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### H-PROJECTIVE CONNECTIONS AND H-PROJECTIVE TRANSFORMATIONS

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#### (Received May 16, 1977)

### Introduction

Let M be an *n*-dimensional complex manifold. We write J for its natural almost complex structure. Let  $\nabla$  be an almost complex affine connection without torsion on M. A curve c(t) in M is called an *H*-planner curve with respect to  $\nabla$  if

$$\nabla_{c'}c' = ac' + bJc'$$

for certain smooth functions a and b. Two almost complex affine connections  $\nabla$  and  $\nabla'$  without torsion are said to be *H*-projectively equivalent if they have their *H*-planner curves in common. From the result of T. Otsuki and Y. Tashiro, this is equivalent to existence of a 1-form  $\rho$  on *M* satisfying

(0.2) 
$$\nabla_X Y - \nabla'_X Y = \rho(X)Y + \rho(Y)X - \rho(JX)JY - \rho(JY)JX$$

for arbitrary vector fields X and Y ([5], [8]). By an *H*-projective transformation of  $\nabla$ , we mean a biholomorphic transformation  $f: M \rightarrow M$  such that  $f^*\nabla$ and  $\nabla$  are *H*-projectively equivalent. For example, let  $P^n(C) = L/L_0$  be the *n*-dimensional complex projective space of lines in  $C^{n+1}$  with the usual connection, where

(0.3) 
$$L = SL(n+1, \mathbf{C}),$$
$$L_0 = \left\{ \begin{pmatrix} a & u \\ 0 & B \end{pmatrix} \in SL(n+1, \mathbf{C}) | B \in GL(n, \mathbf{C}) \right\}.$$

Then L/(center) is the group of all *H*-projective transformations.

In the present paper, we shall study *H*-projective equivalence from the view point of  $L_0$ -structure of second order, studied by N. Tanaka and T. Ochiai. In fact, we shall show that *H*-projective equivalence of  $\nabla$  and  $\nabla'$  is the same as  $P^n(C)$ -equivalence in [6] and [4] (Theorem 1). Therefore, using their results, the family  $\{\nabla\}$  of almost complex affine connections without torsion which are *H*-projectively equivalent to  $\nabla$  uniquely determines a Cartan connection  $\omega$  of type  $P^n(C)$ . This enables us to show that the group of all *H*-projective

transformations of  $\nabla$  is a Lie group of finite dimension (Theorem 2). Then we shall prove that a curve c(t) is an *H*-planner curve with respect to  $\nabla$  if and only if the development of c(t) into  $P^n(C)$  by  $\omega$  is an *H*-planner curve in  $P^n(C)$ (Theorem 3).

An *H*-planner curve c(t) with respect to  $\nabla$  is called an *H*-geodesic of  $\nabla$  if a=0 and b is a constant in (0.1). An almost complex affine connection  $\nabla$  without torsion is said to be *H*-complete if any *H*-geodesic c(t) of  $\nabla$  can be defined for all  $t \in \mathbb{R}$ . When  $\nabla$  is the Kaehler connection of a Kaehler metric  $ds^2$ , *H*-completeness of  $\nabla$  is equivalent to completeness of  $ds^2$  (Theorem 4). An almost complex affine connection without torsion is said to be of Kaehler type if its Ricci tensor is hermitian (i.e., symmetric and *J*-invariant). In this case we shall show that an *H*-planner curve c(t) with a=0 in (0.1) is an *H*-geodesic if the development of c(t) is an *H*-geodesic in  $\mathbf{P}^n(\mathbf{C})$  (Theorem 5). Finally we shall prove

**Theorem 6.** Let  $\nabla$  and  $\nabla'$  be H-complete connections of Kaehler type with parallel Ricci tensors S and S' respectively. Suppose that either S=0 or S has at least one negative eigenvalue at one point, and that  $\nabla$  and  $\nabla'$  are H-projectively equivalent. Then we have  $\nabla = \nabla'$ .

When  $\nabla$  and  $\nabla'$  are the Kaehler connections of complete Kaehler metrics and both S and S' are parallel and negative semi-definite, the above result has been obtained by S. Ishihara and S. Tachibana [1].

Finally we remark that the present paper has been motivated by the paper of N. Tanaka on real projective transformations [7].

I would like to express my gratitude to my thesis advisor, Professor T. Ochiai for his valuable suggestions and encouragement.

### NOTATION

Throughout this paper the following standard conventions will be adopted. **R** (resp. **C**) denotes the real (resp. complex) number field. For  $z \in C$ , Re(z) is the real part of z. We write  $\mathbb{R}^m$  (resp.  $\mathbb{C}^m$ ) for the *m*-dimensional standard real (resp. complex) vector space. An element of  $\mathbb{R}^m$  (resp.  $\mathbb{C}^m$ ) is considered as a column vector. We denote by  $e_1, \dots, e_m$  the canonical basis of  $\mathbb{R}^m$  or  $\mathbb{C}^m$ . For  $x \in \mathbb{R}^m$  or  $\mathbb{C}^m$ , <sup>t</sup>x denotes the transpose of x. The general linear group acting on  $\mathbb{R}^m$  (resp.  $\mathbb{C}^m$ ) and its Lie algebla are denoted, respectively, by  $GL(m, \mathbb{R})$  (resp.  $GL(m, \mathbb{C})$ ) and  $gl(m, \mathbb{R})$  (resp.  $gl(m, \mathbb{C})$ ). We write  $1_m$  for the identity  $m \times m$  matrix. For an  $m \times m$  matrix A, det A denotes the determinant of A.

For a point p of manifold N,  $T_p(N)$  is the tangent space to N at p. For a differentiable mapping f,  $f_*$  and  $f^*$  are the differential and the codifferential of f respectively. For a Lie group G, its Lie algebla is written by the corresponding German letter g. For a G-principal bundle  $Q \rightarrow M$ ,  $R_a$  denotes the right tran-

slation by an element a of G acting on Q. For an element A or g,  $A^*$  denotes the fundamental vector field on Q corresponding to A.

### 1. *H*-projective equivalence

Let M be an m-dimensional manifold. Let us denote by  $j^{r}(f)$  the r-frame at p=f(0) given by a diffeomorphism f of a neighborhood of the origin 0 of  $\mathbb{R}^{m^{r}}$ onto an open subset of M. The set  $G^{r}(m)$  of r-frames at  $0 \in \mathbb{R}^{m}$  is a Lie group with multiplication defined by the composition of jets. The set  $F^{r}(M)$  of rframes of M is a principal bundle over M with natural projection  $\pi^{r}$  satisfying  $\pi^{r}(j^{r}(f))=f(0)$ , and with structure group  $G^{r}(m)$ .  $F^{1}(M)$  is nothing but the bundle of linear frames.

We have a natural inclusion of  $GL(m, \mathbf{R})$  into G'(m), defined by  $g \rightarrow j'(g)$  for  $g \in GL(m, \mathbf{R})$ . In particular  $GL(m, \mathbf{R})$  and  $G^{1}(m)$  are isomorphic by this inclusion. We shall identify  $GL(m, \mathbf{R})$  with  $G^{1}(m)$  and consider  $GL(m, \mathbf{R})$  as a subgroup of G'(m) by this inclusion.

Let f be a diffeomorphism of M onto a manifold N. Then f induces a bundle isomorphism  $f^{(r)}: F^r(M) \rightarrow F^r(N)$  defined by

$$f^{(r)}(j^r(h)) = j^r(f \cdot h)$$
 for  $j^r(h) \in F^r(M)$ .

We have a natural projection  $\nu: F^2(M) \to F^1(M)$  defined by  $\nu(j^2(f)) = j^1(f) \cdot (j^2(f) \in F^2(M))$ . A cross-section s:  $F^1(M) \to F^2(M)$  is said to be *admissible* if we have

$$s(xa) = s(x)a$$
 for  $x \in F^1(M)$  and  $a \in GL(m, \mathbb{R})$ .

The  $\mathbb{R}^m$  (resp.  $gl(m, \mathbb{R})$ )-component of the canonical form  $\Theta$  on  $F^2(M)$  (see [2]) for the meaning of terminology) is denoted by  $\Theta_{-1}$  (resp.  $\Theta_0$ ).

**Proposition 1** (S. Kobayashi [2]). For an admissble crosssection  $s: F^1(M) \rightarrow F^2(M)$ ,  $s^*\Theta_0$  is an affine connection on M without torsion. And this defines a one-to-one correspondence between affine connections on M without torsion and admissible cross-sections.

