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On Modified Bent-Functions and Phragmén-Lindelöf's Principle

By Yoshimi MATSUMUMA

§ 1. Phragmén-Lindelöf's principle is usually stated as follows: Let $f(z)$ be a function, regular in the right half-plane and bounded on the imaginary axis, e. g. we assume

$$\limsup_{z\to iy} |f(z)| \leq 1, \quad z = x + iy, \tag{1. 1}$$

and we shall denote by $M(r)$ the least upper bound of the absolute values $|f(z)|$ on the semi-circle $|z|=r$ and $|\theta|<\frac{\pi}{2}$, i. e. $M(r)=\underset{|\theta|<\pi/2}{\text{l. u. b.}} |f(re^{i\theta})|$, and put for the sake of simplicity

$$\alpha = \liminf_{r\to+\infty} \frac{\log M(r)}{r}, \qquad \beta = \limsup_{r\to+\infty} \frac{\log M(r)}{r}.$$

Then, there may happen two cases: Either the absolute value $|f(z)|$ increases to infinity so that α is positive, or the function $f(z)$ is bounded so that we have $|f(z)|\leq 1$ at every point of the half-plane. Especially, if $\alpha=-\infty$, then $f(z)$ is identically zero [2].

Since E. Phragmén and E. Lindelöf established this famous principle [1], many authors have studied cn this subject. E. and R. Nevanlinna introduced the bentfunction $m(r)=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log^+ |f(re^{i\theta})| \cos\theta d\theta$ [2], [3] and proved the monotonousness of $m(r)/r$. A. Dinghas has obtained this result by using the Poisson representation [6]. L. Ahlfors discussed the same property from a standpoint of a certain differential inequality and proposed a question if we have $\alpha=\beta$ so that the limit of $\frac{\log M(r)}{r}$ for $r\to+\infty$ exists [5]. M. Heins has answered this question, showing that for $0\leq\alpha<+\infty$ we have $\alpha=\beta$ and the case $-\infty<\alpha<0$ does not occur [8].

In this Note, we first introduce *Modified Bent-Functions* as follows:

$$\mu(r)=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log|f(re^{i\theta})| \cos\theta d\theta \tag{1. 2}$$

and we shall prove the montonousness of $\mu(r)/r$, so that we can put

$$(1.3)\qquad \eta=\lim_{r\to+\infty}\frac{\mu(r)}{r},\qquad \eta^{+}=\lim_{r\to+\infty}\frac{m(r)}{r}.$$

Next, in § 4, we shall establish the following *Fundamental Inequality*:

$$(1.4)\qquad \log|f(\rho e^{i\varphi})|\leq\frac{2}{\pi}\eta\cdot\rho\cos\varphi,\qquad |\varphi|<\frac{\pi}{2}$$

from which we can deduce the relations between η and η^{+}, as well as Heins' results containing the Phragmén-Lindelöf principle (Theorem 5):

i) if $\eta=+\infty$, then $\alpha=\beta=\eta=+\infty$.

ii) if $0<\eta<+\infty$, then $\alpha=\beta=\frac{2}{\pi}\eta$.

iii) if $-\infty<\eta\leq 0$, then $\alpha=\beta=0$.

iv) if $\eta=-\infty$, then $\alpha=\beta=\eta=-\infty$.

Simple examples show that the quantity η may be really negative.

§ 2. We use the following inequality as the starting point of our study, which can be derived from the theory of harmonic majoration:

$$(2.1)\qquad \log|f(\rho e^{i\varphi})|\leq\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\log|f(re^{i\theta})|\left\{\frac{r^2-\rho^2}{r^2+\rho^2-2\rho r\cos(\varphi-\theta)}\right.$$

$$\left.-\frac{r^2-\rho^2}{r^2+\rho^2+2\rho r\cos(\theta+\varphi)}\right\}d\theta.$$ [1]

The Poisson's kernels $\frac{r^2-\rho^2}{r^2+\rho^2-2\rho r\cos(\theta-\varphi)}$ and $\frac{r^2-\rho^2}{r^2+\rho^2+2\rho r\cos(\theta+\varphi)}$ are expressible as power series of ρ/r which converge uniformly for $\rho\leq r_1<r$. That is,

$$(2.2)\qquad \frac{r^2-\rho^2}{r^2+\rho^2-2\rho r\cos(\theta-\varphi)}=1+2\sum_{n=1}^{\infty}\left(\frac{\rho}{r}\right)^n\cos n(\theta-\varphi)$$

and

$$(2.3)\qquad \frac{r^2-\rho^2}{r^2+\rho^2+2\rho r\cos(\theta+\varphi)}=1+2\sum_{n=1}^{\infty}\left(\frac{\rho}{r}\right)^n\cos n(\theta+\varphi-\pi).$$

