

Title	On the indecomposability of amalgamated sums
Author(s)	Asashiba, Hideto
Citation	Osaka Journal of Mathematics. 1983, 20(4), p. 701-711
Version Type	VoR
URL	https://doi.org/10.18910/5451
rights	
Note	

Osaka University Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

Osaka University

ON THE INDECOMPOSABILITY OF AMALGAMATED SUMS

HIDETO ASASHIBA

(Received December 20, 1981)

Let R be a semiprimary ring with identity element. In this paper, we study when a factor module of a direct sum of local or colocal R -modules of finite length is indecomposable.

Let $(E): 0 \rightarrow K \xrightarrow{f} \bigoplus_{i=1}^n L_i \xrightarrow{g} M \rightarrow 0$ be a nonsplit exact sequence of right R -modules of finite length and $f_i: K \rightarrow L_i$ be the i^{th} coordinate map of f for each $i=1, \dots, n$. As we will see in the paper, if M is indecomposable, then the following condition holds:

(*) For each $j=1, \dots, n$ and each $h=(h_i)_{i=1}^n: \bigoplus_{i=1}^n L_i \rightarrow L_j$, $hf = \sum_{i=1}^n h_i f_i = 0$ implies that h_i is not an isomorphism for each $i=1, \dots, n$.

The converse is not true in general, but in Tachikawa [3] we see the converse holds under rather strong conditions. Moreover in [1, section 2], we showed that this converse assertion is still true in the case of each of three groups of weaker conditions than those required in [3]. But in [1, Proposition 2.7], the third group of conditions, we assumed a condition on composition lengths of the L_i 's which was not assumed in the other two cases. In this paper, we remove this condition on composition lengths and show that the condition (*) implies the indecomposability of M if each L_i is local and colocal, and each f_i is a monomorphism (see (3.3)).

In section 1, we consider the fundamental properties of the map $f=(f_i)_{i=1}^n$ in the sequence (E) satisfying the condition (*). Section 2 is a generalization of tools used in [1, section 2] (this generalization is not essential to understanding the main results) and in section 3 we give the main results.

Throughout the paper R is a ring with identity element, J the Jacobson radical of R , every module is a unitary right R -module. We denote by $\text{Mod } R$ and by $\text{mod } R$ the category of all R -modules and R -modules of finite length, respectively. We call an R -module M completely indecomposable in case the endomorphism ring $\text{End}_R(M)$ is a local ring. For maps $f: K \rightarrow L$ and $g: L \rightarrow M$, and for a decomposition $D: L = \bigoplus_i L_i$ of L , the notations $(f, D) = (f_i)_i^T$ and

$(D, g) = (g_i)_I$ are matrix expressions of f and g relative to D , respectively. The notation $I_1 \perp \dots \perp I_n = I$ means that the union $I_1 \cup \dots \cup I_n = I$ is disjoint. Finally, the socle of M is denoted by $\text{soc } M$, the m^{th} socle of M is denoted by $\text{soc}_m M$ for each R -module M . (It is well known that if R is a semiprimary ring, $\text{soc}_m M$ is equal to the left annihilator of J^m in M .)

The author would like to thank Professor T. Sumioka for his encouragement and helpful advice.

1. Infusible maps

DEFINITION. Let $f: K \rightarrow L$ be a homomorphism of R -modules, $D: L = \bigoplus_{i=1}^n L_i$ be a decomposition of L and let $(f, D) = (f_i)_{i=1}^n$. Then the pair (f, D) is said to be *infusible* in case for each $j=1, \dots, n$ and each $h = (h_i)_{i=1}^n: \bigoplus_{i=1}^n L_i \rightarrow L_j$; $hf = \sum_{i=1}^n h_i f_i = 0$ implies that h_i is not an isomorphism for each $i=1, \dots, n$. Further the pair (f, D) is called *fusible* in case it is not infusible.

Dually, for a homomorphism $g: L \rightarrow M$ of R -modules and for a decomposition $D: L = \bigoplus_{i=1}^n L_i$ of L , the pair (D, g) is said to be *coinfusible* in case for each $j=1, \dots, n$ and each $h = (h_i)_{i=1}^n: L_j \rightarrow \bigoplus_{i=1}^n L_i$; $gh = 0$ implies that each h_i is not an isomorphism for each $i=1, \dots, n$. Further the pair (D, g) is called *cofusible* if it is not coinfusible.

Finally, assume that the R -module L above can be written as a finite direct sum of completely indecomposable R -modules. Then we simply say that the map f (resp. g) is infusible (resp. coinfusible) in case the pair (f, D) (resp. (D, g)) is infusible (resp. coinfusible) for every decomposition D of L , and the map f (resp. g) is said to be fusible (resp. cofusible) if it is not infusible (resp. coinfusible).