Let  $u^1, \dots, u^m$  be a local coordinate system in M, and let  $y^1, \dots, y^m$  be the natural coordinate system in  $\mathbb{R}^m$ . Each 2-frame u (resp.  $a \in G^2(m)$ ) has a unique polynomial respresentation  $u = j^2(f)$  (resp.  $a = j^2(f)$ ) of the form

$$f^{i}(y) = u^{i} + \sum u^{i}_{j}y^{j} + \frac{1}{2} \sum u^{i}_{jk}y^{j}y^{k}$$
  
(resp.  $f^{i}(y) = \sum a^{i}_{j}y^{j} + \frac{1}{2} \sum a^{i}_{jk}y^{j}y^{k}$ )

,

where  $u_{jk}^{i} = u_{kj}^{i}$  (resp.  $a_{jk}^{i} = a_{kj}^{i}$ ), and  $f^{i}(y)$  is the *i*-th coordinate of f(y) with respect

to  $u^1, \dots, u^m$  (resp.  $y^1, \dots, y^m$ ). We shall consider  $(u^i, u^i_j, u^i_{jk})$  (resp.  $(a^i_j, a^i_{jk})$ ) as a local coordinate system in  $F^2(M)$  (resp. a coordinate system in  $G^2(m)$ ). In the same way, a local coordinate system  $(u^i, u^i_j)$  in  $F^1(M)$  and a coordinate system  $(a^i_j)$  in  $G^1(m)$  are defined. The action of  $G^2(m)$  on  $F^2(M)$  is then given by

(1.1) 
$$(u^{i}, u^{i}_{j}, u^{i}_{jk}) (a^{i}_{j}, a^{i}_{jk}) = (u^{i}, \sum u^{i}_{q}a^{q}_{j}, \sum u^{i}_{q}a^{q}_{jk} + \sum u^{i}_{lr}a^{l}_{j}a^{r}_{k})$$

Let s be the cross-section corresponding by Proposition 1 to an affine connection  $\nabla$  without torsion. Then the local expression of s is

(1.2) 
$$s(u^i, u^i_j) = (u^i, u^i_j, -\sum u^q_j \Gamma^i_{q_l} u^i_k),$$

where  $\Gamma_{q_l}^i$  are the Christoffel's symbols of  $\nabla$  with respect to  $u^1, \dots, u^m$  ([2]).

Let L and  $L_0$  be as in (0.3). We shall consider  $L_0/(\text{center})$  as a subgroup of  $G^2(n)$  as follows. Let  $\pi: \mathbb{C}^{n+1} \to \{0\} \to \mathbb{P}^n(\mathbb{C})$  be the Hopf fibering. Identifying the subset

$$\left\{ \pi \begin{pmatrix} 1 \\ z \end{pmatrix} \in P^n(C) \mid z \in C^n \right\}$$

of  $P^n(C)$  with  $C^n = R^{2n}$ ,  $a \in L_0$  can be considered as a local diffeomorphism of  $R^{2n}$  leaving the origin 0 of  $R^{2n}$  fixed. Here  $C^n$  is identified with  $R^{2n}$  by the correspondence  $(z^1, \dots, z^n) \in C^n \to (x^1, \dots, x^n, y^1, \dots, y^n) \in R^{2n}$ ,  $z^i = x^i + \sqrt{-1}y^i$ ,  $x^i, y^i \in R$ ,  $i=1, \dots, n$ . It can be easily verified that  $j^2(a) = id$  if and only if a is the identity transformation of  $L/L_0$ . Hence  $L_0/(\text{center})$  can be identified with the group of 2-jets  $\{j^2(a) | a \in L_0\}$ . By a straightforward computation we have

Lemma 1.1. The expression of

$$a = \begin{pmatrix} 1 & t_{\mathfrak{p}} \\ 0 & 1_{\mathfrak{n}} \end{pmatrix} (mod \ center) \in L_0/(center)$$

as an element of  $G^2(n)$  is given by  $(\delta_j^i, a_{jk}^i)$  with

(1.3) 
$$a_{jk}^i = \delta_j^i \rho_k + \delta_k^i \rho_j - \phi_j^i \rho_s \phi_k^s - \phi_k^i \rho_s \phi_j^s,$$

where

$$\rho_{k} = \begin{cases} -v^{k} & \text{if } 1 \leq k \leq n \\ v^{k} & \text{if } n+1 \leq k \leq 2n, \end{cases} \begin{pmatrix} \phi_{j}^{i} \end{pmatrix} = \begin{pmatrix} 0 & -1_{n} \\ 1_{n} & 0 \end{pmatrix},$$

 $v^k$  being k-th component of  $v \in C^n = R^{2n}$ .

Let us denote the Lie algebras of L and  $L_0$  by I and  $I_0$  respectively. Subalgebras  $g_{-1}$ ,  $g_0$  and  $g_1$  of I are defined, respectively, as follows:

$$\mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix} \in \mathfrak{l} \, | \, u \in C^n \right\}$$

$$g_0 = \left\{ \begin{pmatrix} z & 0 \\ 0 & A \end{pmatrix} \in \mathfrak{l} | A \in \mathfrak{gl}(n, C) \right\}$$
$$g_1 = \left\{ \begin{pmatrix} 0 & tv \\ 0 & 0 \end{pmatrix} \in \mathfrak{l} | v \in C^n \right\}.$$

In the following,  $g_{-1}$  and,  $g_1$  are identified, respectively, with  $C^n$  and its dual space  $(C^n)^*$ . And  $g_0$  is identified with gl(n, C) by the correspondence

$$\begin{pmatrix} z & 0 \\ 0 & A \end{pmatrix} \in \mathfrak{g}_0 \to A - z \mathfrak{l}_n \in \mathfrak{gl}(n, C)$$

Therefore we can consider GL(n, C) as a subgroup of  $L_0/(\text{center})$  by the injection

$$B \in GL(n, \mathbb{C}) \to \begin{pmatrix} (\det B)^{-1/n+1} & 0\\ 0 & (\det B)^{-1/n+1}B \end{pmatrix} \pmod{\operatorname{center}} \in L_0/(\operatorname{center})$$

Put  $L_1 = \exp \mathfrak{g}_1$ . Then

(1.4)  $L_0/(\text{center}) = GL(n, C) \cdot L_1$  (semi-direct).

For the remainder of this section we suppose that M is a complex manifold of complex dimension n. Let  $\nabla$  be an almost complex affine connection without torsion on M and let  $\gamma$  be its connection form on the bundle C(M) of complex linear frames. By Proposition 1 there exists an admissible cross-section  $l: F^1(M)$  $\rightarrow F^2(M)$  corresponding to  $\nabla$ . Let  $\iota$  denote the inclusion map C(M) into  $F^1(M)$ . Then  $s=l \cdot \iota$  is an imbedding of C(M) into  $F^2(M)$  such that  $s^*\Theta_0 = \gamma$  and s(xa)= s(x)a for  $x \in C(M)$  and  $a \in GL(n, C)$ . Thus C(M) can be considered as a GL(n, C)-subbundle of  $F^2(M)$ . The group extension of C(M) to  $L_0/(\text{center})$ with respect to (1.4) will be denoted by  $Q(\nabla)$ .

**Theorem 1.** Let  $\nabla_1$  and  $\nabla_2$  be two almost complex affine connections without torsion. Then  $\nabla_1$  and  $\nabla_2$  are H-projectively equivalent if and only if  $Q(\nabla_1) = Q(\nabla_2)$ .

Proof. Let  $z^A = x^A + \sqrt{-1} x^{A+n}$ ,  $A = 1, \dots, n$ , be a complex local coordinate system in an open subset U of M. We define the natural almost complex stucture J on M by

$$J(\partial/\partial x^{A}) = \partial/\partial x^{A+n}, J(\partial/\partial x^{A+n}) = -\partial/\partial x^{A}, A = 1, \dots, n.$$

It follows from (1.2) that the injections

 $s_1: C(M) \to Q(\nabla_1)$  and  $s_2: C(M) \to Q(\nabla_2)$ 

corresponding respectively to  $\nabla_1$  and  $\nabla_2$  are expressed as follows:

$$s_1(x^i, x^i_j) = (x^i, x^i_j, -\sum_{m,l=1}^{2n} x^m_j(\Gamma_1)^i_{ml} x^l_k),$$

$$s_2(x^i, x^i_j) = (x^i, x^i_j, -\sum_{m,l=1}^{2n} x^m_j(\Gamma_2)^i_{ml} x^l_k),$$

where  $(\Gamma_1)_{m_l}^i$  and  $(\Gamma_2)_{m_l}^i$  are respectively the Christoffel's symbols of  $\nabla_1$  and  $\nabla_2$  with respect to  $x^1, \dots, x^n, x^{n+1}, \dots, x^{2n}$ . Note that  $(x^i, \delta_j^i) \in C(M)$ .

Assume that  $Q(\nabla_1)=Q(\nabla_2)$ . Then there exists a  $C^{\infty}$ -map  $a: U \to L_0$  such that

(1.5) 
$$s_1(x^i, \delta^i_j) = s_2(x^i, \delta^i_j)a, a = (a^i_j, a^i_{jk})$$

By the above formulas for local expression of  $s_1$  and  $s_2$ , we see that  $a_j^i = \delta_j^i$ . This means  $a(U) \subset L_1$ .

By (1.1) we have

$$-(\Gamma_2)^i_{jk}+a^i_{jk}=-(\Gamma_1)^i_{jk}$$
.

It follows from Lemma 1.1 that there exist real functions  $\rho_1, \dots, \rho_{2n}$  such that

$$(\Gamma_2)^i_{jk} - (\Gamma_1)^i_{jk} = \delta^i_j \rho_k + \delta^i_k \rho_j - \sum_s \phi^i_j \rho_s \phi^s_k - \sum_s \phi^i_k \rho_s \phi^s_j \,.$$

Let  $J_j^i$  be the local expression of J with respect to  $x^1, \dots, x^{2n}$ , then  $J_j^i = \phi_j^i$ . Thus we obtain

(1.6) 
$$(\Gamma_2)^i_{jk} - (\Gamma_1)^i_{jk} = \delta^i_j \rho_k + \delta^i_k \rho_j - \sum_s J^i_j \rho_s J^s_k - \sum_s J^i_k \rho_s J^s_j .$$

This shows that  $(\rho_i)$  is a 1-form. Thus  $\nabla_1$  and  $\nabla_2$  are *H*-projectively equivalent (cf. the definition in Introduction).

