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ONE FLAT 3-MANIFOLDS IN 5-SPACE

Dedicated to Professor Hidetaka Terasaka on his sixtieth birthday

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1. Introduction

The results concerned with closed orientable surfaces in 4–space obtained in [5] will be extended in the paper.

Things will be considered only from the piecewise-linear (or semi-linear) and combinatorial point of view, and manifolds M, W etc., will be combinatorial, orientable with an orientation, maps will be piecewise-linear with respect to (simplicial) subdivisions and generally homeomorphisms between manifolds will be orientation preserving. So that $M \subset M_1$, $M = M_2$ and $\partial M = M_3$ will indicate obvious relations between the orientations of manifolds, if meaningful, together with the usual set theoretic meanings, where ∂M is the boundary of M.

Let M_i be a closed n–manifold in an $(n+2)$–manifold W_i without boundary, $i=1, 2$. Precisely, there are subdivisions K_i and L_i of M_i and W_i respectively such that K_i is a subcomplex of L_i. For convenience, the situation is simply said that $M_i = |K_i|$ is in $W_i = |L_i|$ in the rest of the paper. Then M_1 is *iso-neighboring* to M_2 if there are regular neighborhoods U_i of M_i in W_i, see [4], where $U_i \subset W_i$ and an onto homeomorphism $\psi\colon U_1 \to U_2$ such that $\psi(M_1) = M_2$. By Theorem 1 of [4], the iso-neighboring relation is an equivalence relation.

In § 2 two invariances the collection of singularities and the Stiefel-Whitney class under the iso-neighboring relation will be dealt with. Let a closed n–manifold $M = |K|$ be in an $(n+2)$–manifold $W = |L|$ without boundary. For each point x of M, the links $Lk(x, K)$ $Lk(x, L)$ in K, L are $(n-1)$–, $(n+1)$–spheres respectively. Then M is said to be *p–flat* in W if the link $Lk(x, K)$ bounds an n–cell in $Lk(x, L)$, alternatively the $(n-1, n+1)$–knot $(Lk(x, K), Lk(x, L))$ is trivial, where $x \in M - |K^{p-1}|$ and K^q is the q–skeleton of K (K^{-1} is the empty set). The p–flatness of M in W is clearly invariant under the iso-neighboring relation. A 0–flat M in W is alternatively said to be *locally flat*. For a 1–flat M in W the *collection of singularities* of M in W will be defined, which is an invariance under the iso-neighboring relation.

If K is a full subcomplex of L, then the star neighborhood $N(K', L')$ is a regular neighborhood, which consists of $(n+2)$-cells dual to vertices of K in L, where X' denotes the first barycentric subdivision of X. In general, a regular neighborhood U of M in W carries some properties similar to those of normal bundles in differential topology. So that an invariance ω, called the *Stiefel-Whitney class*, under the iso-neighboring relation may be defined for M in W following the classical arguments due to Seifert [7] and Whitney [8].

In the paper the boundary of a regular neighborhood of M in W is called a *tube* of M in W, and for a mapping $f: X \to Y$, $f^*(f_*)$ denotes the induced homomorphism between cohomology groups of Y and X (homology groups of X and Y).

The following will be established in § 3.

Theorem A. *Let a closed* 3*-manifold* M_i *be* 1*-flat in* 5*-manifold* W_i *without boundary, where* $i=1,2$. *Then* M_1 *and* M_2 *are iso-neighboring if and only if there is an onto homeomorphism* $\phi: M_1 \to M_2$, *such that* $\phi^*(\omega_2) = \omega_1$ *and they have the same collection of singularities.*

By the argument due to [7] the Stiefel-Whitney class ω is the identity if M is in euclidean $(n+2)$-space R^{n+2}. Thus,

Corollary to Theorem A. *Let closed* 3*-manifolds* M_1 *and* M_2 *be* 1*-flat in* 5*-space such that* M_1 *and* M_2 *are homeomorphic and symmetric. Then they are iso-neighboring if and only if they have the same collection of singularities.* (We say that M is *symmetric* if there is an orientation reversing homeomorphism onto itself.)