Proposition 1.1. (a) For a homomorphism $f: K \rightarrow L$ of R -modules and for a decomposition $D: L = \bigoplus_{i=1}^n L_i$ of L , putting $g: L \rightarrow \text{Coker } f (=M)$ the canonical epimorphism and $(D, g) = (g_i)_{i=1}^n$, the following statements are equivalent:

- (1) (f, D) is fusible.
- (2) There is a homomorphism $h: \bigoplus_{i \neq j} L_i \rightarrow L_j$ for some $j=1, \dots, n$ such that $f_j = h(f_i)_{i \neq j}^T$.
- (3) g_j is a split monomorphism for some $j=1, \dots, n$.
- (4) There is a split epimorphism $p: M \rightarrow M' (\neq 0)$ and a nonempty subset $I \subseteq \{1, \dots, n\}$ such that $p(g_i)_I: \bigoplus_I L_i \rightarrow M'$ is an isomorphism.

Further if each L_i is completely indecomposable, then the above conditions are equivalent to

- (5) $\{h \in \text{End}_R(L) \mid hf = 0\} \not\subseteq J(\text{End}_R(L))$ where $J(-)$ denotes the Jacobson

radical of $(-)$.

(a)' For a homomorphism $g: L \rightarrow M$ of R -modules and for a decomposition $D: L = \bigoplus_{i=1}^n L_i$ of L , putting $f: (K=) \text{Ker } g \rightarrow L$ the inclusion map and $(f, D) = (f_i)_{i=1}^n$, the following statements are equivalent:

(1)' (D, g) is cofusible.

(2)' There is a homomorphism $h: L_j \rightarrow \bigoplus_{i \neq j} L_i$ for some $j=1, \dots, n$ such that $g_j = (g_i)_{i \neq j} h$.

(3)' f_j is a split epimorphism for some $j=1, \dots, n$.

(4)' There is a split monomorphism $q: (0 \neq) K' \rightarrow K$ and a nonempty subset $I \subseteq \{1, \dots, n\}$ such that $(f_i)_I^T q: K' \rightarrow \bigoplus_I L_i$ is an isomorphism.

Further if each L_i is completely indecomposable, then the above conditions are equivalent to

(5)' $\{h \in \text{End}_R(L) \mid gh=0\} \not\cong J(\text{End}_R(L))$.

Proof. We prove only (a). (1) \Rightarrow (2). If the pair (f, D) is fusible, then there exist $i_0=1, \dots, n$ and $h: L \rightarrow L_{i_0}$, $h = (h_i)_{i=1}^n$ such that $hf=0$ and h_j is an isomorphism for some $j=1, \dots, n$. Since h_j is an isomorphism, we may assume $i_0=j$ and $h_j=1_{L_j}$. Then $hf=0$ implies that $f_j = (-h_i)_{i \neq j} (f_i)_{i \neq j}^T$.

(2) \Rightarrow (3). Suppose that $f_j = h(f_i)_{i \neq j}^T$ and $h = (h_i)_{i \neq j}$. Taking $h' = (-h_1, \dots, -h_{j-1}, 1_{L_j}, -h_{j+1}, \dots, -h_n)$, we have $h'f=0$. Therefore there is a homomorphism $p: M \rightarrow L_j$ such that $h' = pg$. Let $k_j: L_j \rightarrow L$ be the inclusion map. Then $pg_j = pgk_j = h'k_j = 1_{L_j}$. Thus g_j is a split monomorphism.

(3) \Rightarrow (4). Trivial.

(4) \Rightarrow (1). Suppose (4) holds. Taking $h = p_i(p(g_i)_i)^{-1}pg$ where $p_i: \bigoplus_I L_i \rightarrow L_i$ ($i \in I$) is the canonical projection, we have $hf=0$ and $h_i = 1_{L_i}$ thus (f, D) is fusible.

If each L_i is completely indecomposable, then $J(\text{End}_R(L)) = \{(f_{ij}) \in \text{End}_R(\bigoplus_{i=1}^n L_i) \mid f_{ij} \text{ is not an isomorphism for each } i, j=1, \dots, n\}$. From this fact, equivalence of (1) and (5) is immediate. //

REMARK 1. (a) There is a homomorphism $f: K \rightarrow L$ and decompositions D, D' of L such that (f, D) is fusible but (f, D') is infusible.

(a)' There is a homomorphism $g: L \rightarrow M$ and decompositions D, D' of L such that (D, g) is cofusible but (D, g) is coinfusible.