Conversely assume that  $\nabla_1$  and  $\nabla_2$  are *H*-projectively equivalent, *i.e.*,  $\nabla_1$  and  $\nabla_2$  are related by the formula (1.6). Define  $a=(\delta_j^i, a_{jk}^i) \in L_1$  by (1.3). Then (1.5) holds. Thus we see  $Q(\nabla_1)=Q(\nabla_2)$ . q.e.d.

Let  $\nabla$  be an almost complex affine connection without torsion and let  $s: C(M) \rightarrow Q(\nabla)$  be the cross-section corresponding to  $\nabla$ . For a biholomorphic transformation  $f: M \rightarrow M$ , define an admissible cross-section  $s': C(M) \rightarrow F^2(M)$  by  $s'=(f^{(2)})^{-1} \cdot s \cdot f^{(1)}$ . Since  $f^{(2)}$  leaves  $\Theta$  invariant, s' is the admissible cross-section corresponding to  $\nabla'=f^*\nabla$ . Thus we have  $f^{(2)}(Q(\nabla'))=Q(\nabla)$ . Therefore  $Q(\nabla)=Q(\nabla')$  if and only if  $f^{(2)}(Q(\nabla))=Q(\nabla)$ . Applying Corollary 11-1 in [4] to our case, we obtain

**Theorem 2.** Let  $\nabla$  be an almost complex affine connection without torsion. Then the group of all H-projective transformations of  $\nabla$  is a Lie group of finite dimension.

# 2. The development of an *H*-planner curve with respect to a Cartan connection of type $P^{n}(C)$

Let M be a manifold of dimension n, G a Lie group, K a closed subgroup

of G with dim G/K=n and Q a principal bundle over M with structure group K. A G/K-Cartan connection in the bundle Q is a 1-form  $\omega$  on Q with values in the Lie algebra g of G satisfying the following conditions:

- i)  $R_h^* \omega = Ad(h^{-1})\omega, \quad h \in K$
- ii)  $\omega(A^*) = A$ ,  $A \in \mathfrak{k}$
- iii)  $\omega(X) \neq 0$  for every nonzero vector X of Q.

A G/K-Cartan connection is said to be a  $P^{n}(C)$ -Cartan connection when G = L/(center) and  $K = L_{0}/(\text{center})$ , L and  $L_{0}$  being as in (0.3).

Let P be the group extension of Q to G, i.e.,  $P=Q \times_K G$ . Then a Cartan connection  $\omega$  in Q can be uniquely extended to a connection form on P, denoted by  $\tilde{\omega}$ . Let c(t) be a curve in M and let  $z(t) \in P$  be a horizontal lift of c(t) with respect to  $\tilde{\omega}$  such that  $z(0) \in Q$ . Then there exists a curve  $a(t) \in G$  such that  $z(t)a(t) \in Q$ . The development  $c^*(t)$  of c(t) at c(0) by  $\omega$  is defined by

$$c^*(t) = z(0) \cdot a(t) 0 \in Q \times_{\kappa} G/K$$
,

where 0 denotes the origin of G/K ([3]). We shall often identify  $c^*(t)$  with the curve  $a(t)0 \in G/K$ .

We shall consider the case when G=L/(center) and  $K=L/_0(\text{center})$ , Land  $L_0$  being as in (0.3). We call a curve c(t) in  $P^n(C)$  a projective line if there exists a 2-dimensional complex subspace W of  $C^{n+1}$  such that  $c(t) \in \pi(W-$ (0)). Let M be an *n*-dimensional complex manifold with an almost complex affine connection  $\nabla$  without torsion. Let us denote by  $\theta$  the canonical form on C(M) and by  $\gamma$  the connection form on C(M) corresponding to  $\nabla$ . We see in Section 1 that  $\nabla$  gives rise to a K-structure  $Q(\nabla)$  of second order, i.e., Ksubbundle of  $F^2(M)$ , and the injection  $s: C(M) \rightarrow Q(\nabla)$ . We know that there exists a Cartan connection  $\omega$  on  $Q(\nabla)$  satisfying

(2.1) 
$$s^*\omega_{-1} = \theta$$
 and  $s^*\omega_0 = \gamma$ ,

where  $\omega_{-1}$  and  $\omega_0$  are respectively  $g_{-1}$ -component and  $g_0$ -component of  $\omega$ .

We shall prove

**Proposition 2.1.** Let  $\nabla$  be an almost complex affine connection on a complex manifold M and let  $\omega$  be any Cartan connection on  $Q(\nabla)$  satisfying (2.1). Then, a curve in M is H-planner if and only if its development with respect to  $\omega$  is a projective line.

This follows directly from following Lemmas 2.2 and 2.3.

**Lemma 2.1.** Let c(t) be a curve in M and let x(t) be a horizontal lift of c(t) in C(M). Define  $v(t) \in \mathbb{C}^n$  by

$$(2.2) c'(t) = x(t)v(t)$$

Then

$$(2.3) \qquad \nabla_{c'}c' = ac' + bJc'$$

for certain smooth functions a and b if and only if

(2.4) 
$$v(t) = \exp\left(\int_{0}^{t} (a(t) + \sqrt{-1} b(t))dt\right)v(0).$$

Proof. From the difinition of covariant derivative, we obtain

 $\nabla_{c'(t)}c'(t) = x(t)v'(t) .$ 

By (2.2),

$$a(t)c'(t)+b(t)Jc'(t) = x(t) (a(t)+\sqrt{-1} b(t))v(t)$$
.

Therefore (2.3) holds if and only if

(2.5)  $v'(t) = (a(t) + \sqrt{-1} b(t))v(t)$ .

We have (2.4) if and only if (2.5) holds.

Let c(t) be a regular curve in M and let x(t) (resp. z(t) with z(0)=s(x(0))) be a horizontal lift of c(t) in C(M) (resp. P) with respect to  $\nabla$  (resp.  $\tilde{\omega}$ ). Choose a curve  $a(t) \in L$  satisfying

(2.6) 
$$z(t) [a(t)] = s(x(t)), a(0) = 1_{n+1},$$

where [a(t)] denotes the image of a(t) by the natural projection  $L \rightarrow L/(\text{center})$ . We may assume that a(t) is smooth since the center of L is discrete. We shall denote the (A+1)-th column vector of a(t) by  $a_A(t)$   $(0 \le A \le n)$ .

**Lemma 2.2.**  $a_0(t)$ ,  $a'_0(t)$  and  $a''_0(t)$  are linearly dependent for each t if and only if c(t) is H-planner.

Proof. Differentiating both sides of (2.6), we obtain

$$R_{[a(t)]*}z'(t) + (a(t)^{-1}a'(t))*_{z(t)[a(t)]} = s_*(x'(t)).$$

Hence we have

(2.7) 
$$a(t)^{-1}a'(t) = \tilde{\omega}(s_*x'(t)).$$

Let  $\tilde{\omega}_B^A$   $(0 \leq A, B \leq n)$  denote the (A+1, B+1)-component of  $\tilde{\omega}(s_*(x'(t)))$ . From (2.7) we obtain

$$a_{B'} = \sum_{A=0}^{n} a_A \tilde{\omega}_B^A \qquad 0 \leq B \leq n.$$

Hence

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q.e.d.

$$a_0^{\prime\prime} = \sum_{A=0}^n a_A rac{d\widetilde{\omega}_0^A}{dt} + \sum_{B=0}^n a_{B^\prime} \widetilde{\omega}_0^B = \sum_{A=0}^n a_A \Big( rac{d\widetilde{\omega}_0^A}{dt} + \sum_{B=0}^n \widetilde{\omega}_B^A \widetilde{\omega}_0^B \Big).$$

Since x(t) is horizontal with respect to  $\nabla$ , we have

$$\tilde{\omega}_0^0 = 0$$
 and  $\tilde{\omega}_k^j = 0$   $1 \leq j, k \leq n$ .

Thus we obtain

$$(2.8) a_0' = \sum_{k=1}^n a_k \widetilde{\omega}_0^k,$$

(2.9) 
$$a_0^{\prime\prime} = \sum_{k=1}^n a_0 \widetilde{\omega}_k^0 \widetilde{\omega}_0^k + \sum_{k=1}^n a_k \frac{d\widetilde{\omega}_0^k}{dt}.$$

Now suppose that  $a_0(t)$ ,  $a_0'(t)$  and  $a_0''(t)$  are linearly dependent for each t. Then there exist functions f(t), g(t) and h(t) such that

$$(2.10) fa_0 + ga_0' + ha_0'' = 0$$

and 
$$|f| + |g| + |h| \neq 0$$
.

