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<thead>
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<th>Title</th>
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</thead>
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1. Introduction

By making fundamental use of the Farey shift map and employing infinite (but $\sigma$-finite) measures together with the Chacon-Ornstein ergodic theorem it is possible to find new metrical results for continued fractions. Moreover this offers a unified approach to several existing theorems.

The application of ergodic theory to the study of continued fractions began with the Gauss transformation, $G: [0,1] \mapsto [0,1]$, 

$$G(x) = \begin{cases} \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

which is ergodic with respect to the Gauss measure $\mu_g$, where

$$\mu_g(B) = \frac{1}{\log 2} \int \frac{1}{\log(1+x)} \, dx$$

for any Borel subset $B$ of $[0,1]$. H. Nakada [11] extended $G$ to the 2-dimensional case. Let $\tilde{G}: [0,1] \times [0,1] \mapsto [0,1] \times [0,1]$ be defined to be

$$\tilde{G}(x,y) = \left( G(x), \frac{1}{a_1 + y} \right)$$

where $a_1 = \left\lfloor \frac{1}{x} \right\rfloor$. The absolutely continuous invariant measure of $\tilde{G}$, $\tilde{\mu}_g$, is given by

$$d\tilde{\mu}_g = \frac{1}{\log 2} \frac{dxdy}{(1+xy)^2}.$$
to $\tilde{\mu}$. Many metrical results for regular continued fractions can be proved using the ergodicity of $G$ or $\tilde{G}$. For example, [3] (W. Bosma et al) gave the distribution of the sequences of approximation constants $\{\theta_n\}$.

In this paper we focus on the convergents and the mediants of the Farey (or slow) continued fractions. Define $T$: [0,1]→[0,1] by

$$T(x) = \begin{cases} \frac{x}{1-x}, & 0 \leq x \leq \frac{1}{2}, \\ \frac{1-x}{x}, & \frac{1}{2} \leq x < 1. \end{cases}$$

$T$ is called the Farey shift map (see [10]). $T$ preserves the measure $\nu$ given by

$$d\nu = \frac{1}{\log 2} \frac{dx}{x},$$

which is $\sigma$-finite but not a probability measure, and $T$ is ergodic with respect to this measure (see [12] or [10]). The natural extension of $T$, denoted by $\tilde{T}$, is the transformation on $[0,1] \times [0,1]$ given by

$$\tilde{T}(x,y) = \begin{cases} \left( \frac{x}{1-x}, \frac{y}{1+y} \right), & 0 \leq x < \frac{1}{2} \\ \left( \frac{1-x}{x}, \frac{1}{1+y} \right), & \frac{1}{2} \leq x < 1 \end{cases}$$

The absolutely continuous invariant measure, $\tilde{\nu}$, of $\tilde{T}$ is determined by

$$d\tilde{\nu} = \frac{dx dy}{(x+y-xy)^2} \frac{1}{\log 2}.$$

The ergodicity of $\tilde{T}$ can be established from that of $\tilde{G}$ using an argument of [16] or by direct appeal to a general result given in [4]. Since the $\nu$ or $\tilde{\nu}$ are infinite ($\sigma$-finite, though), the Birkhoff Ergodic Theorem is not applicable for $T$ or $\tilde{T}$. To avoid this disadvantage, Ito considered another transformation $T_1$ induced by $T$, the invariant measure of which is a probability measure. By the ergodicity of $T_1$ and its natural extension, he obtained in [8] many metrical results related to convergents and nearest mediants. In this paper we consider $T$ and $\tilde{T}$ directly. We shall establish an ergodic theorem for $\tilde{T}$ though the Chacon-Ornstein ergodic theorem. In this way the results of [8] can be generalized for we can derive metrical results on Diophantine approximation by all the mediants not only the nearest ones. The results for nearest mediants become a special case. Any other metrical results obtained by applying the Birkhoff ergodic theorem for $G$ or
\( G \) can also be obtained by using the ergodic theorem we build for \( \bar{T} \).

In Section 2, we recall some basic results about Farey and regular continued fractions and give some basic properties of \( T \) and \( \bar{T} \). In Section 3, we establish an ergodic theorem for \( T \) through the Chacon-Ornstein Ergodic Theorem. In Section 4, we apply the ergodic theorem established in Section 3 to prove some old and new metrical results both for the regular continued fractions and the Farey continued fractions.

2. Preliminaries

For an irrational \( x \in [0,1] \) with regular continued fraction expansion

\[
\frac{1}{a_1 + \frac{1}{a_2 + \cdots}} = [0; a_1, a_2, \ldots],
\]

the \( n \)-th convergent is given by

\[
\frac{p_n}{q_n} = \frac{1}{a_1 + \frac{1}{a_2 + \cdots + \frac{1}{a_n}}} = [0; a_1, a_2, \ldots, a_n].
\]

The integers \( p_n, q_n \) can be described inductively by setting

\[
p_{-1} = 1, \quad p_0 = 0, \quad q_{-1} = 0, \quad q_0 = 1,
\]

\[
p_n = a_n p_{n-1} + p_{n-2}, \quad q_n = a_n q_{n-1} + q_{n-2}.
\]

We shall be concerned with a slower sequence \( \{P_n/Q_n\} \) of approximations to \( x \), corresponding to a branch of the Farey tree (see [7], [14] for details). For our present purpose it suffices to know that

\[
P_n = kp_m + p_{m-1}, \quad Q_n = kq_m + q_{m-1},
\]

where

\[
n = a_0 + a_1 + \cdots a_m + k, \quad 0 < k \leq a_{m+1}, \quad (a_0 = 0).
\]

The sequence \( \{p_n/q_n\} \) consists of the convergents of \( x \), while \( \{P_n/Q_n\} \) is the sequence of convergents and mediants.