For example: (a). Let L_1, L_2, L_3 be nonzero R -modules of finite length, $L = L_1 \oplus L_2 \oplus L_3$ and $M = L_1$ and let $g_i: L_i \rightarrow M$ be a homomorphism for each $i=1, 2, 3$ such that g_1 is the identity, g_3 is not a split monomorphism; and put $g = (g_1, g_2, g_3): L \rightarrow M$. Consider the following exact sequence: $0 \rightarrow \text{Ker } g \xrightarrow{f} L \xrightarrow{g} M \rightarrow 0$. Let $D: L = L_1 \oplus (L_2 \oplus L_3)$, $D': L = (L_1 \oplus L_2) \oplus L_3$ be two decompositions of L . Then (f, D) is fusible since g_1 is a split monomorphism but (f, D') is infusible since (g_1, g_2) and g_3 are not split monomorphisms $(1, 1; 3)$.

(a)': Dual.

REMARK 2. Let $D: L = \bigoplus_{i=1}^n L_i$ and $D': L = \bigoplus_{j=1}^m N_j$ be two decompositions of an R -module L where $N_j = \bigoplus_{I_j} L_i$ for each $j=1, \dots, m$ and $\{1, \dots, n\} = I_1 \amalg \dots \amalg I_m$. Then from (1.1; 3 and 3') it holds that

(a) For a map $f: K \rightarrow L$ in $\text{Mod } R$, if (f, D') is fusible, then (f, D) is fusible; and

(a)' For a map $g: L \rightarrow M$ in $\text{Mod } R$, if (D', g) is cofusible, then (D, g) is cofusible.

From the above remark and (1.1; 5 and 5'), we obtain the following

Corollary 1.2. *Suppose that an R -module L has a finite decomposition $D: L = \bigoplus_{i=1}^n L_i$ with each L_i completely indecomposable. Then*

(a) *A homomorphism $f: K \rightarrow L$ in $\text{Mod } R$ is infusible if and only if the pair (f, D) is infusible; and*

(a)' *A homomorphism $g: L \rightarrow M$ in $\text{Mod } R$ is coinfusible if and only if the pair (D, g) is coinfusible.*

REMARK. Let $(E): 0 \rightarrow K \xrightarrow{f} L \xrightarrow{g} M \rightarrow 0$ be a nonsplit exact sequence in $\text{mod } R$. In [1] we called (E) a $(*)$ -sequence iff the map f is infusible.

Corollary 1.3. *Let $(E): 0 \rightarrow K \xrightarrow{f} L \xrightarrow{g} M \rightarrow 0$ be a nonsplit exact sequence in $\text{mod } R$. Then*

(a) *If M is indecomposable, f is infusible; and*

(a)' *If K is indecomposable, g is coinfusible.*

Proof. We only show that (a) holds. If f is fusible, thus (f, D) is fusible for some decomposition $D: L = \bigoplus_{i=1}^n L_i$, then from (1.1; 3) and the fact that the sequence (E) does not split, L_i is a proper nonzero direct summand of M for some $i=1, \dots, n$. //

REMARK. (a) There is a nonsplit monomorphism f in $\text{mod } R$ such that f is infusible but $\text{Coker } f$ is decomposable.

(a)' There is a nonsplit epimorphism g in $\text{mod } R$ such that g is coinfusible but $\text{Ker } g$ is decomposable.

For example: (a) Let the exact sequence $0 \rightarrow K \xrightarrow{f} L \xrightarrow{g} M \rightarrow 0$ be the projective cover of $M = M_1 \oplus M_2$ where M_1 and M_2 are nonprojective indecomposables in $\text{mod } R$. Then f does not split and by (1.1; 3), f is infusible but $\text{Coker } f = M$ is decomposable.

(a)': Dual. //

From (1.2; 2 and 2') we have

Corollary 1.4. (a) Let $K_i \leq L_i$ for each $i = 1, 2$ and $h: K_1 \rightarrow K_2$ be an isomorphism. Define $f_1 = k_1, f_2 = k_2 h$ where $k_i: K_i \rightarrow L_i$ is the inclusion map for each $i = 1, 2$. Then h or h^{-1} is extendable to a homomorphism $L_1 \rightarrow L_2$ or $L_2 \rightarrow L_1$, respectively iff $(f, D) = (f_i)_{i=1,2}^T: K_1 \rightarrow L_1 \oplus L_2$ is fusible.