Moreover, $M \times o$ is locally flat in $M \times R^2$ and its Stiefel-Whitney class ω is the identity, where R^2 is 2-space and o is the origin of R^2. And $M \times C^2$ is a regular neighborhood of $M \times o$ in $M \times R^2$ where C^2 is a 2-cell containing o in its interior. Therefore,

Theorem B. *If a closed* 3*-manifold* M *is locally flat in* 5*-space* R^5. *Then a regular neighborhood* U *of* M *in* R^5 *is the product of* M *and a* 2*-cell.*

Finally, (1). Some results of the paper [4] will be used in this paper. Although they were proved modulo the Schoenflies conjecture, they are verified without the conjecture in virtue of Theorem (2.3) of [6].

(2). The detail of the proofs which were omitted in the paper [5] will be seen in the paper, even if this paper concentrates upon 3-manifolds M.

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2. Invariances

NOTATION A. Let $M=|K|$ be a closed n-manifold in an $(n+2)$-manifold $W=|L|$ without boundary, where it is assumed that K is a full subcomplex of L, that is, the intersection of a simplex of L and $|K|$ is either a simplex of K or empty. By Δ we shall denote a (closed) r-simplex of K. Then by ∇ and $\square$ we denote the $(n-r)$-, $(n-r+2)$-cells dual to Δ in K and L respectively. (∇ and $\square$ are covered by subcomplexes of K' and L' respectively.) By $\mathfrak{K}^q$ we shall denote the polyhedron consisting of dual cells ∇ where Δ ranges over $K-K^{n-q-1}$. Similarly by $\mathfrak{N}^{q+2}$ we denote the polyhedron consisting of dual cells $\square$ where $\Delta \in K-K^{n-q-1}$. Note that $\mathfrak{N}^{n+2}$ is the star neighborhood $N(K', L')$.

If an orientation is assigned to Δ, the orientation of $\nabla(\square)$ is naturally determined such that the intersection number of Δ and $\nabla(\square)$ in $M(W)$ is 1. We shall always assigne an orientation to Δ and use those natural orientation for ∇ and $\square$ throughout the paper. It is obvious that the $(n-1,\ n+1)$-knot $(Lk(x, K), Lk(x, L))$ is trivial if $x \in M-|K^{n-2}|$, and that M is $(n-1)$-flat in W.

Lemma 1. *If M is p-flat in W then for each r-simplex Δ of K the $(n-r-1,\ n-r+1)$-knot $(\partial\nabla, \partial\square)$ is trivial where $r \geq p \geq 0$ and for each $(p-1)$-simplex Δ the $(n-p,\ n-p+2)$-knot $(\partial\nabla, \partial\square)$ is locally flat, where $p \geq 1$.*

Proof. Let x be an interior point of an r-simplex Δ of K. It is elementary to check that that the $(n-1, n+1)$-knots $(Lk(x, K), Lk(x, L))$ and $(\partial\Delta*\partial\nabla,\ \partial\Delta*\partial\square)$ are equivalent, where $X*Y$ is the join of X and Y. If $(\partial\nabla, \partial\square)$ is not trivial then $(Lk(x, K), Lk(x, L))$ is not locally flat, because the latter is the suspension of the former. In particular $(Lk(x, K), Lk(x, L))$ is not trivial, contradiction. Therefore $(\partial\nabla, \partial\square)$ is trivial.

