1 > t_2 > \dots > t_p > 0$ and let u be a poly-temperature of degree m on D satisfying (1), (16), (17) and (18) in Theorem 3. If $u(X_0, t_0) = 0$ for some $(X_0, t_0) \in \mathbf{R}^n \times (t_1, T)$, then $u \equiv 0$ on D .*

Proof. Let u be a poly-temperature of degree m on D satisfying (1). First we remark that u is real analytic on D . In fact, for $T > t > t_1 > \dots > t_m > 0$, applying the mean value property [2, Theorem 1] to the case $c_k = t - t_k (k = 1, \dots, m)$, we have

$$(19) \quad u(X, t) = \sum_{k=1}^m \left(\prod_{j=1, j \neq k}^m \frac{t - t_j}{t_k - t_j} \right) \int_{\mathbf{R}^n} W(X - Y, t - t_k) u(Y, t_k) dY.$$

This representation implies that u is real analytic on D . Since $u = 0$ on $\mathbf{R}^n \times (t_p, t_0)$ by Theorem 3, from the real analyticity it follows that $u \equiv 0$ on D .

For the proof of Theorem 3, we prepare the following

Lemma 3. *Let $n \geq 1$ be an integer, $0 \leq c_0 \leq c_1 < \dots < c_n$ and f be a C^n -function on a neighborhood of $[c_0, c_n]$. Then we have an estimate*

$$\left| \sum_{k=1}^n A_k^n(c_1, \dots, c_n) f(c_k) - \sum_{l=0}^{n-1} \frac{(-c_0)^\ell}{\ell!} f^{(\ell)}(c_0) \right| \leq \frac{c_1 \cdots c_n - c_0^n}{n!} \sup_{c_0 \leq t \leq c_n} |f^{(n)}(t)|.$$

In particular,

$$\lim_{c_1, \dots, c_n \rightarrow c_0} \sum_{k=1}^n A_k^n(c_1, \dots, c_n) f(c_k) = \sum_{l=0}^{n-1} \frac{(-c_0)^\ell}{\ell!} f^{(\ell)}(c_0).$$

Proof. We first remark that

$$\begin{aligned} \sum_{k=1}^n A_k c_k^q &= \sum_{k=1}^n c_k^q \frac{(-1)^{k-1} c_1 \cdots c_k \cdots c_n \prod_{i < j, i, j \neq k} (c_j - c_i)}{\prod_{i < j} (c_j - c_i)} \\ &= \begin{vmatrix} 1 & c_1 & \cdots & c_1^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & c_n & \cdots & c_n^{n-1} \end{vmatrix}^{-1} \begin{vmatrix} c_1^q & c_1 & \cdots & c_1^{n-1} \\ \vdots & \vdots & & \vdots \\ c_n^q & c_n & \cdots & c_n^{n-1} \end{vmatrix} \\ &= \begin{cases} 1, & q = 0, \\ 0, & q = 1, \dots, n - 1. \end{cases} \end{aligned}$$

By using the above, (8), (10) and the Taylor formula

$$f(c_k) = \sum_{\ell=0}^{n-1} \frac{(c_k - c_0)^\ell}{\ell!} f^{(\ell)}(c_0) + \frac{1}{(n-1)!} \int_{c_0}^{c_k} (c_k - t)^{n-1} f^{(n)}(t) dt,$$

we obtain

$$\begin{aligned} \sum_{k=1}^n A_k f(c_k) &= \sum_{q=0}^{n-1} \left(\sum_{k=1}^n A_k c_k^q \right) \left(\sum_{\ell=q}^{n-1} \binom{\ell}{q} \frac{(-c_0)^{\ell-q}}{\ell!} f^{(\ell)}(c_0) \right) \\ &\quad + \sum_{k=1}^n A_k \frac{1}{(n-1)!} \int_{c_0}^{c_k} (c_k - t)^{n-1} f^{(n)}(t) dt \end{aligned}$$

$$\begin{aligned}
 &= \sum_{\ell=0}^{n-1} \binom{\ell}{0} \frac{(-c_0)^\ell}{\ell!} f^{(\ell)}(c_0) \\
 &\quad + \frac{(-1)^{n-1}}{(n-1)!} \left(\sum_{p=0}^{n-1} \sum_{k=p+1}^n \int_{c_p}^{c_{p+1}} A_k(t-c_k)^{n-1} f^{(n)}(t) dt \right) \\
 &= \sum_{\ell=0}^{n-1} \frac{(-c_0)^\ell}{\ell!} f^{(\ell)}(c_0) + (-1)^{n-1} \int_{c_0}^{c_n} \Psi_n(c_1, \dots, c_n; t) f^{(n)}(t) dt.
 \end{aligned}$$

