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METRICAL THEORY FOR
FAREY CONTINUED FRACTIONS

GAVIN BROWN and QINGHE YIN

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1. Introduction

By making fundamental use of the Farey shift map and employing infinite (but $\sigma$-finite) measures together with the Chacon-Ornstein ergodic theorem it is possible to find new metrical results for continued fractions. Moreover this offers a unified approach to several existing theorems.

The application of ergodic theory to the study of continued fractions began with the Gauss transformation, $G: [0,1] \mapsto [0,1]$, $G(x) = \begin{cases} 1 - \left[ \frac{1}{x} \right], & x \neq 0 \\ 0, & x = 0, \end{cases}$ which is ergodic with respect to the Gauss measure $\mu_g$, where

$$\mu_g(B) = \frac{1}{\log 2} \int_B \frac{1}{1 + x} \, dx$$

for any Borel subset $B$ of $[0,1]$. H. Nakada [11] extended $G$ to the 2-dimensional case. Let $\tilde{G}: [0,1] \times [0,1] \mapsto [0,1] \times [0,1]$ be defined to be

$$\tilde{G}(x,y) = \left( G(x), \frac{1}{a_1 + y} \right)$$

where $a_1 = \left[ \frac{1}{x} \right]$. The absolutely continuous invariant measure of $\tilde{G}$, $\tilde{\mu}_g$, is given by

$$d\tilde{\mu}_g = \frac{1}{\log 2} \cdot \frac{dxdy}{(1 + xy)^2}.$$
Many metrical results for regular continued fractions can be proved using the ergodicity of $G$ or $\tilde{G}$. For example, [3] (W. Bosma et al) gave the distribution of the sequences of approximation constants $\{\theta_n\}$.

In this paper we focus on the convergents and the mediants of the Farey (or slow) continued fractions. Define $T: [0,1] \rightarrow [0,1]$ by

$$T(x) = \begin{cases} 
\frac{x}{1-x}, & 0 \leq x \leq \frac{1}{2}, \\
\frac{1-x}{x}, & \frac{1}{2} \leq x < 1.
\end{cases}$$

$T$ is called the Farey shift map (see [10]). $T$ preserves the measure $v$ given by

$$dv = \frac{1}{\log 2} \frac{dx}{x},$$

which is $\sigma$-finite but not a probability measure, and $T$ is ergodic with respect to this measure (see [12] or [10]). The natural extension of $T$, denoted by $\tilde{T}$, is the transformation on $[0,1] \times [0,1]$ given by

$$\tilde{T}(x,y) = \begin{cases} 
\left(\frac{x}{1-x}, \frac{y}{1+y}\right), & 0 \leq x \leq \frac{1}{2}, \\
\left(\frac{1-x}{x}, \frac{1}{1+y}\right), & \frac{1}{2} \leq x < 1
\end{cases}$$

The absolutely continuous invariant measure, $\bar{v}$, of $\tilde{T}$ is determined by

$$d\bar{v} = \frac{dx dy}{(x+y-xy)^2} \cdot \frac{1}{\log 2}.$$

The ergodicity of $\tilde{T}$ can be established from that of $\tilde{G}$ using an argument of [16] or by direct appeal to a general result given in [4]. Since the $v$ or $\bar{v}$ are infinite ($\sigma$-finite, though), the Birkhoff Ergodic Theorem is not applicable for $T$ or $\tilde{T}$. To avoid this disadvantage, Ito considered another transformation $T_1$ induced by $T$, the invariant measure of which is a probability measure. By the ergodicity of $T_1$ and its natural extension, he obtained in [8] many metrical results related to convergents and nearest mediants. In this paper we consider $T$ and $\tilde{T}$ directly. We shall establish an ergodic theorem for $\tilde{T}$ though the Chacon-Ornstein ergodic theorem. In this way the results of [8] can be generalized for we can derive metrical results on Diophantine approximation by all the mediants not only the nearest ones. The results for nearest mediants become a special case. Any other metrical results obtained by applying the Birkhoff ergodic theorem for $G$ or
\( G \) can also be obtained by using the ergodic theorem we build for \( \bar{T} \).

In Section 2, we recall some basic results about Farey and regular continued fractions and give some basic properties of \( T \) and \( \bar{T} \). In Section 3, we establish an ergodic theorem for \( T \) through the Chacon-Ornstein Ergodic Theorem. In Section 4, we apply the ergodic theorem established in Section 3 to prove some old and new metrical results both for the regular continued fractions and the Farey continued fractions.

