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NOTE ON POLY-SUPERTEMPERATURES
ON A STRIP DOMAIN

MASAHARU NISHIO, KATSUNORI SHIMOMURA AND NORIAKI SUZUKI

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0. Introduction

Let $m$ be a positive integer and let

$$D = \{(X, t); X = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, 0 < t < T\}$$

be a strip domain in the $(n + 1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. We consider supersolutions of the $m$-th iterates of the heat operator

$$H = \Delta_X - \frac{\partial}{\partial t}$$
on D. A lower semi-continuous and locally integrable function $u$ on $D$ is called a poly-supertemperature of degree $m$, if $(-H)^m u \geq 0$ on $D$ in the sense of distributions. If $u$ and $-u$ are both poly-supertemperatures of degree $m$, then $u$ is called a poly-temperature of degree $m$.

In our previous paper [2] (see also [1]), we have shown the following super-mean-value property for poly-supertemperatures.

**Theorem A** ([2, Theorem 2]). Let $u$ be a $C^{2m-2}$-function on $D$ satisfying the growth condition

$$(1) \quad |H^k u(X, t)| \leq M e^{a|X|^2}, \quad k = 0, 1, \ldots, m - 1,$$

with some constants $M > 0$ and $a > 0$ (here $H^0 u$ means $u$). If $u$ is a poly-supertemperature of degree $m$ on $D$, then

$$(2) \quad u(X_0, t_0) \geq A[u, c_1, c_2, \ldots, c_m](X_0, t_0)$$

whenever $(X_0, t_0) \in D$ and $0 < c_1 < c_2 \cdots < c_m < \min\{1/4a, t_0\}$. (For notation, see (5) below.)
In the present note, we first point out that the above mean \( A[u, c_1, \cdots, c_m] \) is a decreasing function of each \( c_1, \cdots, c_m \) and converges to \( u(X_0, t_0) \) as \( c_1, \cdots, c_m \) tend to 0 under the condition \( 0 < c_1 < \cdots < c_m \) (Theorem 1). Secondly, in section 2, we show that the lower-regularization \( \hat{\nu} \) of a Borel measurable function \( \nu \) having the super-mean-value property (2) is a poly-supertemperature (Theorem 2). In the final section, we derive a minimum principle for poly-supertemperatures, from the super-mean-value property (Theorem 3). As its corollary, we have some uniqueness results for poly-temperatures. Especially, we obtain the existence and uniqueness of poly-temperatures satisfying the boundary conditions.

1. Monotonicity of the mean

Let \( W \) denote the fundamental solution for the heat equation on \( \mathbb{R}^{n+1} \), that is,

\[
W(X, t) = \begin{cases} 
(4\pi t)^{-\frac{n}{2}} \exp(-\frac{|X|^2}{4t}) & \text{if } t > 0, \\
0 & \text{if } t \leq 0.
\end{cases}
\]

We set \( W^1 := W \) and \( W^k := W^{k-1} * W \) for \( k \geq 2 \), inductively, where * denotes the convolution in \( \mathbb{R}^{n+1} \). Then

\[
W^m(X, t) = \frac{t^{m-1}}{(m-1)!} W(X, t)
\]

and this is the fundamental solution of the equation \((-H)^mu = 0\), that is,

\[
(-H)^m(W^m * \phi) = W^m * ((-H)^m \phi) = \phi
\]

for all \( \phi \in C_0^\infty(D) \) (cf. [2, Proposition 2]).

Now we recall the definition of the mean values \( A[u, c_1, c_2, \cdots, c_m] \):

\[
A[u, c_1, c_2, \cdots, c_m](X_0, t_0) := \sum_{k=1}^{m} A_k W[u, c_k](X_0, t_0),
\]

where

\[
W[u, c_k](X_0, t_0) := \int_{\mathbb{R}^n} u(X - X_0, t_0 - c_k) W(X, c_k) dX
\]

and the coefficients \( A_k, k = 1, 2, \cdots, m \), are given by

\[
A_k = A_k^m(c_1, \cdots, c_m) := \prod_{j=1, j \neq k}^{m} \frac{c_j}{c_j - c_k} (A_1 = 1 \text{ when } m = 1).
\]
Note that

\[ (-1)^{k-1}A_k > 0, \quad k = 1, 2, \ldots, m. \]

For integers \( m, p \) with \( 0 \leq p \leq m \) and real numbers \( c_1, \ldots, c_m \) with \( 0 = c_0 < c_1 < \cdots < c_m < c_{m+1} := \infty \), we consider the following functions:

\[
\psi_p^m(t) = \psi_p^m(c_1, \ldots, c_m; t) := t^{m-1} - \sum_{k=1}^{p} A_k^m(c_1, \ldots, c_m)(t - c_k)^{m-1}
\]

and

\[
\Psi_m(t) = \Psi_m(c_1, \ldots, c_m; t) := \frac{1}{(m-1)!} \sum_{p=0}^{m-1} \psi_p^m(t) \chi(c_p, c_{p+1})(t),
\]

where \( \psi_0^m(t) = t^{m-1} \) and \( \chi(c_p, c_{p+1}) \) denotes the characteristic function of the interval \( (c_p, c_{p+1}] \). We remark that the above functions were already introduced in our previous paper \([2]\) as \( \phi_p(t) = \psi_p^m(t_0 - t) \) and \( \psi_m(t) = (m - 1)!\Psi_m(t) \). We have already obtained the following (\([2, \text{Lemma 1}]\), for the proof see \([1, \text{Lemma 8}]\)): for all integers \( p \) with \( 0 \leq p \leq m - 1 \),

\[
\psi_p^m(t) = \sum_{k=p+1}^{m} A_k(t - c_k)^{m-1},
\]

\[
\psi_p^m(t) \geq 0 \quad \text{for} \quad c_p \leq t \leq c_{p+1},
\]

and

\[
\psi_m^m(t) = 0.
\]

The function \( \Psi_m \) has the following properties.