Since $\Psi_n(c_1, \dots, c_n; t) = t^{n-1}/(n-1)!$ for $0 \leq t \leq c_1$, Lemma 1 (C) gives

$$\int_{c_0}^{c_n} \Psi_n(c_1, \dots, c_n; t) dt = \left(\int_0^{c_n} - \int_0^{c_0} \right) \Psi_n(c_1, \dots, c_n; t) dt = \frac{c_1 \cdots c_n - c_0^n}{n!}.$$

This and the nonnegativity of Ψ_n lead to Lemma 3.

Proof of Theorem 3. Let (Ξ, τ) be fixed in $\mathbf{R}^n \times (t_1, T)$. For sufficiently small $c > 0$, set

$$\begin{cases} c_k := \tau - t_k & \text{for } k = 1, \dots, p, \\ c_{p+\ell} = c_{p+\ell}(c) := \tau - t_p + c \ell & \text{for } \ell = 1, \dots, m-p. \end{cases}$$

For $k = 1, \dots, p-1$, we put

$$(20) \quad A_{p,k}^m := A_k^m(c_1, \dots, c_{p-1}, c_p, \dots, c_p).$$

For $k = 0, \dots, m-p$, we put

$$(21) \quad B_{p,k}^m := \sum_{\ell=0}^{m-p-k} \frac{c_p^{k+\ell}}{(k+\ell)!} \binom{k+\ell}{k} \left(-\frac{\partial}{\partial c_p} \right)^\ell A_p^p(c_1, \dots, c_p).$$

Here recall that $A_p^p(c_1, \dots, c_p) = \prod_{j=1}^{p-1} c_j / (c_j - c_p)$. Then by definition,

$$(22) \quad (-1)^{k-1} A_{p,k}^m > 0 \quad \text{for } k = 1, \dots, p-1,$$

and moreover

$$(23) \quad (-1)^{p-1} \left(-\frac{\partial}{\partial c_p} \right)^k A_p^p(c_1, \dots, c_p) > 0 \quad \text{for } \forall k \geq 0.$$

In fact, $(-1)^{p-1} A_p^p(c_1, \dots, c_p) > 0$ by (5), and

$$-\frac{\partial}{\partial c_p} A_p^p(c_1, \dots, c_p) = \left(\frac{1}{c_p - c_1} + \cdots + \frac{1}{c_p - c_{p-1}} \right) A_p^p(c_1, \dots, c_p)$$

shows the assertion for $k = 1$. For $k \geq 2$, the Leibniz rule gives

$$\begin{aligned} \left(-\frac{\partial}{\partial c_p}\right)^k A_p^p(c_1, \dots, c_p) &= \sum_{\ell=0}^{k-1} \binom{k-1}{\ell} \sum_{j=1}^{p-1} \frac{(k-1-\ell)!}{(c_p - c_j)^{k-\ell}} \left(-\frac{\partial}{\partial c_p}\right)^\ell A_p^p(c_1, \dots, c_p), \end{aligned}$$

from which (23) follows inductively. In consequence of (23), we see easily that

$$(24) \quad (-1)^{p-1} B_{p,k}^m > 0 \quad \text{for } k = 0, \dots, m-p.$$

Now we shall show $u(\Xi, \tau) \geq 0$. It is sufficient to show that

$$(25) \quad \begin{aligned} \lim_{c_j \downarrow 0} A[u, c_1, \dots, c_p, c_{p+1}(c), \dots, c_m(c)] \\ = \sum_{k=1}^{p-1} A_{p,k}^m W[u, c_k] + \sum_{k=0}^{m-p} B_{p,k}^m W[(-H)^k u, c_p] \end{aligned}$$

