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NOTE ON POLY-SUPERTEMPERATURES ON A STRIP DOMAIN

MASAHARU NISHIO, KATSUNORI SHIMOMURA AND NORIAKI SUZUKI

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0. Introduction

Let m be a positive integer and let

$$D = \{(X,t); X = (x_1, x_2, \cdots, x_n) \in \mathbf{R}^n, 0 < t < T\}$$

be a strip domain in the $(n+1)$-dimensional Euclidean space $\mathbf{R}^{n+1}$. We consider supersolutions of the m-th iterates of the heat operator

$$H = \Delta_X - \frac{\partial}{\partial t}$$

on D. A lower semi-continuous and locally integrable function u on D is called a poly-supertemperature of degree m, if $(-H)^m u \geq 0$ on D in the sense of distributions. If u and $-u$ are both poly-supertemperatures of degree m, then u is called a poly-temperature of degree m.

In our previous paper [2] (see also [1]), we have shown the following super-mean-value property for poly-supertemperatures.

Theorem A ([2, Theorem 2]). *Let u be a C^{2m-2}-function on D satisfying the growth condition*

$$|H^k u(X,t)| \leq M e^{a|X|^2}, \quad k = 0, 1, \cdots, m-1, \tag{1}$$

with some constants $M > 0$ and $a > 0$ (here $H^0 u$ means u). If u is a poly-supertemperature of degree m on D, then

$$u(X_0, t_0) \geq A[u, c_1, c_2, \cdots, c_m](X_0, t_0) \tag{2}$$

whenever $(X_0, t_0) \in D$ and $0 < c_1 < c_2 \cdots < c_m < \min\{1/4a, t_0\}$.(For notation, see (5) below.)

In the present note, we first point out that the above mean $A[u, c_1, \cdots, c_m]$ is a decreasing function of each $c_1, \cdots, c_m$ and converges to $u(X_0, t_0)$ as $c_1, \cdots, c_m$ tend to 0 under the condition $0 < c_1 < \cdots < c_m$ (Theorem 1). Secondly, in section 2, we show that the lower-regularization $\hat{v}$ of a Borel measurable function v having the super-mean-value property (2) is a poly-supertemperature (Theorem 2). In the final section, we derive a minimum principle for poly-supertemperatures, from the super-mean-value property (Theorem 3). As its corollary, we have some uniqueness results for poly-temperatures. Especially, we obtain the existence and uniqueness of poly-temperatures satisfying the boundary conditions.

1. Monotonicity of the mean

Let W denote the fundamental solution for the heat equation on $\mathbf{R}^{n+1}$, that is,

$$W(X,t) = \begin{cases} (4\pi t)^{-\frac{n}{2}} \exp(-\frac{|X|^2}{4t}) & \text{if } t > 0, \\ 0 & \text{if } t \leq 0. \end{cases}$$

We set $W^1 := W$ and $W^k := W^{k-1} * W$ for $k \geq 2$, inductively, where $*$ denotes the convolution in $\mathbf{R}^{n+1}$. Then

$$W^m(X,t) = \frac{t^{m-1}}{(m-1)!} W(X,t) \tag{3}$$

and this is the fundamental solution of the equation $(-H)^m u = 0$, that is,

$$(-H)^m (W^m * \phi) = W^m * ((-H)^m \phi) = \phi \tag{4}$$

for all $\phi \in C_0^\infty(D)$ (cf. [2, Proposition 2]).

Now we recall the definition of the mean values $A[u, c_1, c_2, \cdots, c_m]$:

$$A[u, c_1, c_2, \cdots, c_m](X_0, t_0) := \sum_{k=1}^{m} A_k W[u, c_k](X_0, t_0), \tag{5}$$

where

$$W[u, c_k](X_0, t_0) := \int_{\mathbf{R}^n} u(X - X_0, t_0 - c_k) W(X, c_k) dX$$

and the coefficients $A_k, k = 1, 2, \cdots, m$, are given by

$$A_k = A_k^m(c_1, \cdots, c_m) := \prod_{j=1, j\neq k}^{m} \frac{c_j}{c_j - c_k} \quad (A_1 = 1 \text{ when } m = 1).$$

Note that

$$(-1)^{k-1}A_k > 0, \quad k = 1, 2, \cdots, m.$$

For integers m, p with $0 \leq p \leq m$ and real numbers $c_1, \cdots, c_m$ with $0 = c_0 < c_1 < \cdots < c_m < c_{m+1} := \infty$, we consider the following functions:

$$\psi_p^m(t) = \psi_p^m(c_1, \cdots, c_m; t) := t^{m-1} - \sum_{k=1}^{p} A_k^m(c_1, \cdots, c_m)(t - c_k)^{m-1} \tag{6}$$

and

$$\Psi_m(t) = \Psi_m(c_1, \cdots, c_m; t) := \frac{1}{(m-1)!} \sum_{p=0}^{m-1} \psi_p^m(t)\chi_{(c_p, c_{p+1}]}(t), \tag{7}$$

where $\psi_0^m(t) = t^{m-1}$ and $\chi_{(c_p, c_{p+1}]}$ denotes the characteristic function of the interval $(c_p, c_{p+1}]$. We remark that the above functions were already introduced in our previous paper [2] as $\phi_p(t) = \psi_p^m(t_0 - t)$ and $\psi_m(t) = (m-1)!\Psi_m(t)$. We have already obtained the following ([2, Lemma 1], for the proof see [1, Lemma 8]): for all integers p with $0 \leq p \leq m - 1$,

$$\psi_p^m(t) = \sum_{k=p+1}^{m} A_k(t - c_k)^{m-1}, \tag{8}$$

$$\psi_p^m(t) \geq 0 \quad \text{for } c_p \leq t \leq c_{p+1}, \tag{9}$$

and

$$\psi_m^m(t) \equiv 0. \tag{10}$$

The function Ψ_m has the following properties.

Lemma 1. (A) *$\Psi_m(c_1, \cdots, c_m; t)$ is a continuous (for $m \geq 2$) and nonnegative function of $t \geq 0$. Moreover $\Psi_m(c_1, \cdots, c_m; t) > 0$ if $0 < t < c_m$.*
(B) *$\Psi_m(c_1, \cdots, c_m; t)$ is an increasing function of each variable c_j, $j = 1, 2, \cdots, m$, and*
$\lim_{0 < c_1 < \cdots < c_m \to 0} \Psi_m(c_1, \cdots, c_m; t) = 0$.