Substituting (2.8) and (2.9) in (2.10), we have

$$(f(t)+h(t)\sum_{k=1}^{n}\tilde{\omega}_{k}^{0}\tilde{\omega}_{0}^{k})a_{0}+\sum_{j=1}^{n}\left(g(t)\tilde{\omega}_{0}^{j}+h(t)\frac{d\tilde{\omega}_{0}^{j}}{dt}\right)a_{j}=0.$$

Since  $a_0(t), a_1(t), \dots, a_n(t)$  are linearly independent, this is equivalent to the following:

$$f(t)+h(t) = \sum_{k=1}^{n} \tilde{\omega}_{k}^{0} \tilde{\omega}_{0}^{k} = 0,$$
  
$$g(t)\tilde{\omega}_{0}^{j}+h(t)\frac{d\tilde{\omega}_{0}^{j}}{dt} = 0 \text{ for } 1 \leq j \leq n.$$

Since  $c'(t) \neq 0$ , we have  $\tilde{\omega}_0^i \neq 0$  for a certain integer j  $(1 \leq i \leq n)$ . Hence  $h(t) \neq 0$  for each t. Putting

$$F(t) = -g(t)/h(t),$$

we obtain

$$F(t) heta(x'(t)) = d heta(x'(t))/dt$$
,

which shows that F is a differentiable function. Hence

(2.11) 
$$\theta(x'(t)) = \exp\left(\int_0^t F(t)dt\right)v_0, \ v_0 \in C^n$$

i.e.,

$$c'(t) = x(t) \exp\left(\int_0^t F(t)dt\right)v_0.$$

Therefore if follows from Lemma 2.1 that c(t) is *H*-planner. Taking the steps backwards, it is now easy to prove the converse.

**Lemma 2.3.**  $a_0(t)$ ,  $a_0'(t)$  and  $a_0''(t)$  are linearly dependent for each t if and only if there exists a 2-dimensional complex subspace W of  $C^{n+1}$  in which  $a_0(t)$  is contained for every t.

Proof. First note that  $a_0$  and  $a_0'$  are linearly independent for each t. This follows from formula (2.8), because  $\tilde{\omega}_0^k \neq 0$  for a certain integer  $k(1 \leq k \leq n)$  and  $a_0(t)$ ,  $a_1(t), \dots, a_n(t)$  are linearly independent for each t. Let  $b_A (0 \leq A \leq n)$  be the (A+1)-th component of  $a_0$  and define an  $(n+1) \times 3$  matrix B by

$$B = \begin{pmatrix} b_0 & b_0' & b_0'' \\ b_1 & b_1' & b_1'' \\ \vdots & \vdots & \vdots \\ b_n & b_n' & b_n'' \end{pmatrix}.$$

We may assume that in an open interval U containing  $t=t_0$ 

(2.12) 
$$\det \begin{pmatrix} b_0 & b_0' \\ b_1 & b_1' \end{pmatrix} \neq 0.$$

Now suppose that  $a_0$ ,  $a_0'$  and  $a_0''$  are linearly dependent. Since rank B=2,  $b_j(j=2,3,\dots,n)$  are solutions of the following ordinary linear differential equation of second order:

$$\det egin{pmatrix} b_0 & b_0' & b_0'' \ b_1 & b_1' & b_1'' \ x & x' & x'' \end{pmatrix} = 0 \ .$$

It follows that there exist constants  $\alpha_i$ ,  $\beta_i$   $(j=2,\dots,n)$  such that

$$b_j = \alpha_j b_0 + \beta_j b_1 \, .$$

Thus we obtain

$$a_0 = b_0 \begin{pmatrix} 1 \\ 0 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} + b_1 \begin{pmatrix} 0 \\ 1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}.$$

This shows that  $a_0(t)$  ( $t \in U$ ) is contained in the 2-dimensional complex subspace W of  $C^{n+1}$  spanned by

$$\begin{pmatrix} 1 \\ 0 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}.$$

We shall see that such a 2-dimensional subspace is independent of the choice of  $t_0$ . In fact, suppose that there exists a 1-dimensional subspace V of  $C^{n+1}$  such that  $a_0(t) \in V$  for every t in a certain open interval V contained in U. This contradicts (2.12). The proof for the converse is trivial. q.e.d.

EXAMPLE 2.1. S=SU(n+1,C)/(center) acts transitively on  $P^n(C)$  in a natural manner. Let H be the isotropy subgroup of S at

$$0 = \pi \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \boldsymbol{P}^{n}(\boldsymbol{C}).$$

Since each  $f \in S$  is a transformation of  $P^{n}(C)$  and a neighborhood of 0 in  $P^{n}(C)$ is identified with a neighborhood of 0 in  $\mathbb{R}^{2n}$  in a natural way, the 1-jet  $j_0^1(f)$  can be considered as a 1-frame of  $P^{n}(C)$  at f(0). The set of all 1-frames thus obtained defines an H-subbundle of the bundle  $C(P^n(C))$  of complex linear frames, which may be identified with the bundle S over  $P^{n}(C)$ . L and  $L_{0}$  being as in (0.3), let G and K denote L/(center) and  $L_0/(\text{center})$  respectively. Then the set of all 2-frames  $\{j_0^2(f) \mid \in G\}$  defines a K-subbundle of  $F^2(P^n(C))$ , and this can be identified with the bundle G over  $P^n(C)$ . The Maurer-Cartan form  $\omega$  of G is a G/K-Cartan connection in G. Define an injection s:  $C(P^n(C)) \rightarrow F^2(P^n(C))$ by  $s(xa) = \iota(x)a$  for  $x \in S$  and  $a \in GL(n, C)$ ,  $\iota$  being the inclusion map of S into G. Then the bundle G is the group extension of  $C(P^n(C))$  by s to the group K. The 1-form  $s^*\omega|_{g_0}$  on  $C(P^n(C))$ , restriction of values of  $s^*\omega$  to the Lie algebra  $\mathfrak{g}_0$  of  $GL(n, \mathbb{C})$ , corresponds to the Kaehler connection  $\nabla$  on the symmetric space  $P^{n}(C) = S/H$ . Thus  $\omega$  is a Cartan connection corresponding to  $\nabla$  and, in fact,  $\omega$  is the normal Cartan connection (see section 4 for the meaning of terminology) [4].  $\omega$  can be uniquely extended to a connection form  $\tilde{\omega}$  on the bundle  $G \times_{\kappa} G$  over  $P^{n}(C)$ . A horizontal lift of a curve  $c(t) = a(t) 0 \in P^{n}(C)$  $(a(t) \in G)$  with respect to  $\tilde{\omega}$  is  $z(t) = a(t) \cdot a(t)^{-1}a(0) \in G \times_{\kappa} G$ . In fact, noting that  $R_{a(0)^{-1}a(t)}z(t)$  belongs to the subbundle G, we have by the definition of  $\tilde{\omega}$ 

$$\begin{split} \widetilde{\omega}(z'(t)) &= \widetilde{\omega}(R_{a(t)^{-1}a(0)^*}R_{a(0)^{-1}a(t)^*}(z'(t))) \\ &= Ad(a(0)^{-1}a(t))\widetilde{\omega}(R_{a(0)^{-1}a(t)^*}z'(t))) \\ &= Ad(a(0)^{-1}a(t)) \left(\omega(a'(t)) + Ad(a(t)^{-1}a(0)) \left(a(0)^{-1}a(t) \left(a(t)^{-1}a(0)\right)'\right)\right) \\ &= Ad(a(0)^{-1}a(t)) \left(a(t)^{-1}a'(t) + \left(a(t)^{-1}\right)'a(t)\right) = 0 \,. \end{split}$$

Here we may assume a(t) is locally differentiable, since z(t) is independent of

the choice of  $a(t) \in G$ . Thus  $c^*(t) = a(0)^{-1}a(t)0 \in P^n(C)$  is the development of c(t) with respect to  $\omega$ .

Applying Proposition 2.1 to the case when  $M = P^{n}(C)$ , we obtain

**Corollary 2.1.** A curve in  $P^{n}(C)$  is H-planner if and only if it is a projective line.

By Proposition 2.1 and Corollary 2.1 we have

**Theorem 3.** The assumptions and notation being as in Proposition 2.1, a curve in M is H-planner if and only if its development with respect to  $\omega$  is H-planner.

### 3. H-completeness

We have defined an *H*-geodesic and *H*-completeness in Introduction. In this section we shall prove the following:

**Theorem 4.** Let M be a connected Kaehler manifold with a Kaehler metric g and let  $\nabla$  be the Kaehler connection of g. Then H-completeness of  $\nabla$  is equivalent to completeness of g.

Proof. Completeness of g follows from H-completeness of  $\nabla$  since a geodesic of g is clearly an H-geodesic of  $\nabla$ . Assume that g is complete. Let  $c(t) \ 0 \le t < L$  be an H-geodesic, i.e.,

(3.1) 
$$\nabla_{c'}c' = bJc'$$
 b: constant.