Hence we have

$$\frac{r^2-\rho^2}{r^2+\rho^2-2\rho r\cos(\theta-\varphi)}-\frac{r^2-\rho^2}{r^2+\rho^2+2\rho r\cos(\theta+\varphi)}.$$

1) The detailed proof of this inequality is found in [2], p. 5-18.

$$= 4 \sum_{n=1}^{\infty}{}' \left(\frac{\rho}{r}\right)^n \sin n\left(\frac{\pi}{2}-\varphi\right) \sin n\left(\frac{\pi}{2}-\theta\right),$$

which converges uniformly for $\rho \leq r_1 < r$. Consequently, we can interchange the signs $\sum'$ and $\int$ in (2. 1):

$$\text{(2. 4)} \qquad \log |f(\rho e^{i\varphi})| \leq \frac{2}{\pi} \sum_{n=1}^{\infty}{}' \left(\frac{\rho}{r}\right)^n \sin n\left(\frac{\pi}{2}-\varphi\right)$$

$$\times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(re^{i\theta})| \sin n\left(\frac{\pi}{2}-\theta\right) d\theta .$$

By multiplying both sides of the inequality by $\cos\varphi$ and integrating them with respect to φ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, we obtain for $\rho < r$,

$$\text{(2. 5)} \qquad \frac{1}{\rho} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(\rho e^{i\varphi})| \cos\varphi d\varphi \leq \frac{1}{r} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(re^{i\theta})| \cos\theta d\theta ,$$

Thus we have

THEOREM 1. *$\mu(r)/r$ is a non-decreasing function of r.*

The monotonousness of $m(r)/r$ was given by Nevanlinna, Ahlfors and Dinghas.

From (2. 5), we have for $\rho < r$,

$$\text{(2. 6)} \qquad \frac{\mu(\rho)}{\rho} \leq \frac{m(\rho)}{\rho} \leq 2 \frac{\log M(r)}{r} .$$

Let r tend to infinity, then $\frac{\mu(\rho)}{\rho} \leq \frac{m(\rho)}{\rho} \leq 2\alpha$.

Now, as $\mu(\rho)/\rho$ increases monotonously, $\lim\limits_{\rho\to+\infty} \mu(\rho)/\rho$ must exist. Hence we have the following theorem.

THEOREM 2. *For $\alpha \neq +\infty$, η exists and we have furthermore*

$$\text{(2. 7)} \qquad \eta \leq \eta^{+} \leq 2\alpha .$$

§ 3. Now let $\underset{|\varphi|<\pi/2}{\text{l. u. b.}} |f(\rho e^{i\varphi})|$ be attained at a point $z = \rho e^{i\varphi_1(\rho)}$, $\rho < r$, $|\varphi_1(\rho)| < \frac{\pi}{2}$, that is

$$M(\rho) = |f(\rho e^{i\varphi_1(\rho)})| = \underset{|\varphi|<\pi/2}{\text{l. u. b.}} |f(\rho e^{i\varphi})|,^{1)}$$

then from (2. 4), we have

1) In case $|\varphi_1(\rho)| = \pi/2$, by the hypothesis (1. 1), $|f(z)| \leq 1$ for $|z| = \rho$. And then by the maximal principle, we can proceed our discussion as in § 5. Hence we obtain $-\infty < \eta \leq 0$ and $\alpha = \beta = 0$.

$$\log M(\rho) \leq \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\frac{\rho}{r}\right)^n \sin n\left(\frac{\pi}{2} - \varphi_1(\rho)\right) \tag{3.1}$$
$$\times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(re^{i\theta})| \, n\left(\frac{\pi}{2} - \theta\right) d\theta .$$