**Lemma 1.** (A) \( \Psi_m(c_1, \ldots, c_m; t) \) is a continuous (for \( m \geq 2 \)) and nonnegative function of \( t \geq 0 \). Moreover \( \Psi_m(c_1, \ldots, c_m; t) > 0 \) if \( 0 < t < c_m \).

(B) \( \Psi_m(c_1, \ldots, c_m; t) \) is an increasing function of each variable \( c_j, \ j = 1, 2, \ldots, m \), and

\[
\lim_{0 < c_1 < \cdots < c_m \rightarrow 0} \Psi_m(c_1, \ldots, c_m; t) = 0.
\]

(C) \[
\int_0^{c_m} \Psi_m(c_1, \ldots, c_m; t) \, dt = \frac{c_1 \cdots c_m}{m!}.
\]
Proof. (A) The continuity of $\Psi_m$ follows from the facts $\psi^m_p(c_p) = \psi^{m-1}_p(c_p)$, $p = 1, 2, \ldots, m$. Inequalities (9) show the nonnegativity of $\Psi_m$. The positivity of $\Psi_m$ is obtained immediately in the case of $c_{m-1} < t < c_m$ because

$$\Psi_m(c_1, \ldots, c_m; t) = \frac{1}{(m-1)!}(-1)^{m-1}A_m(c_m-t)^{m-1}$$

and $(-1)^{m-1}A_m > 0$. As will be seen in the below, the proof of (B) is independent of the positivity of $\Psi_m$. Therefore the general case follows from the case of $c_{m-1} < t < c_m$ because of (B).

(B) For the proof, we use the following fact: Let $m \geq 2$. We define

$$\rho^m_p(t) = \rho^m_p(c_1, \ldots, c_m; t) := \sum_{k=1}^{p} c_k A^m_k(c_1, \ldots, c_m)(t-c_k)^{m-2}$$

where $\rho^m_0 := 0$. Then we have

$$(11) \quad \rho^m_p(c_1, \ldots, c_m; t) \geq 0 \quad \text{for} \quad c_p \leq t \leq c_{p+1}$$

and

$$(12) \quad \rho^m_m(c_1, \ldots, c_m; t) = 0.$$

This can be proved by the quite same manner as in [1, Lemma 8], so we omit the proof.

Now we consider the first part of (B). Though the method of the proof is also similar to that of [1, lemma 8], we give the proof, because it is a little more complicated.

In the case $m = 1$, assertion (B) is clear, because $\Psi_1(c_1; t) = \chi(0, c_1)(t)$. Since for $m \geq 2$, $\Psi_m(c_1, \ldots, c_m; t)$ is a continuous function of $t$, it is sufficient to show that for $p = 0, 1, \ldots, m$ and $j = 1, \ldots, m$,

$$\frac{\partial \psi^m_p}{\partial c_j}(c_1, \ldots, c_m; t) \geq 0 \quad \text{if} \quad c_p \leq t \leq c_{p+1}.$$

In the sequel, for $m \geq 1$, $0 \leq p \leq m$ and $1 \leq j \leq m$, we say that the assertion $(m, p, j)$ holds if we have (13) for all real numbers $0 < c_1 < c_2 < \cdots < c_m$. We shall prove the assertions $(m, p, j)$ for all $m, p, j$ by the induction on $m$, and at each step we consider the induction with respect to $p$. First remark that assertions $(m, 0, j)$ and $(m, m, j)$ hold for all $m$ and $j$, because $\partial \psi^m_0/\partial c_j = \partial \psi^m_m/\partial c_j = 0$. In particular, the assertions $(1, 0, 1)$ and $(1, 1, 1)$ hold, and hence the step $m = 1$ is obtained. Let $m \geq 2$ and assume that the assertions at the step $m = 1$ is valid. Since

$$\frac{\partial}{\partial c_j} A^m_k(c_1, \ldots, c_m) = \frac{-c_k}{c_j(c_j - c_k)} A^m_k(c_1, \ldots, c_m)$$
for \( k \neq j \), it follows from (6) and (8) that

\[
\frac{\partial \psi^m_p}{\partial c_j}(c_1, \ldots, c_m; t) = \begin{cases} 
\sum_{k=1}^{p} \frac{c_k}{c_j(c_j - c_k)} A^m_k(c_1, \ldots, c_m)(t - c_k)^{m-1} & \text{for } p = 0, \ldots, j - 1, \\
\sum_{k=p+1}^{m} \frac{-c_k}{c_j(c_j - c_k)} A^m_k(c_1, \ldots, c_m)(t - c_k)^{m-1} & \text{for } p = j, \ldots, m.
\end{cases}
\]

Now let \( j \) be fixed. First we deal with the case of \( 0 < p < j - 1 \). The assertion \((m, 0, j)\) has been obtained in the above. Assume \( 1 < p < j - 1 \). We shall show that the assertion \((m, p, j)\) follows from induction assumptions \((m, p-1, j)\), \((m-1, p, j-1)\) and \((m-1, p-1, j-1)\). Assume that the function \( f(t) := \frac{\partial \psi^m_p}{\partial c_j}(c_1, \ldots, c_m; t) \) attains its minimum on \([c_p, c_{p+1}]\) at \( \tau_0 \). It is sufficient to show \( f(\tau_0) \geq 0 \). If \( \tau_0 = c_p \), then

\[
f(\tau_0) = \frac{\partial \psi^m_p}{\partial c_j}(c_1, \ldots, c_m; c_p) = \sum_{k=1}^{p} \frac{c_k}{c_j(c_j - c_k)} A^m_k(c_1, \ldots, c_m)(c_p - c_k)^{m-1} = \frac{\partial \psi^m_{p-1}}{\partial c_j}(c_1, \ldots, c_m; c_p) \geq 0
\]