We shall show that this *H*-geodesic can be extended beyond *L*. Let x(t) be a horizontal lift of c(t) in the unitary frame bundle with respect to *g*. We can choose such a horizontal lift because  $\nabla$  is the Kaeler connection of *g*. Then c'(t)=x(t)v(t), where  $v(t)=\exp(\sqrt{-1} bt)v(0)$  by Lemma 2.1. Let  $\{t_k\}$  be an infinite sequence such that  $t_k \rightarrow L \ (k \rightarrow \infty)$ . Then

$$d(c(t_k), c(t_l)) \leq |\int_{t_k}^{t_l} g(c'(t), c'(t)dt| \\ = |t_k - t_l| |v(0)|,$$

where d denotes the distance function defined by g and |v(0)| denotes the usual norm of v(0) in  $\mathbb{C}^n$ . This shows that  $\{c(t_k)\}$  is a Canchy sequence in M with respect to d and hence converges to a point, say p. The limit point is independent of the choice of a sequence  $\{t_k\}$  converging to L. Let  $x^1, x^2, \dots, x^{2n}$  be a local coordinate system in a relatively compact coordinate neighborhood U of p. The local expression of (3.1) in U is

(3.2) 
$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = bJ^i_j \frac{dx^j}{dt}.$$

The exists a positive number  $\delta$  such that  $\{c(s)|L-\delta \leq s < L\} \subset U$ . Since the length of c' is constant,  $\{dx^{j}/dt(s)|L-\delta < s < L\}$  are bounded. It follows from (3.2) that  $\{|d^{2}x^{j}/dt^{2}(s)||L-\delta < s < L\}$  are also bounded, and less than a constant N. Let  $\{s_{k}\}$  be an infinite sequence such that  $s_{k} \rightarrow L$   $(k \rightarrow \infty)$ . Then

$$\left|\frac{dx^{j}}{dt}(s_{m})-\frac{dx^{j}}{dt}(s_{l})\right|=\left|\int_{s_{l}}^{s_{m}}\frac{d^{2}x^{j}}{dt^{2}}dt\right|\leq N|s_{m}-s_{l}|.$$

This shows that  $\{dx^i/dt(s_k)\}\$  is a Cauchy sequence in R, hence converges to a real number. The limit is independent of the choice of a sequence  $\{s_k\}\$  converging to L. Since c(t) and  $dx^i/dt$  converge when  $t \rightarrow L$ , the solution of (3.2) can be extended beyond L. This completes the proof of Theorem 3.

### 4. A connection of Kaehler type

In this section we shall prove a certain property of a connection of Kaehler type defined in Introduction. The result will be used to prove Theorem 5 and Theorem 6 in the following sections.

Let  $\nabla$  be an almost complex affine connection without torsion on a complex manifold M of complex dimension n. And let Q and  $s: C(M) \rightarrow Q$  be the corresponding  $L_0/(\text{center})$ -structure and the injection. We know that there exists a  $P^n(C)$ -Cartan connection  $\omega$  satisfying (2.1) for any almost complex affine connection without torsion which is H-projectively equivalent to  $\nabla$ ([4]). Define a subspace  $H_q$  of the tangent space  $T_q(Q)$  at  $q \in Q$  by

$$H_{q} = \{X \in T_{q}(Q) | \omega_{0}(X) = 0, \, \omega_{1}(X) = 0\}$$

Then  $\omega_{-1}: H_q \rightarrow \mathfrak{g}_{-1}$  is a linear isomorphism. Put

$$\Omega = d\omega + [\omega, \omega]/2.$$

Decompose  $\Omega$  into  $\Omega = \Omega_{-1} \oplus \Omega_0 \oplus \Omega_1$ ,  $\Omega_{-1}$ ,  $\Omega_0$  and  $\Omega_1$  being  $\mathfrak{g}_{-1}$ ,  $\mathfrak{g}_0$ - and  $\mathfrak{g}_1$ components of  $\Omega$  respectively. Let  $\{v_i\}_{i=1,2,\cdots,2n}$  be a real basis of  $\mathfrak{g}_{-1}$  and let  $\{z^i\}$  be its dual basis in  $\mathfrak{g}_1$  with respect to the Killing-Cartan form B of  $\mathfrak{L}$  which is non-singular on  $\mathfrak{g}_{-1} \times \mathfrak{g}_1$ . Choose  $X_i \in H_q$  such that  $\omega_{-1}(X_i) = v_i$ . We shall call  $\omega$  a  $P^n(C)$ -nomal Cartan connection if  $\Omega_0$  satisfies

$$\sum z^i \Omega_0(X_i, Y) = 0$$
 at each point  $q \in Q$ .

If  $n \ge 2$ , there exists uniquely a  $P^n(C)$ -normal Cartan connection ([4]).

For the  $P^n(C)$ -normal Cartan connection, define  $E_x: \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_1 (x \in C(M))$  by

(4.1) 
$$E_{\mathfrak{s}}(\theta(Y)) = \mathfrak{s}^*\omega_1(Y) \qquad Y \in T_{\mathfrak{s}}(C(M)).$$

 $E_x$  is well-defined. In fact, if  $\theta_x(Y)=0$ , there exists  $A \in \mathfrak{gl}(n, \mathbb{C})$  such that  $Y=(A^*)_x$ . Hence

$$(s^*\omega_1)(Y) = \omega_1(s_*(A^*)_x) = \omega_1((A^*)_{s(x)}) = 0.$$

Let us denote by  $C^{p,q}$   $(-1 \le p \le 3)$  the set of all  $\mathfrak{g}_{p-1}$ -valued q-skew-symmetric multilinear form on  $\mathfrak{g}_{-1}$ , where  $\mathfrak{g}_{-2} = \{0\}$  and  $\mathfrak{g}_2 = \{0\}$ . Define d:  $C^{p,q} \rightarrow C^{p-1,q+1}$  by

$$dc(y_1, \dots, y_{q+1}) = \sum_{i=1}^{q+1} (-1)^{i+1} [y^i, C(y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_{q+1})],$$

 $y_1, \cdots, y_{q+1} \in \mathfrak{g}_{-1}$ . And define  $d^* \colon C^{p,q} \to C^{p+1,q-1}$  by

$$(d^*c)(y_1,\cdots,y_{q-1}) = \sum_{i=1}^{2\pi} [z^i, c(v_i,y_1,\cdots,y_{q-1})],$$

 $y_1, \dots, y_{q-1} \in \mathfrak{g}_{-1}$ , where  $\{v_i\}$  denotes a basis of  $\mathfrak{g}_{-1}$  and  $\{z^i\}$  denotes the dual basis of  $\{v_i\}$  in  $\mathfrak{g}_1$  with respect to the Killing-Cartan form B of  $\mathfrak{L}$ .

We shall denote by S the Ricci tensor field of  $\nabla$ . Define  $S_x: \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \rightarrow \mathbb{R}$ and  $T_x: \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_1$  for  $x \in C(M)$  by

(4.2) 
$$S_x(u, v) = S(xu, xv) \text{ and}$$
$$B(T_x(u), v) = S_x(u, v)$$

respectively. Then

(4.3) 
$$T_{x} = -d^{*}dE_{x} ([4]).$$

For  $z \in g_1$  and  $v \in g_{-1}$  we shall denote by  $\langle z, v \rangle$  the real part of zv.

**Lemma 4.1.** Let  $\nabla$  be a connection of Kaehler type on an n-dimensional complex manifold  $(n \ge 2)$ . Then

$$\langle E_x(u), v \rangle = -S(xu, xv)/2(n+1)$$

or equivalently

$$E_x(u)v = -\{S(xu, xv) - \sqrt{-1} S(xu, Jxv)\}/2(n+1).$$

In particular,  $E_x(v)v$  is real valued.

Proof. We write E for  $E_x$  for simplicity. From the definition of the Killing-Cartan form of  $\mathfrak{A}$ , we obtain

$$(4.4) B(X, Y)/4(n+1) = \operatorname{Re}(\text{the trace of } XY),$$

for X,  $Y \in \mathfrak{A}$ . Hence we consider  $\mathfrak{A}$  as a real Lie algebra. Since  $\{{}^{i}e_{i}/4(n+1), -\sqrt{-1}{}^{i}e_{i}/4(n+1)\}_{i=1,2,\cdots,n}$  is the dual basis of  $\mathfrak{g}_{1}$  corresponding to a real basis  $\{e_{i}, \sqrt{-1} e_{i}\}_{i=1,\cdots,n}$  of  $\mathfrak{g}_{-1}$  with respect to B, we have

(4.5) 
$$d^*dE(v) = \sum_{i=1}^n \frac{1}{4(n+1)} [{}^{i}e_i, dE(e_i, v)] + \sum_{i=1}^n \frac{1}{4(n+1)} [-\sqrt{-1} {}^{i}e_i, dE(\sqrt{-1} e_i, v)]$$

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$$=\sum_{i=1}^{n} \frac{1}{4(n+1)} \{ [{}^{i}e_{i}, [e_{i}, E(v)] - [v, E(e_{i})]] + [-\sqrt{-1} {}^{t}e_{i}, [\sqrt{-1} {}^{e}e_{i}, E(v)] - [v, E(\sqrt{-1} {}^{e}e_{i})]] \} .$$

On the other hand, for  $v \in \mathfrak{g}_{-1}$ ,  $z \in \mathfrak{g}_1$  and  $A \in \mathfrak{g}_0$ ,

$$[v, z] = vz + (zv)1_n,$$
  
$$[z, A] = zA.$$

Applying these formulas to (4.5), we obtain

(4.6) 
$$d^*dE(v) = \frac{1}{4(n+1)} \sum_{i=1}^n \left\{ 2E(v) + 2^i e_i E(v) e_i - ({}^t e_i v E(e_i) + {}^t e_i E(e_i) v) + (\sqrt{-1} {}^t e_i v E(\sqrt{-1} {}^e_i) + \sqrt{-1} {}^t e_i E(\sqrt{-1} {}^e_i) v \right\} .$$

By virtue of (4.2), (4.3), (4.4) and (4.6),

$$(4.7) \qquad -S_{x}(u,v) = 2(n+1)\langle E(u),v\rangle - \sum_{i=1}^{n} \langle e_{i}uE(e_{i}) + e_{i}E(e_{i})u,v\rangle \\ + \sum_{i=1}^{n} \langle \sqrt{-1} e_{i}uE(\sqrt{-1} e_{i}) + \sqrt{-1} e_{i}E(\sqrt{-1} e_{i})u,v\rangle.$$

Since  $S_x$  is symmetric, we have by (4.7)

(4.8) 
$$\langle E(u), v \rangle = \langle E(v), u \rangle$$
 for any  $u, v \in g_{-1}$ .