2. Preliminaries

For an irrational \( x \in [0,1] \) with regular continued fraction expansion

\[
\frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n}}}} = [0; a_1, a_2, \ldots],
\]

the \( n \)-th convergent is given by

\[
\frac{p_n}{q_n} = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n}}}} = [0; a_1, a_2, \ldots, a_n].
\]

The integers \( p_n, q_n \) can be described inductively by setting

\[
\begin{align*}
    p_{-1} &= 1, & p_0 &= 0, & q_{-1} &= 0, & q_0 &= 1, \\
    p_n &= a_n p_{n-1} + p_{n-2}, & q_n &= a_n q_{n-1} + q_{n-2}.
\end{align*}
\]

We shall be concerned with a slower sequence \( \{p_n/q_n\} \) of approximations to \( x \), corresponding to a branch of the Farey tree (see [7], [14] for details). For our present purpose it suffices to know that

\[
P_n = kp_m + p_{m-1}, \quad Q_n = kq_m + q_{m-1},
\]

where

\[
n = a_0 + a_1 + \cdots a_m + k, \quad 0 < k \leq a_{m+1}, \quad (a_0 = 0).
\]

The sequence \( \{p_n/q_n\} \) consists of the convergents of \( x \), while \( \{P_n/Q_n\} \) is the sequence of convergents and mediants.

The Farey shift map \( T: [0,1] \rightarrow [0,1] \), defined in the introduction, may be characterised as follows. For \( x = [0; a_1, a_2, \ldots] \), we have

\[
T(x) = \begin{cases} 
    [0; a_1 - 1, a_2, \ldots], & a \geq 2 \\
    [0; a_2, a_3, \ldots], & a_1 = 1.
\end{cases}
\]
Moreover, for \( x = [0; a_1, a_2, \ldots], \ y = [0; b_1, b_2, \ldots] \) we have
\[
\bar{T}(x, y) = \begin{cases} 
(0; a_1 - 1, a_2, \ldots, [0; b_1 + 1, b_2, \ldots]) & a \geq 2 \\
([0; a_2, a_3, \ldots], [0; b_1, b_2, \ldots]) & a_1 = 1.
\end{cases}
\]

We make basic use of the numbers \( X_n, Y_n \) defined by \( \bar{T}^n(x, 1) = (X_n, Y_n) \). Note, in particular, that
\[
Y_n = \begin{cases} 
[0; k + 1, a_m, \ldots, a_1], & m \geq 1 \\
[0; k + 1], & m = 0
\end{cases}
\]
where
\[
n = a_0 + a_1 + \cdots + a_m + k, \quad 0 \leq k < a_{m+1}.
\]

It is well-known that for \( \bar{G} \), defined in the introduction,
\[
\bar{G}^m(x, 0) = (x_m, y_m) = ([0; b_{m+1}, b_m, \ldots], [0; a_m, \ldots, a_1]),
\]
where \( x_m = G^m x, \ y_m = q_{m-1}/q_m \).

Let us write
\[
(1) \quad \Theta_n = Q_n^2 \left| x - \frac{P_n}{Q_n} \right|
\]
and
\[
(2) \quad \theta_m = q_m^2 \left| x - \frac{P_m}{q_m} \right|.
\]

Then we have
\[
(3) \quad \theta_m = x_m(1 + x_m y_m)^{-1}
\]
and this is contained in the following formula, given in [2],
\[
(4) \quad \Theta_n = (1 - k x_m)(k + y_m)(1 + x_m y_m)^{-1},
\]
where
\[
n = a_0 + a_1 + \cdots + a_m + k, \quad 0 < k \leq a_{m+1}.
\]

(4) allows discussion of \( \Theta_n \) via \( \bar{G} \) but our strategy is to work directly with \( \bar{T} \) so we set about expressing the quantities \( Q_{n-1}/Q_n \) and \( \Theta_n \) in terms of \( X_n, Y_n \).

**Lemma 1.** For \( n = 1, 2, \cdots \), we have
\[ \frac{Q_n}{Q_{n+1}} = (1 + Y_{n-1})^{-1} = \max\{Y_n, 1 - Y_n\}. \]

Proof. We know that \( Y_n \) equals \( Y_{n-1}(1 + Y_{n-1})^{-1} \) or \( (1 + Y_{n-1})^{-1} \) and that \( 0 < Y_{n-1} \leq 1 \). Hence \( (1 + Y_{n-1})^{-1} \) equals \( \max\{Y_n, 1 - Y_n\} \).

Next we use induction. When \( n=1 \), we have \( Y_1 = 1/2 \), \( Q_1 = 1 \). And we always have \( Q_2 = 2 \). Hence
\[
\frac{Q_1}{Q_2} = 1/2 = Y_1 = \max\{Y_1, 1 - Y_1\}.
\]

Suppose for \( l \leq n \) we have
\[
\frac{Q_l}{Q_{l+1}} = \max\{Y_l, 1 - Y_l\}.
\]