by the assumption \((m, p-1, j)\). Next, if \( \tau_0 = c_{p+1} \), then

\[
f(\tau_0) = \frac{\partial \psi^m_p}{\partial c_j}(c_1, \ldots, c_m; c_{p+1}) = \sum_{k=1}^{p} \frac{c_k}{c_j(c_j - c_k)} A^m_{k}(c_1, \ldots, c_m)(c_{p+1} - c_k)^{m-2} = \sum_{k=1}^{p} \frac{c_k \cdot c_{p+1}}{c_j(c_j - c_k)} A^m_{k-1}(c_1, \ldots, \hat{c}_{p+1}, \ldots, c_m)(c_{p+1} - c_k)^{m-2} = \begin{cases} 
\frac{\partial \psi^m_{p-1}}{\partial c_j}(c_1, \ldots, \hat{c}_{p+1}, \ldots, c_m; c_{p+1}) & \text{if } p < j - 1, \\
\frac{\partial \psi^m_{p-1}}{\partial c_j}(c_1, \ldots, \hat{c}_{p+1}, \ldots, c_m; c_{p+1}) & \text{if } p = j - 1
\end{cases}
\]

by the assumption \((m-1, p, j-1)\) and (11); here by \( \hat{c}_{p+1} \) we indicate that the factor
\( c_{p+1} \) is missing. Finally, if \( \tau_0 \in (c_p, c_{p+1}) \), then \( f'(\tau_0) = 0 \), that is,

\[
\frac{\partial^2 \psi^m_p}{\partial t \partial c_j}(c_1, \ldots, c_m; \tau_0) = (m - 1) \sum_{k=1}^p \frac{c_k}{c_j(c_j - c_k)} A_k^m(c_1, \ldots, c_m)(\tau_0 - c_k)^{m-2} = 0.
\]

Hence we have

\[
f(\tau_0) = \frac{\partial \psi^m_p}{\partial c_j}(c_1, \ldots, c_m; \tau_0)
\]

\[
= \sum_{k=1}^{p-1} \frac{c_k}{c_j(c_j - c_k)} A_k^m(c_1, \ldots, c_m)(\tau_0 - c_k)^{m-2} + \frac{c_p}{c_j(c_j - c_p)} A_p^m(c_1, \ldots, c_m)(\tau_0 - c_p)^{m-2}(\tau_0 - c_p)
\]

\[
= \sum_{k=1}^{p-1} \frac{c_k}{c_j(c_j - c_k)} A_k^m(c_1, \ldots, c_m)(\tau_0 - c_k)^{m-2}(c_p - c_k)
\]

\[
= \sum_{k=1}^{p-1} \frac{c_k \cdot c_p}{c_j(c_j - c_k)} A_k^{m-1}(c_1, \ldots, c_p, \ldots, c_m)(\tau_0 - c_k)^{m-2}
\]

\[
= c_p \frac{\partial \psi^{m-1}_{p-1}}{\partial c_{j-1}}(c_1, \ldots, c_p, \ldots, c_m; \tau_0) \geq 0
\]

by the assumption \((m - 1, p - 1, j - 1)\). Therefore we have the assertions \((m, p, j)\) for \(0 \leq p \leq j - 1\) by the induction with respect to \(p\). Next we deal with the case of \(j < p < m\). In this case, likewise remarking that for \(j \leq p \leq m - 1\),

\[
\frac{\partial \psi^m_p}{\partial c_j}(c_1, \ldots, c_m; c_{p+1}) = \frac{\partial \psi^m_{p+1}}{\partial c_j}(c_1, \ldots, c_m; c_{p+1}),
\]

\[
\frac{\partial \psi^m_p}{\partial c_j}(c_1, \ldots, c_m; c_p) = \begin{cases} 
\frac{\partial \psi^{m-1}_{p-1}}{\partial c_j}(c_1, \ldots, c_p, \ldots, c_m; c_p) & \text{if } p > j \\
\partial^{m-1}_{p}(c_1, \ldots, c_p, \ldots, c_m; c_p) & \text{if } p = j
\end{cases}
\]

and that if

\[
\frac{\partial^2 \psi^m_p}{\partial t \partial c_j}(c_1, \ldots, c_m; \tau_0) = (m - 1) \sum_{k=p+1}^m \frac{-c_k}{c_j(c_j - c_k)} A_k^m(c_1, \ldots, c_m)(\tau_0 - c_k)^{m-2} = 0,
\]

then

\[
\frac{\partial \psi^m_p}{\partial c_j}(c_1, \ldots, c_m; \tau_0) = c_{p+1} \frac{\partial \psi^{m-1}_{p+1}}{\partial c_j}(c_1, \ldots, c_{p+1}, \ldots, c_m; \tau_0),
\]
we can obtain the assertion \((m, p, j)\) from induction assumptions \((m, p + 1, j)\), \((m - 1, p - 1, j)\) and \((m - 1, p, j)\). Here we note that the induction on \(p\) goes downward from \((m, m, j)\).

In the end, we have the assertion \((m, p, j)\) for all \(m \geq 1, 0 \leq p \leq m, 1 \leq j \leq m\). Clearly \(\lim_{t_0 < c_1 < \cdots < c_m \to 0} \Psi_m(t) = 0\) for \(t > 0\), and thus we achieve the proof of (B).

