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On Modified Bent-Functions and Phragmén-Lindelöf's Principle

By Yoshimi MATSUMUMA

§ 1. Phragmén-Lindelöf's principle is usually stated as follows:

Let $f(z)$ be a function, regular in the right half-plane and bounded on the imaginary axis, e. g. we assume

$$(1.1) \quad \limsup_{z \rightarrow iy} |f(z)| \leq 1, \quad z = x + iy,$$

and we shall denote by $M(r)$ the least upper bound of the absolute values $|f(z)|$ on the semi-circle $|z| = r$ and $|\theta| < \frac{\pi}{2}$, i. e. $M(r) = \text{l. u. b. } |f(re^{i\theta})|$, and put for the sake of simplicity

$$\alpha = \liminf_{r \rightarrow +\infty} \frac{\log M(r)}{r}, \quad \beta = \limsup_{r \rightarrow +\infty} \frac{\log M(r)}{r}.$$

Then, there may happen two cases: Either the absolute value $|f(z)|$ increases to infinity so that α is positive, or the function $f(z)$ is bounded so that we have $|f(z)| \leq 1$ at every point of the half-plane. Especially, if $\alpha = -\infty$, then $f(z)$ is identically zero [2].

Since E. Phragmén and E. Lindelöf established this famous principle [1], many authors have studied on this subject. E. and R.

Nevanlinna introduced the bentfunction $m(r) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log^+ |f(re^{i\theta})| \cos \theta d\theta$

[2], [3] and proved the monotonousness of $m(r)/r$. A. Dinghas has obtained this result by using the Poisson representation [6]. L. Ahlfors discussed the same property from a standpoint of a certain differential inequality and proposed a question if we have $\alpha = \beta$ so that the limit of $\frac{\log M(r)}{r}$ for $r \rightarrow +\infty$ exists [5]. M. Heins has answered this question, showing that for $0 \leq \alpha < +\infty$ we have $\alpha = \beta$ and the case $-\infty < \alpha < 0$ does not occur [8].

In this Note, we first introduce *Modified Bent-Functions* as follows:

$$(1.2) \quad \mu(r) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(re^{i\theta})| \cos \theta d\theta$$

and we shall prove the monotonousness of $\mu(r)/r$, so that we can put

$$(1.3) \quad \eta = \lim_{r \rightarrow +\infty} \frac{\mu(r)}{r}, \quad \eta^+ = \lim_{r \rightarrow +\infty} \frac{m(r)}{r}.$$

Next, in § 4, we shall establish the following *Fundamental Inequality*:

$$(1.4) \quad \log |f(\rho e^{i\varphi})| \leq \frac{2}{\pi} \eta \cdot \rho \cos \varphi, \quad |\varphi| < \frac{\pi}{2}$$

from which we can deduce the relations between η and η^+ , as well as Heins' results containing the Phragmén-Lindelöf principle (Theorem 5):

- i) if $\eta = +\infty$, then $\alpha = \beta = \eta = +\infty$.
- ii) if $0 < \eta < +\infty$, then $\alpha = \beta = \frac{2}{\pi} \eta$.
- iii) if $-\infty < \eta \leq 0$, then $\alpha = \beta = 0$.
- iv) if $\eta = -\infty$, then $\alpha = \beta = \eta = -\infty$.

Simple examples show that the quantity η may be really negative.

§ 2. We use the following inequality as the starting point of our study, which can be derived from the theory of harmonic majoration:

$$(2.1) \quad \log |f(\rho e^{i\varphi})| \leq \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(re^{i\theta})| \left\{ \frac{r^2 - \rho^2}{r^2 + \rho^2 - 2\rho r \cos(\varphi - \theta)} - \frac{r^2 - \rho^2}{r^2 + \rho^2 + 2\rho r \cos(\theta + \varphi)} \right\} d\theta. \quad 1)$$

The Poisson's kernels $\frac{r^2 - \rho^2}{r^2 + \rho^2 - 2\rho r \cos(\theta - \varphi)}$ and