Assume that \( n = a_0 + a_1 + a_2 + \cdots + a_m + k, \quad 0 \leq k < a_{m+1} \). If \( k = 0 \), then \( Q_n = q_m \), \( Q_{n+1} = 1 \cdot q_m + q_{m-1} \) and \( Y_n = [0; a_m, a_{m-1}, \ldots, a_1] \geq 1/2 \). Hence \( Q_n / Q_{n+1} = Y_n \). If \( a_{m+1} \geq 2 \), then \( Q_{n+2} = 2 \cdot q_m + q_{m-1} = Q_n + Q_{n+1} \). Thus
\[
Q_{n+1} / Q_{n+2} = \frac{(1 + Q_n / Q_{n+1})^{-1}}{(1 + Y_n)^{-1}}.
\]

If \( a_{m+1} = 1 \), then \( Q_{n+1} = q_{m+1} \) and \( Q_{n+2} = q_m + q_{m+1} \). Again we have \( Q_{n+2} = Q_n + Q_{n+1} \) and again we get
\[
Q_{n+1} / Q_{n+2} = (1 + Y_n)^{-1}.
\]

When \( 0 < k < a_{m+1} - 1 \), we have \( Q_n = kq_m + q_{m-1} \), \( Q_{n+1} = (k+1)q_m + q_{m-1} \) and \( Q_{n+2} = (k+2)q_m + q_{m-1} \), and \( Y_n \), which equals \([0; k+1]\) or \([0; k+1, a_m, \ldots, a_1]\), is at most \( 1/2 \). Hence
\[
Q_{n+1} / Q_{n+2} = Q_n(2q_m + Q_n)^{-1}
= Q_n (2q_m / Q_{n+1} + (1 - Y_n))^{-1}
= Q_n (2(k+1 + Y_m))^{-1} + (1 - Y_n))^{-1}
= (2Y_n + 1 - Y_n)^{-1} = (1 + Y_n)^{-1}.
\]

Lastly we consider \( k = a_{m+1} - 1 > 0 \). We have \( Q_n = kq_m + q_{m-1} = q_{m+1} - q_m \), \( Q_{n+1} = q_{m+1} \) and \( Q_{n+2} = q_{m+1} + q_m \). We also have \( Y_n = y_{m+1} \leq 1/2 \). Hence, once more, \( Q_{n+1} / Q_{n+2} = (1 + Y_n)^{-1} \).

Lemma 2. For \( n=2,3,\ldots \), we have
\[
\Theta_n = \begin{cases} (1 - Y_n)(X_n + Y_n - X_n)Y_n^{-1}, & Y_n < 1/2, \\ X_nY_n(X_n + Y_n - X_n)Y_n^{-1}, & Y_n > 1/2, \end{cases}
\]

\[
\Theta_{n+1} = (1 - X_n)(X_n + Y_n - X_n)Y_n^{-1}.
\]

Proof. For \( n=a_0 + a_1 + a_2 + \cdots + a_m \geq 2 \), we have \( X_n = x_m \) and \( Y_n = (1 + y_m)^{-1} \). An application of (3) gives
\[ \Theta_n = \theta_m = X_n(1 + X_n(Y^{-1} - 1))^{-1} = X_n Y_n(X_n + Y_n - X_n Y_n)^{-1}. \]

For \( n = a_1 + \cdots + a_m + k \geq 2 \) where \( 0 < k < a_{m+1} \), we apply (4) after noting that
\[ X_n = (x_m^{-1} - k)^{-1}, \quad Y_n = (k + 1 + y_m)^{-1} < 1/2. \]

This gives
\[ \Theta_n = (1 - Y_n)(X_n + Y_n - X_n Y_n)^{-1}. \]

It is now easy to see that \( Y_{n+1} > 1/2 \) if and only if \( X_n > 1/2 \). When \( X_n > 1/2 \)
\[ (X_{n+1}, Y_{n+1}) = \tilde{T}(X_n, Y_n) = (X_n^{-1}(1 - X_n) (1 + Y_n)^{-1}). \]

In this case we see that
\[ \Theta_{n+1} = X_{n+1} Y_{n+1}(X_n + Y_n - X_n Y_n)^{-1} \]
\[ = (1 - X_n)(X_n + Y_n - X_n Y_n)^{-1} \]

For the case \( X_n < 1/2 \), we substitute \( X_{n+1} = (1 - X_n^{-1}) X_n, Y_{n+1} = Y_n(1 + Y_n)^{-1} \) in the formula
\[ \Theta_{n+1} = (1 - Y_n)(X_{n+1} + Y_{n+1} - X_{n+1} Y_{n+1})^{-1} \]
to obtain the required result.

3. Ergodicity and ergodic theorem for \( \tilde{T} \)

We begin this section by showing that \( \tilde{G} \) can be induced from \( \tilde{T} \).

**Theorem 1.** The dynamical system \((\Omega, \mathcal{B}, \mu_g, \tilde{G})\) is (isomorphic with) the system induced from \((\Omega, \mathcal{B}, \tilde{v}, \tilde{T})\) on the set \( E = \{(x, y) : y > 1/2 \} \).