(C) By a direct calculation, we have

\[
\int_0^{c_m} \Psi_m(c_1, \ldots, c_m; t) \, dt = \frac{1}{(m-1)!} \left( \int_0^{c_m} t^{m-1} \, dt - \sum_{p=1}^{m-1} \int_{c_p}^{c_{p+1}} \frac{1}{c_p} \sum_{k=1}^{p} A_k^m(c_1, \ldots, c_m)(t - c_k)^{m-1} \, dt \right) = \frac{1}{(m-1)!} \left( c_m \frac{m}{m} \sum_{k=1}^{m-1} A_k^m(c_1, \ldots, c_m)(c_m - c_k)^{m-1} \right).
\]

Since for \(1 \leq k \leq m - 1\), \(A_k^m(c_1, \ldots, c_m)(c_m - c_k) = c_m A_k^{m-1}(c_1, \ldots, c_{m-1})\), and since \(\psi_{m-1}^{m-1}(c_1, \ldots, c_{m-1}; t) \equiv 0\) by (10), we have

\[
\int_0^{c_m} \Psi_m(c_1, \ldots, c_m; t) \, dt = \frac{c_m}{m!} \left( c_m - \sum_{k=1}^{m-1} A_k^{m-1}(c_1, \ldots, c_{m-1})(c_m - c_k)^{m-1} \right) = \frac{c_m}{m!} \left( (m-1) \int_{c_{m-1}}^{c_m} \psi_{m-1}^{m-1}(c_1, \ldots, c_{m-1}; t) \, dt + c_m^{m-1} - \sum_{k=1}^{m-1} A_k^{m-1}(c_1, \ldots, c_{m-1})(c_m - c_k)^{m-1} \right) = \frac{c_m}{m!} \int_0^{c_{m-1}} \Psi_{m-1}(c_1, \ldots, c_{m-1}; t) \, dt,
\]

which shows (C), because \(\int_0^{c_1} \Psi_1(c_1; t) \, dt = c_1\). This completes the proof of Lemma 1.

**Theorem 1.** Let \(u\) be the same as in Theorem A and let \((X_0, t_0) \in D\). Suppose that \(u\) is a poly-supertemperature of degree \(m\) on \(D\) and \(0 < c_1 < \cdots < c_m < \min\{1/4a, t_0\}\). Then the mean value \(A[u, c_1, c_2, \cdots, c_m](X_0, t_0)\) is a decreasing function of each \(c_j\) \((1 \leq j \leq m)\) and converges to \(u(X_0, t_0)\) as \(c_m\) tends to 0.

**Proof.** Put \(\mu := (-H)^m u\). Then by [2, Theorems 1 and 2] and their proofs, we
have
\[
\begin{align*}
    u(X_0, t_0) - A[u, c_1, c_2, \ldots, c_m](X_0, t_0) \\
    = W^m \ast \mu(X_0, t_0) - A[W^m \ast \mu, c_1, c_2, \ldots, c_m](X_0, t_0) \\
    = \int_{\mathbb{R}^{n+1}} \Psi_m(c_1, \ldots, c_m; t_0 - t) W(X_0 - X, t_0 - t) \, d\mu(X, t).
\end{align*}
\]
Hence Theorem 1 follows from Lemma 1 (B).

2. Lower-regularization

For a Borel measurable function \( v \) on \( D \), its lower-regularization \( \hat{v} \) is defined by
\[
\hat{v}(X, t) := \min \left\{ \liminf_{(Y,s) \to (X,t)} v(Y, s), v(X, t) \right\}.
\]
Remark that \( \hat{v} \) is lower semi-continuous on \( D \). Our result is the following

**Theorem 2.** Let \( v \) be a Borel measurable function on \( D \) satisfying the growth condition
\[
|v(X, t)| \leq Me^{a|X|^2}, \quad \forall (X, t) \in D
\]
with some constants \( M > 0 \) and \( a > 0 \). Suppose that \( v \) has the super-mean-value property, that is,
\[
\begin{align*}
    v(X, t) &\geq A[v, c_1, \ldots, c_m](X, t) \\
    &\text{for all } (X, t) \in D \text{ and } 0 < c_1 < \cdots < c_m < \min\{1/4a, t\}. \quad \text{Then } \hat{v} \text{ is a poly-}
\end{align*}
\]
supertemperature of degree \( m \) and is equal to \( v \) a.e. on \( D \).

We make some preparations for the proof of Theorem 2. The following assertion was noted in [2, Theorem 4] without proof. It can be shown by the similar manner to [1, Lemma 6], but we here give the proof for the sake of completeness.

**Proposition 1.** Let \( v \) be a Borel measurable function on \( D \) satisfying the growth condition (14). Then
\[
\lim_{0 < c_1 < \cdots < c_m \to 0} \frac{m!}{c_1 \cdots c_m} \left( v - A[v, c_1, c_2, \ldots, c_m] \right) = (-H)^m v
\]
in the sense of distributions.
Proof. Let \( \phi \in C_0^\infty(D) \) be fixed. Then for sufficiently small \( c_m > 0 \), we have

\[
\iint_D \{v(X,t) - A[v,c_1,\ldots,c_m](X,t)\phi(X,t)\} dX dt
= \iint_D v(X,t)\{\phi(X,t) - A^*[\phi,c_1,\ldots,c_m](X,t)\} dX dt,
\]

where

\[
A^*[\phi,c_1,\ldots,c_m](X,t) := \sum_{k=1}^m A_k W^*[\phi,c_k](X,t)
\]

and

\[
W^*[\phi,c](X,t) = \int_{\mathbb{R}^n} W(Y,c)\phi(X-Y,t+c) dY.
\]

Put \( \psi(X,t) = \phi(X,T-t) \). Then \( \psi \in C_0^\infty(D) \) and hence \( \psi = W^n * (-H)^m \psi \) by (4). Since \( A^*[\phi,c_1,\ldots,c_m](X,t) = A[\psi,c_1,\ldots,c_m](X,T-t) \), an argument in [2, Proof of Theorem 2] gives

\[
\phi(X,t) - A^*[\phi,c_1,\ldots,c_m](X,t)
= \psi(X,T-t) - A[\psi,c_1,\ldots,c_m](X,T-t)
= W^m * ((-H)^m \psi)(X,T-t) - A[W^m * ((-H)^m \psi),c_1,\ldots,c_m](X,T-t)
= \int_{T-t-c_m}^{T-t} \left( \Psi_m(c_1,\ldots,c_m;T-t-s) \right) \times \int_{\mathbb{R}^n} W(X-Y,T-t-s)((-H)^m \psi)(Y,s) dY ds
= \int_{T-t-c_m}^{T-t} \left( \Psi_m(c_1,\ldots,c_m;T-t-s) \right) \times \int_{\mathbb{R}^n} W(X-Y,T-t-s)((-H^*)^m \phi)(Y,T-s) dY ds
= \int_{\mathbb{R}^n} W(X-Y,\tau)((-H^*)^m \phi)(Y,t+\tau) dY d\tau,
\]

where \( H^* = \Delta_X + \partial/\partial t \) is the adjoint operator of \( H \). Remarking the growth condition (14), Lemma 1 (C) and

\[
\lim_{\tau \to 0} \int_{\mathbb{R}^n} W(X-Y,\tau)((-H^*)^m \phi)(Y,t+\tau) dY = (-H^*)^m \phi(X,t),
\]