The Farey shift map \( T: [0,1] \to [0,1] \), defined in the introduction, may be characterised as follows. For \( x = [0; a_1, a_2, \ldots] \), we have

\[
T(x) = \begin{cases} 
[0; a_1 - 1, a_2, \ldots], & a \geq 2 \\
[0; a_2, a_3, \ldots], & a_1 = 1.
\end{cases}
\]
Moreover, for $x = [0; a_1, a_2, \ldots]$, $y = [0; b_1, b_2, \ldots]$ we have

$$
\tilde{T}(x, y) = \begin{cases} 
(0; a_1 - 1, a_2, \ldots), [0; b_1 + 1, b_2, \ldots] & a \geq 2 \\
(0; a_2, a_3, \ldots), [0; 1, b_1, b_2, \ldots] & a_1 = 1.
\end{cases}
$$

We make basic use of the numbers $X_n, Y_n$ defined by $\tilde{T}^n(x, 1) = (X_n, Y_n)$. Note, in particular, that

$$
Y_n = \begin{cases} 
[0; k + 1, a_m, \ldots, a_1] & m \geq 1 \\
[0; k + 1] & m = 0
\end{cases}
$$

where

$$
n = a_0 + a_1 + \cdots + a_m + k, \quad 0 \leq k < a_{m+1}.
$$

It is well-known that for $G$, defined in the introduction,

$$
G^n(x, 0) = (x_{m}, y_{m}) = ([0; a_{m+1}, a_{m+2}, \ldots], [0; a_{m}, \ldots, a_1]),
$$

where $x_{m} = G^{m}x$, $y_{m} = q_{m-1} / q_{m}$.

Let us write

(1) $$
\Theta_n = Q_n^2 \left| x - \frac{P_n}{Q_n} \right|
$$

and

(2) $$
\theta_m = q_m^2 \left| x - \frac{P_m}{q_m} \right|.
$$

Then we have

(3) $$
\theta_m = x_m (1 + x_m y_m)^{-1}
$$

and this is contained in the following formula, given in [2],

(4) $$
\Theta_n = (1 - k x_m)(k + y_m)(1 + x_m y_m)^{-1},
$$

where

$$
n = a_0 + a_1 + \cdots + a_m + k, \quad 0 \leq k \leq a_{m+1}.
$$

(4) allows discussion of $\Theta_n$ via $G$ but our strategy is to work directly with $\tilde{T}$ so we set about expressing the quantities $Q_{n-1} / Q_n$ and $\Theta_n$ in terms of $X_n, Y_n$.

**Lemma 1.** For $n = 1, 2, \ldots$, we have
$$Q_n / Q_{n+1} = (1 + Y_{n-1})^{-1} = \max \{Y_n, 1 - Y_n\}.$$  

Proof. We know that $Y_n$ equals $(1 + Y_{n-1})^{-1}$ or $(1 + Y_{n-1})^{-1}$ and that $0 < Y_{n-1} \leq 1$. Hence $(1 + Y_{n-1})^{-1}$ equals $\max \{Y_n, 1 - Y_n\}$.

Next we use induction. When $n = 1$, we have $Y_1 = 1/2$, $Q_1 = 1$. And we always have $Q_2 = 2$. Hence

$$Q_1 / Q_2 = 1/2 = Y_1 = \max \{1, 1 - 1\}.$$  

Suppose for $1 \leq n$ we have

$$Q_l / Q_{l+1} = \max \{Y_l, 1 - Y_l\}.$$  

Assume that $n = a_0 + a_1 + a_2 + \cdots + a_m + k$, $0 \leq k < a_{m+1}$. If $k = 0$, then $Q_n = q_m$, $Q_{n+1} = 1 \cdot q_m + q_{m-1}$ and $Y_l = [0; 1, a_m, a_{m-1}, \ldots, a_1] \geq 1/2$. Hence $Q_n / Q_{n+1} = Y_n$. If $a_{m+1} \geq 2$, then $Q_{n+2} = 2 \cdot q_m + q_{m-1} = Q_n + Q_{n+1}$. Thus

$$Q_{n+1} / Q_{n+2} = (1 + Q_n / Q_{n+1})^{-1} = (1 + Y_n)^{-1}.$$  

If $a_{m+1} = 1$, then $Q_{n+1} = q_{m+1}$ and $Q_{n+2} = q_m + q_{m+1}$. Again we have $Q_{n+2} = Q_n + Q_{n+1}$ and again we get $Q_{n+1} / Q_{n+2} = (1 + Y_n)^{-1}$.

When $0 < k < a_{m+1} - 1$, we have $Q_n = k q_m + q_{m-1}$, $Q_{n+1} = (k + 1) q_m + q_{m-1}$ and $Q_{n+2} = (k + 2) q_m + q_{m-1}$, and $Y_n$, which equals $[0; k + 1]$ or $[0; k + 1, a_m, \ldots, a_1]$, is at most $1/2$. Hence

$$Q_{n+1} / Q_{n+2} = Q_n + 1 / 2.$$  

Lastly we consider $k = a_{m+1} - 1 > 0$. We have $Q_n = k q_m + q_{m-1} = q_{m+1} - q_m$, $Q_{n+1} = q_{m+1}$ and $Q_{n+2} = q_{m+1} + q_m$. We also have $Y_n = y_{m+1} \leq 1/2$. Hence, once more, $Q_{n+1} / Q_{n+2} = (1 + Y_n)^{-1}$.