(C) $$\int_0^{c_m} \Psi_m(c_1, \cdots, c_m; t)\, dt = \frac{c_1 \cdots c_m}{m!}.$$

Proof. (A) The continuity of Ψ_m follows from the facts $\psi_p^m(c_p) = \psi_{p-1}^m(c_p)$, $p = 1, 2, \cdots, m$. Inequalities (9) show the nonnegativity of Ψ_m. The positivity of Ψ_m is obtained immediately in the case of $c_{m-1} < t < c_m$ because

$$\Psi_m(c_1, \ldots, c_m; t) = \frac{1}{(m-1)!}(-1)^{m-1} A_m (c_m - t)^{m-1}$$

and $(-1)^{m-1} A_m > 0$. As will be seen in the below, the proof of (B) is independent of the positivity of Ψ_m. Therefore the general case follows from the case of $c_{m-1} < t < c_m$ because of (B).

(B) For the proof, we use the following fact: Let $m \geq 2$. We define

$$\rho_p^m(t) = \rho_p^m(c_1, \cdots, c_m; t) := \sum_{k=1}^{p} c_k A_k^m(c_1, \cdots, c_m)(t - c_k)^{m-2}$$

where $\rho_0^m := 0$. Then we have

$$\rho_p^m(c_1, \cdots, c_m; t) \geq 0 \quad \text{for} \quad c_p \leq t \leq c_{p+1} \tag{11}$$

and

$$\rho_m^m(c_1, \cdots, c_m; t) \equiv 0. \tag{12}$$

This can be proved by the quite same manner as in [1, Lemma 8], so we omit the proof.

Now we consider the first part of (B). Though the method of the proof is also similar to that of [1, lemma 8], we give the proof, because it is a little more complicated.

In the case $m = 1$, assertion (B) is clear, because $\Psi_1(c_1; t) = \chi_{(0,c_1]}(t)$. Since for $m \geq 2$, $\Psi_m(c_1, \cdots, c_m; t)$ is a continuous function of t, it is sufficient to show that for $p = 0, 1, \cdots, m$ and $j = 1, \cdots, m$,

$$\frac{\partial \psi_p^m}{\partial c_j}(c_1, \cdots, c_m; t) \geq 0 \quad \text{if } c_p \leq t \leq c_{p+1}. \tag{13}$$

In the sequel, for $m \geq 1$, $0 \leq p \leq m$ and $1 \leq j \leq m$, we say that the assertion (m, p, j) holds if we have (13) for all real numbers $0 < c_1 < c_2 < \cdots < c_m$. We shall prove the assertions (m, p, j) for all m, p, j by the induction on m, and at each step we consider the induction with respect to p. First remark that assertions $(m, 0, j)$ and (m, m, j) hold for all m and j, because $\partial\psi_0^m/\partial c_j = \partial\psi_m^m/\partial c_j = 0$. In particular, the assertions $(1, 0, 1)$ and $(1, 1, 1)$ hold, and hence the step $m = 1$ is obtained. Let $m \geq 2$ and assume that the assertions at the step $m - 1$ is valid. Since

$$\frac{\partial}{\partial c_j} A_k^m(c_1, \cdots, c_m) = \frac{-c_k}{c_j(c_j - c_k)} A_k^m(c_1, \cdots, c_m)$$

for $k \neq j$, it follows from (6) and (8) that

$$\begin{aligned}&\frac{\partial \psi_p^m}{\partial c_j}(c_1, \cdots, c_m; t)\\ = &\begin{cases} \displaystyle\sum_{k=1}^{p} \frac{c_k}{c_j(c_j - c_k)} A_k^m(c_1, \cdots, c_m)(t - c_k)^{m-1} & \text{for } p = 0, \cdots, j-1, \\ \displaystyle\sum_{k=p+1}^{m} \frac{-c_k}{c_j(c_j - c_k)} A_k^m(c_1, \cdots, c_m)(t - c_k)^{m-1} & \text{for } p = j, \cdots, m. \end{cases}\end{aligned}$$

Now let j be fixed. First we deal with the case of $0 \leq p \leq j-1$. The assertion $(m, 0, j)$ has been obtained in the above. Assume $1 \leq p \leq j-1$. We shall show that the assertion (m, p, j) follows from induction assumptions $(m, p-1, j), (m-1, p, j-1)$ and $(m-1, p-1, j-1)$. Assume that the function $f(t) := (\partial \psi_p^m / \partial c_j)(c_1, \cdots, c_m; t)$ attains its minimum on $[c_p, c_{p+1}]$ at τ_0. It is sufficient to show $f(\tau_0) \geq 0$. If $\tau_0 = c_p$, then

$$\begin{aligned} f(\tau_0) &= \frac{\partial \psi_p^m}{\partial c_j}(c_1, \cdots, c_m; c_p) \\ &= \sum_{k=1}^{p} \frac{c_k}{c_j(c_j - c_k)} A_k^m(c_1, \cdots, c_m)(c_p - c_k)^{m-1} \\ &= \frac{\partial \psi_{p-1}^m}{\partial c_j}(c_1, \cdots, c_m; c_p) \geq 0 \end{aligned}$$

by the assumption $(m, p-1, j)$. Next, if $\tau_0 = c_{p+1}$, then

$$\begin{aligned} f(\tau_0) &= \frac{\partial \psi_p^m}{\partial c_j}(c_1, \cdots, c_m; c_{p+1}) \\ &= \sum_{k=1}^{p} \frac{c_k}{c_j(c_j - c_k)} A_k^m(c_1, \cdots, c_m)(c_{p+1} - c_k)(c_{p+1} - c_k)^{m-2} \\ &= \sum_{k=1}^{p} \frac{c_k \cdot c_{p+1}}{c_j(c_j - c_k)} A_k^{m-1}(c_1, \cdots, \check{c}_{p+1}, \cdots, c_m)(c_{p+1} - c_k)^{m-2} \\ &= \begin{cases} c_{p+1} \dfrac{\partial \psi_p^{m-1}}{\partial c_{j-1}}(c_1, \cdots, \check{c}_{p+1}, \cdots, c_m; c_{p+1}) & \text{if } p < j-1 \\ c_{p+1}^{-1} \rho_p^m(c_1, \cdots, c_m; c_{p+1}) & \text{if } p = j-1 \end{cases} \\ &\geq 0 \end{aligned}$$