Let x be a point of $\partial\nabla$ where Δ is a $(p-1)$-simplex of K. Then x is an interior point of a q-simplex η of K where $q > p-1$. Let $\mathcal{E}$ be the face opposite to Δ in η and c be the barycenter of $\mathcal{E}$. Since the $(n-1,\ n+1)$-knot $(\partial(\Delta*c)*Lk(x, \partial\nabla),\ \partial(\Delta*c)*Lk(x, \partial\square))$ is equivalent to $(Lk(x, K), Lk(x, L))$ which is trivial, the $(n-p-1,\ n-p+1)$-knot $(Lk(x, \partial\nabla), Lk(x, \partial\square))$ may not be non-trivial. Hence $(Lk(x, \partial\nabla), Lk(x, \partial\square))$ is trivial for each x of $\partial\nabla$ and $(\partial\nabla, \partial\square)$ is locally flat.

DEFINITION 1. Let $M=|K|$ be 1-flat in $W=|L|$. By Lemma 1 only the $(n-1,\ n+1)$-knot $(\partial\nabla, \partial\square)$ may not be trivial where Δ is a vertex of K. We say that a vertex Δ is a *non-singular* point or *singular* point of M in W according as the $(n-1,\ n+1)$-knot class k containing $(\partial\nabla, \partial\square)$

is a trivial class 0 or $k\neq 0$. If $k\neq 0$, we say that the *singularity* of M on W at Δ is of type k. If M in W has singular points $\Delta_1,\cdots,\Delta_s$ of type $k_1,\cdots,k_s$, then the unordered set of classes $k_1,\cdots,k_s$ will be called the *collection of singularities* of M in W, which is invariant under the iso-neighboring relation as easily seen.

Now let us define the Stiefel-Whitney class ω for M in W following the argument due to Seifert [7] and Whitney [8]. Let $\mathfrak{H}$ be a subpolyhedron of $\mathfrak{K}^n$, we say that a map $\kappa:\mathfrak{H}\to\partial N(K', L')$ is a *cross section* over $\mathfrak{H}$ if $\kappa(\nabla)\subset\partial\Box$ for all ∇ of $\mathfrak{H}$.

(a) *Let $\sigma:\mathfrak{H}\to\partial N(K', L')$ be a cross section, then σ may be extended to a cross section κ over $\mathfrak{H}\cup\mathfrak{K}^1$, where $\mathfrak{H}$ may be empty.*

Proof. Define $\kappa^0(\nabla)=$ a vertex of $\partial\Box$ if $\nabla\in\mathfrak{K}^0-\mathfrak{H}$ and $\kappa^0|\mathfrak{H}=\sigma|\mathfrak{H}$, then $\kappa^0:\mathfrak{H}\cup\mathfrak{K}^0\to\partial N(K', L')$ is a cross section over $\mathfrak{H}\cup\mathfrak{K}^0$. Let Δ be an $(n-1)$-simplex then ∇ and $\Box$ are 1-, 3-cells respectively. Let Δ_1, Δ_2 be n-simplexes incident to Δ then $\Box_1\cup\Box_2$ is a regular neighborhood of the 0-sphere $\nabla_1\cup\nabla_2$ in the 2-sphere $\partial\Box$, and $\partial\Box-\mathit{Int}\ (\Box_1\cup\Box_2)$ is the cylinder $S^1\times I$ where *Int* M is the interior of M and I is the closed unit interval. Then there is a homeomorphism $\kappa_\nabla:\nabla\to\partial\Box-\mathit{Int}\ (\Box_1\cup\Box_2)$ such that $\kappa_\nabla|\nabla_1\cup\nabla_2=\kappa^0|\nabla_1\cup\nabla_2$. Define $\kappa|\nabla=\kappa_\nabla$ if $\nabla\in\mathfrak{K}^1-\mathfrak{H}$ and $\kappa|\nabla=\sigma|\nabla$ if $\nabla\in\mathfrak{H}$, then $\kappa:\mathfrak{H}\cup\mathfrak{K}^1\to\partial N(K', L')$ is the required cross section.

(b) Using a cross section $\kappa:\mathfrak{K}^1\to\partial N(K', L')$, let us define an integral 2-cochain W_κ of M as follows.