(a)' Let $K_i \leq L_i$ for each $i = 1, 2$ and $h: L_1/K_1 \rightarrow L_2/K_2$ be an isomorphism. Define $g_1 = h p_1, g_2 = p_2$ where $p_i: L_i \rightarrow L_i/K_i$ is the projection for each $i = 1, 2$. Then h or h^{-1} is liftable to a homomorphism $L_1 \rightarrow L_2$ or $L_2 \rightarrow L_1$, respectively iff $(D, g) = (g_i)_{i=1,2}: L_1 \oplus L_2 \rightarrow L_2/K_2$ is cofusible. ||

The forms of (1.1; 4 and 4') are mainly used to show that (f, D) is fusible and (D, g) is cofusible below.

2. Covering property

Recall that a functor r of $\text{Mod } R$ to itself is called a preradical in case it is a subfunctor of the identity functor of $\text{Mod } R$, that is, for any $M \in \text{Mod } R$; $rM \leq M$ and for any map $f: M \rightarrow N$ in $\text{Mod } R$, $rf: rM \rightarrow rN$ is the restriction map of f . A preradical r is called a radical (resp. an idempotent preradical) in case for any $M \in \text{Mod } R$, $r(M/rM) = 0$ (resp. $r(rM) = rM$). For any $N \leq M$ in $\text{Mod } R$ the notations $N \ll M$ and $N \leq_e M$ mean “ N is small in M ” and “ N is essential in M ”, respectively. An R -module M is called local (resp. colocal) in case M has the unique maximal (resp. minimal) submodule. We denote the composition length of M by $|M|$.

Lemma 2.1. Let $L = \bigoplus_{i=1}^n L_i$ and $M = M_1 \oplus M_2$ be decompositions of R -modules such that each L_i is completely indecomposable and let $p_j: M \rightarrow M_j$ denote the projections. If $f: L \rightarrow M$ is an isomorphism, then there exists a partition $\{1, \dots, n\} = I_1 \sqcup I_2$ such that the restriction map $p_j f: \bigoplus_{I_j} L_i \rightarrow M_j$ is an isomorphism for each $j = 1, 2$.

Proof. See [1, Lemma 1.4].

REMARK. By induction, this Lemma holds more generally. Let $M = \bigoplus_{i=1}^n L_i = \bigoplus_{j=1}^r M_j$ be direct decompositions of an R -module M with completely indecomposable modules L_i . Then there exists a partition $\{1, \dots, n\} = I_1 \sqcup \dots \sqcup I_r$ such that the induced map $p_j: N_j \rightarrow M_j$ is an isomorphism for each $j = 1, \dots, r$ where $N_j = \bigoplus_{I_j} L_i$ and $p_j: M \rightarrow M_j$ is the projection.

DEFINITION. Let $D: L = \bigoplus_{i=1}^n L_i$ be a decomposition of an R -module L and $g: L \rightarrow M, f: M \rightarrow L$ be homomorphisms. Then g is said to have the *covering* (resp. *cocovered*) *property* and f is said to have the *covered* (resp. *cocovering*) *property* in case for any decomposition $M = M_1 \oplus M_2$ there is a decomposition

$L=N_1\oplus N_2$ such that the restriction maps $p_jg: N_j\rightarrow M_j$ and g itself are epimorphisms (resp. monomorphisms) and the restriction maps $q_jf: M_j\rightarrow N_j$ and f itself are epimorphisms (resp. monomorphisms), respectively where $p_j: M\rightarrow M_j$ and $q_j: L\rightarrow N_j$ are the projections. In addition, if in the definitions above, N_j is always given by the form $N_j=\bigoplus_{I_j} L_i$ for each $j=1, 2$ where $I_1\perp I_2=\{1, \dots, n\}$, then we say that g has the D -compatible covering (resp. cocovered) property and so on.

Proposition 2.2. *Let $0\rightarrow K \xrightarrow{f} L \xrightarrow{g} M\rightarrow 0$ be a nonsplit exact sequence in mod R , $D: L=\bigoplus_{i=1}^n L_i$ a decomposition of L and $(f, D)=(f_i)_{i=1}^n$, $(D, g)=(g_i)_{i=1}^n$. Then*

(a) *If K is simple, then M is indecomposable iff (f, D) is infusible and g has the D -compatible covering property; and*

(b) *If f_1 is monic and $\text{Coker } f_1$ is simple, then M is indecomposable iff (f, D) is infusible and $(g_i)_{i\neq 1}$ has the $\bigoplus_{i\neq 1} L_i$ -compatible cocovered property.*

The dual statements also hold.