Lemma 2. For $n = 2, 3, \ldots$, we have

$$\Theta_n \begin{cases} 
(1 - Y_n)(X_n + Y_n - X_n Y_n)^{-1}, & Y_n < 1/2, \\
X_n Y_n(X_n + Y_n - X_n Y_n)^{-1}, & Y_n > 1/2,
\end{cases}$$  

(5)  

$$\Theta_{n+1} = (1 - X_n)(X_n + Y_n - X_n Y_n)^{-1}.$$  

(6)  

Proof. For $n = a_0 + a_1 + a_2 + \cdots + a_m \geq 2$, we have $X_n = x_m$ and $Y_n = (1 + y_m)^{-1} > 1/2$. An application of (3) gives
\[ \Theta_n = \theta_m = X_n(1 + X_n(Y_{n-1} - 1))^{-1} = X_n Y_n(X_n + Y_n - X_n Y_n)^{-1}. \]

For \( n = a_1 + \cdots + a_m + k \geq 2 \) where \( 0 < k < a_{m+1} \), we apply (4) after noting that
\[ X_n = (x_m^{-1} - k)^{-1}, \quad Y_n = (k + 1 + y_m)^{-1} < 1/2. \]

This gives
\[ \Theta_n = (1 - Y_n)(X_n + Y_n - X_n Y_n)^{-1}. \]

It is now easy to see that \( Y_{n+1} > 1/2 \) if and only if \( X_n > 1/2 \). When \( X_n > 1/2 \)
\[ (X_{n+1}, Y_{n+1}) = \tilde{T}(X_n, Y_n) = (X_n^{-1}(1 - X_n)(1 + Y_n)^{-1}). \]

In this case we see that
\[ \Theta_{n+1} = X_{n+1} Y_{n+1}(X_{n+1} + Y_{n+1} - X_{n+1} Y_{n+1})^{-1} = (1 - X_n)(X_n + Y_n - X_n Y_n)^{-1} \]

For the case \( X_n < 1/2 \), we substitute \( X_{n+1} = (1 - X_n)^{-1} X_n, \ Y_{n+1} = Y_n(1 + Y_n)^{-1} \)
in the formula
\[ \Theta_{n+1} = (1 - Y_{n+1})(X_{n+1} + Y_{n+1} - X_{n+1} Y_{n+1})^{-1} \]
to obtain the required result. \( \square \)

3. Ergodicity and ergodic theorem for \( \tilde{T} \)

We begin this section by showing that \( \tilde{G} \) can be induced from \( \tilde{T} \).

Theorem 1. The dynamical system \((\Omega, \mathcal{A}, \mu, \tilde{G}) \) is (isomorphic with) the system induced from \((\Omega, \mathcal{B}, \nu, \tilde{T}) \) on the set \( E = \{ (x,y): y > 1/2 \} \).

Proof. Recall that for \( x = [0; a_1, a_2, \ldots], \ y = [0; b_1, b_2, \ldots] \) we have
\[ \tilde{T}(x,y) = \begin{cases} ([0; a_1 - 1, a_2, \ldots], [0; b_1 + 1, b_2, \ldots]), & a_1 \geq 2, \\ ([0; a_2, a_3, \ldots], [0; 1, b_1, b_2, \ldots]), & a_1 = 1. \end{cases} \]

In particular the second coordinate of \( \tilde{T}(x,y) \) is greater than 1/2 if and only if \( a_1 = 1 \). Accordingly the induced map \( \tilde{T}_E \) is given by
\[ \tilde{T}_E(x,y) = (y^{-1} - a_1, (1 + (y^{-1} + a_1 - 1)^{-1})^{-1}). \]

Now consider the map \( \phi: E \to \Omega \) given by
The map transforms to \((x,y)\mapsto(x^{-1} - a_1, (y + a_1)^{-1})\) and the measure \(dxdy(x+y-xy)^{-2}\) transforms to \(dxdy(1+xy)^{-2}\).

By Theorem 1 and the ergodicity of \(\tilde{G}\) together with a result of [16], we obtain the ergodicity of \(\tilde{T}\). Theorem 1 also gives an abstract justification of the statement that any result derived from \(G\) or \(G\) can be obtained from \(\tilde{T}\).

The map \(\tilde{T}\) is ergodic and invertible and \(\tilde{\nu}\) is non-atomic so it follows (see [5]) that \(\tilde{T}\) is conservative. Therefore we may apply the Chacon-Ornstein theorem (cf. [13]) on the system \((\Omega, \mathcal{B}, \tilde{\nu}, \tilde{T})\) (we use \(\Omega\) to denote \([0,1] \times [0,1]\) for the remainder of the paper) to derive the following result.

**Theorem 2.** For any \(f,g \in L^1(\Omega, \mathcal{B}, \tilde{\nu})\) with \(\int gd\tilde{\nu} \neq 0\), one has

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(\tilde{T}^k(x,y)) = \int f \, d\tilde{\nu} \quad \text{a.e.}
\]

Next we show that under Lipschitz conditions on \(f,g\), the points \(\tilde{T}^k(x,y)\) in Theorem 2 can be replaced by \((X_k, Y_k) = \tilde{T}^k(x,1)\).