By using the inequality $n \sin \theta \geq |\sin n\theta|$, $0 \leq \theta \leq \pi$, we can estimate $\frac{1}{r^n} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(re^{i\theta})| \sin n\left(\frac{\pi}{2} - \theta\right) d\theta$ as follows:

$$\left| \frac{1}{r^n} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(re^{i\theta})| \sin n\left(\frac{\pi}{2} - \theta\right) d\theta \right|$$
$$\leq \frac{1}{r^n} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(re^{i\theta})| \sin n\left(\frac{\pi}{2} - \theta\right) d\theta$$
$$\leq \frac{n}{r^n} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(re^{i\theta})| \cos \theta d\theta$$
$$= \frac{n}{r^n} \left\{ 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log^+ |f(re^{i\theta})| \cos \theta d\theta - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(re^{i\theta})| \cos \theta d\theta \right\} .$$

If α is finite, we can see from (2.7) that $m(r)/r$ and $\mu(r)/r$ are bounded for $r \to +\infty$. Thus, if we let r tend to $+\infty$, ρ fixed, all the terms but the first in the right-hand side of (3.1) vanish. Consequently, we have, if α is finite,

$$\frac{\log M(\rho)}{\rho} \leq \frac{2}{\pi} \eta \cos \varphi_1(\rho) .$$

If $\eta \geq 0$, then $\log M(\rho)/\rho \leq \frac{2}{\pi} \eta$. Finally, be making ρ tend to infinity, we have

$$\frac{\pi}{2} \limsup_{\rho \to +\infty} \frac{\log M(\rho)}{\rho} \leq \eta . \tag{3.2}$$

From the inequalities (2.7) and (3.2), we have

THEOREM 3. *If η is not negative and α is finite, then*

$$\frac{\pi}{2} \beta \leq \eta \leq 2 \alpha .$$

COROLLARY. *Under the same conditions as* THEOREM 3,

$$0 \leq \beta \leq \frac{4}{\pi} \alpha .$$

REMARK. If η is $+\infty$, then α and β are also $+\infty$, and conversely, if α is $+\infty$, then η is also $+\infty$.

§ 4. For a fixed ρ, let r tend to infinity in the inequality (2. 4), then

$$\log |f(\rho e^{i\varphi})| \leq \frac{2}{\pi} \eta\rho \cos \varphi . \tag{1. 4}$$

If η is not negative, then we have

$$\log |f(\rho e^{i\varphi})| \leq \frac{4}{\pi} \alpha\rho \cos \varphi \leq \frac{4}{\pi} \alpha\rho , \tag{4. 1}$$

by virtue of (2. 6). Consequently, we obtain

THEOREM 4. *If η is not negative, then*

$$\frac{\log M(\rho)}{\rho} \leq \frac{4}{\pi} \alpha ,$$

for any positive ρ. Accordingly, if the equality holds identically in the above inequality, then $f(z)$ must have the form $Ce^{\frac{4}{\pi}\alpha z}$, where C is a constant having absolute valve 1.

From (1. 4), we have the following inequality for any positive ρ and φ

$$|f(\rho e^{i\varphi})| \leq e^{\frac{2}{\pi}\eta\rho\cos\varphi} , \qquad \left(|\varphi| < \frac{\pi}{2}\right). \tag{4. 2}$$

Consequently, if η is not positive, then we have for any positive ρ and φ

$$|f(\rho e^{i\varphi})| \leq 1 , \qquad \left(|\varphi| < \frac{\pi}{2}\right). \tag{4. 4}$$

These results prove the famous Phragmén-Lindelöf principle.

REMARK. From (4. 1), we can see that, when α is not positive, we have also $|f(z)| \leq 1$ identically.

§ 5. We are in a position to make the Theorem 3 more precise and to deduce Heins' results.

We shall express $f(z)$ as follows, that is

$$f(z) \equiv e^{\frac{2}{\pi}\eta z} g(z) . \tag{5. 1}$$

By the relation (1.4) and (1.1), $g(z)$ is analytic and of modulus less than 1 in the half-plane $\Re z>0$, and further $\limsup_{z\to iy} |g(z)|\leq 1$, that is $g(z)$ is a function of Phragmén-Lindelöf's type.