by the assumption $(m-1, p, j-1)$ and (11); here by $\check{c}_{p+1}$ we indicate that the factor

c_{p+1} is missing. Finally, if $\tau_0 \in (c_p, c_{p+1})$, then $f'(\tau_0) = 0$, that is,

$$\frac{\partial^2 \psi_p^m}{\partial t \partial c_j}(c_1, \cdots, c_m; \tau_0) = (m-1) \sum_{k=1}^{p} \frac{c_k}{c_j(c_j - c_k)} A_k^m(c_1, \cdots, c_m)(\tau_0 - c_k)^{m-2} = 0.$$

Hence we have

$$\begin{aligned}
f(\tau_0) &= \frac{\partial \psi_p^m}{\partial c_j}(c_1, \cdots, c_m; \tau_0) \\
&= \sum_{k=1}^{p-1} \frac{c_k}{c_j(c_j - c_k)} A_k^m(c_1, \cdots, c_m)(\tau_0 - c_k)^{m-2}(\tau_0 - c_k) \\
&\qquad + \frac{c_p}{c_j(c_j - c_p)} A_p^m(c_1, \cdots, c_m)(\tau_0 - c_p)^{m-2}(\tau_0 - c_p) \\
&= \sum_{k=1}^{p-1} \frac{c_k}{c_j(c_j - c_k)} A_k^m(c_1, \cdots, c_m)(\tau_0 - c_k)^{m-2}(c_p - c_k) \\
&= \sum_{k=1}^{p-1} \frac{c_k \cdot c_p}{c_j(c_j - c_k)} A_k^{m-1}(c_1, \cdots, \check{c}_p, \cdots, c_m)(\tau_0 - c_k)^{m-2} \\
&= c_p \frac{\partial \psi_{p-1}^{m-1}}{\partial c_{j-1}}(c_1, \cdots, \check{c}_p, \cdots, c_m; \tau_0) \geq 0
\end{aligned}$$

by the assumption $(m-1, p-1, j-1)$. Therefore we have the assertions (m, p, j) for $0 \leq p \leq j-1$ by the induction with respect to p. Next we deal with the case of $j \leq p \leq m$. In this case, likewise remarking that for $j \leq p \leq m-1$,

$$\begin{aligned}
\frac{\partial \psi_p^m}{\partial c_j}(c_1, \cdots, c_m; c_{p+1}) &= \frac{\partial \psi_{p+1}^m}{\partial c_j}(c_1, \cdots, c_m; c_{p+1}), \\
\frac{\partial \psi_p^m}{\partial c_j}(c_1, \cdots, c_m; c_p) &= \begin{cases} c_p \dfrac{\partial \psi_{p-1}^{m-1}}{\partial c_j}(c_1, \cdots, \check{c}_p, \cdots, c_m; c_p) & \text{if } p > j \\ c_p^{-1} \rho_p^m(c_1, \cdots, c_m; c_p) & \text{if } p = j \end{cases}
\end{aligned}$$

and that if

$$\frac{\partial^2 \psi_p^m}{\partial t \partial c_j}(c_1, \cdots, c_m; \tau_0) = (m-1) \sum_{k=p+1}^{m} \frac{-c_k}{c_j(c_j - c_k)} A_k^m(c_1, \cdots, c_m)(\tau_0 - c_k)^{m-2} = 0,$$

then

$$\frac{\partial \psi_p^m}{\partial c_j}(c_1, \cdots, c_m; \tau_0) = c_{p+1} \frac{\partial \psi_p^{m-1}}{\partial c_j}(c_1, \cdots, \check{c}_{p+1}, \cdots, c_m; \tau_0),$$

we can obtain the assertion (m,p,j) from induction assumptions $(m,p+1,j)$, $(m-1,p-1,j)$ and $(m-1,p,j)$. Here we note that the induction on p goes downward from (m,m,j).

In the end, we have the assertion (m,p,j) for all $m \geq 1, 0 \leq p \leq m, 1 \leq j \leq m$. Clearly $\lim_{0<c_1<\cdots<c_m\to 0} \Psi_m(t) = 0$ for $t > 0$, and thus we achieve the proof of (B).

(C) By a direct calculation, we have

$$\begin{aligned}
&\int_0^{c_m} \Psi_m(c_1, \cdots, c_m; t)\, dt \\
&= \frac{1}{(m-1)!}\Big(\int_0^{c_m} t^{m-1}\, dt - \sum_{p=1}^{m-1} \int_{c_p}^{c_{p+1}} \sum_{k=1}^{p} A_k^m(c_1, \cdots, c_m)(t-c_k)^{m-1}\, dt\Big) \\
&= \frac{1}{(m-1)!}\Big(\frac{c_m^m}{m} - \sum_{k=1}^{m-1} A_k^m(c_1, \cdots, c_m) \sum_{p=k}^{m-1} \int_{c_p}^{c_{p+1}} (t-c_k)^{m-1}\, dt\Big) \\
&= \frac{1}{m!}\Big(c_m^m - \sum_{k=1}^{m-1} A_k^m(c_1, \cdots, c_m)(c_m - c_k)^m\Big).
\end{aligned}$$

Since for $1 \leq k \leq m-1$, $A_k^m(c_1, \cdots, c_m)(c_m - c_k) = c_m\, A_k^{m-1}(c_1, \cdots, c_{m-1})$, and since $\psi_{m-1}^{m-1}(c_1, \cdots, c_{m-1}; t) \equiv 0$ by (10), we have

$$\begin{aligned}
\int_0^{c_m} \Psi_m(c_1, \cdots, c_m; t)\, dt &= \frac{c_m}{m!}\Big(c_m^{m-1} - \sum_{k=1}^{m-1} A_k^{m-1}(c_1, \cdots, c_{m-1})(c_m - c_k)^{m-1}\Big) \\
&= \frac{c_m}{m!}\Big((m-1)\int_{c_{m-1}}^{c_m} \psi_{m-1}^{m-1}(c_1, \cdots, c_{m-1}; t)\, dt \\
&\quad + c_{m-1}^{m-1} - \sum_{k=1}^{m-1} A_k^{m-1}(c_1, \cdots, c_{m-1})(c_{m-1} - c_k)^{m-1}\Big) \\
&= \frac{c_m}{m}\int_0^{c_{m-1}} \Psi_{m-1}(c_1, \cdots, c_{m-1}; t)\, dt,
\end{aligned}$$

which shows (C), because $\int_0^{c_1} \Psi_1(c_1; t)\, dt = c_1$. This completes the proof of Lemma 1.