**Proof.** Recall that for \( x = [0; a_1, a_2, \ldots], y = [0; b_1, b_2, \ldots] \) we have
\[ \tilde{T}(x, y) = \begin{cases} ([0; a_1 - 1, a_2, \ldots], [0; b_1 + 1, b_2, \ldots]), & a \geq 2, \\ ([0; a_2, a_3, \ldots], [0; 1, b_1, b_2, \ldots]), & a_1 = 1. \end{cases} \]

In particular the second coordinate of \( \tilde{T}(x, y) \) is greater than \( 1/2 \) if and only if \( a_1 = 1 \). Accordingly the induced map \( \tilde{T}_E \) is given by
\[ \tilde{T}_E(x, y) = (x^{-1} - a_1, (1 + (y^{-1} + a_1 - 1)^{-1})^{-1}). \]

Now consider the map \( \phi : E \to \Omega \) given by
The map transforms to $(x, y) \mapsto (x^{-1} - a_1, (y + a_1)^{-1})$ and the measure $dx dy (x + y - xy)^{-2}$ transforms to $dx dy (1 + xy)^{-2}$. 

By Theorem 1 and the ergodicity of $\tilde{G}$ together with a result of [16], we obtain the ergodicity of $\tilde{T}$. Theorem 1 also gives an abstract justification of the statement that any result derived from $G$ or $\tilde{G}$ can be obtained from $\tilde{T}$.

The map $\tilde{T}$ is ergodic and invertible and $\tilde{v}$ is non-atomic so it follows (see [5]) that $\tilde{T}$ is conservative. Therefore we may apply the Chacon-Ornstein theorem (cf. [13]) on the system $(\Omega, \mathcal{B}_2, \tilde{v}, \tilde{T})$ (we use $\Omega$ to denote $[0,1] \times [0,1]$ for the remainder of the paper) to derive the following result.

**Theorem 2.** For any $f, g \in L^1(\Omega, \mathcal{B}_2, \tilde{v})$ with $\int g d\tilde{v} \neq 0$, one has

$$\lim_{n \to \infty} \frac{\sum_{k=0}^{n-1} f(\tilde{T}^k(x,y))}{\sum_{k=0}^{n-1} g(\tilde{T}^k(x,y))} = \frac{\int f d\tilde{v}}{\int g d\tilde{v}} \quad \text{a.e.}$$

Next we show that under Lipschitz conditions on $f, g$, the points $\tilde{T}^k(x,y)$ in Theorem 2 can be replaced by $(X_k, Y_k) = \tilde{T}^k(x,1)$.

**Theorem 3.** Suppose that $f, g \in L^1(\Omega, \mathcal{B}, \tilde{v})$ satisfy

$$|f(x,y) - f(x,y')| \leq L|y - y'|^\alpha$$

$$|g(x,y) - g(x,y')| \leq L|y - y'|^\alpha$$

where $L > 0$, $\alpha > 0$ are constants. If $\int g d\tilde{v} \neq 0$, then for almost all $(x,y) \in \Omega$ one has

$$\lim_{n \to \infty} \frac{\sum_{k=0}^{n-1} f(\tilde{T}^k(x,y))}{\sum_{k=0}^{n-1} g(\tilde{T}^k(x,y))} = \lim_{n \to \infty} \frac{\sum_{k=0}^{n-1} f(X_k, Y_k)}{\sum_{k=0}^{n-1} g(X_k, Y_k)}$$

**Proof.** Let $Z_i$ be the second coordinate of $\tilde{T}^i(x,y)$, i.e. $\tilde{T}^i(x,y) = (X_i, Z_i)$. We claim that for almost all $x \in [0,1]$ and all $y \in (0,1]$, we have

$$\sum_{i=0}^{\infty} |Z_i - Y_i|^\alpha < +\infty.$$

In fact, for $x = [0; a_1, a_2, \ldots]$, $i = a_1 + a_2 + \cdots + a_m + k$, $0 \leq k < a_m + 1$, one has
for some constant \( c \), where \( q_m \) is the denominator of the \( m \)-th convergent \( \frac{p_m}{q_m} \) of the regular continued fraction expansion of \( x \) (cf. [1] p.42]. Therefore we get that

\[
\sum_{i=0}^{\infty} |Z_i - Y_i|^x \leq \sum_{m=0}^{\infty} \sum_{k=1}^{a_m+1} \left( \frac{c^2}{q_m^2} \right)^x = c^2 \sum_{m=0}^{\infty} \frac{a_{m+1}}{q_m^{2x}}.
\]

By induction we can see that

\[ q_m \geq 2^{(m-1)/2}. \]

We need the following theorem (see [15]).

**Theorem A.** Let \( F(n) > 1 \), for \( n = 1, 2, \cdots \), and suppose that \( \sum_{n=1}^{\infty} \frac{1}{F(n)} < \infty \).

Then the set

\[ A = \{ x \in [0,1], \ a_k(x) > F(k) \text{ infinitely many times} \} \]

has Lebesgue measure 0.