Let Δ_j be an $(n-2)$-simplex of K then ∇_j, $\Box_j$ are 2-, 4-cells respectively. Then we have a knot $(\partial\nabla_j, \partial\Box_j)$ and the tube (=torus) $T_j=\partial(\bigcup_k\Box_{jk})$, where $\bigcup_k\Box_{jk}$ is a regular neighborhood of $\partial\nabla_j$ in $\partial\Box_j$ when Δ_{jk} ranges over $(n-1)$-simplexes of K incident to Δ_j by Lemma 4 of [4]. By the knot theory the longitude b_j and the meridian a_j of the torus T_j are well defined up to homology such that $a_j\sim\partial\Box_p$ in T_j where $\Delta_p(\subset M)$ is an n-simplex having Δ_j as a face, and such that $b_j\sim\partial\nabla_j$ in $\bigcup_k\Box_{jk}$ and $b_j\sim 0$ in $\partial\Box_j-\mathit{Int}\ (\bigcup_k\Box_{jk})$ where $\sim$ means to be homologous. By w_j we denote the looping coefficient of $\kappa_*(\partial\nabla_j)$ and b_j in $\partial\Box_j$. That is, $\kappa_*(\partial\nabla_j)\sim w_ja_j+b_j$ in T_j. Then an integral 2-cochain W_κ of M is defined by taking $W_\kappa(\nabla_j)=w_j$ for each ∇_j.

The following (c), (d) and (e) are the modification of the arguments due to Seifert and Whitney.

(c) *Let κ, σ be cross sections over $\mathfrak{K}^1$ then W_κ is cohomologous to W_σ in M.* See [8, p. 120].

(d) *Let M and W be spheres. Then there is a cross section σ over*

$\mathfrak{R}^1$ *such that* $W_\sigma(\nabla_j)=0$ *for each* ∇_j. See [7, pp. 6–7].

(e) W_κ *is a cocycle.* See [8, p. 121].

(f) *The cohomology class* ω *containing the cocyle* W_κ *is independent of the subdivisions* K *and* L.

Proof. As usual it is sufficient to prove that W_L is cohomologous to W_Z where W_L and W_Z are the cocyles obtained from the subdivisions K, L and Y, Z of M, W respectively such that Z is a subdivision of L and Y is the subcomplex of Z covering M.

Then it may be assumed that L is transformed to Z by a simple subdivision (γ, d) of L where γ is a 1-simplex of L and d is an interior point of γ, see [1, p. 302]. The proof is separated in two cases. That is, $\gamma \notin K$ and $\gamma \in K$. Since the both cases may be treated similarly, we shall prove the second one.

Suppose that $\gamma \in K$. Let $\gamma = ab$ where a and b are vertices of K. At first we construct an onto map $\theta: N(Y', Z') \to N(K', L')$ which takes the dual cells of d in Y, Z onto the dual cells of γ in K, L and the dual cells of c in Y, Z onto the dual cells of itself in K, L respectively, where c is a vertex of Y other than d. Let $\square_Z$ be the dual cell of d in Z, $\partial\square_Z$ consists of simplexes written $v_0v_1\cdots v_q$ where v_j is the barycenter of the simplex $dc_0c_1\cdots c_j$ and c_i is a vertex of L lying in $Lk(d, L)$. Define $\theta(v_0\cdots v_q)=u_0\cdots u_q$ if $c_0=a$ (or b), where u_i is the barycenter of $bc_0\cdots c_q$ $(ac_0\cdots c_q)$, $\theta(v_0\cdots v_q)=u_0\cdots u_q$ if $c_0 \neq a$ and b, where u_i is the barycenter of $abc_0\cdots c_q$, $\theta(d)=e$, the barycenter of γ, and $\theta \mid Lk(d, L)=$identity. Since a simplex of Z' in the star $St(d, L)$ is either the join of a simplex lying in $Lk(d, L)$ and a simplex in $\partial\square_Z$, or the join of a simplex in $\partial\square_Z$ and d, the map θ may be extended over the star $St(d, L)$ and then over $N(Y', Z')$ by taking identity on $N(Y', Z')-St(d, L)$. Then the map θ is the required one.