**Theorem 3.** Suppose that \(f,g \in L^1(\Omega, \mathcal{B}, \tilde{\nu})\) satisfy

\[
|f(x,y) - f(x,y')| \leq L|y - y'|^\alpha
\]

\[
|g(x,y) - g(x,y')| \leq L|y - y'|^\alpha
\]

where \(L > 0, \alpha > 0\) are constants. If \(\int gd\tilde{\nu} \neq 0\), then for almost all \((x,y)\in\Omega\) one has

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(\tilde{T}^k(x,y)) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(X_k, Y_k)
\]

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} g(\tilde{T}^k(x,y)) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} g(X_k, Y_k)
\]

Proof. Let \(Z_i\) be the second coordinate of \(\tilde{T}^i(x,y)\), i.e. \(\tilde{T}^i(x,y) = (X_i, Z_i)\). We claim that for almost all \(x \in [0,1]\) and all \(y \in (0,1]\), we have

\[
\sum_{i=0}^\infty |Z_i - Y_i|^\alpha < +\infty.
\]

In fact, for \(x = [0;a_1, a_2, \ldots], \, i = a_1 + a_2 + \cdots + a_m + k, \, 0 \leq k < a_{m+1}, \) one has
and

\[ Y_i = [0; k+1, a_m, \ldots, a_2, a_1]. \]

Hence we have

\[ |Y_i - Z_i| \leq c/q_m^2 \]

for some constant \( c \), where \( q_m \) is the denominator of the \( m \)-th convergent \( p_m/q_m \) of the regular continued fraction expansion of \( x \) (cf. [1] p.42]. Therefore we get that

\[ \sum_{i=0}^{\infty} |Z_i - Y_i|^q \leq \sum_{m=0}^{\infty} \sum_{k=1}^{m+1} \left( \frac{c}{q_m^2} \right)^q = c^q \sum_{m=0}^{\infty} \frac{a_{m+1}}{q_m^{2q}}. \]

By induction we can see that

\[ q_m \approx 2^{(m-1)/2}. \]

We need the following theorem (see [15]).

**Theorem A.** Let \( F(n) > 1 \), for \( n = 1, 2, \ldots \), and suppose that \( \sum_{n=1}^{\infty} \frac{1}{F(n)} < \infty \).

Then the set

\[ A = \{ x \in [0,1], a_k(x) > F(k) \text{ infinitely many times} \} \]

has Lebesgue measure 0.

Now we choose \( F(n) = 2^{n/2} \). By the above theorem we see that the set

\[ E = \{ x \in [0,1], a_k(x) > F(k) \text{ only finitely many times} \} \]

has Lebesgue measure 1. Hence for almost all \( x \in [0,1] \),

\[ \sum_{m=0}^{\infty} \frac{a_{m+1}}{q_m^{2q}} \leq C(x) + \sum_{m=0}^{\infty} 2^{n(2-m)/2} < \infty. \]

Where \( C(x) = \sum_{a_m > F(m)} \frac{a_{m+1}}{q_m^{2q}} \). The required result follows easily when we bear in mind the fact that \( \Sigma g(T^q(x,y)) \) diverges almost everywhere because \( T \) is conservative and ergodic. \( \blacksquare \)

For some functions \( f(x,y) \) though we do not have
for all $y, y' \in [0,1]$, it is still true that

$$|f(T^n(x,y)) - f(X_n, Y_n)| \leq L|Z_i - Y_i|^\alpha, \quad \alpha > 0,$$

for almost all $x \in [0,1]$ and $i$ large enough.

**Example.** Let

$$f(x,y) = \begin{cases} \log(1-y) & y \in [0,1) \\ 0 & y = 1 \end{cases}$$

Then $f \in L^1(\Omega)$. We do not have $(\ast)$ for all $y, y' \in [0,1]$.

For $x = [0; a_1, a_2, \ldots]$, let

$$i = a_1 + a_2 + \cdots + a_m + k, \quad 0 \leq k < a_{m+1}$$

where $m \geq 2$. Then

$$|f(T^n(x,y)) - f(X_n, Y_n)|$$

$$= |\log(1-Z_i) - \log(1-Y_i)|$$

$$= \frac{1}{|1-\xi_i|} |Z_i - Y_i|$$

where $\xi_i$ is in between $y_i$ and $Y_i$. It is easy to see that $1 - \xi_i \geq 1/q_m$, i.e.

$$\frac{1}{1-\xi_i} \leq q_m \leq c |Y_i - Z_i|^{1/2}.$$ 

Therefore, $|f(T^n(x,y)) - f(X_n, Y_n)| \leq c |Y_i - Z_i|^{1/2}$.

4. Applications

In this section we apply the ergodic theorems for $\tilde{T}$ to obtain metrical results for convergents and medians of regular continued fractions. For all the functions $f, g$ involved in this section it is valid to replace $T^n(x,y)$ by $(X_k, Y_k)$ as in Theorem 3 but omit the tedious verification.

For an irrational $x = [0; a_1, a_2, \ldots]$ we shall call

$$\frac{kp_n + p_{n-1}}{kq_n + q_{n-1}} \quad \text{and} \quad \frac{(a_{n+1} - k)p_n + p_{n-1}}{(a_{n+1} - k)q_n + q_{n-1}}$$
the $k$-th mediants of $x$, when $a_{n+1} \geq 2k$. We let $P_n^{(k)}/Q_n^{(k)}$ denote the sequence which consists of all convergents and $i$-th mediants of $x$ for all $i \leq k$. When $k=0$, we recover the convergents $\{p_m/q_m\}$ and, when $k=1$, we obtain the so-called nearest mediants of Ito, [8]. It is easy to see that the event \"$P_n/Q_n$ appears as some $P_{n+i}^{(k)}/Q_{n+i}^{(k)}$ is characterised by $X_n > 1/(k+1)$ or $Y_n > 1/(k+2)$ while \"$P_{n+1}/Q_{n+1}$ appears as some $P_{n+i}^{(k)}/Q_{n+i}^{(k)}$ is by $X_n > 1/(k+2)$ or $Y_n > 1/(k+1)$.