Now we choose \( F(n) = 2^{n/2} \). By the above theorem we see that the set

\[ E = \{ x \in [0,1], \ a_k(x) > F(k) \text{ only finitely many times} \} \]

has Lebesgue measure 1. Hence for almost all \( x \in [0,1] \),

\[ \sum_{m=0}^{\infty} \frac{a_{m+1}}{q_m^{2x}} \leq C(x) + \sum_{m=0}^{\infty} 2^{(2-m)/2} < \infty. \]

Where \( C(x) = \sum_{a_m > F(m)} \frac{a_{m+1}}{q_m^{2x}} \). The required result follows easily when we bear in mind the fact that \( \sum_{k=1}^{F^2(x,y)} \) diverges almost everywhere because \( \tilde{T} \) is conservative and ergodic.

For some functions \( f(x,y) \) though we do not have
\[(*)\]
\[|f(x,y) - f(x,y')| \leq L|y - y'|^2\]
for all \(y, y' \in [0,1]\), it is still true that
\[|f(T^n(x,y)) - f(X_i, Y_i)| \leq L|Z_i - Y_i|^\alpha, \quad \alpha > 0,\]
for almost all \(x \in [0,1]\) and \(i\) large enough.

**Example.** Let
\[f(x,y) = \begin{cases} \log(1-y), & y \in [0,1) \\ 0, & y = 1 \end{cases}\]
Then \(f \in L^1(\Omega)\). We do not have \((*)\) for all \(y, y' \in [0,1]\).

For \(x = [0; a_1, a_2, \ldots]\), let
\[i = a_1 + a_2 + \cdots + a_m + k, \quad 0 \leq k < a_{m+1}\]
where \(m \geq 2\). Then
\[|f(T^n(x,y)) - f(X_i, Y_i)| = |\log(1 - Z_i) - \log(1 - Y_i)| = \frac{1}{|1 - \xi_i|} |Z_i - Y_i|\]
where \(\xi_i\) is in between \(y_i\) and \(Y_i\). It is easy to see that \(1 - \xi_i \geq 1/q_m\), i.e.
\[\frac{1}{1 - \xi_i} \leq q_m \leq c|Y_i - Z_i|^{1/2}\]
Therefore, \(|f(T^n(x,y)) - f(X_i, Y_i)| \leq c|Y_i - Z_i|^{1/2}\).

**4. Applications**

In this section we apply the ergodic theorems for \(\tilde{T}\) to obtain metrical results for convergents and medians of regular continued fractions. For all the functions \(f, g\) involved in this section it is valid to replace \(\tilde{T}^n(x,y)\) by \((X_k, Y_k)\) as in Theorem 3 but omit the tedious verification.

For an irrational \(x = [0; a_1, a_2, \ldots]\) we shall call
\[\frac{kp_n + p_{n-1}}{kq_n + q_{n-1}} \quad \text{and} \quad \frac{(a_{n+1} - k)p_n + p_{n-1}}{(a_{n+1} - k)q_n + q_{n-1}}\]
the $k$-th mediants of $x$, when $a_{n+1} \geq 2k$. We let $P^{(k)}_n/Q^{(k)}_n$ denote the sequence which consists of all convergents and $i$-th mediants of $x$ for all $i \leq k$. When $k=0$, we recover the convergents $\{p_m/q_m\}$ and, when $k=1$, we obtain the so-called nearest mediants of Ito, [8]. It is easy to see that the event "$P^n/Q_n$ appears as some $P^{(k)}_i/Q^{(k)}_i$" is characterised by $X_n > 1/(k+1)$ or $Y_n > 1/(k+2)$ while "$P^{n+1}/Q^{n+1}$ appears as some $P^{(k)}_i/Q^{(k)}_i$" is by $X_n > 1/(k+2)$ or $Y_n > 1/(k+1)$.

Let us write also

$$\Theta^{(k)}_n = (Q^{(k)}_n)^2|x - P^{(k)}_n/Q^{(k)}_n|.$$

Our main theorem can now be stated.

**Theorem 4.** For almost all $x$ we have

(i). for $k = 0, 1, 2, \cdots$,

$$\lim_{n \to \infty} \frac{1}{n} \log Q^{(k)}_n = \frac{\pi^2}{12 \log(2k+2)},$$

(ii). for $k = 0, 1, 2, \cdots$,

$$\lim_{n \to \infty} \frac{1}{n} \log \left| x - \frac{P^{(k)}_n}{Q^{(k)}_n} \right| = -\frac{\pi^2}{6 \log(2k+2)},$$

(iii). for $k = 1, 2, \cdots$,

$$\lim_{n \to \infty} \frac{1}{n} \#\{i: i \leq n, \Theta^{(k)}_i < z\} = \left\{ \begin{array}{l} z, \quad 0 \leq z < 1 \\ 1 + \log z, \quad 1 \leq z < \frac{k+1}{2} \\ 2 - \frac{2z}{k+1} + \log \frac{2z^2}{k+1}, \quad \frac{k+1}{2} \leq z < k+1 \\ \log(2k+2), \quad k+1 < z \end{array} \right.$$