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we obtain
\[
\lim_{\epsilon_1 < \cdots < \epsilon_m \to 0} \frac{m!}{c_1 \cdots c_m} \int_D (v(X, t) - A[v, c_1, \cdots, c_m](X, t)) \phi(X, t) dX dt
\]
\[
= \int_D v(X, t)((-H^*)^m \phi)(X, t) dX dt
\]
by the Lebesgue dominated convergence theorem. This completes the proof.

The following lemma is the key in our argument.

**Lemma 2.** Let \( v \) be a Borel measurable function on \( D \) satisfying (14) and (15).

(A) If \((X_0, t_0) \in D\) and \(0 < c_0 < c_1 < \cdots < c_m < \min\{1/4a, t_0\}\), then
\[
A[v, c_0, \cdots, c_m](X_0, t_0) \geq A[v, c_1, \cdots, c_m](X_0, t_0).
\]

(B) If \((X_0, t_0) \in D\) and \(0 < d_1 < \cdots < d_m < c_1 < \cdots < c_m < \min\{1/4a, t_0\}\), then
\[
A[v, d_1, \cdots, d_m](X_0, t_0) \geq A[v, c_1, \cdots, c_m](X_0, t_0).
\]

**Proof.** Before giving the proof, we remark that Theorem 1 is not applicable to this case directly, because we do not assume the condition (1) for \( v \).

Integrating both sides of \( v(Y, t_0 - c_0) > A[v, c_1 - c_0, \cdots, c_m - c_0](Y, t_0 - c_0) \) with respect to \( W(X_0 - Y, c_0) dY \), we have
\[
W[v, c_0](X_0, t_0) \geq \sum_{k=1}^{m} A_k^m(c_1 - c_0, \cdots, c_m - c_0) W[v, c_k](X_0, t_0).
\]

Hence the fact \( A_1^m(c_0, \cdots, c_{m-1}) > 0 \) implies
\[
A[v, c_0, \cdots, c_{m-1}](X_0, t_0)
\]
\[
= A_1^m(c_0, \cdots, c_{m-1}) W[v, c_0](X_0, t_0) + \sum_{k=1}^{m-1} A_{k+1}^m(c_0, \cdots, c_{m-1}) W[v, c_k](X_0, t_0)
\]
\[
\geq A_1^m(c_0, \cdots, c_{m-1}) \sum_{k=1}^{m} A_k^m(c_1 - c_0, \cdots, c_m - c_0) W[v, c_k](X_0, t_0)
\]
\[
+ \sum_{k=1}^{m-1} A_{k+1}^m(c_0, \cdots, c_{m-1}) W[v, c_k](X_0, t_0)
\]
\[
= \sum_{k=1}^{m} A_k^m(c_1, \cdots, c_m) W[v, c_k](X_0, t_0),
\]
because $A_t^m(c_0, \ldots, c_{m-1})A_m^m(c_1 - c_0, \ldots, c_m - c_0) = A_m^m(c_1, \ldots, c_m)$ and for $k = 1, \ldots, m - 1$,

$$A_t^m(c_0, \ldots, c_{m-1})A_k^m(c_1 - c_0, \ldots, c_m - c_0) + A_{k+1}^m(c_0, \ldots, c_{m-1})$$

$$= \prod_{j=1}^{m-1} \frac{c_j}{c_j - c_0} \prod_{j=1, j \neq k}^{m} \frac{c_j - c_0}{c_j - c_k} + \frac{c_0}{c_0 - c_k} \prod_{j=1, j \neq k}^{m-1} \frac{c_j}{c_j - c_k}$$

$$= \prod_{j=1, j \neq k}^{m-1} \frac{c_j}{c_j - c_k} \left( \frac{c_k}{c_k - c_0} \frac{c_m - c_0}{c_m - c_k} + \frac{c_0}{c_0 - c_k} \right)$$

$$= A_k^m(c_1, \ldots, c_m).$$

This shows the assertion (A). The assertion (B) follows from (A) immediately.

Now we shall prove Theorem 2.

Proof of Theorem 2. Let $1 \leq d_1 < d_2 < \cdots < d_m \leq 2$ be fixed and $\rho \in C_0^\infty(0, \infty)$ satisfy $\rho \geq 0$, $\operatorname{supp}[\rho] \subset [1,2]$ and $\int_1^2 \rho(t) \, dt = 1$. For each integer $j \geq 1$, we put

$$A_j(X, t) := \sum_{k=1}^{m} \frac{4^j A_k}{d_0^j} \rho(\frac{4^j t}{d_0^j}) W(X, t).$$

Then for $t > 4^{1-j}$,

$$A_j * v(X, t) = \int A[v, 4^{-j}d_1 \tau, \ldots, 4^{-j}d_m \tau](X, t) \rho(\tau) \, d\tau.$$

Next we consider the function $Rv$ defined by

$$Rv(X, t) := \sup_{0 < c_1 < \cdots < c_m} A[v, c_1, \cdots, c_m](X, t).$$

Then Lemma 2 (B) shows

$$Rv(X, t) = \lim_{j \to \infty} A_j * v(X, t) \left( = \lim_{0 < c_1 < \cdots < c_m} A[v, c_1, \cdots, c_m](X, t) \right)$$

Since $\{A_j * u\}$ is an increasing sequence of continuous functions, $Rv$ is lower semicontinuous on $D$, so that

$$v(X, t) \geq \hat{v}(X, t) \geq Rv(X, t) \quad \text{on } D.$$
Moreover Proposition 1 gives \( v = Rv \) a.e. and \((-H)^m v \geq 0\) in the sense of distributions. These mean that \( v = \hat{v} \) a.e. and \( \hat{v} \) is poly-supertemperature of degree \( m \), which completes the proof.