Theorem 1. *Let u be the same as in Theorem A and let $(X_0, t_0) \in D$. Suppose that u is a poly-supertemperature of degree m on D and $0 < c_1 < \cdots < c_m < \min\{1/4a, t_0\}$. Then the mean value $A[u, c_1, c_2, \cdots, c_m](X_0, t_0)$ is a decreasing function of each c_j $(1 \leq j \leq m)$ and converges to $u(X_0, t_0)$ as c_m tends to 0.*

Proof. Put $\mu := (-H)^m u$. Then by [2, Theorems 1 and 2] and their proofs, we have

$$\begin{aligned} & u(X_0,t_0) - A[u,c_1,c_2,\cdots,c_m](X_0,t_0) \\ = \; & W^m * \mu(X_0,t_0) - A[W^m * \mu, c_1, c_2, \cdots, c_m](X_0,t_0) \\ = \; & \iint_{\mathbf{R}^{n+1}} \Psi_m(c_1,\cdots,c_m;t_0-t)W(X_0-X,t_0-t)\,d\mu(X,t). \end{aligned}$$

Hence Theorem 1 follows from Lemma 1 (B).

2. Lower-regularization

For a Borel measurable function v on D, its lower-regularization $\hat{v}$ is defined by

$$\hat{v}(X,t) := \min\Big\{\liminf_{(Y,s)\to(X,t)} v(Y,s), v(X,t)\Big\}.$$

Remark that $\hat{v}$ is lower semi-continuous on D. Our result is the following

Theorem 2. *Let v be a Borel measurable function on D satisfying the growth condition*

$$|v(X,t)| \le Me^{a|X|^2}, \quad \forall (X,t) \in D \tag{14}$$

with some constants $M > 0$ and $a > 0$. Suppose that v has the super-mean-value property, that is,

$$v(X,t) \ge A[v,c_1,\cdots,c_m](X,t) \tag{15}$$

for all $(X,t) \in D$ and $0 < c_1 < \cdots < c_m < \min\{1/4a, t\}$. Then $\hat{v}$ is a poly-supertemperature of degree m and is equal to v a.e. on D.

We make some preparations for the proof of Theorem 2. The following assertion was noted in [2, Theorem 4] without proof. It can be shown by the similar manner to [1, Lemma 6], but we here give the proof for the sake of completeness.

Proposition 1. *Let v be a Borel measurable function on D satisfying the growth condition* (14). *Then*

$$\lim_{\substack{0<c_1<\cdots<c_m \\ c_m\to 0}} \frac{m!}{c_1\cdots c_m}\Big(v - A[v,c_1,c_2,\cdots,c_m]\Big) = (-H)^m v$$

in the sense of distributions.

Proof. Let $\phi \in C_0^\infty(D)$ be fixed. Then for sufficiently small $c_m > 0$, we have

$$\begin{aligned} & \iint_D \{v(X,t) - A[v, c_1, \cdots, c_m](X,t)\}\phi(X,t)\, dX\, dt \\ = & \iint_D v(X,t)\{\phi(X,t) - A^*[\phi, c_1, \cdots, c_m](X,t)\}\, dX\, dt, \end{aligned}$$

where

$$A^*[\phi, c_1, \cdots, c_m](X,t) := \sum_{k=1}^{m} A_k W^*[\phi, c_k](X,t)$$

and

$$W^*[\phi, c](X,t) = \int_{\mathbf{R}^n} W(Y,c)\phi(X-Y, t+c)\, dY.$$

Put $\psi(X,t) = \phi(X, T-t)$. Then $\psi \in C_0^\infty(D)$ and hence $\psi = W^n * ((-H)^m\psi)$ by (4). Since $A^*[\phi, c_1, \cdots, c_m](X,t) = A[\psi, c_1, \cdots, c_m](X, T-t)$, an argument in [2, Proof of Theorem 2] gives

$$\begin{aligned} & \phi(X,t) - A^*[\phi, c_1, \cdots, c_m](X,t) \\ = \; & \psi(X, T-t) - A[\psi, c_1, \cdots, c_m](X, T-t) \\ = \; & W^m * ((-H)^m\psi)(X, T-t) - A[W^m * ((-H)^m\psi), c_1, \cdots, c_m](X, T-t) \\ = \; & \int_{T-t-c_m}^{T-t} \Big(\Psi_m(c_1, \cdots, c_m; T-t-s) \\ & \times \int_{\mathbf{R}^n} W(X-Y, T-t-s)((-H)^m\psi)(Y,s)\, dY\Big)\, ds \\ = \; & \int_{T-t-c_m}^{T-t} \Big(\Psi_m(c_1, \cdots, c_m; T-t-s) \\ & \times \int_{\mathbf{R}^n} W(X-Y, T-t-s)((-H^*)^m\phi)(Y, T-s)\, dY\Big)\, ds \\ = \; & \int_0^{c_m} \Psi_m(c_1, \cdots, c_m; \tau) \int_{\mathbf{R}^n} W(X-Y, \tau)((-H^*)^m\phi)(Y, t+\tau)\, dY\, d\tau, \end{aligned}$$

where $H^* = \Delta_X + \partial/\partial t$ is the adjoint operator of H. Remarking the growth condition (14), Lemma 1 (C) and

$$\lim_{\tau \to 0} \int_{\mathbf{R}^n} W(X-Y, \tau)((-H^*)^m\phi)(Y, t+\tau)\, dY = (-H^*)^m\phi(X,t),$$

we obtain

$$\begin{aligned}&\lim_{\substack{0<c_1<\cdots<c_m\\ c_m\to 0}} \frac{m!}{c_1\cdots c_m}\iint_D (v(X,t)-A[v,c_1,c_2,\cdots,c_m](X,t))\phi(X,t)\,dX\,dt\\ =\ &\iint_D v(X,t)((-H^*)^m\phi)(X,t)\,dX\,dt\end{aligned}$$

by the Lebesgue dominated convergence theorem. This completes the proof.