By virtue of (1.4) and (5.4), we have

$$|g(z)|\leq 1. \tag{5.2}$$

Now let $E_r(\theta)$ be the set of angles, θ, of the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, for which

$$\log|g(re^{i\theta})|\leq -\varepsilon r, \quad (\rho \text{ fixed}), \tag{5.3}$$

for an arbitrary given positive number ε.

$$\log|f(re^{i\theta})|=\frac{2}{\pi}\eta r\cos\theta+\log|g(re^{i\theta})|.$$

Multiplying the both side by $\cos\theta$ and integrating with θ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, it follows that

$$\begin{aligned}\frac{\mu(r)}{r}&=\eta+\frac{1}{r}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\log|g(re^{i\theta})|\cos\theta d\theta\\&=\eta+\frac{1}{r}\int_{E_r(\theta)}\log|g(re^{i\theta})|\cos\theta d\theta\\&\quad+\frac{1}{r}\int_{\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-E_r(\theta)}\log|g(re^{i\theta})|\cos\theta d\theta\\&\leq\eta+\frac{1}{r}\int_{E_r(\theta)}\log|g(re^{i\theta})|\cos\theta d\theta.\end{aligned}$$

By (5.3)

$$\frac{\mu(r)}{r}\leq\eta-\frac{\varepsilon r}{r}\int_{E_r(\theta)}d\sin\theta.$$

Therefore we have $\mu(r)/r+\varepsilon\,\text{meas}_{\text{sine}}\,E_r(\theta)\leq\eta$, where we denote by $\text{meas}_{\text{sine}}$ A, the measure of A with respect to the mass distribution $d\sin\theta$:

$$\text{meas}_{\text{sine}}\,A=\int_A d\sin\theta.$$

Let r tend to infinity, then we have

$$\lim_{r\to+\infty} \text{meas}_{\text{sine}}\, E_r(\theta) \leq 0 \text{ i. e. } \lim_{r\to+\infty} \text{meas}_{\text{sine}}\, E_r(\theta) = 0\,.$$

Hence, given any positive number ε', there exist a positive number δ and a large R, for $r > R$ we have

$$\text{meas}\Big(E_r(\theta) \cap \Big[\frac{\pi}{2}-\delta,\ \frac{\pi}{2}\Big]\Big) < \frac{\varepsilon'}{4} \text{ and}$$

$$\text{meas}\Big(E_r(\theta) \cap \Big[-\frac{\pi}{2},\ -\frac{\pi}{2}+\delta\Big]\Big).^{1)}$$

On the other hand,

$$\text{meas}_{\text{sine}}\Big(E_r(\theta) \cap \Big[-\frac{\pi}{2}+\delta,\ \frac{\pi}{2}-\delta\Big]\Big) = \int_{E_r(\theta)\cap[-\frac{\pi}{2}+\delta,\ \frac{\pi}{2}-\delta]} d\sin\theta \geq \cos\Big(\frac{\pi}{2}-\delta\Big)$$

$$\times \int_{E_r(\theta)\cap[-\frac{\pi}{2}+\delta,\ \frac{\pi}{2}-\delta]} d\theta = \cos\Big(\frac{\pi}{2}-\delta\Big)\ \text{meas}\ \Big(E_r(\theta) \cap \Big[-\frac{\pi}{2}+\delta,\ \frac{\pi}{2}-\delta\Big]\Big).$$

Hence we obtain

$$\lim_{r\to\infty} \text{meas}\ E_r(\theta) = 0\,. \tag{5.4}$$

Case A. $0 < \eta < +\infty$. In this case, for any sufficiently small positive number ε'', there exists an angle $\theta_0(r)$ such that $|\theta_0(r)| < \varepsilon''$ and $\theta_0(r) \bar{\in} E_r(\theta)$.[2] For this angle $\theta_0(r)$,

$$\log M(r) \geq \log |f(re^{i\theta_0(r)})| = \frac{2}{\pi}\eta r \cos\theta_0(r) + \log |g(re^{i\theta_0(r)})|\,.$$

From (5. 3) and the definition of $\theta_0(r)$,

$$\frac{\log M(r)}{r} > \frac{2}{\pi}\,\eta\,\cos\,\varepsilon'' - \varepsilon\,.$$

Let r tend to positive infinity, we have $\alpha \geq \frac{2}{\pi}\eta\cos\varepsilon'' - \varepsilon$. As ε and ε'' are arbitrary, we have $\alpha \geq \frac{2}{\pi}\eta$ which proves $\alpha = \beta = \frac{2}{\pi}\eta$.

Case B. $-\infty < \eta \leq 0$. In this case it is clear that α and β are not positive. For any sufficiently small positive number δ', there would exist an angle $\theta(r)$ such that $\frac{\pi}{2} > \theta(r) > \frac{\pi}{2} - \delta'$ and $\theta(r) \bar{\in} E_r(\theta)$.

For this angle $\theta(r)$,

1) This measure is used in the Lebesgue's sense.