**Remark 1.** In the theorem, if \( v \) is continuous, then we see \( Rv = v \) without difficulty. But unfortunately, in case that \( v \) is lower semi-continuous (that is, \( v \equiv \hat{v} \)), we do not know whether \( Rv = v \) everywhere or not.

### 3. Minimum principle

From the super-mean-value property, we obtain the following minimum principle.

**Theorem 3.** Let \( u \) be a \( C^{2m-2} \)-function on \( D \) satisfying the growth condition (1). We assume further

\[
a \leq \frac{1}{4T}.
\]

Let \( p \) be an integer with \( 1 \leq p \leq m \) and \( \{t_j\}_{j=1}^p \) be real numbers such that \( T > t_1 > \cdots > t_p > 0 \). If \( u \) is a poly-supertemperature of degree \( m \) on \( D \) and if \( u \) satisfies

\[
(-1)^{k-1} u(Y, t_k) \geq 0, \quad \forall k = 1, \cdots, p \text{ and } \forall Y \in \mathbb{R}^n,
\]

\[
(-1)^{p-1}(-H)^k u(Y, t_p) \geq 0, \quad \forall k = 1, \cdots, m-p \text{ and } \forall Y \in \mathbb{R}^n,
\]

then \( u(\Xi, \tau) \geq 0 \) for \( (\Xi, \tau) \in \mathbb{R}^n \times (t_1, T) \).

In addition, if \( u(X_0, t_0) = 0 \) for some \( (X_0, t_0) \in \mathbb{R}^n \times (t_1, T) \), then \( u = 0 \) on \( \mathbb{R}^n \times (t_p, t_0) \).

**Corollary 1.** Let \( T > t_1 > t_2 > \cdots > t_p > 0 \) and let \( u \) be a poly-temperature of degree \( m \) on \( D \) satisfying (1), (16), (17) and (18) in Theorem 3. If \( u(X_0, t_0) = 0 \) for some \( (X_0, t_0) \in \mathbb{R}^n \times (t_1, T) \), then \( u \equiv 0 \) on \( D \).

**Proof.** Let \( u \) be a poly-temperature of degree \( m \) on \( D \) satisfying (1). First we remark that \( u \) is real analytic on \( D \). In fact, for \( T > t > t_1 > \cdots > t_m > 0 \), applying the mean value property [2, Theorem 1] to the case \( c_k = t - t_k (k = 1, \cdots, m) \), we have

\[
u(X, t) = \sum_{k=1}^m \left( \prod_{j=1, j \neq k}^m \frac{t - t_j}{t_k - t_j} \right) \int_{\mathbb{R}^n} W(X - Y, t - t_k) u(Y, t_k) dY.
\]
This representation implies that \( u \) is real analytic on \( D \). Since \( u = 0 \) on \( \mathbb{R}^n \times (t_p, t_0) \) by Theorem 3, from the real analyticity it follows that \( u \equiv 0 \) on \( D \).

For the proof of Theorem 3, we prepare the following

**Lemma 3.** Let \( n \geq 1 \) be an integer, \( 0 \leq c_0 \leq c_1 < \cdots < c_n \) and \( f \) be a \( C^n \)-function on a neighborhood of \([c_0, c_n]\). Then we have an estimate

\[
\left| \sum_{k=1}^{n} A_k^n(c_1, \ldots, c_n) f(c_k) - \sum_{\ell=0}^{n-1} \frac{(-c_0)^\ell}{\ell!} f^{(\ell)}(c_0) \right| \leq \frac{c_1 \cdots c_n - c_0^n}{n!} \sup|f^{(n)}(t)|.
\]

In particular,

\[
\lim_{c_1, \ldots, c_n \to c_0} \sum_{k=1}^{n} A_k^n(c_1, \ldots, c_n) f(c_k) = \sum_{\ell=0}^{n-1} \frac{(-c_0)^\ell}{\ell!} f^{(\ell)}(c_0).
\]

**Proof.** We first remark that

\[
\sum_{k=1}^{n} A_k^n f(c_k) = \sum_{k=1}^{n} c_k^n \prod_{i<j, i \neq k} (c_j - c_i)^{-1} = \begin{vmatrix}
1 & c_1 & \cdots & c_1^{n-1} \\
& \ddots & \ddots & \ddots \\
& & 1 & c_n & \cdots & c_n^{n-1} \\
& & & \vdots & \ddots & \ddots \\
& & & & 1 & c_n & \cdots & c_n^{n-1}
\end{vmatrix}^{-1}
\]

By using the above, (8), (10) and the Taylor formula

\[
f(c_k) = \sum_{\ell=0}^{n-1} \frac{(c_k - c_0)^\ell}{\ell!} f^{(\ell)}(c_0) + \frac{1}{(n-1)!} \int_{c_0}^{c_k} (c_k - t)^{n-1} f^{(n)}(t) \, dt,
\]

we obtain

\[
\sum_{k=1}^{n} A_k f(c_k) = \sum_{q=0}^{n-1} \left( \sum_{k=1}^{n} A_k c_k^q \right) \left( \sum_{\ell=q}^{n-1} \frac{(-c_0)^{\ell-q}}{\ell!} f^{(\ell)}(c_0) \right) + \sum_{k=1}^{n} A_k \frac{1}{(n-1)!} \int_{c_0}^{c_k} (c_k - t)^{n-1} f^{(n)}(t) \, dt.
\]
Since \( \Phi_n(c_1, \ldots, c_n; t) = t^{n-1}/(n-1)! \) for \( 0 \leq t \leq c_1 \), Lemma 1 (C) gives
\[
\int_{c_0}^{c_n} \Psi_n(c_1, \ldots, c_n; t) dt = \left( \int_0^{c_n} - \int_0^{c_0} \right) \Psi_n(c_1, \ldots, c_n; t) dt = \frac{c_1 \cdots c_n - c_0^n}{n!}.
\]
This and the nonnegativity of \( \Psi_n \) lead to Lemma 3.