Proof. We prove only (a) and (b). (a). (\Rightarrow). By (1.3), (f, D) is infusible and it is trivial that g has the D -compatible covering property. (\Leftarrow). Suppose $M=M_1\oplus M_2$; $M_1, M_2\neq 0$. Then there is a decomposition $L=N_1\oplus N_2$ such that the induced maps $N_j\rightarrow M_j$ are epimorphisms where N_j 's are direct sums of some L_i 's. But since K is simple, $|N_j|=|M_j|$ for some $j=1, 2$ which means that the map $N_j\rightarrow M_j$ is an isomorphism. Hence (f, D) is fusible by (1.1).

(b). (\Rightarrow). Trivial. (\Leftarrow). Note that $(g_i)_{i\neq 1}$ is monic, since f_1 is monic and $\text{Ker } g\leq \text{Im } f$. We put $g_0=(g_i)_{i\neq 1}$ and $L_0=\bigoplus_{i\neq 1} L_i$. Suppose $M=M_1\oplus M_2$; $M_1, M_2\neq 0$. Then there is a decomposition $L_0=N_1\oplus N_2$ such that the maps $N_j\rightarrow M_j$ induced by g_0 are monomorphisms where N_j 's are direct sums of some L_i 's. But since $\text{Coker } f_1$ is simple and $\text{Coker } g_0=(g_1L_1+g_0L_0)/g_0L_0\cong g_1L_1/(g_1L_1\cap g_0L_0)=g_1L_1/g_1f_1K$; we have $|\text{Coker } g_0|=1$ or 0 . Hence $|N_j|=|M_j|$ for some $j=1, 2$ thus (f, D) is fusible. //

The following proposition is a generalization of [1, Lemma 2.4].

Proposition 2.3. *Let $D: L=\bigoplus_{i=1}^n L_i$ be a decomposition of L in Mod R . Then we have*

(a) *For an epimorphism $g: L\rightarrow M$ in Mod R , if there is a radical r such that L_i/rL_i is completely indecomposable for each $i=1, \dots, n$ and $\text{Ker } g\leq rL\ll L$, then g has the D -compatible covering property;*

(a)' *For a monomorphism $f: M\rightarrow L$ in Mod R , if there is an idempotent preradical r such that rL_i is completely indecomposable for each $i=1, \dots, n$ and $rL\leq \text{Im } f$, $rL\leq_e L$, then f has the D -compatible cocovering property;*

(b) *For a monomorphism $g: L\rightarrow M$ in Mod R , if there is an idempotent*

preradical r such that rL_i is completely indecomposable for each $i=1, \dots, n$ and $rM \leq \text{Im } g, rM \leq_e M$, then g has the D -compatible cocovered property; and

(b)' For an epimorphism $f: M \rightarrow L$ in $\text{Mod } R$, if there is a radical r such that L_i/rL_i is completely indecomposable for each $i=1, \dots, n$ and $\text{Ker } f \leq rM \ll M$, then f has the D -compatible covered property.

Proof. (a). Since $\text{Ker } g \leq rL$ and r is a radical, $r(L/\text{Ker } g) = rL/\text{Ker } g$. Accordingly, we get the commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & r(L/\text{Ker } g) & \rightarrow & L/\text{Ker } g & \rightarrow & L/rL \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & rM & \longrightarrow & M & \longrightarrow & M/rM \rightarrow 0 \end{array}$$

with exact rows where all vertical maps are isomorphisms, in particular the induced map $\bar{g}: L/rL \rightarrow M/rM$ is an isomorphism. Now let $M = M_1 \oplus M_2, p_j: M \rightarrow M_j$ be the canonical projections and $s: L \rightarrow L/rL, t: M \rightarrow M/rM$ the canonical epimorphisms. Then we have $L/rL = \bigoplus_{i=1}^n s(L_i)$ and $M/rM = t(M_1) \oplus t(M_2)$ where $s(L_i) \cong L_i/rL_i$ is completely indecomposable for each $i=1, \dots, n$. From (2.1), we obtain that there is a partition $\{1, \dots, n\} = I_1 \sqcup I_2$ such that the restriction maps $\bar{p}_j \bar{g}: \bigoplus_{I_j} s(L_i) \rightarrow t(M_j)$ are isomorphisms. But since $\text{Ker } g \leq rL \ll L$, we have $rM \ll M$ thus t is a small epimorphism. Hence the restriction maps $p_j g: \bigoplus_{I_j} L_i \rightarrow M_j$ are epimorphisms. The rest of the proof is similar to (a). //

The following corollary is just [1, Lemma 2.4] and its dual. But for completeness, we shall rewrite it below.