in the sense of distribution. In fact, since both functions $u(X, t)$ and

$$\sum_{k=1}^{p-1} A_{p,k}^m W[u, c_k](X, t) + \sum_{k=0}^{m-p} B_{p,k}^m W[(-H)^k u, c_p](X, t)$$

are continuous on $\mathbf{R}^n \times (c_1, T)$, it follows from (2), (17), (18), (22), (24) and (25) that

$$\begin{aligned} u(\Xi, \tau) &\geq \sum_{k=1}^{p-1} A_{p,k}^m W[u, c_k](\Xi, \tau) + \sum_{k=0}^{m-p} B_{p,k}^m W[(-H)^k u, c_p](\Xi, \tau) \\ &= \sum_{k=1}^{p-1} A_{p,k}^m \int_{\mathbf{R}^n} W(\Xi - Y, \tau - t_k) u(Y, t_k) dY \\ &\quad + \sum_{k=0}^{m-p} B_{p,k}^m \int_{\mathbf{R}^n} W(\Xi - Y, \tau - t_p) (-H)^k u(Y, t_p) dY \\ &\geq 0. \end{aligned}$$

We thus devote ourselves to the proof of (25). Let $\phi \in C_0^\infty(\mathbf{R}^{n+1})$. Write out

$$\begin{aligned} A^*[\phi, c_1, \dots, c_m](Y, s) &:= \sum_{k=0}^m A_k^m(c_1, \dots, c_m) W^*[\phi, c_k](Y, s) \\ &= \sum_{k=0}^{p-1} A_k^m(c_1, \dots, c_m) W^*[\phi, c_k](Y, s) + \sum_{\ell=1}^{m-p+1} A_\ell^{m-p+1}(c_p, \dots, c_m) f(c_{p-1+\ell}), \end{aligned}$$

where $f(t) := A_p^p(c_1, \dots, c_{p-1}, t)W^*[\phi, t](Y, s)$. Applying Lemma 3 to this function f , $n = m - p + 1$ and $c_0 = c_p$, we see that

$$\begin{aligned} & \lim_{c \downarrow 0} A^*[\phi, c_1, \dots, c_p, c_{p+1}(c), \dots, c_m(c)](Y, s) \\ &= \sum_{k=1}^{p-1} A_{p,k}^m W^*[\phi, c_k](Y, s) + \sum_{q=0}^{m-p} \frac{(-c_p)^q}{q!} f^{(q)}(c_p). \end{aligned}$$

Hence observing

$$\begin{aligned} \sum_{q=0}^{m-p} \frac{(-c_p)^q}{q!} f^{(q)}(c_p) &= \sum_{k=0}^{m-p} B_{p,k}^m \left(-\frac{\partial}{\partial c_p}\right)^k W^*[\phi, c_p](Y, s) \\ &= \sum_{k=0}^{m-p} B_{p,k}^m W^*[(-H)^* \phi, c_p](Y, s), \end{aligned}$$

we get

$$\begin{aligned} & \lim_{c \downarrow 0} \iint A[u, c_1, \dots, c_p, c_{p+1}(c), \dots, c_m(c)](Y, s) \phi(Y, s) dY ds \\ &= \lim_{c \downarrow 0} \iint u(Y, s) A^*[\phi, c_1, \dots, c_p, c_{p+1}(c), \dots, c_m(c)](Y, s) dY ds \\ &= \iint u(Y, s) \left(\sum_{k=1}^{p-1} A_{p,k}^m W^*[\phi, c_k](Y, s) + \sum_{k=0}^{m-p} B_{p,k}^m W^*[(-H)^* \phi, c_p](Y, s) \right) dY ds \\ &= \iint \left(\sum_{k=1}^{p-1} A_{p,k}^m W[u, c_k](Y, s) + \sum_{k=0}^{m-p} B_{p,k}^m W[(-H)^k u, c_p](Y, s) \right) \phi(Y, s) dY ds, \end{aligned}$$

which implies (25).