(iv). \(\lim_{n \to \infty} \frac{1}{\log Q_n} \#\{i: i \leq n, \Theta_i < z\} = \frac{12}{\pi^2} \left\{ \begin{array}{l} z, \quad 0 < z \leq 1, \\ 1 + \log z, \quad 1 < z. \end{array} \right.\)

**Remark.** The case $k=0$ of (i) and (ii) are the basic results of Levy (see [1]), and the case $k=1$ of (i), (ii) and (iii) give results of Ito, [8]. The important result of Bosma et al in [3] corresponds to the case $k=0$ of (ii) and the proof which
follows could be simplified to yield that special case. Nevertheless a suitable interpretation of the three terms corresponding to $1/2 \leq z < 1$ yields the appropriate distribution. We take the signed sum, i.e.

$$
\lim_{n \to \infty} \frac{1}{n} \sum_{i \leq n, \, \theta_n < z} \begin{cases} 
z, & 0 \leq z < \frac{1}{2} \\
1 - z + \log(2z), & \frac{1}{2} \leq z < 1 \\
\log 2, & 1 \leq z \end{cases}
$$

where

$$
1 - z + \log(2z) = z - (1 + \log z) + [2 - 2z + \log(2z^2)].
$$

Proof. For (i) we take

$$
f(x, y) = \log(\max\{y, 1 - y\}),
g(x, y) = \begin{cases} 1, & \text{when } x > (k + 1)^{-1}, \text{ or } y > (k + 2)^{-1} \\
0, & \text{otherwise.}
\end{cases}
$$

Then

$$
\int f d\bar{y} = -\frac{\pi^2}{12 \log 2}, \quad \int g d\bar{y} = \frac{\log(2k + 2)}{\log 2}.
$$

For $n = a_0 + a_1 + \cdots + a_m + k, \quad 0 \leq k < a_{m+1}$, we have, by Lemma 1,

$$
\log \frac{Q_n}{Q_{n+1}} = f(X_n, Y_n)
$$

while

$$
\sum_{i=1}^n g(X_n, Y_n) = \#\{i: i \leq n, \, Q_i \text{ appears as some } Q^{(k)}_j\}.
$$

Therefore

$$
\sum_{i=1}^n f(X_n, Y_n) / \sum_{i=1}^n g(X_n, Y_n) = -s^{-1}(\log Q_{n+1} - \log Q_1).
$$

where $s$ is determined by $Q^{(k)}_s \leq Q_n < Q^{(k)}_{s+1}$. By the ergodic theorem of the last section

$$
\lim_{s \to \infty} \frac{\log Q^{(k)}_s}{s} = -\lim_{n \to \infty} \frac{\sum_{i=1}^n f(X_n, Y_i)}{\sum_{i=1}^n g(X_n, Y_i)} = -\int f d\bar{y} / \int g d\bar{y},
$$
and the required result follows.

For (ii) noting that \( \Theta_{n}^{(k)} = (Q_{n}^{(k)})^2 |x - P_{n}^{(k)}/Q_{n}^{(k)}| \), by (i) it suffices to show that

\[
\lim_{n \to \infty} \frac{\log \Theta_{n}^{(k)}}{n} = 0 \quad \text{a.e.}
\]

Remember that \( \Theta_{n+1} \) is one of \( \Theta_{s}^{(k)} \) if and only if \( X_n > (k+2)^{-1} \) or \( Y_n > (k+1)^{-1} \). Then when \( \Theta_{n+1} \) is one of \( \Theta_{s}^{(k)} \) we have

\[
\Theta_{n+1} = \frac{1 - X_n}{X_n + Y_n - X_n Y_n} < k + 1.
\]

On the other hand,

\[
\Theta_{n+1} \geq 1 - X_n \geq 1 - [0; 1, a_{m+1}, \ldots] > 1/(a_{m+1} + 1) \geq 1/2a_{m+1},
\]

where \( m \) is determined by \( Q_n < q_m < Q_{n+1} \). Therefore,

\[
\frac{- \log a_{m+1} + \log 2}{s} \leq \frac{\Theta_{s}^{(k)}}{s} \leq \frac{\log(k + 1)}{s}.
\]

Noting that \( m \leq s \leq (2k + 1)m \) we obtain

\[
\frac{\log a_{m+1}}{s} \leq \frac{\log a_{m+1}}{m} = \left( \frac{a_{m+1}}{m^s} \right)^{1/\alpha} \cdot \frac{\log a_{m+1}}{a_{m+1}^{1/\alpha}} \to 0 \quad \text{a.e.}
\]

by Theorem A, where \( \alpha > 1 \) is a constant. Therefore

\[
\lim_{n \to \infty} \frac{\log \Theta_{n}^{(k)}}{n} = 0 \quad \text{a.e.}
\]