By (d) and (a) we may construct a cross section κ defining W_Z such that $W_Z(\nabla_Y^2)=0$ for each 2-cell ∇_Y^2 which is on $\partial\nabla_Y$ and ∇_Y is the cell dual to d in Y and such that for each point x of $\mathfrak{R}^1$ the set $\theta^{-1}(x)$ is mapped by $\theta\kappa$ to a point of $\partial N(K', L')$. Then the mapping $\theta\kappa\theta^{-1}$ is well defined which is a cross section over $\mathfrak{R}^1$ defining W_L such that $W_L(\nabla_K^2)=0$ for each 2-cell ∇_K^2 which is on $\partial\nabla_K$ and ∇_k is the cell dual to γ in K, and such that $W_Z(\nabla)=W_L(\nabla)$ for each $(n-2)$-simplex Δ of K, which does not contain γ. Hence W_Z is cohomologous to W_L.

DEFINITION 2. Let a closed n-manifold $M=|K|$ be in an $(n+2)$-manifold $W=|L|$ without boundary. Then by (a), (b), (c), (d), (e) and (f) a 2-dimensional cohomology class ω of M is defined, called the *Stiefel-Whitney class* of M in W.

We gather the above in the following:

Lemma 2. *The Stiefel-Whitney class ω of M in W is invariant under the iso-neighboring relation. Moreover, ω is the identity if W is $(n+2)$-space R.*

3. The dual skeletonwise extension scheme

NOTATION B. Let $M_i=|K_i|$ be a closed n-manifold in an $(n+2)$-manifold $W_i=|L_i|$ without boundary, $i=1, 2$. Suppose that $\phi: M_1\to M_2$ is a homeomorphism which is simplicial relative to K_1 and K_2. Then by Δ_i, Δ_{ij} we shall denote simplexes of K_i such that $\phi(\Delta_1)=\Delta_2$, $\phi(\Delta_{1j})=\Delta_{2j}$. Since ϕ induces an isomorphism between complexes K_1 and K_2 and the correspondence between Δ_i and ∇_i is one-to-one, ϕ also induces an isomorphism, written ϕ, between $\mathfrak{K}_1^q$ and $\mathfrak{K}_2^q$ by taking $\phi(\nabla_1)=\nabla_2$. Since the correspondence between Δ_i and $\Box_i$ is one-to-one, ϕ also induces a one-to-one correspondence ψ between cells of $\mathfrak{N}_1^{q+2}$ and cells of $\mathfrak{N}_2^{q+2}$, by taking $\psi(\Box_1)=\Box_2$.

(0) *Let $M_i=|K_i|$ be a closed n-manifold in an $(n+2)$-manifold $W_i=|L_i|$ without boundary. Let $\phi: M_1\to M_2$ be a homeomorphism which is simplicial relative to K_1 and K_2. Then there is a homeomorphism $\psi^0: \mathfrak{N}_1^2\to\mathfrak{N}_2^2$ such that $\psi^0|\mathfrak{K}_1^0=\phi$, and $\psi^0(\Box_1)=\Box_2$ for each n-simplex Δ_i of K_i.*

Proof. For each n-simplex $\Delta_i(\subset M_i)$ of K_i, $\partial\Box_i$ is a 1-sphere and we have a homeomorphism $\psi'': \partial\Box_1\to\partial\Box_2$. Since ∇_i is the point such that $\Box_i$ is the join $\nabla_i*(\partial\Box_i)$, there is a homeomorphism $\psi': \Box_1\to\Box_2$ such that $\psi'|\partial\Box_1=\psi''$ and $\psi'|\nabla_1=\phi|\nabla_1$. Since all $\Box_i$ are disjoint, $\psi^0: \mathfrak{N}_1^2\to\mathfrak{N}_2^2$ defined by $\psi^0|\Box_1=\psi'$ is a homeomorphism such that $\psi^0|\mathfrak{K}_1^0=\phi$ and $\psi^0(\Box_1)=\Box_2$ for each Δ_i, proving (0).