Put  $u=e_j$  and  $v=e_k$  in (4.7). Then we obtain

$$-S_{\mathbf{x}}(e_{j}, e_{k}) = (2n+1) \langle E(e_{j}), e_{k} \rangle - \langle E \langle (e_{k}), e_{j} \rangle + \langle \sqrt{-1} E(\sqrt{-1} e_{j}), e_{k} \rangle + \langle \sqrt{-1} E(\sqrt{-1} e_{k}), e_{j} \rangle.$$

Thus, by (4.8)

(4.9) 
$$-S_{x}(e_{j},e_{k}) = 2n\langle E(e_{j}),e_{k}\rangle + 2\langle \sqrt{-1} E(\sqrt{-1} e_{j}),e_{k}\rangle.$$

Analogously, we have

$$(4.10) \qquad -S_x(\sqrt{-1}\,e_j,\sqrt{-1}\,e_k) = 2n\langle\sqrt{-1}\,E(\sqrt{-1}\,e_j),e_k\rangle + 2\langle E(e_j),e_k\rangle,$$

$$(4.11) \qquad -S_{\mathbf{x}}(e_{j}, \sqrt{-1} e_{k}) = 2n\langle E(e_{j}), \sqrt{-1} e_{k} \rangle - 2\langle E(\sqrt{-1} e_{j}), e_{k} \rangle$$

$$(4.12) \quad -S_x(\sqrt{-1}\,e_j,e_k) = 2n\langle E(\sqrt{-1}\,e_j),e_k\rangle - 2\langle E(e_j),\sqrt{-1}\,e_k\rangle.$$

Since  $S(e_j, e_k) = S(\sqrt{-1} e_j, \sqrt{-1} e_k)$ , (4.9) and (4.10) give

$$2(n-1)\langle \sqrt{-1} E(\sqrt{-1} e_j), e_k \rangle = 2(n-1)\langle E(e_j), e_k \rangle.$$

Since  $n \ge 2$  by assumption, we have

(4.13) 
$$\langle E(e_j), e_k \rangle = \langle \sqrt{-1} E(\sqrt{-1} e_j), e_k \rangle.$$

In a similar fashion, (4.11) and (4.12) give

(4.14) 
$$\langle E(e_j), \sqrt{-1} e_k \rangle = \langle \sqrt{-1} E(\sqrt{-1} e_j), \sqrt{-1} e_k \rangle.$$

By virtue of (4.13) and (4.14),

$$E(e_j) = \sqrt{-1} E(\sqrt{-1} e_j) \, .$$

Applying this to (4.7), we obtain

$$-S_x(u, v) = 2(n+1)\langle E(u), v \rangle$$
.

The second formula in Lemma 4.1 is now easy to show, because the imaginary part of E(u)v is  $-\langle E(u), \sqrt{-1}v \rangle$ . This completes the proof of Lemma 4.1.

# 5. The development of an *H*-geodesic with respect to the $P^{n}(C)$ -normal Cartan connection

Let  $\nabla$  be a connection of Kaehler type on a complex manifold M. Let us denote by  $\{\nabla\}$  the family of almost complex affine connections without torsion which are *H*-projectively equivalent to  $\nabla$ . We see in Section 4 that  $\{\nabla\}$  determines uniquely a  $P^n(C)$ -normal Cartan connection. We shall prove

**Proposition 5.1.** Assume that the development of a curve c(t) with respect to the normal Cartan connection is contained in  $\pi(W-\{0\})$  for a 2-dimensional real subspace W of  $C^{n+1}$ . Then, under a certain change of parameter, c(t) is an H-geodesic.

Proof. By Theorem 3 c(t) is an *H*-planner curve. Hence c(t) satisfies  $\nabla_{c'}c' = ac' + bJc'$  for cetrin real functions a and b. Define a curve  $\tilde{c}$  by

(5.0) 
$$\tilde{c}(T) = c(t), \quad T = \int_0^t \exp\left(\int_0^t a(t)dt\right) dt \, .$$

Then we have

 $abla \widetilde{c}' \widetilde{c}' = \widetilde{b} J \widetilde{c}'$ ,  $\widetilde{b}$ : a real function.

Since  $\tilde{c}(t)$  satisfies the assumption of Proposition 5.1, we may assume  $\nabla_{c'}c' = bJc'$ . Let x(t) be a horizontal lift in C(M). Then by Lemma 2.1,

$$c'(t) = x(t) (\exp \sqrt{-1} \int_0^t b dt) v$$
,

v being a certain vector in  $C^n$ . This is equivalent to

$$d\theta(x'(t))/dt = \sqrt{-1} b\theta(x'(t)).$$

Here  $\theta$  denotes the canonical form on C(M). The notation being as in Lemma 2.2, put

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$$f(t) = -\sum_{k=1}^{n} \widetilde{\omega}_{k}^{0} \widetilde{\omega}_{0}^{k}$$
.

By the definition of  $E_{x(t)}: \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_1$  given in (4.1), we see

(5.1) 
$$f(t) = -E_{x(t)}(\theta(x'(t)))\theta(x'(t)).$$

It follows from Lemma 4.1 that f(t) is a real-valued function. Let a(t) be as in (2.6). Then by (2.8) and (2.9) in Lemma 2.2, we have

(5.2) 
$$a_0'' - \sqrt{-1} b a_0' + f a_0 = 0$$
.

Let  $c_1$  and  $c_2$  be the solutions of

(5.3) 
$$c'' - \sqrt{-1} bc' + fc = 0$$

with initial values, respectively,

$$\begin{cases} c_1(0) = 1 \\ c_1'(0) = 0 \end{cases} \begin{cases} c_2(0) = 0 \\ c_2'(0) = 1 \end{cases}.$$

Then

$$a_0 = \begin{pmatrix} c_1 \\ c_2 v \end{pmatrix}.$$

Let W be a 2-dimensional real subspace of  $C^{n+1}$  such that

$$\pi\binom{c_1}{c_2 v} \subset \pi(W - \{0\}).$$

Since  $c_1(0) = 1$  and  $c_2(0) = 0$ ,

$$\pi\binom{1}{0} \in \pi(W - \{0\}).$$

So there exists a constant  $s \in C^* = C - \{0\}$  such that

$$s\binom{1}{0} \in W - \{0\}, i.e., \binom{1}{0} \in s^{-1}W - \{0\}.$$

Therefore we may assume

$$\binom{1}{0} \in W - \{0\} .$$

Lemma 5.1. There exists a differentiable function h such that

$$h\binom{c_1}{c_2v} \in W - \{0\} .$$

in an open interval U in which  $c_2 \neq 0$ .

Proof of Lemma 5.1. Let

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \tilde{\alpha} = \begin{pmatrix} \alpha^0 \\ \alpha \end{pmatrix} \quad (\alpha^0 \in \boldsymbol{C}, \, \alpha \in \boldsymbol{C}^n)$$

be a basis of W. Putting

$$d(t) = \begin{pmatrix} c_1(t) \\ c_2(t)v \end{pmatrix},$$

we have  $d=z(u_1e_1+u_2\tilde{\alpha})$  for certain real valued functions  $u_1$  and  $u_2$ , and a complex valued non-zero function z.  $u_2 \neq 0$  follows from the assumption  $c_2 \neq 0$ . We only have to put  $h=1/zu_2$  to complete the proof.

By Lemma 5.1 we see that  $h(t_0)d(t_0)$  and  $e_1$  for  $t_0 \in U$  is a basis of W. So

$$h\binom{c_1}{c_2v} = A\binom{1}{0} + Bh(t_0)\binom{c_1(t_0)}{c_2(t_0)v}$$

for certain real-valued functions A and B. Hence

$$c_1/c_2 = A/Bh(t_0)c_2(t_0) + c_1(t_0)/c_2(t_0)$$

Put

(5.4) 
$$D = c_1/c_2, G = A/B \text{ and } K = 1/h(t_0)c_2(t_0)$$
.

Then

$$(5.5) D' = G'K$$

**Lemma 5.2.** Let D be as in (5.4) and let U be an open interval in which  $c_2(t) \neq 0$ . Then

(5.6) 
$$D' = \frac{D'(t_0) (c_2(t_0))^2}{(c_2(t))^2} \exp(\sqrt{-1} \int_{t_0}^t b dt) \quad t_0 \in U.$$

Proof of Lemma 5.2. Since  $c_1$  is a solution of (5.3), i.e.,  $c_1'' - \sqrt{-1} bc_1' + fc_1 = 0$ , substituting  $c_1 = Dc_2$  in this equation, we have  $D''c_2 + (2c_2' - \sqrt{-1} bc_2)D' = 0$ . Hence

$$D'' + (2c_2'/c_2 - \sqrt{-1} b)D' = 0$$

Solving this equation on D', we obtain (5.6). This completes the proof of Lemma 5.2.

