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A NOTE ON DIRECT SUMS OF CYCLIC MODULES OVER COMMUTATIVE REGULAR RINGS

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Throughout this note R is a commutative ring with identity and all modules are unital.

We denote the maximal ring of quotients of R by $Q(R)$ and the ring generated by the set of all idempotents of $Q(R)$ over R by $C(R)$. In case R is semi-prime, $C(R)$ coincides with the Baer hull of R in the sense of *A. C.* Mewborn ([3, Proposition 2.5]). For an R-module M, we denote its injective hull by $E_R(M)$. It is well known (e.g. [1]) that if R is semi-prime, then $Q(R)=E_R(R)$.

Let M be an R-module. We put

$$\begin{aligned} T(M) &= \{x\in M \mid \mathrm{Hom}_R(Rx, E_R(R)) = 0\} \\ &= \{x\in M \mid (0:x) \text{ is a dense}^{\dagger} \text{ ideal of } R\} \end{aligned}$$

where $(0:x)=\{r\in R \mid rx=0\}$ (see [7]). M is said to be torsion if $T(M)=M$ and torsion free if $T(M)=0$.

Now, for an R-module M, we shall consider the following condition studied in [4]:

(*) M is embedded in a direct sum of cyclic R-modules as an essential R-submodule.

In [4] the author proved the following

Theorem 1. *Let R be a regular ring. Then the following conditions are equivalent*:

(a) $Q(R)=C(R)$.

(b) *Every finitely generated torsion free R-module M satisfies the condition* (*)

The purpose of this note is to prove the following two theorems.

Theorem 2. *Let R be a regular ring. Then the following conditions are equivalent*:

(a) $Q(R/I)=C(R/I)$ *for every dense ideal I of R.*

† An ideal I of R is said to be dense provided that for any r and r' in R with $r'\neq 0$, there exists s in R such that $sr\in I$ and $sr'\neq 0$.

(b) *Every finitely generated torsion R-module M satisfies the condition* (*).

Theorem 3. *Let R be a regular ring. Then the following conditions are equivalent*:

(a) $Q(R/I)=C(R/I)$ *for every ideal I of R.*

(b) *Every finitely generated R-module M satisfies the condition* (*).

To prove the theorems above, we use the following lemmas.

Lemma 4. *For any R-module M with n generators, there exist n elements* $x_1, \cdots, x_n$ *in* $E_R(M)$ *such that* $E_R(M)$ *is a direct sum of* $E_R(Rx_1), \cdots, E_R(Rx_n)$. *Moreover we can take* $Rx_1+\cdots+Rx_n$ *to be torsion if M is torsion.*

Proof. We proceed the proof by induction on the number n of the generators of M. If $n=1$, our assertion is obvious. We assume that the lemma is true for every R-module with $n(<m)$ generators, and let us suppose that M has m generators, say $M=Ra_1+\cdots+Ra_m$.

By a usual property of injectivity of $E_R(Ra_m)$, we have that

$$E_R(Ra_m)+M = E_R(Ra_m)\oplus S\subseteq E_R(M)$$

for some submodule S of $E_R(Ra_m)+M$. Putting $a_i=b_i+s_i$, $b_i\in E_R(Ra_m)$, $s_i\in S$, $i=1, 2, \cdots, m-1$, we obtain

$$M+E_R(Ra_m) = (\textstyle\sum_{i=1}^{m-1}Rs_i)\oplus E_R(Ra_m)\,.$$

Hence by making use of our induction hypothesis on $\sum_{i=1}^{m-1}Rs_i$, there are $m-1$ elements $x_1, \cdots, x_{m-1}$ in $E_R(\sum_{i=1}^{m-1}Rs_i)\subseteq E_R(M)$ such that

$$E_R(\textstyle\sum_{i=1}^{m-1}Rs_i) = E_R(Rx_1)\oplus\cdots\oplus E_R(Rx_{m-1})\,.$$

Hence it follows that

$$E_R(M) = E_R(Rx_1)\oplus\cdots\oplus E_R(Rx_{m-1})\oplus E_R(Ra_m)\,.$$

Moreover, if M is torsion, so is each Ra_i and hence each Rs_i is also torsion. This implies that $(\sum_{i=1}^{m-1}Rx_i)+Ra_m$ is torsion.