Corollary 2.4. Let R be a semiprimary ring, $D: L = \bigoplus_{i=1}^n L_i$ a decomposition of L in $\text{Mod } R$. Then it holds that

(a) For an epimorphism $g: L \rightarrow M$ in $\text{Mod } R$, if all L_i 's are local then g has the D -compatible covering property;

(a)' For a monomorphism $f: M \rightarrow L$ in $\text{Mod } R$, if all L_i 's are colocal then f has the D -compatible cocovering property;

(b) For a monomorphism $g: L \rightarrow M$ in $\text{Mod } R$, if all L_i 's are colocal then g has the D -compatible cocovered property; and

(b)' For an epimorphism $f: M \rightarrow L$ in $\text{Mod } R$, if all L_i 's are local then f has the D -compatible covered property.

Proof. (a). In the proof of (2.3), put $r = J$ (Jacobson radical) and note that $MJ \ll M$ and there is a subset $I \subseteq \{1, \dots, n\}$ such that the induced map $\bigoplus_I s(L_i) \rightarrow M/MJ$ is an isomorphism since L/LJ is semisimple. The rest of the proof is similar. //

The following proposition is a generalization of [1, Theorem 2.5 and Proposition 2.6].

Proposition 2.5. *Let $0 \rightarrow K \xrightarrow{f} L \xrightarrow{g} M \rightarrow 0$ be a nonsplit exact sequence in mod R and $D: L = \bigoplus_{i=1}^n L_i$ a decomposition of L . Then it holds that*

(a) *If K is simple and there is a radical r such that $L_i|_rL_i$ is indecomposable for each $i=1, \dots, n$ and $\text{Ker } g \leq rL \ll L$, then M is indecomposable iff (f, D) is infusible;*

(a)' *If M is simple and there is an idempotent preradical r such that rL_i is indecomposable for each $i=1, \dots, n$ and $rL \leq \text{Im } f$, $rL \leq_e L$, then K is indecomposable iff (D, g) is coinfusible;*

(b) *If f_1 is monic, $\text{Coker } f_1$ is simple and there is an idempotent preradical r such that rL_i is indecomposable for each $i=2, \dots, n$ and $rM \leq \text{Im } (g_i)_{i \neq 1}$, $rM \leq_e M$, then M is indecomposable iff (f, D) is infusible; and*

(b)' *If g_1 is epic, $\text{Ker } g_1$ is simple and there is a radical r such that $L_i|_rL_i$ is indecomposable for each $i=2, \dots, n$ and $\text{Ker } (f_i)_{i \neq 1}^T \leq rM \ll M$, then K is indecomposable iff (D, g) is coinfusible, where $(f, D) = (f_i)_{i=1}^T$ and $(D, g) = (g_i)_{i=1}^n$.*

Proof. Clear from (2.2), (2.3) and the fact that every indecomposable module of finite length is completely indecomposable. //

The following corollary is just [1, Theorem 2.5 and Proposition 2.6] and their duals.

Corollary 2.6. *Let R be a semiprimary ring and $0 \rightarrow K \xrightarrow{f} L \xrightarrow{g} M \rightarrow 0$ be a nonsplit exact sequence in mod R , and let $D: L = \bigoplus_{i=1}^n L_i$ be a decomposition of L . Then it holds that*

(a) *If K is simple and all L_i 's are local, then M is indecomposable iff (f, D) is infusible;*

(a)' *If M is simple and all L_i 's are colocal, then K is indecomposable iff (D, g) is coinfusible;*

(b) *If f_1 is monic, $\text{Coker } f_1$ is simple and L_i is colocal for each $i=2, \dots, n$, then M is indecomposable iff (f, D) is infusible; and*

(b)' *If g_1 is epic, $\text{Ker } g_1$ is simple and L_i is local for each $i=2, \dots, n$, then K is indecomposable iff (D, g) is coinfusible, where $(f, D) = (f_i)_{i=1}^T$ and $(D, g) = (g_i)_{i=1}^n$. //*

REMARK. (2.6.a) is also a generalization of [2, Theorem 3.7] in the semi-primary case.

3. Main results

Throughout this section, we assume that R is a semiprimary ring and every

module is of finite length. For a map $u: M \rightarrow N$ in $\text{mod } R$, the notation $u: M \twoheadrightarrow N$ (resp. $u: M \rightarrowtail N$) means that u is a monomorphism (resp. an epimorphism). We denote by $h(M)$ the height (= Loewy length) of M for each M in $\text{mod } R$, namely $h(M) = \min\{n \in \mathbb{N} \cup \{0\} \mid MJ^n = 0\} = \min\{m \in \mathbb{N} \cup \{0\} \mid \text{soc}_m M = M\}$ where we put $MJ^0 = M$, $\text{soc}_0 M = 0$.