To prove (iii) and (iv) we also consider \( \Theta_{n+1} \) instead of \( \Theta_{n} \). Let

\[
E_{z,t} = \{(x,y) \in \Omega: \frac{1-x}{x+y-xy} < z, \frac{t}{1+t} < x \}\]

\[
\cup \{(x,y) \in \Omega: \frac{1-x}{x+y-xy} < z, y > t\} = E_1 \cup E_2
\]

where \( 0 < z, 0 \leq t \leq 1/2 \) and

\[
f(x,y) = \begin{cases} 
1, & (x,y) \in E_{z,t} \\
0, & \text{otherwise.}
\end{cases}
\]
We recast the inequality

\[(1-x)(x+y-xy)^{-1} < z\]

in the form

\[x > (1-zy)(1+z(1-y))^{-1} = u(y),\] say,

and note that \((1-zy)(1+z(1-y))^{-1} > t/(1+t)\) if and only if \(y > z^{-1} - t\). Therefore

\[E_1 = \begin{cases} 
{x > u(y),} & {0 < y < 1}, \\
{x > \frac{t}{1+t},} & {0 < y < 1},
\end{cases}\]

\[0 < z < \frac{1}{1+t}\]

Suppose first that \(0 < z < 1/(1+t)\). Then \(E_{z,t} = E_1 = \{x > u(y), \ 0 > y > 1\}\). Hence

\[\log 2 \int f d\bar{y} = \int_0^1 dy \int_{u(y)}^1 (x+y-xy)^{-2} dx = z.\]

Now consider the case \(1/(1+t) \leq z < 1\). Note that \(t \leq 1-t < z^{-1} - t\). Then again \(E_{z,t} = \{x > u(y), \ 0 < y < 1\}\) and \(\log 2 \int f d\bar{y} = z\).

When \(1 < z < 1/2t\) we also have \(z^{-1} - t > t\). Remember that \(u(y) > 0\) when \(y > z^{-1}\). Thus

\[E_{z,t} = \{(x, y) \in \Omega: x > u(y), \ 0 < y < 1\}\]

\[= \{x > u(y), \ 0 < y < z^{-1}\} \cup \{x > 0, \ z^{-1} < y < 1\},\]

and

\[\log 2 \int f d\bar{y} = 1 + \log z.\]

If \(1/2t < z < 1/t\), then \(0 < z^{-1} - t < t\) and

\[E_{z,t} = \{x > u(y), \ 0 < y < \frac{1}{z} - t\} \cup \{x > \frac{t}{1+t}, \ \frac{1}{z} - t < y < t\}\]

\[\cup \{x > u(y), \ t < y < \frac{1}{z}\} \cup \{x > 0, \ \frac{1}{z} < y < 1\}\]
Calculate that

$$\log 2 \int f \, d\nu = 2 - 2tz + \log(2tz^2).$$

Lastly, when \( z > 1/t \) we have

$$E_{x,t} = \{x > t/(1+t), \ 0 < y < t\} \cup \{x > 0, \ t < y < 1\}$$

and \( \log 2 \int f \, d\nu = \log 2 - \log t \).

It is now easy to piece together the result of (iii) by taking \( t = (k+1)^{-1} \) and

$$g(x,y) = \begin{cases} 1, & \text{when } x > (k+2)^{-1} \text{ or } y > (k+1)^{-1} \\ 0, & \text{otherwise.} \end{cases}$$

To obtain (iv) we set \( t = 0 \) and replace \( g \) by the function \( \log(\max\{y, 1-y\}) \) used as \( f \) in proving (i). This completes the proof. ■

**Remark.** Part (iii) of the theorem (in some sense a limiting case as \( k \to \infty \)) shows that \( \frac{1}{n} \# \{i: i \leq n, \ \Theta_n < z\} \to 0 \) as \( n \to \infty \) and hence that \( \{\Theta_n\} \) does not have a distribution function. We can obtain some more information about \( \Theta_n \). We have

For almost all \( x \in [0,1] \) and any \( \epsilon > 0 \),

(i). \( \lim_{n \to \infty} \frac{\Theta_n}{m^{1+\epsilon}} = 0 \), where \( q_m \leq \mathcal{Q}_n < q_{m+1}; \)

(ii). \( \lim_{n \to \infty} \frac{1}{n} \sum_{1}^{n} \Theta_i = \infty \); and

(iii). \( \lim_{n \to \infty} \frac{1}{n^{2+\epsilon}} \sum_{1}^{n} \Theta_i = 0. \)

In fact, for \( n = a_0 + a_1 + \cdots + a_m + k \), where \( 0 \leq k < a_{m+1} \), by Lemma 2

\( \Theta_{n+1} < Y_n = [k+1; a_m, \cdots, a_1] < k+2. \)

Then (i) follows from Theorem A. Using (*) we can get the following estimation:

\[ \frac{1}{16} \left( \sum_{i=1}^{m} a_i^2 + k^2 - n \right) - (m+1) < \sum_{i=1}^{n} \Theta_i < \sum_{i=1}^{m} a_i^2 + k^2. \]

Then follow (ii) and (iii).