$\frac{r^2 - \rho^2}{r^2 + \rho^2 + 2\rho r \cos(\theta + \varphi)}$ are expressible as power series of ρ/r which converge uniformly for $\rho \leq r_1 < r$. That is,

$$(2.2) \quad \frac{r^2 - \rho^2}{r^2 + \rho^2 - 2\rho r \cos(\theta - \varphi)} = 1 + 2 \sum_{n=1}^{\infty} \left(\frac{\rho}{r} \right)^n \cos n(\theta - \varphi)$$

and

$$(2.3) \quad \frac{r^2 - \rho^2}{r^2 + \rho^2 + 2\rho r \cos(\theta + \varphi)} = 1 + 2 \sum_{n=1}^{\infty} \left(\frac{\rho}{r} \right)^n \cos n(\theta + \varphi - \pi).$$

Hence we have

$$\frac{r^2 - \rho^2}{r^2 + \rho^2 - 2\rho r \cos(\theta - \varphi)} - \frac{r^2 - \rho^2}{r^2 + \rho^2 + 2\rho r \cos(\theta + \varphi)}.$$

1) The detailed proof of this inequality is found in [2], p. 5-18.

$$= 4 \sum_{n=1}^{\infty} \left(\frac{\rho}{r}\right)^n \sin n\left(\frac{\pi}{2} - \varphi\right) \sin n\left(\frac{\pi}{2} - \theta\right),$$

which converges uniformly for $\rho \leq r_1 < r$. Consequently, we can interchange the signs \sum and \int in (2.1):

$$(2.4) \quad \log |f(\rho e^{i\varphi})| \leq \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\frac{\rho}{r}\right)^n \sin n\left(\frac{\pi}{2} - \varphi\right) \\ \times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(re^{i\theta})| \sin n\left(\frac{\pi}{2} - \theta\right) d\theta.$$

By multiplying both sides of the inequality by $\cos \varphi$ and integrating them with respect to φ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, we obtain for $\rho < r$,

$$(2.5) \quad \frac{1}{\rho} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(\rho e^{i\varphi})| \cos \varphi d\varphi \leq \frac{1}{r} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(re^{i\theta})| \cos \theta d\theta,$$

Thus we have

THEOREM 1. $\mu(r)/r$ is a non-decreasing function of r .

The monotonousness of $m(r)/r$ was given by Nevanlinna, Ahlfors and Dinghas.

From (2.5), we have for $\rho < r$,

$$(2.6) \quad \frac{\mu(\rho)}{\rho} \leq \frac{m(\rho)}{\rho} \leq 2 \frac{\log M(r)}{r}.$$

Let r tend to infinity, then $\frac{\mu(\rho)}{\rho} \leq \frac{m(\rho)}{\rho} \leq 2\alpha$.

Now, as $\mu(\rho)/\rho$ increases monotonously, $\lim_{\rho \rightarrow +\infty} \mu(\rho)/\rho$ must exist. Hence we have the following theorem.

THEOREM 2. For $\alpha \neq +\infty$, η exists and we have furthermore

$$(2.7) \quad \eta \leq \eta^+ \leq 2\alpha.$$

§ 3. Now let 1. u. b. $|f(\rho e^{i\varphi})|$ be attained at a point $z = \rho e^{i\varphi_1(\rho)}$,

$\rho < r$, $|\varphi_1(\rho)| < \frac{\pi}{2}$, that is

$$M(\rho) = |f(\rho e^{i\varphi_1(\rho)})| = 1. \text{ u. b. } |f(\rho e^{i\varphi})|, \quad |\varphi| < \pi/2$$

then from (2.4), we have

1) In case $|\varphi_1(\rho)| = \pi/2$, by the hypothesis (1.1), $|f(z)| \leq 1$ for $|z| = \rho$. And then by the maximal principle, we can proceed our discussion as in § 5. Hence we obtain $-\infty < \eta \leq 0$ and $\alpha = \beta = 0$.