The following lemma is the key in our argument.

Lemma 2. *Let v be a Borel measurable function on D satisfying* (14) *and* (15). (A) *If $(X_0,t_0)\in D$ and $0<c_0<c_1<\cdots<c_m<\min\{1/4a,t_0\}$, then*

$$A[v,c_0,\cdots,c_{m-1}](X_0,t_0)\geq A[v,c_1,\cdots,c_m](X_0,t_0).$$

(B) *If $(X_0,t_0)\in D$ and $0<d_1<\cdots<d_m<c_1<\cdots<c_m<\min\{1/4a,t_0\}$, then*

$$A[v,d_1,\cdots,d_m](X_0,t_0)\geq A[v,c_1,\cdots,c_m](X_0,t_0).$$

Proof. Before giving the proof, we remark that Theorem 1 is not applicable to this case directly, because we do not assume the condition (1) for v.

Integrating both sides of $v(Y,t_0-c_0)\geq A[v,c_1-c_0,\cdots,c_m-c_0](Y,t_0-c_0)$ with respect to $W(X_0-Y,c_0)dY$, we have

$$W[v,c_0](X_0,t_0)\geq\sum_{k=1}^m A_k^m(c_1-c_0,\cdots,c_m-c_0)W[v,c_k](X_0,t_0).$$

Hence the fact $A_1^m(c_0,\cdots,c_{m-1})>0$ implies

$$\begin{aligned}&A[v,c_0,\cdots,c_{m-1}](X_0,t_0)\\ =\ &A_1^m(c_0,\cdots,c_{m-1})W[v,c_0](X_0,t_0)+\sum_{k=1}^{m-1}A_{k+1}^m(c_0,\cdots,c_{m-1})W[v,c_k](X_0,t_0)\\ \geq\ &A_1^m(c_0,\cdots,c_{m-1})\sum_{k=1}^m A_k^m(c_1-c_0,\cdots,c_m-c_0)W[v,c_k](X_0,t_0)\\ &+\sum_{k=1}^{m-1}A_{k+1}^m(c_0,\cdots,c_{m-1})W[v,c_k](X_0,t_0)\\ =\ &\sum_{k=1}^m A_k^m(c_1,\cdots,c_m)W[v,c_k](X_0,t_0),\end{aligned}$$

because $A_1^m(c_0,\cdots,c_{m-1})A_m^m(c_1-c_0,\cdots,c_m-c_0)=A_m^m(c_1,\cdots,c_m)$ and for $k=1,\cdots,m-1$,

$$\begin{aligned}
&A_1^m(c_0,\cdots,c_{m-1})A_k^m(c_1-c_0,\cdots,c_m-c_0)+A_{k+1}^m(c_0,\cdots,c_{m-1})\\
=\ &\prod_{j=1}^{m-1}\frac{c_j}{c_j-c_0}\cdot\prod_{j=1,j\neq k}^{m}\frac{c_j-c_0}{c_j-c_k}+\frac{c_0}{c_0-c_k}\cdot\prod_{j=1,j\neq k}^{m-1}\frac{c_j}{c_j-c_k}\\
=\ &\prod_{j=1,j\neq k}^{m-1}\frac{c_j}{c_j-c_k}\left(\frac{c_k}{c_k-c_0}\cdot\frac{c_m-c_0}{c_m-c_k}+\frac{c_0}{c_0-c_k}\right)\\
=\ &A_k^m(c_1,\cdots,c_m).
\end{aligned}$$

This shows the assertion (A). The assertion (B) follows from (A) immediately.

Now we shall prove Theorem 2.

Proof of Theorem 2. Let $1\le d_1<d_2<\cdots<d_m\le 2$ be fixed and $\rho\in C_0^\infty(0,\infty)$ satisfy $\rho\ge 0$, $\mathrm{supp}[\rho]\subset[1,2]$ and $\int_1^2\rho(t)\,dt=1$. For each integer $j\ge 1$, we put

$$\mathcal{A}_j(X,t):=\sum_{k=1}^{m}\frac{4^jA_k}{d_k}\rho\left(\frac{4^jt}{d_k}\right)W(X,t).$$

Then for $t>4^{1-j}$,

$$\mathcal{A}_j*v(X,t)=\int A[v,4^{-j}d_1\tau,\cdots,4^{-j}d_m\tau](X,t)\rho(\tau)\,d\tau.$$

Next we consider the function Rv defined by

$$Rv(X,t):=\sup_{0<c_1<\cdots<c_m}A[v,c_1,\cdots,c_m](X,t).$$

Then Lemma 2 (B) shows

$$Rv(X,t)=\lim_{j\to\infty}\mathcal{A}_j*v(X,t)\ \Big(=\lim_{\substack{0<c_1<\cdots<c_m\\ c_m\to 0}}A[v,c_1,\cdots,c_m](X,t)\Big)$$

Since $\{\mathcal{A}_j*u\}$ is an increasing sequence of continuous functions, Rv is lower semicontinuous on D, so that

$$v(X,t)\ge\hat{v}(X,t)\ge Rv(X,t)\quad\text{on } D.$$

Moreover Proposition 1 gives $v = Rv$ a.e. and $(-H)^m v \geq 0$ in the sense of distributions. These mean that $v = \hat{v}$ a.e. and $\hat{v}$ is poly-supertemperature of degree m, which completes the proof.

REMARK 1. In the theorem, if v is continuous, then we see $Rv \equiv v$ without difficulty. But unfortunately, in case that v is lower semi-continuous (that is, $v \equiv \hat{v}$), we do not know whether $Rv = v$ everywhere or not.