Let us write also

$$\Theta_n^{(k)} = (Q_n^{(k)})^2 |x - P_n^{(k)}/Q_n^{(k)}|.$$

Our main theorem can now be stated.

**Theorem 4.** For almost all $x$ we have

(i). for $k=0,1,2,\cdots$,

$$\lim_{n \to \infty} \frac{1}{n} \log Q_n^{(k)} = \frac{\pi^2}{12 \log(2k+2)},$$

(ii). for $k=0,1,2,\cdots$,

$$\lim_{n \to \infty} \frac{1}{n} \log |x - P_n^{(k)}/Q_n^{(k)}| = -\frac{\pi^2}{6 \log(2k+2)},$$

(iii). for $k=1,2,\cdots$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\{i \leq n, \Theta_i^{(k)} < z\} = \begin{cases} z, & 0 \leq z < 1 \\ 1 + \log z, & 1 \leq z < \frac{k+1}{2} \\ \frac{2 - 2z}{k+1} + \log \frac{2z^2}{k+1}, & \frac{k+1}{2} \leq z < k+1 \\ \log(2k+2), & k+1 \leq z \end{cases}$$

(iv). $\lim_{n \to \infty} \frac{1}{\log Q_n} \sum_{i=1}^{n} \mathbb{1}\{i \leq n, \Theta_i < z\} = \begin{cases} z, & 0 < z \leq 1, \\ \frac{\pi^2}{1 + \log z}, & 1 < z. \end{cases}$

**Remark.** The case $k=0$ of (i) and (ii) are the basic results of Levy (see [1]), and the case $k=1$ of (i), (ii) and (iii) give results of Ito, [8]. The important result of Bosma *et al* in [3] corresponds to the case $k=0$ of (ii) and the proof which
follows could be simplified to yield that special case. Nevertheless a suitable interpretation of the three terms corresponding to $1/2 \leq z < 1$ yields the appropriate distribution. We take the signed sum, i.e.

$$\lim_{n \to \infty} \frac{1}{n} \#\{i: \theta_n \leq z\} = \begin{cases} 
z, & 0 \leq z < \frac{1}{2} \\
1 - z + \log(2z), & \frac{1}{2} \leq z < 1 \\
\log 2, & 1 \leq z \end{cases}$$

where

$$1 - z + \log(2z) = z - (1 + \log z) + [2 - 2z + \log(2z^2)].$$

Proof. For (i) we take

$$f(x,y) = \log(\max\{y, 1 - y\}),$$

$$g(x,y) = \begin{cases} 1, & \text{when } x > (k + 1)^{-1}, \text{ or } y > (k + 2)^{-1} \\
0, & \text{otherwise.} \end{cases}$$

Then

$$\int f \, d\bar{y} = -\frac{n^2}{12 \log 2} \quad \text{and} \quad \int g \, d\bar{y} = \frac{\log(2k + 2)}{\log 2}.$$  

For $n = a_0 + a_1 + \cdots + a_m + k, \ 0 \leq k < a_{m+1}$, we have, by Lemma 1,

$$\log \frac{Q_n}{Q_{n+1}} = f(X_n, Y_n)$$

while

$$\sum_{i=1}^{n} g(X_n, Y_i) = \#\{i: \theta_i \leq z, Q_i \text{ appears as some } Q_j^{(k)}\}.$$  

Therefore

$$\sum_{i=1}^{n} f(X_n, Y_i) = \sum_{i=1}^{n} g(X_n, Y_i) = -s^{-1}(\log Q_{n+1} - \log Q_1),$$

where $s$ is determined by $Q_s^{(k)} \leq Q_n < Q_{s+1}^{(k)}$. By the ergodic theorem of the last section

$$\lim_{s \to \infty} \frac{\log Q_s^{(k)}}{s} = -\lim_{n \to \infty} \left[ \sum_{i=1}^{n} f(X_n, Y_i) / \sum_{i=1}^{n} g(X_n, Y_i) \right] = -\int f \, d\bar{y} / \int g \, d\bar{y}.$$
and the required result follows.

For (ii) noting that \( \Theta_n^{(k)} = (Q_n^{(k)})^2 |x - p_n^{(k)} / Q_n^{(k)}| \), by (i) it suffices to show that

\[
\lim_{n \to \infty} \frac{\log \Theta_n^{(k)}}{n} = 0 \text{ a.e.}
\]

Remember that \( \Theta_{n+1} \) is one of \( \Theta^{(k)} \) if and only if \( X_n > (k+2)^{-1} \) or \( Y_n > (k+1)^{-1} \). Then when \( \Theta_{n+1} \) is one of \( \Theta^{(k)} \) we have

\[
\Theta_{n+1} = \frac{1 - X_n}{X_n + Y_n - X_n Y_n} < k + 1.
\]

On the other hand,

\[
\Theta_{n+1} = 1 - X_n > 1 - [0; 1, a_{m+1}, \ldots] > 1 / (a_{m+1} + 1) \geq 1 / 2a_{m+1},
\]

where \( m \) is determined by \( Q_n \leq q_m < Q_{n+1} \). Therefore,

\[
\frac{- \log a_{m+1} + \log 2 + \Theta_n^{(k)}}{s} < \frac{\log(k + 1)}{s}.
\]