(0)$\to$(1). *Under the situation of* (0), *furthermore we suppose that $\phi^*(\omega_2)=\omega_1$ where ω_i is the Stiefel-Whitney class of M_i in W_i. Then there is a homeomorphism $\rho: \mathfrak{N}_1^3\to\mathfrak{N}_2^3$ such that $\rho|\mathfrak{K}_1^1=\phi$ and $\rho(\Box_1)=\Box_2$ for each $(n-1)$-simplex Δ_i of K_i and such that for each $(n-2)$-simplex Δ_{ij} of K_i, $\rho_*a_{1j}\sim a_{2j}$, $\rho_*b_{1j}\sim b_{2j}$ on the tube T_{2j}, see* (b) *in* §2.

Proof. Let Δ_{ia}, Δ_{ib} be n-simplexes incident to an $(n-1)$-simplex Δ_i. Then $\Box_{ia}\cup\Box_{ib}$ is a regular neighborhood of the 0-sphere $\partial\nabla_i$ in the 2-sphere $\partial\Box_i$ by [4]. Since $\Box_{ia}\cup\Box_{ib}$ consists of disjoint 2-cells and $\phi: M_1\to M_2$ is orientation preserving, there is an onto homeomorphism $\psi'': \partial\Box_1\to\partial\Box_2$ such that $\psi''(\partial\nabla_1)=\partial\nabla_2$ and $\psi''|\Box_{1a}\cup\Box_{1b}=\psi^0|\Box_{1a}\cup\Box_{1b}$. Since $\Box_i$ is the join $c_i*(\partial\Box_i)$ and $\nabla_i=c_i*(\partial\nabla_i)$ where c_i is the barycenter of Δ_i, we have an onto homeomorphism $\psi': \square_1 \to \square_2$ such that $\psi'|\partial\square_1 = \psi''$ for each Δ_i. Define ψ^1 by taking $\psi^1|\square_1 = \psi'$ for each $(n-1)$-simplex Δ_1, we have a homeomorphism $\psi^1: \mathfrak{N}_1^3 \to \mathfrak{N}_2^3$ such that $\psi^1|\mathfrak{K}_1^1 = \phi$ and $\psi^1_* a_{1j} \sim a_{2j}$ on T_{2j} for each $(n-2)$-simplex Δ_{ij} of K_i. Let $\kappa: \mathfrak{K}_1^1 \to \partial N(K_1', L_1')$ be a cross section we, have integers w_{1j}, w_{2j} such that $\kappa^*(\partial\nabla_{1j}) \sim w_{1j}a_{1j} + b_{1j}$ on T_{1j} and $\psi^1_* \kappa_*(\partial\nabla_{1j}) \sim w_{2j}a_{2j} + b_{2j}$ on T_{2j}.

Let $W_i(\nabla_{ij}) = w_{ij}$ then W_i is a 2-cocycle contained in ω_i. Since $\phi^*(\omega_2) = \omega_1$, $W_1 - \phi^* W_2 = \delta X$ for a 1-cochain X of M_1, where δ is the coboundary operator. Since $\partial\square_i - Int\ (\square_{ia} \cup \square_{ib})$ is the finite cylinder C_i, there is an onto homeomorphism $\eta: \square_1 \to \square_2$ such that $\eta|\square_{1a} \cup \square_{1b} = \psi^1|\square_{1a} \cup \square_{1b}$ and $(\eta\kappa)_*(\nabla_1) - \psi^1_*\kappa_*(\nabla_1) = (X\cdot\nabla_1)a_{2j}$ on T_{2j}, where $X\cdot\nabla_1$ is the coefficient of ∇_1 in X. Define ρ by taking $\rho|\square_1 = \eta$ if $X\cdot\nabla_1 \neq 0$, and $\rho|\square_{1j} = \psi^1|\square_1$ otherwise. Since $W_1 - \phi^* W_2 = \delta X$, $(\rho\kappa)_*(\partial\nabla_{1j}) \sim (w_{2j} + X\cdot\partial\nabla_{1j}) a_{2j} + b_{2j} = w_{1j}a_{2j} + b_{2j}$ on T_{2j} for each Δ_{ij}. Since $\rho_*\kappa_*(\partial\nabla_{1j}) \sim \rho_*(w_{1j}a_{1j} + b_{1j}) \sim w_{1j}a_{2j} + \rho_*(b_{1j})$, $\rho_*(b_{1j}) \sim b_{2j}$ on T_{2j}.