3. Minimum principle

From the super-mean-value property, we obtain the following minimum principle.

Theorem 3. *Let u be a C^{2m-2}-function on D satisfying the growth condition* (1). *We assume further*

$$a \leq \frac{1}{4T}. \tag{16}$$

Let p be an integer with $1 \leq p \leq m$ and $\{t_j\}_{j=1}^p$ be real numbers such that $T > t_1 > \cdots > t_p > 0$. If u is a poly-supertemperature of degree m on D and if u satisfies

$$(-1)^{k-1} u(Y, t_k) \geq 0, \quad \forall k = 1, \cdots, p \text{ and } \forall Y \in \boldsymbol{R}^n, \tag{17}$$

$$(-1)^{p-1} (-H)^k u(Y, t_p) \geq 0, \quad \forall k = 1, \cdots, m-p \text{ and } \forall Y \in \boldsymbol{R}^n, \tag{18}$$

then $u(\Xi, \tau) \geq 0$ for $(\Xi, \tau) \in \boldsymbol{R}^n \times (t_1, T)$.

In addition, if $u(X_0, t_0) = 0$ for some $(X_0, t_0) \in \boldsymbol{R}^n \times (t_1, T)$, then $u = 0$ on $\boldsymbol{R}^n \times (t_p, t_0)$.

Corollary 1. *Let $T > t_1 > t_2 > \cdots > t_p > 0$ and let u be a poly-temperature of degree m on D satisfying* (1), (16), (17) *and* (18) *in* Theorem 3. *If $u(X_0, t_0) = 0$ for some $(X_0, t_0) \in \boldsymbol{R}^n \times (t_1, T)$, then $u \equiv 0$ on D.*

Proof. Let u be a poly-temperature of degree m on D satisfying (1). First we remark that u is real analytic on D. In fact, for $T > t > t_1 > \cdots > t_m > 0$, applying the mean value property [2, Theorem 1] to the case $c_k = t - t_k (k = 1, \cdots, m)$, we have

$$u(X, t) = \sum_{k=1}^m \left(\prod_{j=1, j \neq k}^m \frac{t - t_j}{t_k - t_j} \right) \int_{\mathbf{R}^n} W(X - Y, t - t_k) u(Y, t_k) \, dY. \tag{19}$$

This representation implies that u is real analytic on D. Since $u = 0$ on $\mathbf{R}^n \times (t_p, t_0)$ by Theorem 3, from the real analyticity it follows that $u \equiv 0$ on D.

For the proof of Theorem 3, we prepare the following

Lemma 3. *Let $n \geq 1$ be an integer, $0 \leq c_0 \leq c_1 < \cdots < c_n$ and f be a C^n–function on a neighborhood of $[c_0, c_n]$. Then we have an estimate*

$$\left| \sum_{k=1}^{n} A_k^n(c_1, \ldots, c_n) f(c_k) - \sum_{l=0}^{n-1} \frac{(-c_0)^\ell}{\ell!} f^{(\ell)}(c_0) \right| \leq \frac{c_1 \cdots c_n - c_0^n}{n!} \sup_{c_0 \leq t \leq c_n} |f^{(n)}(t)|.$$

In particular,

$$\lim_{c_1, \cdots, c_n \to c_0} \sum_{k=1}^{n} A_k^n(c_1, \cdots, c_n) f(c_k) = \sum_{l=0}^{n-1} \frac{(-c_0)^\ell}{\ell!} f^{(\ell)}(c_0).$$

Proof. We first remark that

$$\begin{aligned}
\sum_{k=1}^{n} A_k c_k^q &= \sum_{k=1}^{n} c_k^q \frac{(-1)^{k-1} c_1 \cdots \check{c}_k \cdots c_n \prod_{i<j, i,j \neq k} (c_j - c_i)}{\prod_{i<j} (c_j - c_i)} \\
&= \begin{vmatrix} 1 & c_1 & \cdots & c_1^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & c_n & \cdots & c_n^{n-1} \end{vmatrix}^{-1} \begin{vmatrix} c_1^q & c_1 & \cdots & c_1^{n-1} \\ \vdots & \vdots & & \vdots \\ c_n^q & c_n & \cdots & c_n^{n-1} \end{vmatrix} \\
&= \begin{cases} 1, & q = 0, \\ 0, & q = 1, \ldots, n-1. \end{cases}
\end{aligned}$$

By using the above, (8), (10) and the Taylor formula

$$f(c_k) = \sum_{\ell=0}^{n-1} \frac{(c_k - c_0)^\ell}{\ell!} f^{(\ell)}(c_0) + \frac{1}{(n-1)!} \int_{c_0}^{c_k} (c_k - t)^{n-1} f^{(n)}(t)\, dt,$$

we obtain

$$\begin{aligned}
\sum_{k=1}^{n} A_k f(c_k) &= \sum_{q=0}^{n-1} \left(\sum_{k=1}^{n} A_k c_k^q \right) \left(\sum_{\ell=q}^{n-1} \binom{\ell}{q} \frac{(-c_0)^{\ell-q}}{\ell!} f^{(\ell)}(c_0) \right) \\
&\quad + \sum_{k=1}^{n} A_k \frac{1}{(n-1)!} \int_{c_0}^{c_k} (c_k - t)^{n-1} f^{(n)}(t)\, dt
\end{aligned}$$

$$= \sum_{\ell=0}^{n-1} \begin{pmatrix} \ell \\ 0 \end{pmatrix} \frac{(-c_0)^\ell}{\ell!} f^{(\ell)}(c_0)$$
$$+ \frac{(-1)^{n-1}}{(n-1)!} \left(\sum_{p=0}^{n-1} \sum_{k=p+1}^{n} \int_{c_p}^{c_{p+1}} A_k (t-c_k)^{n-1} f^{(n)}(t)\, dt \right)$$
$$= \sum_{\ell=0}^{n-1} \frac{(-c_0)^\ell}{\ell!} f^{(\ell)}(c_0) + (-1)^{n-1} \int_{c_0}^{c_n} \Psi_n(c_1, \dots, c_n; t) f^{(n)}(t)\, dt.$$

Since $\Psi_n(c_1, \dots, c_n; t) = t^{n-1}/(n-1)!$ for $0 \le t \le c_1$, Lemma 1 (C) gives

$$\int_{c_0}^{c_n} \Psi_n(c_1, \dots, c_n; t)\, dt = \left(\int_0^{c_n} - \int_0^{c_0} \right) \Psi_n(c_1, \dots, c_n; t)\, dt = \frac{c_1 \cdots c_n - c_0^n}{n!}.$$

This and the nonnegativity of Ψ_n lead to Lemma 3.