$$(3.1) \quad \log M(\rho) \leq \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\frac{\rho}{r}\right)^n \sin n\left(\frac{\pi}{2} - \varphi_1(\rho)\right) \\ \times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(re^{i\theta})| n\left(\frac{\pi}{2} - \theta\right) d\theta.$$

By using the inequality $n \sin \theta \geq |\sin n\theta|$, $0 \leq \theta \leq \pi$, we can estimate $\frac{1}{r^n} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(re^{i\theta})| \sin n\left(\frac{\pi}{2} - \theta\right) d\theta$ as follows:

$$\begin{aligned} & \left| \frac{1}{r^n} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(re^{i\theta})| \sin n\left(\frac{\pi}{2} - \theta\right) d\theta \right| \\ & \leq \frac{1}{r^n} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(re^{i\theta})| \sin n\left(\frac{\pi}{2} - \theta\right) d\theta \\ & \leq \frac{n}{r^n} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(re^{i\theta})| \cos \theta d\theta \\ & = \frac{n}{r^n} \left\{ 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log^+ |f(re^{i\theta})| \cos \theta d\theta - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(re^{i\theta})| \cos \theta d\theta \right\}. \end{aligned}$$

If α is finite, we can see from (2.7) that $m(r)/r$ and $\mu(r)/r$ are bounded for $r \rightarrow +\infty$. Thus, if we let r tend to $+\infty$, ρ fixed, all the terms but the first in the right-hand side of (3.1) vanish. Consequently, we have, if α is finite,

$$\frac{\log M(\rho)}{\rho} \leq \frac{2}{\pi} \eta \cos \varphi_1(\rho).$$

If $\eta \geq 0$, then $\log M(\rho)/\rho \leq \frac{2}{\pi} \eta$. Finally, by making ρ tend to infinity, we have

$$(3.2) \quad \frac{\pi}{2} \limsup_{\rho \rightarrow +\infty} \frac{\log M(\rho)}{\rho} \leq \eta.$$

From the inequalities (2.7) and (3.2), we have

THEOREM 3. *If η is not negative and α is finite, then*

$$\frac{\pi}{2} \beta \leq \eta \leq 2\alpha.$$

COROLLARY. *Under the same conditions as THEOREM 3,*

$$0 \leq \beta \leq \frac{4}{\pi} \alpha.$$

REMARK. If η is $+\infty$, then α and β are also $+\infty$, and conversely, if α is $+\infty$, then η is also $+\infty$.

§ 4. For a fixed ρ , let r tend to infinity in the inequality (2.4), then

$$(1.4) \quad \log |f(\rho e^{i\varphi})| \leq \frac{2}{\pi} \eta \rho \cos \varphi.$$

If η is not negative, then we have

$$(4.1) \quad \log |f(\rho e^{i\varphi})| \leq \frac{4}{\pi} \alpha \rho \cos \varphi \leq \frac{4}{\pi} \alpha \rho,$$

by virtue of (2.6). Consequently, we obtain

THEOREM 4. *If η is not negative, then*

$$\frac{\log M(\rho)}{\rho} \leq \frac{4}{\pi} \alpha,$$

for any positive ρ . Accordingly, if the equality holds identically in the above inequality, then $f(z)$ must have the form $Ce^{\frac{4}{\pi}\alpha z}$, where C is a constant having absolute value 1.

From (1.4), we have the following inequality for any positive ρ and φ

$$(4.2) \quad |f(\rho e^{i\varphi})| \leq e^{\frac{2}{\pi} \eta \rho \cos \varphi}, \quad \left(|\varphi| < \frac{\pi}{2} \right).$$

Consequently, if η is not positive, then we have for any positive ρ and φ

$$(4.4) \quad |f(\rho e^{i\varphi})| \leq 1, \quad \left(|\varphi| < \frac{\pi}{2} \right).$$

These results prove the famous Phragmén-Lindelöf principle.

REMARK. From (4.1), we can see that, when α is not positive, we have also $|f(z)| \leq 1$ identically.

§ 5. We are in a position to make the Theorem 3 more precise and to deduce Heins' results.