(1)$\to$(2). *Under the situation of* (0)$\to$(1), *suppose that* M_i *is* $(n-2)$-*flat in* W_i. *Then there is an onto homeomorphism* $\psi^2: \mathfrak{N}_1^4 \to \mathfrak{N}_2^4$ *such that* $\psi^2|\mathfrak{K}_1^2 = \phi$ *and* $\psi^2(\square_1) = \square_2$ *for each* $(n-2)$-*simplex* Δ_i *of* K_i.

Proof. Let Δ_{ij} be $(n-1)$-simplexes of K_i incident to Δ_i. Then $\bigcup_j \square_{ij}$ is a regular neighborhood of the 1-sphere $\partial\nabla_i$ in the 3-sphere $\partial\square_i$ by [4]. By Lemma 1, the knot $(\partial\nabla_i, \partial\square_i)$ is trivial, and then there is an onto homeomorphism $\theta: \partial\square_2 \to \partial\square_1$ such that $\theta|\partial\nabla_2 = \phi^{-1}|\partial\nabla_2$ and such that $\theta(\bigcup_j \square_{2j}) = \bigcup_j \square_{1j}$ by Theorem 1 of [4]. So we have a homeomorphism $\rho\theta: \bigcup_j \square_{2j} \to \bigcup_j \square_{2j}$ such that $(\rho\theta)_* a_2 \sim a_2$ and $(\rho\theta)_* b_2 \sim b_2$ on $T_2 (= \partial(\bigcup_j \square_{2j}))$. By the argument due to Baer [2] $\rho\theta| T_2: T_2 \to T_2$ is isotopic to the identity. And then by Theorem 4 of [4] there is an onto homeomorphism $\alpha: \partial\square_2 \to \partial\square_2$ such that $\alpha|\bigcup_j \square_{2j} = \rho\theta|\bigcup_j \square_{2j}$.

Taking $\psi'' = \alpha\theta^{-1}$, then $\psi'': \partial\square_1 \to \partial\square_2$ is an onto homeomorphism such that $\psi''|\bigcup_j \square_{1j} = \rho$. Since $\square_i$ is the join $c_i * (\partial\square_i)$ and $\nabla_i = c_i * (\partial\nabla_i)$ where c_i is the barycenter of Δ_i, we have an onto homeomorphism $\psi': \square_1 \to \square_2$ such that $\psi^1|\partial\square = \psi''$. Then $\psi^2: \mathfrak{N}_1^4 \to \mathfrak{N}_2^4$ defined by $\psi^2|\square_1^4 = \psi'$, is an onto homeomorphism such that $\psi^2|\mathfrak{K}_1^2 = \phi$ and $\psi^2(\square_1) = \square_2$ for each $(n-2)$-simplex Δ_i, proving (1)$\to$(2).

Under the conditions that there is a homeomorphism $\phi: M_1 \to M_2$ which is simplicial relative to K_1 and K_2, $\phi^*(\omega_2) = \omega_1$, and M_i is $(n-2)$-flat in W_i, we have proved the following (m) for $m \leq 2$.