Lemma 3.1. *Consider the maps*

$$\begin{array}{ccc} D & \xrightarrow{u} & C_1 \xleftarrow{v} E \oplus F \\ E & \xrightarrow{u'} & C_2 \xleftarrow{v'} D \end{array}$$

in $\text{mod } R$ and put $h = h(F)$. Then it holds that

- (a) *If F is colocal, $DJ^{h-1} \leq_e D$ and $EJ^{h-1} \leq_e E$, then v or v' is an isomorphism; and*
- (b) *If $DJ^h \leq_e D$, then v' is an isomorphism.*

Proof. (a). From the maps above we get

$$\begin{array}{ccc} DJ^{h-1} & \xrightarrow{u} & C_1 J^{h-1} \xleftarrow{v} EJ^{h-1} \oplus FJ^{h-1} \\ EJ^{h-1} & \xrightarrow{u'} & C_2 J^{h-1} \xleftarrow{v'} DJ^{h-1}. \end{array}$$

Since F is colocal, $FJ^{h-1} = \text{soc } F$ is simple and hence $|EJ^{h-1}| \leq |DJ^{h-1}| \leq |EJ^{h-1}| + 1$. Therefore $|DJ^{h-1}| = |EJ^{h-1}|$ or $|DJ^{h-1}| = |EJ^{h-1}| + 1$. Thus $v' \mid DJ^{h-1}$ or $v \mid (EJ^{h-1} \oplus FJ^{h-1})$ is a monomorphism. But since $DJ^{h-1} \leq_e D$ and $EJ^{h-1} \oplus FJ^{h-1} \leq_e E \oplus F$, we see that v' or v is a monomorphism hence an isomorphism.

- (b) Similar to (a). //

Dually we have

Lemma 3.1'. *Consider the maps*

$$\begin{array}{ccc} D & \xleftarrow{u} C_1 \xrightarrow{v} E \oplus F \\ E & \xleftarrow{u'} C_2 \xrightarrow{v'} D \end{array}$$

in $\text{mod } R$ and put $h = h(F)$. Then it holds that

- (a) *If F is local, $\text{soc}_{h-1} D \ll D$ and $\text{soc}_{h-1} E \ll E$, then v or v' is an isomorphism; and*
- (b) *If $\text{soc}_h D \ll D$, then v' is an isomorphism.* //

Now we state our main results.

Theorem 3.2. *Let $0 \rightarrow K \xrightarrow{f} L \xrightarrow{g} M \rightarrow 0$ be a nonsplit exact sequence in*

mod R where $L = \bigoplus_{i=1}^n L_i$ ($n \geq 2$), f_1 , the first coordinate map of f , is a monomorphism and L_1, \dots, L_n are local and colocal. If it holds either

- (a) $h(L_1) = \min_{i=1}^n h(L_i)$, L_1 is local and colocal; or
- (b) $h(L_1) < \min_{i \neq 1} h(L_i)$, L_1 is local,

then M is indecomposable iff f is infusible.

Proof. (\Rightarrow). By (1.3). (\Leftarrow). Let D be the decomposition: $L = \bigoplus_{i=1}^n L_i$ and $M = M_1 \oplus M_2$; $M_1, M_2 \neq 0$, $p_i: M \rightarrow M_i$ be the canonical projections. Put $(D, g) = (g_i)_{i=1}^n$. Since all L_i 's are local, g has the D -compatible covering property. On the other hand noting that $(g_i)_{i \neq 1}$ is monic since f_1 is, we see that $(g_i)_{i \neq 1}$ has the $\bigoplus_{i \neq 1} L_i$ -compatible cocovered property since all L_i 's ($i \neq 1$) are colocal. Hence there are partitions $\{1, \dots, n\} = I_1 \amalg I_2$ and $\{2, \dots, n\} = J_1 \amalg J_2$ such that

$$\begin{aligned} \bigoplus_{I_1} L_i &\xrightarrow{p_1 g} M_1 \xleftarrow{p_1 g} \bigoplus_{I_2} L_i \\ \bigoplus_{J_2} L_i &\xrightarrow{p_2 g} M_2 \xleftarrow{p_2 g} \bigoplus_{J_1} L_i. \end{aligned}$$