By (5.5) and (5.6) we have

$$\frac{D'(t_0)(c_2(t_0))^2}{(c_2)^2}\exp(\sqrt{-1}\int_{t_0}^t bdt) = G'K.$$

Put  $K/D'(t_0) (c_2(t_0))^2 = l \exp(\sqrt{-1} \psi)$ ,  $c_2 = r_2 \exp(\sqrt{-1} \theta_2)$ , where  $l, \psi, r_2$  and  $\theta_2$  are real functions. Then

$$\exp \{\sqrt{-1} (-2\theta_2 + \int_{t_0}^t b dt - \psi)\} = G' l(r_2)^2.$$

Since G', l and  $r_2$  are continuous real functions, we have

(5.7) 
$$-2\theta_2 + \int_{t_0}^t bdt - \psi = 0 \pmod{\pi}.$$

Differentiating (5.7), we obtain

$$(5.8) \qquad \qquad \theta_2' = b/2$$

Let

$$(5.9) c_2 = r_2 \exp(\sqrt{-1} \theta_2)$$

be the expression by polar coordinates. Since  $c_2$  is a solution of (5.3), i.e.,  $c_2'' - \sqrt{-1} bc_2' + fc_2 = 0$ , putting (5.9) in this equation, we have

$$\exp(\sqrt{-1}\,\theta_2)\,\{(r_2''-r_2(\theta_2')^2+br_2\theta_2'+fr_2)+\sqrt{-1}\,(2r_2'\theta_2'+r_2\theta_2''-br_2')\}=0\,.$$

Hence

(5.10) 
$$2r_2'\theta_2' + r_2\theta_2'' - br_2' = 0.$$

Substituting (5.8) in (5.10), we obtain  $r_2b_2'=0$ . Since  $r_2 \neq 0$ , we have b'=0. This holds in an open interval in which  $c_2 \neq 0$ . However, since  $c_2$  is a solution of an ordinary linear differential equation of second order, the zero points of  $c_2$ are discrete. Thus b is constant, namely c(t) is an H-geodesic. This completes the proof of Proposition 5.1.

**Proposition 5.2.** Let  $\nabla$  be a connection of Kaehler type whose Ricci tensor is parallel, and let c(t) be an H-geodesic with respect to  $\nabla$  under a certain change of parameter. Then there exists a 2-dimensional real subspace W of  $C^{n+1}$  such that the development of c(t) with respect to the normal Cartan connection is contained in  $\pi(W - \{0\})$ .

Proof. We may assume that c(t) is an *H*-geodesic, since existence of such a 2-dimensional real subspace W of  $C^{n+1}$  as above is independent of the choice of a parameter. Let x(t) be a horizontal lift in C(M). Then, by Lemma 2.1,

$$c'(t) = x(t) \exp(\sqrt{-1} bt)v, \qquad v \in C'$$

Since c(t) is an *H*-geodesic, b is a real constant. The notation being as in the proof of Proposition 5.1, we have

$$a_0'' - \sqrt{-1} b a_0' + f a_0 = 0$$
.

Lemma 4.1 shows that f is a real constant, because the Ricci tensor of  $\nabla$  is

parallel. We shall denote this constant by -k. Let  $c_1$  and  $c_2$  be the solutions of

$$c'' - \sqrt{-1} bc' - kc = 0$$

with initial values, respectively,

$$\begin{cases} c_1(0) = 1 \\ c_1'(0) = 0 \end{cases} \begin{cases} c_2(0) = 0 \\ c_2'(0) = 1 \end{cases}.$$

Then

$$a_0 = \binom{c_1}{c_2 v}.$$

We only have to prove existence of a 2-dimensional real subspace W of  $C^{n+1}$  satisfying  $\pi(a_0(t)) \subset \pi(W - \{0\})$ . Since b and k are real constants, the solutions  $c_1$  and  $c_2$  can be obtained explicitly as follows:

i) If 
$$D = -b^2 + 4k \neq 0$$
, then

$$c_{1} = \frac{1}{2\sqrt{D}} \exp(\sqrt{-1} bt/2) \left\{ (-\sqrt{-1} b + \sqrt{D}) \exp(\sqrt{D} t/2) + (\sqrt{-1} b + \sqrt{D}) \exp(-\sqrt{D} t/2) \right\},$$
  
$$c_{2} = \frac{1}{\sqrt{D}} \exp(\sqrt{-1} bt/2) \left\{ \exp(\sqrt{D} t/2) - \exp(-\sqrt{D} t/2) \right\}.$$

- ii) If  $-b^2+4jk=0$  and  $k \neq 0$ , then  $c_1 = (-\sqrt{-1} bt/2) \exp(\sqrt{-1} bt/2) + \exp(\sqrt{-1} bt/2)$ ,  $c_2 = t \exp(\sqrt{-1} bt/2)$ .
- iii) If b=0 and k=0, then

 $c_1=1, \qquad c_2=t.$ 

Thus we can choose a real basis  $\{\alpha, \beta\}$  of W as follows:

i) If D > 0, then

$$lpha = egin{pmatrix} -\sqrt{-1} \ b+D \ 2 \ v \end{pmatrix} egin{pmatrix} eta = egin{pmatrix} \sqrt{-1} \ b+D \ 2 \ -v \end{pmatrix}, \ eta = egin{pmatrix} \sqrt{-1} \ b+D \ 2 \ -v \end{pmatrix},$$

because

$$\pi(a_0(t)) = \pi\left(\exp\left(\frac{\sqrt{D}}{2}t\right)\alpha + \exp\left(\frac{-\sqrt{D}}{2}t\right)\beta\right).$$

i)' If 
$$D < 0$$
, then

$$\alpha = \binom{\sqrt{D}}{0} \qquad \beta = \binom{-\sqrt{-1} b}{2v},$$

because

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$$\pi(a_0(t)) = \pi\left(\cos\left(\frac{\sqrt{-D}}{2}t\right)\alpha + \sqrt{-1}\sin\left(\frac{\sqrt{-D}}{2}t\right)\beta\right).$$

ii) If D=0 and  $k\neq 0$ , then

$$\alpha = \begin{pmatrix} \frac{-\sqrt{-1}}{2} b \\ v \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

because

$$\pi(a_0(t))=\pi(t\alpha+\beta).$$

iii) If b=0 and k=0, then

$$\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} 0 \\ v \end{pmatrix}. \qquad \text{q.e.d.}$$

From Propositions 5.1 and 5.2 follows

**Corollary 5.1.** Let  $\nabla$  be a connection of Kaehler type whose Ricci tensor is parallel. Then a curve c(t) is an H-geodesic with respect to  $\nabla$  under a certain change of parameter if and only if there exists a 2-dimensional real subspace W of  $C^{n+1}$  such that the development of c(t) with respect to the normal Cartan connection is contained in  $\pi(W - \{0\})$ .

We have detailed the development of a curve in  $P^n(C)$  in Example 2.1. Applying Corollary 5.1 to  $M=P^n(C)$ , we obtain

**Corollary 5.2.** A curve c(t) in  $P^n(C)$  is an H-geodesic under a certain change of parameter if and only if there exists a 2-dimensional real subspace W of  $C^{n+1}$  such that c(t) is contained in  $\pi(W-\{0\})$ .

By Proposition 5.1 and Corollary 5.2 we have

**Theorem 5.** Let  $\nabla$  be a connection of Kaehler type. Then a curve c(t) is an H-geodesic with respect to  $\nabla$  under a certain change of parameter, if the development of c(t) with respect to the normal Cartan connection is an H-geodesic in  $P^n(C)$ .

### 6. Proof of Theorem 6

In this section we shall prove Theorem 6.

**Lemma 6.1.** Let  $c_1$  and  $c_2$  be the solutions of the following differential equation

(6.1) 
$$u'' - \sqrt{-1} bu' - ku = 0$$

with initial conditions

(6.2) 
$$c_1(0) = 1, c_1'(0) = 0 \text{ and } c_2(0) = 0, c_2'(0) = 1,$$

where b and k are real constants. Then we have the following:

- a) If  $-b^2+4k>0$ , then  $|\lim_{t\to\infty} c_2/c_1| = 1/\sqrt{k}$ .
- b) If  $-b^2+4k < 0$ , then  $\lim_{t \to \infty} c_2/c_1$  does not exist.
- c) If  $-b^2+4k=0$  and  $k \neq 0$ , then  $\lim_{t \to \infty} c_2/c_1 = 1/\sqrt{k}$ . d) If b=0 and k=0, then  $\lim_{t \to \infty} c_1/c_2 = 0$ .

Proof. We have obtained the solutions  $c_1$  and  $c_2$  explicitly in the proof of Proposition 5.2. Lemma 6.1 follows directly from these results. q.e.d.