(m) *There is an onto homeomorphism* $\psi^m: \mathfrak{N}_1^{m+2} \to \mathfrak{N}_2^{m+2}$ *such that* $\psi^m|\mathfrak{K}_1^m = \phi$ *and* $\psi^m(\square_1^{m+2}) = \square_2^{m+2}$ *for each* $(n-m)$-*simplex* Δ_i *of* K_i.

Proof of Theorem A. The necessity follows from §2. Let $\phi: M_1 \to M_2$ be a given homeomorphism, it may be assumed that ϕ is simplicial with respect to K_1 and K_2. Furthermore since any point x of a closed manifold may be mapped into a given point y of the manifold by a homeomorphism of the manifold onto itself, it is assumed that the $(n-m-1,\ n-m+1)$-knots $(\partial\nabla_1,\ \partial\Box_1)$ and $(\partial\nabla_2,\ \partial\Box_2)$ belong to the same class for every pair of m-simplexes Δ_1 and Δ_2 by Lemma 1 and the assumption of Theorem A.

Then (0), (0)$\to$(1), (1)$\to$(2) hold. Suppose that all (m) are proved for $m \leq n$. Since $\mathfrak{K}_i^n = M_i$ and $\mathfrak{N}_i^{n+2} = N(K_i',\ L_i')$, the homeomorphism $\psi^n: \mathfrak{N}_1^{n+2} \to \mathfrak{N}_2^{n+2}$ is the required homeomorphism ψ. Therefore it remains to prove that the proposition (2) implies the proposition (3).

Let Δ_{ij} be a 1-simplex of K_i incident to a vertex Δ_i. Then $\psi^2(\bigcup_j \Box_{1j}) = \bigcup_j \Box_{2j}$ where $\bigcup_j \Box_{ij} \subset \partial\Box_i$ and $\bigcup_j \Box_{ij}$ is a regular neighborhood of the 2-sphere $\partial\nabla_i$ in the 4-sphere $\partial\Box_i$ by Lemma 4 of [4]. Since $\partial\nabla_i$ is locally flat in $\partial\Box_i$ by Lemma 1, the tube $T_i\,(=\partial(\bigcup_j \Box_{ij}))$ is homeomorphic to $S^2 \times S^1$ by Theorem B of [5]. Since the corresponding (2, 4)-knots $(\partial\nabla_1,\ \partial\Box_1)$ and $(\partial\nabla_2,\ \partial\Box_2)$ belong to the same class, there is a homeomorphism $\theta: \partial\Box_2 \to \partial\Box_1$ such that $\theta|\partial\nabla_2 = \phi^{-1}|\partial\nabla_2$ and $\theta(\bigcup_j \Box_{2j}) = \bigcup_j \Box_{1j}$ by Theorem 1 of [4]. Then $\psi^2\theta|T_2: T_2 \to T_2$ is an onto homeomorphism such that

$$(\psi^2\theta)_* S^2 \sim S^2$$

and

$$(\psi^2\theta)_* S^1 \sim S^1\,.$$

Therefore $\psi^2\theta|T_2: T_2 \to T_2$ is isotopic to either the identity or the homeomorphism $\boldsymbol{T}$, see [3 p. 320]. Since $\boldsymbol{T}$ may not be extended over $\bigcup_j \Box_{2j}$, $\psi^2\theta|T_2$ is isotopic to the identity by [3 p. 323]. Then, by Theorem 4 of [4], there is a homeomorphism $\alpha: \partial\Box_2 \to \partial\Box_2$ such that $\alpha|\bigcup_j \Box_{2j} = \psi^2\theta|\bigcup_j \Box_{2j}$. Then $\alpha\theta^{-1}: \partial\Box_1 \to \partial\Box_2$ is a homeomorphism such that $\alpha\theta^{-1}|\bigcup_j \Box_{1j} = \psi^2|\bigcup_j \Box_{1j}$. Then, by the similar argument in (1)$\to$(2), we may obtain the required homeomorphism ψ^3.

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