2) $|\theta|$ means the magnitude of the angle θ.

$$\log M(r) \geq \frac{2}{\pi} \eta r \cos \theta(r) + \log |g(re^{i\theta(r)})|$$
$$> \frac{2}{\pi} \eta r \cos \theta(r) - \varepsilon r .$$

Hence

$$\frac{\log M(r)}{r} > \frac{2}{\pi} \eta \cos\left(\frac{\pi}{2} - \delta'\right) - \varepsilon .$$

Let r tend to infinity, then we have, by the definitions of ε and δ', $\alpha \geq 0$. Consequenity, $\alpha = \beta = 0$.

Case C. $\eta = +\infty$. In this case it is clear by the Remark in § 4 that $\alpha = \beta = \eta = +\infty$.

Case D. $\eta = -\infty$. This case cannot occure except if $f(z) \equiv 0$. Consequently, $\alpha = \beta = \eta = -\infty$.

Hence we have

THEOREM 5. $\lim\limits_{r \to +\infty} \dfrac{\log M(r)}{r}$ *exists and further*

i) *if* $\eta = +\infty$, *then* $\alpha = \beta = \eta = +\infty$.

ii) *if* $0 < \eta < +\infty$, *then* $\alpha = \beta = \dfrac{2}{\pi} \eta$.

iii) *if* $-\infty < \eta \leq 0$, *then* $\alpha = \beta = 0$ *and* $|f(z)| \leq 1$ *for all* $\Re z > 0$.

iv) *if* $\eta = -\infty$, *then* $\alpha = \beta = \eta = -\infty$ *and in this case* $f(z)$ *must be identically zero.*

COROLLARY. *For a sufficiently large* $|z|$, *the functions* $f(z)$, *which are of the Phragmén-Lindelöf's type, are expressible as* $Ce^{\frac{2}{\pi}\eta z}$ *except for a set of almost measure zero, where* C *is a constant of modulus 1.*

§ 6. Let $u(z)$ be a harmonic or a subharmonic function in the half-plane $\Re z > 0$, and further suppose that $u(z)$ is not positive on the imaginary axis. Then we are able to replace $\log |f(z)|$ by $u(z)$ in the inequality (2. 1). Consequently, we obtain the following theorems.

THEOREM 7. $\dfrac{1}{r} \displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u(re^{i\theta}) \cos \theta d\theta$ *is non-decreasing function of* r *and the limit exists for* $r \to +\infty$.

We denote this limit by K.

THEOREM 8. *Let* $M(r)$ *be maximum of* $u(re^{i\theta})$ *on the semi-circle*

$|z|=r,\ |\theta|<\frac{\pi}{2}$. *Then* $\lim\limits_{r\to+\infty} M(r)/r$ *exists and if* K *is not negative, then*

$$\frac{\pi}{2}\lim_{r\to+\infty}\frac{M(r)}{r}=K.$$

THEOREM 9. *If* K *is finite, then*

$$\lim_{r\to+\infty}\frac{u(re^{i\theta})}{r}=K\cos\theta$$

for almost all θ.

References

[1] E. Phragmén and E. Lindelöf: Sur un extension d'un principe classique de l'analyse et sur quelque propriétés des fonctions monogènes dans le voisinage d'un point singulier, Acta Math. 31 (1908), 381-406.

[2] F. and R. Nevanlinna: Über die Eigenschaften analytischer Funktionen in der Umgebung einer singulären Stelle oder Linie, Acta Soc. Sci. Fenn. 50 (1922), No. 5.

[3] R. Nevanlinna: Über die Eigenschaften meromorpher Funktionen in einem Winkelraum, Acta Soc. Sci. Fenn. 50 (1922), No. 12.

[4] R. Nevanlinna: Eindeutige Analytische Funktionen, Berlin (1935)

[5] L. Ahlfors: On Phragmén-Lindelöf's principle, Trans. Amer. Math. Soc. 41 (1937), pp. 1-8.

[6] A. Dinghas: Zur Theorie der meromorphen Funktionen in einem Winkelraum, Sitzungsber. Preuss. Akad. Wiss. (1935), 576-596.

[7] A. Dinghas: Über das Phragmén-Lindelöfsche Prinzip und der Julia-Carathéodorysche Satz, Sitzungsber. Preuss. Akad. Wiss. (1938), 32-48.

[8] M. Heins: On the Phragmén-Lindölof principle, Trans. Amer. Math. Soc. 60 (1946), 238-244.

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