Here we may assume $1 \in I_1$. Put $G_j = I_j \cap J_j$ and $H_j = J_j - G_j$ for each $j = 1, 2$. Then noting that $I_1 - G_1 = H_2 \amalg \{1\}$ and $I_2 - G_2 = H_1$, the following diagram is induced:

$$\begin{aligned} \bigoplus_{H_1} L_i &\xrightarrow{\quad} M'_1 \xleftarrow{\quad} (\bigoplus_{H_2} L_i) \oplus L_1 \\ \bigoplus_{H_2} L_i &\xrightarrow{\quad} M'_2 \xleftarrow{\quad} \bigoplus_{H_1} L_i \end{aligned}$$

where $M'_j = M_j / p_j g(\bigoplus_{G_j} L_i)$, since for any $G \subseteq I \subseteq \{1, \dots, n\}$ and any $u: \bigoplus_I L_i \rightarrow C$, we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bigoplus_G L_i & \longrightarrow & \bigoplus_I L_i & \longrightarrow & \bigoplus_{I-G} L_i \longrightarrow 0 \\ & & \parallel & & \parallel & & \downarrow \cong \\ 0 & \longrightarrow & \bigoplus_G L_i & \longrightarrow & \bigoplus_I L_i & \longrightarrow & \bigoplus_I L_i / \bigoplus_G L_i \longrightarrow 0 \\ & & \downarrow & & \downarrow u & & \downarrow \\ 0 & \longrightarrow & u(\bigoplus_G L_i) & \longrightarrow & C & \longrightarrow & C / u(\bigoplus_G L_i) \longrightarrow 0 \end{array}$$

with exact rows from which we see that the induced map $v: \bigoplus_{I-G} L_i \rightarrow C / u(\bigoplus_G L_i)$ is monic (resp. epic) if u is monic (resp. epic).

(i) In case (a) is satisfied. Put $h = h(L_1)$. Then for any subset $I \subseteq \{1, \dots, n\}$, we have that $\text{soc}(\bigoplus_I L_i) \leq (\bigoplus_I L_i) J^{h-1}$ since $\text{soc} L_i = L_i J^{h(L_i)-1} \leq L_i J^{h-1}$ for all $i \in I$. Hence from (3.1.a), we obtain that $|M'_1| = |(\bigoplus_{H_2} L_i) \oplus L_1|$ or $|M'_2| = |\bigoplus_{H_1} L_i|$. Therefore $|M_1| = |\bigoplus_{I_1} L_i|$ or $|M_2| = |\bigoplus_{I_2} L_i|$ since $p_j(g_i)_{G_j}$ are monomorphisms. Thus f is fusible.

(ii) In case (b) holds. Similarly we can use (3.1.b) to get $|M_2| = |\bigoplus_{H_1} L_i|$ thus $|M_2| = |\bigoplus_{I_2} L_i|$ and f is fusible. //

Corollary 3.3. Let $0 \rightarrow K \xrightarrow{f} L \xrightarrow{g} M \rightarrow 0$ be a nonsplit exact sequence in mod R , $D: L = \bigoplus_{i=1}^n L_i$ ($n \geq 2$) a decomposition of L and $(f, D) = (f_i)_{i=1}^n$. Suppose that each L_i is local and colocal, and each f_i is a monomorphism. Then M is indecomposable iff f is infusible //

Dually we obtain

Theorem 3.2'. Let $0 \rightarrow K \xrightarrow{f} L \xrightarrow{g} M \rightarrow 0$ be a nonsplit exact sequence in mod R , $D: L = \bigoplus_{i=1}^n L_i$ ($n \geq 2$) a decomposition of L and $(D, g) = (g_i)_{i=1}^n$. Suppose that g_1 is an epimorphism and L_2, \dots, L_n are local and colocal. If it holds either
 (a) $h(L_1) = \min_{i=1}^n h(L_i)$, L_1 is local and colocal; or
 (b) $h(L_1) < \min_{i=1}^n h(L_i)$, L_1 is colocal,
 then K is indecomposable iff g is coinfusible. //

Corollary 3.3'. In the same situation as above. Suppose that all L_i 's are local and colocal, and all g_i 's are epimorphisms. Then K is indecomposable iff g is coinfusible. //

REMARK. (3.2) is also a generalization of [2, Theorem 4.2 and Proposition 4.3] in the semiprimary case.

References

[1] H. Asashiba and T. Sumioka: *On Krull-Schmidt's theorem and the indecomposability of amalgamated sums*, Osaka J. Math. **20** (1983), 321-329.
 [2] E. Green: *On the decomposability of amalgamated sums*, J. Pure Appl. Algebra **14** (1979), 259-272.
 [3] H. Tachikawa: *On rings for which every indecomposable right module has a unique maximal submodule*, Math. Z. **71** (1959), 200-222.

Department of Mathematics
 Osaka City University
 Osaka 558, Japan.