**Proof of Theorem 3.** Let \((\Xi, \tau)\) be fixed in \(\mathbb{R}^n \times (t_1, T)\). For sufficiently small \(c > 0\), we set
\[
 successors = \begin{cases}
  c_k := \tau - t_k & \text{for } k = 1, \ldots, p, \\
  c_{p+\ell} := c_{p+\ell}(c) := \tau - t_p + c \ell & \text{for } \ell = 1, \ldots, m - p.
\end{cases}
\]
For \(k = 1, \ldots, p - 1\), we put
\[
A_{p,k}^m := A_k^m(c_1, \ldots, c_{p-1}, c_p, \ldots, c_p).
\]
For \(k = 0, \ldots, m - p\), we put
\[
B_{p,k}^m := \sum_{\ell=0}^{m-p-k} \frac{c_{p+\ell}}{(k+\ell)!} \binom{k+\ell}{k} \left( -\frac{\partial}{\partial c_p} \right)^\ell A_p^m(c_1, \ldots, c_p).
\]
Here recall that \(A_p^p(c_1, \ldots, c_p) = \prod_{j=1}^{p-1} c_j/(c_j - c_p)\). Then by definition,
\[
(-1)^{k-1} A_{p,k}^m > 0 \quad \text{for } k = 1, \ldots, p - 1,
\]
and moreover
\[
(-1)^{p-1} \left( -\frac{\partial}{\partial c_p} \right)^k A_p^p(c_1, \ldots, c_p) > 0 \quad \text{for } \forall k \geq 0.
\]
In fact, \((-1)^{p-1} A_p^p(c_1, \ldots, c_p) > 0\) by (5), and
\[
-\frac{\partial}{\partial c_p} A_p^p(c_1, \ldots, c_p) = \left( \frac{1}{c_p - c_1} + \cdots + \frac{1}{c_p - c_{p-1}} \right) A_p^p(c_1, \ldots, c_p)
\]
shows the assertion for $k = 1$. For $k \geq 2$, the Leibniz rule gives

$$
\left(-\frac{\partial}{\partial c_p}\right)^k A_p^k(c_1, \ldots, c_p)
= \sum_{\ell=0}^{k-1} \binom{k-1}{\ell} \sum_{j=1}^{p-1} \frac{(k-1-\ell)!}{(c_p-c_j)^{k-\ell}} \left(-\frac{\partial}{\partial c_p}\right)^\ell A_p^\ell(c_1, \ldots, c_p),
$$

from which (23) follows inductively. In consequence of (23), we see easily that

$$(-1)^{p-1} B_{p,k}^m > 0 \quad \text{for } k = 0, \ldots, m-p.$$  

Now we shall show $u(\Xi, \tau) \geq 0$. It is sufficient to show that

$$\lim_{\xi \to 0} A[u, c_1, \cdots, c_p, c_{p+1}(c), \cdots, c_m(c)]$$

in the sense of distribution. In fact, since both functions $u(X, t)$ and

$$\sum_{k=1}^{p-1} A_{p,k}^m W[u, c_k](X, t) + \sum_{k=0}^{m-p} B_{p,k}^m W[(-H)^k u, c_p] (X, t)$$

are continuous on $\mathbb{R}^n \times (c_1, T)$, it follows from (2), (17), (18), (22), (24) and (25) that

$$u(\Xi, \tau) \geq \sum_{k=1}^{p-1} A_{p,k}^m W[u, c_k](\Xi, \tau) + \sum_{k=0}^{m-p} B_{p,k}^m W[(-H)^k u, c_p](\Xi, \tau)$$

$$= \sum_{k=1}^{p-1} A_{p,k}^m \int_{\mathbb{R}^n} W(\Xi - Y, \tau - t_k) u(Y, t_k) dY$$

$$+ \sum_{k=0}^{m-p} B_{p,k}^m \int_{\mathbb{R}^n} W(\Xi - Y, \tau - t_p) (-H)^k u(Y, t_p) dY$$

$$\geq 0.$$  

We thus devote ourselves to the proof of (25). Let $\phi \in C_0^\infty(\mathbb{R}^{n+1})$. Write out

$$A^*[\phi, c_1, \ldots, c_m](Y, s) := \sum_{k=0}^{m} A_{k}^m(c_1, \cdots, c_m) W^*[\phi, c_k](Y, s)$$

$$= \sum_{k=0}^{p-1} A_{k}^m(c_1, \cdots, c_m) W^*[\phi, c_k](Y, s) + \sum_{\ell=1}^{m-p+1} A_{\ell}^{m-p+1}(c_p, \ldots, c_m) f(c_{p-1+\ell}),$$
where \( f(t) := A^p(c_1, \cdots, c_{p-1}, t) W^*[\phi, t](Y, s) \). Applying Lemma 3 to this function \( f, n = m - p + 1 \) and \( c_0 = c_p \), we see that

\[
\lim_{c \to 0} A^*[\phi, c_1, \ldots, c_p, c_{p+1}(c), \ldots, c_m(c)](Y, s)
\]

\[
= \sum_{k=1}^{p-1} A_{p,k}^m W^*[\phi, c_k](Y, s) + \sum_{q=0}^{m-p} \frac{(-c_p)^q}{q!} f^{(q)}(c_p).
\]