Proof of Theorem 3. Let (Ξ, τ) be fixed in $\mathbf{R}^n \times (t_1, T)$. For sufficiently small $c > 0$, set

$$\begin{cases} c_k := \tau - t_k & \text{for } k = 1, \cdots, p, \\ c_{p+\ell} = c_{p+\ell}(c) := \tau - t_p + c\,\ell & \text{for } \ell = 1, \cdots, m-p. \end{cases}$$

For $k = 1, \cdots, p-1$, we put

$$A_{p,k}^m := A_k^m(c_1, \dots, c_{p-1}, c_p, \dots, c_p). \tag{20}$$

For $k = 0, \cdots, m-p$, we put

$$B_{p,k}^m := \sum_{\ell=0}^{m-p-k} \frac{c_p^{k+\ell}}{(k+\ell)!} \begin{pmatrix} k+\ell \\ k \end{pmatrix} \left(-\frac{\partial}{\partial c_p} \right)^\ell A_p^p(c_1, \dots, c_p). \tag{21}$$

Here recall that $A_p^p(c_1, \cdots, c_p) = \prod_{j=1}^{p-1} c_j/(c_j - c_p)$. Then by definition,

$$(-1)^{k-1} A_{p,k}^m > 0 \quad \text{for } k = 1, \cdots, p-1, \tag{22}$$

and moreover

$$(-1)^{p-1} \left(-\frac{\partial}{\partial c_p} \right)^k A_p^p(c_1, \cdots, c_p) > 0 \quad \text{for } \forall k \ge 0. \tag{23}$$

In fact, $(-1)^{p-1} A_p^p(c_1, \cdots, c_p) > 0$ by (5), and

$$-\frac{\partial}{\partial c_p} A_p^p(c_1, \cdots, c_p) = \left(\frac{1}{c_p - c_1} + \cdots + \frac{1}{c_p - c_{p-1}} \right) A_p^p(c_1, \cdots, c_p)$$

shows the assertion for $k=1$. For $k \geq 2$, the Leibniz rule gives

$$\begin{aligned}\left(-\frac{\partial}{\partial c_p}\right)^k & A_p^p(c_1,\cdots,c_p) \\ &= \sum_{\ell=0}^{k-1}\binom{k-1}{\ell}\sum_{j=1}^{p-1}\frac{(k-1-\ell)!}{(c_p-c_j)^{k-\ell}}\left(-\frac{\partial}{\partial c_p}\right)^{\ell} A_p^p(c_1,\cdots,c_p),\end{aligned}$$

from which (23) follows inductively. In consequence of (23), we see easily that

$$(24) \qquad (-1)^{p-1}B^m_{p,k} > 0 \quad \text{for } k=0,\cdots,m-p.$$

Now we shall show $u(\Xi,\tau) \geq 0$. It is sufficient to show that

$$(25) \qquad \begin{aligned} &\lim_{c\downarrow 0} A[u,c_1,\cdots,c_p,c_{p+1}(c),\cdots,c_m(c)] \\ &= \sum_{k=1}^{p-1} A^m_{p,k} W[u,c_k] + \sum_{k=0}^{m-p} B^m_{p,k} W[(-H)^k u, c_p]\end{aligned}$$

in the sense of distribution. In fact, since both functions $u(X,t)$ and

$$\sum_{k=1}^{p-1} A^m_{p,k} W[u,c_k](X,t) + \sum_{k=0}^{m-p} B^m_{p,k} W[(-H)^k u, c_p](X,t)$$

are continuous on $\mathbf{R}^n \times (c_1,T)$, it follows from (2), (17), (18), (22), (24) and (25) that

$$\begin{aligned} u(\Xi,\tau) &\geq \sum_{k=1}^{p-1} A^m_{p,k} W[u,c_k](\Xi,\tau) + \sum_{k=0}^{m-p} B^m_{p,k} W[(-H)^k u, c_p](\Xi,\tau) \\ &= \sum_{k=1}^{p-1} A^m_{p,k} \int_{\mathbf{R}^n} W(\Xi-Y,\tau-t_k)\ u(Y,t_k)\,dY \\ &\quad + \sum_{k=0}^{m-p} B^m_{p,k} \int_{\mathbf{R}^n} W(\Xi-Y,\tau-t_p)\ (-H)^k u(Y,t_p)\,dY \\ &\geq 0.\end{aligned}$$

We thus devote ourselves to the proof of (25). Let $\phi \in C_0^\infty(\mathbf{R}^{n+1})$. Write out

$$\begin{aligned} A^*[\phi,c_1,\ldots,c_m](Y,s) &:= \sum_{k=0}^{m} A^m_k(c_1,\cdots,c_m) W^*[\phi,c_k](Y,s) \\ = \sum_{k=0}^{p-1} A^m_k(c_1,\ldots,c_m) W^*[\phi,c_k](Y,s) &+ \sum_{\ell=1}^{m-p+1} A^{m-p+1}_{\ell}(c_p,\ldots,c_m) f(c_{p-1+\ell}),\end{aligned}$$

where $f(t) := A_p^p(c_1, \cdots, c_{p-1}, t)W^*[\phi, t](Y, s)$. Applying Lemma 3 to this function f, $n = m - p + 1$ and $c_0 = c_p$, we see that

$$\begin{aligned}
&\lim_{c\downarrow 0} A^*[\phi, c_1, \ldots, c_p, c_{p+1}(c), \cdots, c_m(c)](Y, s) \\
&= \sum_{k=1}^{p-1} A_{p,k}^m W^*[\phi, c_k](Y, s) + \sum_{q=0}^{m-p} \frac{(-c_p)^q}{q!} f^{(q)}(c_p).
\end{aligned}$$