Noting that \( m \leq s \leq (2k + 1)m \) we obtain

\[
\frac{\log a_{m+1}}{s} \leq \frac{\log a_{m+1}}{m} = \left( \frac{a_{m+1}}{m^s} \right)^{1/\alpha} \cdot \frac{\log a_{m+1}}{a_{m+1}^{1/\alpha}} \to 0 \text{ a.e.}
\]

by Theorem A, where \( \alpha > 1 \) is a constant. Therefore

\[
\lim_{n \to \infty} \frac{\log \Theta_n^{(k)}}{n} = 0 \text{ a.e.}
\]

To prove (iii) and (iv) we also consider \( \Theta_{n+1} \) instead of \( \Theta_n \). Let

\[
E_{z,t} = \{(x,y) \in \Omega: \frac{1-x}{x+y-xy} < z, x > t \}
\]

\( \cup \{(x,y) \in \Omega: \frac{1-x}{x+y-xy} < z, y > t \} = E_1 \cup E_2 \)

where \( 0 < z, 0 \leq t \leq 1/2 \) and

\[
f(x,y) = \begin{cases} 
1, & (x,y) \in E_{z,t} \\
0, & \text{otherwise.}
\end{cases}
\]
We recast the inequality

\[(1-x)(x + y - xy)^{-1} < z\]

in the form

\[x > (1 - zy)(1 + z(1 - y))^{-1} = u(y), \text{ say,}\]

and note that \((1 - zy)(1 + z(1 - y))^{-1} > t/(1 + t)\) if and only if \(y > z^{-1} - t\). Therefore

\[
E_1 = \begin{cases} 
\{x > u(y), \ 0 < y < 1\}, & 0 < z < \frac{1}{1+t} \\
\{x > u(y), \ 0 < y < \frac{1}{z - t}\} \cup \{x > \frac{t}{1+t}, \ \frac{1}{z} - t < y < 1\}, & \frac{1}{1+t} < z < \frac{1}{t} \\
\{x > \frac{t}{1+t}, \ 0 < y < 1\}, & \frac{1}{t} < z.
\end{cases}
\]

Suppose first that \(0 < z < 1/(1+t)\). Then \(E_{z,t} = E_1 = \{x > u(y), \ 0 > y > 1\}\).

Hence

\[
\log 2 \int f d\tilde{\nu} = \int_0^1 dy \int_{u(y)}^1 (x + y - xy)^{-2} dx = z.
\]

Now consider the case \(1/(1+t) \leq z < 1\). Note that \(t \leq 1 - t < z^{-1} - t\). Then again \(E_{z,t} = \{x > u(y), \ 0 < y < 1\}\) and \(\log 2 \int f d\tilde{\nu} = z\).

When \(1 < z < 1/2t\) we also have \(z^{-1} - t > t\). Remember that \(u(y) > 0\) when \(y > z^{-1}\). Thus

\[
E_{z,t} = \{(x,y) \in \Omega: x > u(y), \ 0 < y < 1\}
= \{x > u(y), \ 0 < y < z^{-1}\} \cup \{x > 0, \ z^{-1} < y < 1\},
\]

and

\[
\log 2 \int f d\tilde{\nu} = 1 + \log z.
\]

If \(1/2t < z < 1/t\), then \(0 < z^{-1} - t < t\) and

\[
E_{z,t} = \{x > u(y), \ 0 < y < \frac{1}{z - t}\} \cup \{x > \frac{t}{1+t}, \ \frac{1}{z} - t < y < t\}
\]

\[
\cup \{x > u(y), \ t < y < \frac{1}{z}\} \cup \{x > 0, \ \frac{1}{z} < y < 1\}
\]
Calculate that
\[ \log 2 \int f \, d\nu = 2 - 2tz + \log(2tz^2). \]

Lastly, when \( z > 1/t \) we have
\[ E_{x,t} = \{ x > t/(1+t), \ 0 < y < t \} \cup \{ x > 0, \ t < y < 1 \} \]
and \( \log 2 \int f \, d\nu = \log 2 - \log t \).

It is now easy to piece together the result of (iii) by taking \( t = (k+1)^{-1} \) and
\[ g(x,y) = \begin{cases} 1, & \text{when } x > (k+2)^{-1} \text{ or } y > (k+1)^{-1} \\ 0, & \text{otherwise.} \end{cases} \]

To obtain (iv) we set \( t = 0 \) and replace \( g \) by the function \( \log(\max\{y, 1-y\}) \) used as \( f \) in proving (i). This completes the proof. \( \blacksquare \)

**REMARK.** Part (iii) of the theorem (in some sense a limiting case as \( k \to \infty \)) shows that \( \frac{1}{n} \# \{ i \leq n, \ \Theta_{n+i} < z \} \to 0 \) as \( n \to \infty \) and hence that \( \{ \Theta_n \} \) does not have a distribution function. We can obtain some more information about \( \Theta_n \). We have

For almost all \( x \in [0, 1] \) and any \( \varepsilon > 0 \),

(i). \( \lim_{n \to \infty} \frac{\Theta_n}{m^1 + \varepsilon} = 0 \), where \( q_m < Q_n < q_{m+1} \);

(ii). \( \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \Theta_i = \infty \); and

(iii). \( \lim_{n \to \infty} \frac{1}{n^2 + \varepsilon} \sum_{i=1}^{n} \Theta_i = 0. \)

In fact, for \( n = a_0 + a_1 + \cdots + a_m + k \), where \( 0 \leq k < a_{m+1} \), by Lemma 2

\[ \Theta_{n+1} < Y_n = [k+1; a_m, \cdots, a_1] < k + 2. \]

Then (i) follows from Theorem A. Using (\*) we can get the following estimation:

\[ \frac{1}{16} \left[ \sum_{i=1}^{m} a_i^2 + k^2 - n \right] = (m + 1) \Theta \leq \sum_{i=1}^{m} a_i^2 + k^2. \]

Then follow (ii) and (iii).