Hence observing

\[
\sum_{q=0}^{m-p} \frac{(-c_p)^q}{q!} f^{(q)}(c_p) = \sum_{k=0}^{m-p} B_{p,k}^m \left( -\frac{\partial}{\partial c_p} \right)^k \phi(\phi, c_p)(Y, s)
\]

\[
= \sum_{k=0}^{m-p} B_{p,k}^m W^*[-H]^k \phi, c_p](Y, s),
\]

we get

\[
\lim_{c \to 0} \iint A[u, c_1, \ldots, c_p, c_{p+1}(c), \ldots, c_m(c)](Y, s) \phi(Y, s) dY ds
\]

\[
= \lim_{c \to 0} \iint u(Y, s) A^*[\phi, c_1, \ldots, c_p, c_{p+1}(c), \ldots, c_m(c)](Y, s) dY ds
\]

\[
= \iint u(Y, s) \left( \sum_{k=1}^{p-1} A_{p,k}^m W^*[\phi, c_k](Y, s) + \sum_{k=0}^{m-p} B_{p,k}^m W^*[-H]^k \phi, c_p](Y, s) \right) dY ds
\]

\[
= \iint \left( \sum_{k=1}^{p-1} A_{p,k}^m W[u, c_k](Y, s) + \sum_{k=0}^{m-p} B_{p,k}^m W([-H]^k u, c_p](Y, s) \right) \phi(Y, s) dY ds,
\]

which implies (25).

By the above argument, we know that if \( u \) is poly-temperature of degree \( m \) on \( \mathbb{R}^n \times (t_p, \tau) \), then

\[
\sum_{k=1}^{p-1} A_{p,k}^m W[u, c_k](\Xi, \tau) + \sum_{k=0}^{m-p} B_{p,k}^m W([-H]^k u, c_p](\Xi, \tau), \quad \forall \Xi \in \mathbb{R}^n.
\]

We use this fact in the proof of second assertion of Theorem 3.

Now we assume that \( u(X_0, t_0) = 0 \) for some point \((X_0, t_0) \in \mathbb{R}^n \times (t_1, T) \). Since

\[
u(X_0, t_0) \geq \sum_{k=1}^{p-1} A_{p,k}^m W[u, c_k](X_0, t_0) + \sum_{k=0}^{m-p} B_{p,k}^m W([-H]^k u, c_p](X_0, t_0) \geq 0,
\]
where \( c_k = t_0 - t_k, \) \( k = 1, \ldots, p, \) we conclude by (17), (18), (22) and (24),

\[
(27) \quad u(Y, t_k) = 0, \quad \forall k = 1, \ldots, p \text{ and } \forall Y \in \mathbb{R}^n
\]

and

\[
(28) \quad (-H)^k u(Y, t_p) = 0, \quad \forall k = 1, \ldots, m - p \text{ and } \forall Y \in \mathbb{R}^n.
\]

Moreover as in the proof of Theorem 1, putting \( \mu := (-H)^m u \) and \( \Psi_{m,p}(t) = \Psi_{m,p}(c_1, \ldots, c_p; t) := \lim_{c \to 0} \Psi_m(c_1, \ldots, c_p, c_{p+1}(c), \ldots, c_m(c); t), \) we have

\[
\begin{align*}
&\quad u(X_0, t_0) - \sum_{k=1}^{p-1} A_{p,k}^n W[u, c_k](X_0, t_0) - \sum_{k=0}^{m-p} B_{p,k}^n W[(-H)^k u, c_p](X_0, t_0) \\
&= \int_{\mathbb{R}^{n+1}} \Psi_{m,p}(c_1, \ldots, c_p; t_0 - s) W(X_0 - Y, t_0 - s) d\mu(Y, s) \geq 0.
\end{align*}
\]

Since \( \Psi_{m,p}(c_1, \ldots, c_p; t_0 - s) W(X_0 - Y, t_0 - s) > 0 \) for \( (Y, s) \in D_0 := \mathbb{R}^n \times (t_p, t_0) \) by Lemma 1(A), we also conclude that \( \mu \) vanishes there, that is, \( u \) is a poly-temperature of degree \( m \) on \( D_0. \) Thus (26), (27) and (28) give \( u(\Xi, \tau) = 0 \) for \( \Xi \in \mathbb{R}^n \) and \( \tau < \tau < t_0. \) Since \( u \) is real analytic on \( D_0 \) (see (19)), it vanishes there, as desired. This completes the proof of Theorem 3.

In particular, we see

**Corollary 2.** (A) Assume that \( u \in C^{2m-2}(\overline{D}). \) Under the conditions (1) and (16), if \( u \) is a poly-supertemperature of degree \( m \) on \( D \) and if \( (-H)^k u(X,0) \geq 0, \forall X \in \mathbb{R}^n \) for \( k = 0, \ldots, m - 1, \) then \( u \geq 0 \) on \( D. \) Moreover, discussing \( (-H)^k u \) in place of \( u, \) we also see that \( (-H)^k u \geq 0 \) on \( D \) for \( k = 1, \ldots, m - 1. \)

(B) Let \( f_k, \) \( k = 0, 1, \ldots, m - 1, \) be continuous functions on \( \mathbb{R}^n \) satisfying the growth condition \( |f_k(X)| \leq M e^{|x|^2} \) with (16). Then the boundary value problem

\[
\begin{cases}
(-H)^m h = 0 & \text{on } D \\
(-H)^k h(\cdot, 0) = f_k & \text{on } \mathbb{R}^n, \quad k = 0, 1, \ldots, m - 1
\end{cases}
\]

has a unique solution on \( D, \) which is given by

\[
(30) \quad h(X, t) = \sum_{k=0}^{m-1} \frac{t^k}{k!} \int_{\mathbb{R}^n} W(X - Y, t)f_k(Y) dY
\]

(cf. [3, p.266 or French summary p.328]).
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