For the remainder of this section, let  $\nabla$  be an *H*-complete connection of Kaehler type on a complex manifold M whose Ricci tensor S is parallel. Let  $Q(\nabla)$  and  $s:C(M) \rightarrow Q(\nabla)$  be, as explained in Section 2, the  $L_0/(\text{center})$ -structure and the injection corresponding to  $\nabla$  respectively. Let  $E_x: \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_1 (x \in C(M))$ be as in (4.1). Define a subset  $\Phi_{E_x}$  of  $P^n(C)$  by

(6.3) 
$$\Phi_{E_x} = \left\{ \pi \begin{pmatrix} v^0 \\ v \end{pmatrix} \in \boldsymbol{P}^n(\boldsymbol{C}) | - |v^0|^2 + E_x(v)v = 0, v^0 \in \boldsymbol{C}, v \in \boldsymbol{C}^n \right\}.$$

**Lemma 6.2.** Let c(t) and x(t) be an H-geodesic of  $\nabla$  and its horizontal lift in C(M) respectively. Put x=x(0). And let  $a(t) \in L$  be as in (2.6). If lim a(t)0 exists, it belong to  $\Phi_{E_x}$ .

Proof. By Lemma 2.1

$$c'(t) = x(t) \exp \left(\int_0^t F(t)dt\right)v$$
,

for a certain function F and a vector  $v \in C^n$ . We see by the definition of an *H*-geodesic  $F(t) = \sqrt{-1} b$ , b being a constant. Thus  $\theta(x'(t)) = \exp((\sqrt{-1} bt)v)$ . On the ohter hand, by Lemma 4.1 and by the assumption that the Ricci tensor field is parallel, we easily see that  $E_{x(t)}(u)w$  is constant for any u and  $w \in g_{-1}$ . Thus  $f(t) = -E_{x(t)}(v)v$  in (5.1) is a constant, which we shall denote by -k.

Let  $a_0$  denote the first column vector of a(t). Then by (5.2)  $a_0$  is the solution of

$$a_0'' - \sqrt{-1} \ b \ a_0' - ka_0 = 0$$

with initial conditions

$$a_0(0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad a_0'(0) = \begin{pmatrix} 0 \\ v \end{pmatrix}.$$

Let  $c_1$  and  $c_2$  be the solutions of (6.1) with initial conditions (6.2), then

$$a_0(t) = \begin{pmatrix} c_1(t) \\ c_2(t)v \end{pmatrix}.$$

Thus

$$a(t)0 = \pi(a_0(t)) = \pi \begin{pmatrix} c_1(t) \\ c_2(t)v \end{pmatrix}$$

Lemma 6.2 now follows from Lemma 6.1 and the definition of  $\Phi_{E_x}$  in (6.3). q.e.d.

**Lemma 6.3.** For any  $\tilde{v} \in \Phi_{E_x}$ , there exists a geodesic c(t) with  $c(0) = \pi^1(x)$  such that

$$\lim_{t\to\infty}a(t)0=\tilde{v},$$

a(t) being defined in (2.6).

Proof. By the difinition of  $\Phi_{E_x}$ ,

$$ilde{v}=\pi{v^0\choose v}$$

for some  $v^0 \in C$  and  $v \in C^n$  with  $-|v^0|^2 + E_x(v)v = 0$ . In the case when  $E_x(v)v > 0$ , take a geodesic with initial conditions  $c(0) = \pi^1(x)$ ,  $c'(0) = x(v/v^0)$ . Then by the same argument as in Lemma 6.2,

(6.4) 
$$a(t)0 = \pi \binom{c_1(t)}{c_2(t)v/v^0},$$

where  $c_1$  and  $c_2$  are the solutions of u''-ku=0  $(k=E_x(v/v^0)v/v^0)$  with initial conditions (6.2). By i) with b=0 in the proof of Proposition 5.2,

$$\lim_{t \to \infty} c_2/c_1 = 1/\sqrt{k} = |v^0|/\sqrt{E_s(v)v} = 1.$$

Thus we have

$$\lim_{t\to\infty}a(t)0=\pi\binom{v^0}{v}.$$

In the case when  $E_x(v)v=0$ , i.e.,  $v^0=0$ , take a geodesic with initial conditions  $c(0)=\pi^1(x)$ , c'(0)=xv. Then by the same argument as above

(6.5) 
$$a(t)0 = \pi \binom{c_1(t)}{c_2(t)v},$$

where  $c_1$  and  $c_2$  are solutions of u''=0 with initial conditions (6.2). By d) in Lemma 6.1,

$$\lim_{t\to\infty}c_1/c_2=0$$

Hence

$$\lim_{t\to\infty}a(t)0=\pi\binom{0}{v}.$$

This completes the proof of Lemma 6.3.

Define a subset  $\Phi(p)$  of  $Q(\nabla) \times_{L_0} P^n(C)$  for  $p \in M$  by  $\Phi(p) = s(x) \Phi_{E_x}$  with  $\pi^1(x) = p$ . This is independent of the cohice of  $x \in C(M)$ .

Let  $\overline{\nabla}$  be another *H*-complete connection of Kaehler type on *M* whose Ricci tensor  $\overline{S}$  is parallel. Then  $\overline{s}: C(M) \to Q(\overline{\nabla}), \ \overline{E}_x: \mathfrak{g}_{-1} \to \mathfrak{g}_1, \ \overline{\Phi}_{\overline{E}_x} \subset \mathbf{P}^n(\mathbf{C})$ and  $\overline{\Phi}(p)$  can be defined in the same way as above. Assume that  $\overline{\nabla}$  is *H*-projectively equivalent to  $\nabla$ . Then  $Q(\nabla) = Q(\overline{\nabla})$  by Theorem 1. Further we obtain the following:

**Lemma 6.4.**  $\Phi(p) = \overline{\Phi}(p)$ .

Proof. Let  $q^*$  be an arbitrary element in  $\Phi(p)$ . Then, by Lemma 6.3, there exists a geodesic c(t) with respect to  $\nabla$  such that the limit point of its development is  $q^*$ . By Proposition 5.2 and Corollary 5.1 we see that c(t) is an *H*-geodesic of  $\nabla$  under a certain change of parameter. Taking into consideration (5.0) which shows how to change parameter, we have  $q^* \in \overline{\Phi}(p)$  by lemma 6.2. Thus  $\Phi(p) \subset \overline{\Phi}(p)$ . In a similar fashion we have  $\overline{\Phi}(p) \subset \Phi(p)$ , and the proof is complete.

In view of (1.4) we can define  $F: C(M) \rightarrow \mathfrak{g}_1$  by  $\overline{\mathfrak{s}}(x) = \mathfrak{s}(x) \exp(F(x))$ . Then we have

**Lemma 6.5.**  $(v^0, Y) \in C \times T_p(M)$  satisfies

(A) 
$$|v^0|^2 + S_p(Y, Y)/2(n+1) = 0$$

if and only if it satisfies

(B) 
$$|v^0 - F(y)v|^2 + \bar{S}_p(Y, Y)/2(n+1) = 0$$
,

for  $y \in C(M)$  and  $v \in C^n$  such that Y = yv.

Proof. Lemma 4.1 shows that (A) (resp. (B)) is equivalent to

(6.6) 
$$\pi \binom{v^0}{v} \in \Phi_{E_y}$$

(6.7) 
$$\left(\operatorname{resp.} \pi \left( \begin{array}{c} v^0 - F(y)v \\ v \end{array} \right) \in \overline{\Phi}_{\overline{E}_y} \right).$$

We have by Lemma 6.4

(6.8) 
$$\exp(-F(y))\Phi_{E_y} = \overline{\Phi}_{\overline{E}_y}$$

Since

$$\exp(-F(y))\pi\binom{v^{0}}{v} = \pi\binom{1 - F(y)}{0 - 1}\binom{v^{0}}{v} = \pi\binom{v^{0} - F(y)v}{v}$$

(A) is equivalent to (B) by (6.6), (6.7) and (6.8).

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q.e.d.

Proof of Theorem 6. Let p be an arbitrary point in M. In the case when  $S \neq 0$ ,  $S_p(Y, Y) < 0$  for some  $Y \in T_p(M)$ . Choose  $v^0 \in \mathbf{R}$  such that

(6.9) 
$$(v^0)^2 + S_p(Y, Y)/2(n+1) = 0.$$

Then we have also

(6.10) 
$$(v^0)^2 + S_p(-Y, -Y)/2(n+1) = 0.$$

Applying Lemma 6.5 to (6.9) and (6.10), we obtain

$$|v^{0} - F(y)v|^{2} + \bar{S}_{p}(Y, Y)/2(n+1) = 0$$
  
$$|v^{0} + F(y)v|^{2} + \bar{S}_{p}(-Y, -Y)/2(n+1) = 0$$

for  $y \in C(M)$  and  $v \in C^n$  such that Y = yv. By these two formulas Re(F(y)v) = 0. On the other hand, the set

$$\{v \in \mathfrak{g}_{-1} | S_{p}(yv, yv) < 0\}$$

is open in  $\mathfrak{g}_{-1}$ . Thus the **R**-linear map  $L:\mathfrak{g}_1 \to \mathbf{R}$  defined by  $L(v) = \operatorname{Re}(F(y)v)$ is zero. Since  $F(y)v = \operatorname{Re}(F(y)v) - \sqrt{-1} \operatorname{Re}(F(y)\sqrt{-1}v)$ , the map  $N:\mathfrak{g}_{+1} \to \mathbf{C}$ defined by N(v) = F(y)v is zero. Thus F=0, because p is an arbitrary point. Also in the case when S=0, we obtain F=0 in a similar fashion. This completes the proof of Theorem 6.

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