We shall express $f(z)$ as follows, that is

$$(5.1) \quad f(z) \equiv e^{\frac{2}{\pi} \eta z} g(z).$$

By the relation (1.4) and (1.1), $g(z)$ is analytic and of modulus less than 1 in the half-plane $\Re z > 0$, and further $\limsup_{z \rightarrow iy} |g(z)| \leq 1$, that is $g(z)$ is a function of Phragmén-Lindelöf's type.

By virtue of (1.4) and (5.4), we have

$$(5.2) \quad |g(z)| \leq 1.$$

Now let $E_r(\theta)$ be the set of angles, θ , of the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, for which

$$(5.3) \quad \log |g(re^{i\theta})| \leq -\varepsilon r, \quad (\rho \text{ fixed}),$$

for an arbitrary given positive number ε .

$$\log |f(re^{i\theta})| = \frac{2}{\pi} \eta r \cos \theta + \log |g(re^{i\theta})|.$$

Multiplying the both side by $\cos \theta$ and integrating with θ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, it follows that

$$\begin{aligned} \frac{\mu(r)}{r} &= \eta + \frac{1}{r} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |g(re^{i\theta})| \cos \theta d\theta \\ &= \eta + \frac{1}{r} \int_{E_r(\theta)} \log |g(re^{i\theta})| \cos \theta d\theta \\ &\quad + \frac{1}{r} \int_{[-\frac{\pi}{2}, \frac{\pi}{2}] - E_r(\theta)} \log |g(re^{i\theta})| \cos \theta d\theta \\ &\leq \eta + \frac{1}{r} \int_{E_r(\theta)} \log |g(re^{i\theta})| \cos \theta d\theta. \end{aligned}$$

By (5.3)

$$\frac{\mu(r)}{r} \leq \eta - \frac{\varepsilon r}{r} \int_{E_r(\theta)} d \sin \theta.$$

Therefore we have $\mu(r)/r + \varepsilon \text{meas}_{\sin} E_r(\theta) \leq \eta$, where we denote by $\text{meas}_{\sin} A$, the measure of A with respect to the mass distribution $d \sin \theta$:

$$\text{meas}_{\sin} A = \int_A d \sin \theta.$$

Let r tend to infinity, then we have

$$\lim_{r \rightarrow +\infty} \text{meas}_{\sin} E_r(\theta) \leq 0 \text{ i. e. } \lim_{r \rightarrow +\infty} \text{meas}_{\sin} E_r(\theta) = 0.$$

Hence, given any positive number ε' , there exist a positive number δ and a large R , for $r > R$ we have

$$\begin{aligned} \text{meas} \left(E_r(\theta) \cap \left[\frac{\pi}{2} - \delta, \frac{\pi}{2} \right] \right) &< \frac{\varepsilon'}{4} \text{ and} \\ \text{meas} \left(E_r(\theta) \cap \left[-\frac{\pi}{2}, -\frac{\pi}{2} + \delta \right] \right) &^{1)} \end{aligned}$$

On the other hand,

$$\begin{aligned} \text{meas}_{\sin} \left(E_r(\theta) \cap \left[-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta \right] \right) &= \int_{E_r(\theta) \cap \left[-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta \right]} d \sin \theta \geq \cos \left(\frac{\pi}{2} - \delta \right) \\ &\times \int_{E_r(\theta) \cap \left[-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta \right]} d\theta = \cos \left(\frac{\pi}{2} - \delta \right) \text{meas} \left(E_r(\theta) \cap \left[-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta \right] \right). \end{aligned}$$

Hence we obtain

$$(5.4) \quad \lim_{r \rightarrow \infty} \text{meas } E_r(\theta) = 0.$$

Case A. $0 < \eta < +\infty$. In this case, for any sufficiently small positive number ε'' , there exists an angle $\theta_0(r)$ such that $|\theta_0(r)| < \varepsilon''$ and $\theta_0(r) \in E_r(\theta)$.²⁾ For this angle $\theta_0(r)$,

$$\log M(r) \geq \log |f(re^{i\theta_0(r)})| = \frac{2}{\pi} \eta r \cos \theta_0(r) + \log |g(re^{i\theta_0(r)})|.$$