The following lemma is due to R. S. Pierce [6, Corollary 23.7].

Lemma 5. *Let R be a regular ring, I an ideal of R and M an R-module with* $IM=0$. *Then M is injective as an R-module if and only if M is injective as an R/I-module.*

Remark. Let R be a regular ring and I an ideal of R. If M is an R-module and $IM=0$, then it is easily seen that $IE_R(M)=0$. In particular we have that $IE_R(R/I)=0$ and hence, by the lemma above, $E_R(R/I)=E_{R/I}(R/I)$ $(=Q(R/I))$.

Proof of Theorem 3. (a)$\Rightarrow$(b). Let $M=Ra_1+\cdots+Ra_n$ be a finitely generated R-module. By Lemma 4 and the above remark, there are ideals $I_1,\cdots,I_n$ of R such that M is embedded in the external direct sum of $Q(R/I_1),\cdots,Q(R/I_n)$ as an essential submodule. Let us write

$$\begin{aligned} a_1 &= x_{11}\times\cdots\times x_{1n}\,, \\ a_2 &= x_{21}\times\cdots\times x_{2n}\,, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ a_n &= x_{n1}\times\cdots\times x_{nn}\,, \end{aligned}$$

where $x_{ij}\in Q(R/I_j)$, $i, j=1, 2, \cdots, n$, and denote $\sum_{i=1}^{n} Rx_{ij}$ by $A_j, j=1, 2, \cdots, n$. Then A_j is a finitely generated R/I_j-submodule of $Q(R/I_j), j=1, 2, \cdots, n$ and M is embedded in $A_1\oplus\cdots\oplus A_n$ as an essential R-submodule. Here, applying Theorem 1, each A_j satisfies the condition $(*)$ as an R/I_j-module and so does as an R-module. Hence M satisfies the condition $(*)$ as an R-module.

In the proof above, we can take each I_i to be dense ideal of R if M is torsion. Therefore this yields the proof of (a)$\Rightarrow$(b) in Theorem 2 at the same time.

(b)$\Rightarrow$(a). Let I be an ideal of R. By Theorem 1, to prove that $Q(R/I)=C(R/I)$, we may show that every finitely generated torsion free R/I-module satisfies the condition $(*)$ as an R/I-module. But this is evident, since every finitely generated torsion free R/I-module satisfies the condition $(*)$ as an R-module and so does as an R/I-module.

Note that if I is a dense ideal and M is an R/I-module, then M is torsion as an R-module since $IM=0$. Hence we also obtain the proof of (b)$\Rightarrow$(a) in Theorem 2.

Corollary 6. *Let R be a regular ring such that $Q(R/I)=C(R/I)$ for every dense ideal I of R. Then every finitely generated torsion injective R-module is a direct sum of cyclic R-modules.*

Corollary 7. *Let R be a regular ring such that $Q(R/I)=C(R/I)$ for every ideal I of R. Then every finitely generated injective R-module is a direct sum of cyclic R-modules.*

Corollary 7 was shown by R. S. Pierce [6, Theorem 23.5] for the ring of all global sections of the simple F-sheaf over a Boolean space where F is a finite field. Let us note that a regular ring R is isomorphic to such a regular ring if and only if there exist finite elements, say $r_1, \cdots, r_n$, in R with the property that all $R/\mathfrak{m}$, $\mathfrak{m}\in \mathrm{Spec}(R)$ are fields with just n elements $r_1+\mathfrak{m}, \cdots, r_n+\mathfrak{m}$ and $R/\mathfrak{n}\simeq R/\mathfrak{n}'$ for any $\mathfrak{n}$, $\mathfrak{n}'$ in Spec (R) by the canonical mapping: $r_i+\mathfrak{n}\to r_i+\mathfrak{n}'$ (cf. [5, Proposition 2.1]). This class of regular rings contains Boolean rings and more generally p-rings in the sense of McCoy and Montgomery [2] ([6, p. 53]).

On the other hand, since this class of regular rings is clearly closed under homomorphic images, it is contained, by [5, Theorem 2.4], in the class of those regular rings R with $Q(R/I)=C(R/I)$ for every ideal I of R. So, Corollary 7 can be seen as a generalization of the result due to R. S. Pierce.

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