Hence observing

$$\begin{aligned}
\sum_{q=0}^{m-p} \frac{(-c_p)^q}{q!} f^{(q)}(c_p) &= \sum_{k=0}^{m-p} B_{p,k}^m \left(-\frac{\partial}{\partial c_p}\right)^k W^*[\phi, c_p](Y, s) \\
&= \sum_{k=0}^{m-p} B_{p,k}^m W^*[(-H)^*\phi, c_p](Y, s),
\end{aligned}$$

we get

$$\begin{aligned}
&\lim_{c\downarrow 0} \iint A[u, c_1, \cdots, c_p, c_{p+1}(c), \cdots, c_m(c)](Y, s)\phi(Y, s)dYds \\
&= \lim_{c\downarrow 0} \iint u(Y, s)A^*[\phi, c_1, \cdots, c_p, c_{p+1}(c), \cdots, c_m(c)](Y, s)dYds \\
&= \iint u(Y, s)\left(\sum_{k=1}^{p-1} A_{p,k}^m W^*[\phi, c_k](Y, s) + \sum_{k=0}^{m-p} B_{p,k}^m W^*[(-H)^*\phi, c_p](Y, s)\right)dYds \\
&= \iint \left(\sum_{k=1}^{p-1} A_{p,k}^m W[u, c_k](Y, s) + \sum_{k=0}^{m-p} B_{p,k}^m W[(-H)^k u, c_p](Y, s)\right) \phi(Y, s)dYds,
\end{aligned}$$

which implies (25).

By the above argument, we know that if u is poly-temperature of degree m on $\mathbf{R}^n \times (t_p, \tau)$, then

$$u(\Xi, \tau) = \sum_{k=1}^{p-1} A_{p,k}^m W[u, c_k](\Xi, \tau) + \sum_{k=0}^{m-p} B_{p,k}^m W[(-H)^k u, c_p](\Xi, \tau), \ \forall \Xi \in \mathbf{R}^n. \tag{26}$$

We use this fact in the proof of second assertion of Theorem 3.

Now we assume that $u(X_0, t_0) = 0$ for some point $(X_0, t_0) \in \mathbf{R}^n \times (t_1, T)$. Since

$$u(X_0, t_0) \geq \sum_{k=1}^{p-1} A_{p,k}^m W[u, c_k](X_0, t_0) + \sum_{k=0}^{m-p} B_{p,k}^m W[(-H)^k u, c_p](X_0, t_0) \geq 0,$$

where $c_k = t_0 - t_k,\ k = 1, \cdots, p$, we conclude by (17), (18), (22) and (24),

$$u(Y, t_k) = 0, \quad \forall k = 1, \cdots, p \text{ and } \forall Y \in \mathbf{R}^n \tag{27}$$

and

$$(-H)^k u(Y, t_p) = 0, \quad \forall k = 1, \cdots, m-p \ \text{ and } \forall Y \in \mathbf{R}^n. \tag{28}$$

Moreover as in the proof of Theorem 1, putting $\mu := (-H)^m u$ and $\Psi_{m,p}(t) = \Psi_{m,p}(c_1, \cdots, c_p; t) := \lim_{c \downarrow 0} \Psi_m(c_1, \cdots, c_p, c_{p+1}(c), \cdots, c_m(c); t)$, we have

$$\begin{aligned} & u(X_0, t_0) - \sum_{k=1}^{p-1} A^m_{p,k} W[u, c_k](X_0, t_0) - \sum_{k=0}^{m-p} B^m_{p,k} W[(-H)^k u, c_p](X_0, t_0) \\ = & \int_{\mathbf{R}^{n+1}} \Psi_{m,p}(c_1, \cdots, c_p; t_0 - s) W(X_0 - Y, t_0 - s)\, d\mu(Y, s) \geq 0. \end{aligned}$$

Since $\Psi_{m,p}(c_1, \cdots, c_p; t_0 - s) W(X_0 - Y, t_0 - s) > 0$ for $(Y, s) \in D_0 := \mathbf{R}^n \times (t_p, t_0)$ by Lemma 1(A), we also conclude that μ vanishes there, that is, u is a poly-temperature of degree m on D_0. Thus (26), (27) and (28) give $u(\Xi, \tau) = 0$ for $\Xi \in \mathbf{R}^n$ and $t_1 < \tau < t_0$. Since u is real analytic on D_0 (see (19)), it vanishes there, as desired. This completes the proof of Theorem 3.

In particular, we see

Corollary 2. (A) *Assume that $u \in C^{2m-2}(\bar{D})$. Under the conditions* (1) *and* (16), *if u is a poly-supertemperature of degree m on D and if $(-H)^k u(X, 0) \geq 0,\ \forall X \in \boldsymbol{R}^n$ for $k = 0, \cdots m-1$, then $u \geq 0$ on D. Moreover, discussing $(-H)^k u$ in place of u, we also see that $(-H)^k u \geq 0$ on D for $k = 1, \cdots, m-1$.*

(B) *Let $f_k,\ k = 0, 1, \cdots, m-1$, be continuous functions on $\boldsymbol{R}^n$ satisfying the growth condition $|f_k(X)| \leq M e^{a|X|^2}$ with* (16). *Then the boundary value problem*

$$\begin{cases} (-H)^m h = 0 & \text{on } D \\ (-H)^k h(\cdot, 0) = f_k & \text{on } \boldsymbol{R}^n, \quad k = 0, 1, \cdots, m-1 \end{cases} \tag{29}$$

has a unique solution on D, which is given by

$$h(X, t) = \sum_{k=0}^{m-1} \frac{t^k}{k!} \int_{\boldsymbol{R}^n} W(X - Y, t) f_k(Y)\, dY \tag{30}$$

(cf. [3, p.266 or French summary p.328]).

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Masaharu Nishio
Department of Mathematics
Osaka City University
Sugimoto, Sumiyoshi
Osaka 558, Japan

Katsunori Shimomura
Department of Mathematical Sciences
Ibaraki University
Mito, Ibaraki 310, Japan

Noriaki Suzuki
Graduate School of Mathematics
Nagoya University
Chikusa-ku, Nagoya, 464-01, Japan