From (5.3) and the definition of $\theta_0(r)$,

$$\frac{\log M(r)}{r} > \frac{2}{\pi} \eta \cos \varepsilon'' - \varepsilon.$$

Let r tend to positive infinity, we have $\alpha \geq \frac{2}{\pi} \eta \cos \varepsilon'' - \varepsilon$. As ε and ε'' are arbitrary, we have $\alpha \geq \frac{2}{\pi} \eta$ which proves $\alpha = \beta = \frac{2}{\pi} \eta$.

Case B. $-\infty < \eta \leq 0$. In this case it is clear that α and β are not positive. For any sufficiently small positive number δ' , there would exist an angle $\theta(r)$ such that $\frac{\pi}{2} > \theta(r) > \frac{\pi}{2} - \delta'$ and $\theta(r) \in E_r(\theta)$.

For this angle $\theta(r)$,

1) This measure is used in the Lebesgue's sense.

2) $|\theta|$ means the magnitude of the angle θ .

$$\begin{aligned}\log M(r) &\geq \frac{2}{\pi} \eta r \cos \theta(r) + \log |g(re^{i\theta(r)})| \\ &> \frac{2}{\pi} \eta r \cos \theta(r) - \varepsilon r.\end{aligned}$$

Hence

$$\frac{\log M(r)}{r} > \frac{2}{\pi} \eta \cos\left(\frac{\pi}{2} - \delta'\right) - \varepsilon.$$

Let r tend to infinity, then we have, by the definitions of ε and δ' , $\alpha \geq 0$. Consequently, $\alpha = \beta = 0$.

Case C. $\eta = +\infty$. In this case it is clear by the Remark in § 4 that $\alpha = \beta = \eta = +\infty$.

Case D. $\eta = -\infty$. This case cannot occur except if $f(z) \equiv 0$. Consequently, $\alpha = \beta = \eta = -\infty$.

Hence we have

THEOREM 5. $\lim_{r \rightarrow +\infty} \frac{\log M(r)}{r}$ exists and further

- i) if $\eta = +\infty$, then $\alpha = \beta = \eta = +\infty$.
- ii) if $0 < \eta < +\infty$, then $\alpha = \beta = \frac{2}{\pi} \eta$.
- iii) if $-\infty < \eta \leq 0$, then $\alpha = \beta = 0$ and $|f(z)| \leq 1$ for all $\Re z > 0$.
- iv) if $\eta = -\infty$, then $\alpha = \beta = \eta = -\infty$ and in this case $f(z)$ must be identically zero.

COROLLARY. For a sufficiently large $|z|$, the functions $f(z)$, which are of the Phragmén-Lindelöf's type, are expressible as $Ce^{\frac{2}{\pi}\eta z}$ except for a set of almost measure zero, where C is a constant of modulus 1.

§ 6. Let $u(z)$ be a harmonic or a subharmonic function in the half-plane $\Re z > 0$, and further suppose that $u(z)$ is not positive on the imaginary axis. Then we are able to replace $\log |f(z)|$ by $u(z)$ in the inequality (2.1). Consequently, we obtain the following theorems.

THEOREM 7. $\frac{1}{r} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u(re^{i\theta}) \cos \theta d\theta$ is non-decreasing function of r and the limit exists for $r \rightarrow +\infty$.

We denote this limit by K .

THEOREM 8. Let $M(r)$ be maximum of $u(re^{i\theta})$ on the semi-circle

$|z| = r$, $|\theta| < \frac{\pi}{2}$. Then $\lim_{r \rightarrow +\infty} M(r)/r$ exists and if K is not negative, then

$$\frac{\pi}{2} \lim_{r \rightarrow +\infty} \frac{M(r)}{r} = K.$$

THEOREM 9. If K is finite, then

$$\lim_{r \rightarrow +\infty} \frac{u(re^{i\theta})}{r} = K \cos \theta$$

for almost all θ .

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