Next let us compare Theorem 4 (iv) with some results of P. Erdös [6] and
J. Blom [2]. Let

\[ \Theta \left( \frac{p}{q}, x \right) = q^2 \left| x - \frac{p}{q} \right| . \]

Define

\[ U(x, z, n) = \# \{ (p, q) \in \mathbb{Z} \times \mathbb{N} : (p, q) = 1, \ \Theta \left( \frac{p}{q}, x \right) \leq z, \ q \leq n \}, \]

\[ U_1(x, z, n) = \# \{ (p, q) \in \mathbb{Z} \times \mathbb{N} : (p, q) = 1, \ \Theta \left( \frac{p}{q}, x \right) \leq z, \ q \leq n, \frac{p}{q} \text{ is a convergent of } x \} \]

and

\[ U_2(x, z, n) = \# \{ (p, q) \in \mathbb{Z} \times \mathbb{N} : (p, q) = 1, \ \Theta \left( \frac{p}{q}, x \right) \leq z, \ q \leq n, \frac{p}{q} \text{ is a best approximant of } x \} . \]

By best approximant we mean that if there is a fraction \( \frac{a}{b} \) different from \( \frac{p}{q} \) such that

\[ \frac{a}{b} \neq \frac{p}{q} \text{ and } \frac{a}{b} \leq \frac{p}{q} \]

then \( b > q \). Erdös [6]) proved that for any \( z \geq 0 \)

\[ \lim_{n \to \infty} \frac{U(x, z, n)}{n} = \frac{12}{\pi^2} z \quad \text{a.e.} \]

Blom [2] gave that

\[ \lim_{n \to \infty} \frac{U_1(x, z, n)}{n} = \frac{12}{\pi^2} (f)z \quad \text{a.e.} \]

and

\[ \lim_{n \to \infty} \frac{U_2(x, z, n)}{n} = \frac{12}{\pi^2} (f(z) + h(z)) \quad \text{a.e.} \]

where
When $z \leq 1$ Theorem 4(iii) corresponds to the result of Erdős [6]. [8] and [9] also gave new proofs in this case. When $z > 1$ this result takes a different form. This fact tells us that, for $z > 1$, there is no result for convergents and mediants analogous to the theorem of Legendre for $z = 1/2$ or the theorem of Fatou and Koksma for $z = 1$ (see [9]).

The result of Blom [2] can also be proved by Theorem 2 or 3. In fact for irrational $x \in [0,1)$, a best approximant is an element of $\left\{ \frac{p_n}{q_n} \right\}$ characterized by

$$Y_n > \frac{1}{2} \quad \text{or} \quad Y_n^{-1} > X_n^{-1} + 1, \quad n \geq 2.$$  

Thus we can prove these results by choosing appropriate functions $f$ and $g$.

Jager [9] considered the two sequences

$$\left\{ \frac{p_n+p_{n-1}}{q_n+q_{n-1}} \right\}, \quad \left\{ \frac{(a_{n+1}-1)p_n+p_{n-1}}{(a_{n+1}-1)q_n+q_{n-1}} \right\}$$

separately and obtained some metrical results. If $P_N/Q_N$ appears as $(p_n+p_{n-1})/(q_n+q_{n-1})$ then we have

$$X_N = [0; a_{n+1}-1, a_{n+2}, \ldots], \quad Y_N = [0; 2, a_n, \ldots, a_1] \quad \text{when } a_{n+1} \geq 2,$$

or

$$X_N = [0; a_{n+2}, \ldots], \quad Y_N = [0; 1, 1, a_n, \ldots, a_1] \quad \text{when } a_{n+1} = 1.$$

Thus the first sequence is characterized by $1/3 < Y_n < 2/3$. The second one is more complicated. If $a_{n+1} \geq 2$, then $((a_{n+1}-1)p_n+p_{n-1})/((a_{n+1}-1)q_n+q_{n-1})$ corresponds

$$X_N = [0; 1, a_{n+2}, \ldots] \in (1/2,1), \quad Y_N = [0; a_{n+1}, \ldots, a_1] \in (0,1/2).$$
When $a_{n+1} = 1$ we get $p_{n-1}/q_{n-1}$ which corresponds to

$$
\begin{align*}
X_n &= [0; a_n, 1, a_{n+2}, \ldots] \in \left(\frac{1}{a_n+1}, \frac{2}{2a_n+1}\right), \\
Y_n &= [0, 1, a_n, \ldots, a_1] \in (1/2, 1).
\end{align*}
$$

Hence the second one is characterised by

$$
X_n \in (1/2, 1), \ Y_n \in (0, 1/2) \quad \text{or} \quad X_n \in \bigcup_{i=1}^{\infty} \left(\frac{1}{i+1}, \frac{2}{2i+1}\right), \ Y_n \in (1/2, 1).
$$