By the above argument, we know that if u is poly-temperature of degree m on $\mathbf{R}^n \times (t_p, \tau)$, then

$$(26) \quad u(\Xi, \tau) = \sum_{k=1}^{p-1} A_{p,k}^m W[u, c_k](\Xi, \tau) + \sum_{k=0}^{m-p} B_{p,k}^m W[(-H)^k u, c_p](\Xi, \tau), \quad \forall \Xi \in \mathbf{R}^n.$$

We use this fact in the proof of second assertion of Theorem 3.

Now we assume that $u(X_0, t_0) = 0$ for some point $(X_0, t_0) \in \mathbf{R}^n \times (t_1, T)$. Since

$$u(X_0, t_0) \geq \sum_{k=1}^{p-1} A_{p,k}^m W[u, c_k](X_0, t_0) + \sum_{k=0}^{m-p} B_{p,k}^m W[(-H)^k u, c_p](X_0, t_0) \geq 0,$$

where $c_k = t_0 - t_k$, $k = 1, \dots, p$, we conclude by (17), (18), (22) and (24),

$$(27) \quad u(Y, t_k) = 0, \quad \forall k = 1, \dots, p \text{ and } \forall Y \in \mathbf{R}^n$$

and

$$(28) \quad (-H)^k u(Y, t_p) = 0, \quad \forall k = 1, \dots, m - p \text{ and } \forall Y \in \mathbf{R}^n.$$

Moreover as in the proof of Theorem 1, putting $\mu := (-H)^m u$ and $\Psi_{m,p}(t) = \Psi_{m,p}(c_1, \dots, c_p; t) := \lim_{c \downarrow 0} \Psi_m(c_1, \dots, c_p, c_{p+1}(c), \dots, c_m(c); t)$, we have

$$\begin{aligned} & u(X_0, t_0) - \sum_{k=1}^{p-1} A_{p,k}^m W[u, c_k](X_0, t_0) - \sum_{k=0}^{m-p} B_{p,k}^m W[(-H)^k u, c_p](X_0, t_0) \\ &= \int_{\mathbf{R}^{n+1}} \Psi_{m,p}(c_1, \dots, c_p; t_0 - s) W(X_0 - Y, t_0 - s) d\mu(Y, s) \geq 0. \end{aligned}$$

Since $\Psi_{m,p}(c_1, \dots, c_p; t_0 - s) W(X_0 - Y, t_0 - s) > 0$ for $(Y, s) \in D_0 := \mathbf{R}^n \times (t_p, t_0)$ by Lemma 1(A), we also conclude that μ vanishes there, that is, u is a poly-temperature of degree m on D_0 . Thus (26), (27) and (28) give $u(\Xi, \tau) = 0$ for $\Xi \in \mathbf{R}^n$ and $t_1 < \tau < t_0$. Since u is real analytic on D_0 (see (19)), it vanishes there, as desired. This completes the proof of Theorem 3.

In particular, we see

Corollary 2. (A) Assume that $u \in C^{2m-2}(\bar{D})$. Under the conditions (1) and (16), if u is a poly-supertemperature of degree m on D and if $(-H)^k u(X, 0) \geq 0, \forall X \in \mathbf{R}^n$ for $k = 0, \dots, m - 1$, then $u \geq 0$ on D . Moreover, discussing $(-H)^k u$ in place of u , we also see that $(-H)^k u \geq 0$ on D for $k = 1, \dots, m - 1$.

(B) Let $f_k, k = 0, 1, \dots, m - 1$, be continuous functions on \mathbf{R}^n satisfying the growth condition $|f_k(X)| \leq M e^{a|X|^2}$ with (16). Then the boundary value problem

$$(29) \quad \begin{cases} (-H)^m h = 0 & \text{on } D \\ (-H)^k h(\cdot, 0) = f_k & \text{on } \mathbf{R}^n, \quad k = 0, 1, \dots, m - 1 \end{cases}$$

has a unique solution on D , which is given by

$$(30) \quad h(X, t) = \sum_{k=0}^{m-1} \frac{t^k}{k!} \int_{\mathbf{R}^n} W(X - Y, t) f_k(Y) dY$$

(cf. [3, p.266 or French summary p.328]).

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