However, those two sequences are not "pure" nearest mediants. We shall consider the sequences

$$
\begin{align*}
\left\{\frac{p_n + p_{n-1}}{q_n + q_{n-1}}, a_{n+1} \geq 2\right\} \quad \text{and} \quad \left\{\frac{(a_{n+1}-1)p_n + p_{n-1}}{(a_n+1-1)q_n + q_{n-1}}, a_{n+1} \geq 2\right\}
\end{align*}
$$

which are characterised by $1/3 < Y_n < 1/2$ and $X_n > 1/2, Y_n < 1/2$ respectively. In general we use $\{a^{(k)}/b^{(k)}\}$ and $\{c^{(k)}/d^{(k)}\}$ to denote the "pure" $k$-th mediant sequences for each of the two directions

$$
\begin{align*}
\left\{\frac{kp_n + p_{n-1}}{kq_n + q_{n-1}}, a_{n+1} \geq 2k\right\} \quad \text{and} \quad \left\{\frac{(a_{n+1}-k)p_n + p_{n-1}}{(a_n+1-k)q_n + q_{n-1}}, a_{n+1} \geq 2k\right\}
\end{align*}
$$

respectively, where $k \leq 1$. It is not hard to see that $\{a^{(k)}/b^{(k)}\}$ is the subsequence of $\{P_n/Q_n\}$ determined by $X_n < 1/k$ and $(k+2)^{-1} < Y_n < (k+1)^{-1}$ while $\{c^{(k)}/d^{(k)}\}$ determined by $(k+1)^{-1} < X_n < k^{-1}$ and $Y_n < (k+1)^{-1}$. Define

$$
\sigma^{(k)}_n = b^{(k)}_n|a^{(k)}_n - c^{(k)}_n|
$$

and

$$
\rho^{(k)}_n = a^{(k)}_n|a^{(k)}_n - c^{(k)}_n|.
$$

**Theorem 5.** For almost all $x \in [0,1]$ and $k = 1, 2, \ldots$, one has

(i). \( \lim_{n \to \infty} \frac{\log b^{(k)}_n}{n} = \lim_{n \to \infty} \frac{\log a^{(k)}_n}{n} = \frac{\pi^2}{12(\log(2k+1)-\log(2k))} \),

(ii). \( \lim_{n \to \infty} \frac{1}{n} \log \left| x - \frac{a^{(k)}_n}{b^{(k)}_n} \right| = \lim_{n \to \infty} \frac{1}{n} \log \left| x - \frac{c^{(k)}_n}{d^{(k)}_n} \right| = -\frac{\pi^2}{6(\log(2k+1)-\log(2k))} \),

(iii). \( \lim_{n \to \infty} \frac{1}{n} \# \{ a^{(k)}_n < z, \ i \leq n \} = \lim_{n \to \infty} \frac{1}{n} \# \{ \rho^{(k)}_n < z, \ i \leq n \} \)
\[
= \frac{1}{\log(2k+1) - \log(2k)} \begin{cases}
\frac{2z}{k} - 1 - \log \frac{2z}{k}, & 0 < z < k^{-1}, \frac{2z}{k} \leq k^2 + k \\
\frac{z}{k^2 + k} - \log \left(1 + \frac{1}{2k + 1}\right), & k^2 + k \leq z < k \\
1 - \frac{z}{k + 1} - \log \frac{2(k^2 + k)}{(2k + 1)^2}, & k < z \leq k + 1 \\
\log(2k+1) - \log(2k), & k + 1 < z.
\end{cases}
\]

Proof. For (i) we take \( f \) to be the same function as in the proof of Theorem 4(i). We let

\[
g_1(x,y) = \begin{cases}
1, & 0 < x < k^{-1}, (k + 2)^{-1} < y < (k + 1)^{-1} \\
0, & \text{otherwise}
\end{cases}
\]

for the first one and

\[
g_2(x,y) = \begin{cases}
1, & (k + 1)^{-1} < y < k^{-1}, 0 < x < (k + 1)^{-1} \\
0, & \text{otherwise}.
\end{cases}
\]

Then

\[
\int g_1 d\bar{y} = \int g_2 d\bar{y} = \frac{\log(2k+1) - \log(2k)}{\log 2}.
\]

Therefore we get (i).

(ii) can be proved by a similar argument as the proof of Theorem 4(ii).

As for (iii), we take \( \sigma_n^{(k)} \) as an example. Since we are concerned \( X_n < 1/k \) and \((k + 2)^{-1} < Y_n < (k + 1)^{-1}\), we have

\[
\Theta_n = (1 - Y_n)(X_n + Y_n - X_n Y_n)^{-1}
\]

by Lemma 2. It is easy to see that for the \( \Theta_n \) in consideration we have

\[
k/2 < \Theta_n < k + 1.
\]

Let

\[
f(x,y) = \begin{cases}
1, & \frac{1-y}{x+y-xy} < z, \frac{1}{k^2 + 2} < y < \frac{1}{k+1} \\
0, & \text{otherwise}.
\end{cases}
\]
FAREY CONTINUED FRACTIONS

Then $\sum_{i=1}^{k} f(X_i, Y_i)$ counts the number of $\Theta_i, i \leq n$ appearing as some $\sigma_i^{(0)}$ and <z. The non-zero regions of $f$ with respect to different values of $z$ are as follows:

$$\left\{ \begin{array}{l}
1-zx < y < \frac{1}{k+1}, \\
1+z-zx < x < \frac{1}{k}
\end{array} \right\}, \text{when } k \frac{2}{2k+1} < z \leq k^2+k$$

$$\left\{ \begin{array}{l}
1-zx < y < \frac{1}{k+1}, \\
1+z-zx < x < \frac{k+1-z}{(k+1)z}
\end{array} \right\}, \text{when } \frac{k^2+k}{2k+1} < z \leq k$$

$$\left\{ \begin{array}{l}
1-zx < y < \frac{1}{k+1}, \\
1+z-zx < x < \frac{k+1-z}{(k+1)z}
\end{array} \right\}, \text{when } k < z \leq k+1.$$}

The proof is completed by calculating several integrals and taking $g$ as $g_1$ in the proof of (i). 

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