Next let us compare Theorem 4 (iv) with some results of P. Erdös [6] and
J. Blom [2]. Let

\[ \Theta \left( \frac{p}{q}, x \right) = q^2 \left| x - \frac{p}{q} \right|. \]

Define

\[ U(x, z, n) = \# \{(p, q) \in \mathbb{Z} \times \mathbb{N} : (p, q) = 1, \Theta \left( \frac{p}{q}, x \right) \leq z, q \leq n \}, \]

\[ U_1(x, z, n) = \# \{(p, q) \in \mathbb{Z} \times \mathbb{N} : (p, q) = 1, \Theta \left( \frac{p}{q}, x \right) \leq z, q \leq n, \frac{p}{q} \text{ is a convergent of } x \} \]

and

\[ U_2(x, z, n) = \# \{(p, q) \in \mathbb{Z} \times \mathbb{N} : (p, q) = 1, \Theta \left( \frac{p}{q}, x \right) \leq z, q \leq n, \frac{p}{q} \text{ is a best approximant of } x \}. \]

By best approximant we mean that if there is a fraction \( \frac{a}{b} \) different from \( \frac{p}{q} \) such that

\[ \left| \frac{a}{b} - x \right| \leq \left| \frac{p}{q} - x \right| \]

then \( b > q \). Erdős [6]) proved that for any \( z \geq 0 \)

\[ \lim_{n \to \infty} \frac{U(x, z, n)}{n} = \frac{12}{\pi^2} z \quad \text{a.e.} \]

Blom [2] gave that

\[ \lim_{n \to \infty} \frac{U_1(x, z, n)}{n} = \frac{12}{\pi^2} (f) z \quad \text{a.e.} \]

and

\[ \lim_{n \to \infty} \frac{U_2(x, z, n)}{n} = \frac{12}{\pi^2} (f(z) + h(z)) \quad \text{a.e.} \]

where
When \( z \leq 1 \) Theorem 4(iii) corresponds to the result of Erdős [6]. [8] and [9] also gave new proofs in this case. When \( z > 1 \) this result takes a different form. This fact tells us that, for \( z > 1 \), there is no result for convergents and mediants analogous to the theorem of Legendre for \( z = 1/2 \) or the theorem of Fatou and Koksma for \( z = 1 \) (see [9]).

The result of Blom [2] can also be proved by Theorem 2 or 3. In fact for irrational \( x \in [0, 1) \), a best approximant is an element of \( \{P_n/Q_n\} \) characterized by

\[
Y_n > \frac{1}{2} \text{ or } Y_n^{-1} > X_n^{-1} + 1, \quad n \geq 2.
\]

Thus we can prove these results by choosing appropriate functions \( f \) and \( g \).

Jager [9] considered the two sequences

\[
\left\{ \frac{p_n + p_{n-1}}{q_n + q_{n-1}} \right\}, \quad \left\{ \frac{(a_n + 1 - 1)p_n + p_{n-1}}{(a_n + 1 - 1)q_n + q_{n-1}} \right\}
\]

separately and obtained some metrical results. If \( P_N/Q_N \) appears as \( (p_n + p_{n-1})/(q_n + q_{n-1}) \) then we have

\[
X_N = [0; a_{n+1} - 1, a_{n+2}, \ldots], \quad Y_N = [0; 2, a_n, \ldots, a_1] \quad \text{when } a_{n+1} \geq 2,
\]

or

\[
X_N = [0; a_n + 2, \ldots], \quad Y_N = [0; 1, 1, a_n, \ldots, a_1] \quad \text{when } a_{n+1} = 1.
\]

Thus the first sequence is characterised by \( 1/3 < Y_n < 2/3 \). The second one is more complicated. If \( a_{n+1} \geq 2 \), then \( ((a_n + 1 - 1)p_n + p_{n-1})/((a_n + 1 - 1)q_n + q_{n-1}) \) corresponds

\[
X_N = [0; 1, a_n + 2, \ldots] \in (1/2, 1), \quad Y_N = [0; a_{n+1}, \ldots, a_1] \in (0, 1/2).
\]
When $a_{n+1} = 1$ we get $p_{n-1}/q_{n-1}$ which corresponds to

$$X_n = [0; a_n, \ldots, a_1] \in \left(\frac{1}{a_n+1}, \frac{2}{2a_n+1}\right),$$

$$Y_n = [0, 1, a_{n-1}, \ldots, a_1] \in (1/2, 1).$$

Hence the second one is characterised by

$$X_n \in (1/2, 1), Y_n \in (0, 1/2) \quad \text{or} \quad X_n \in \bigcup_{i=1}^{\infty} \left(\frac{1}{i+1}, \frac{2}{2i+1}\right), Y_n \in (1/2, 1).$$

However, those two sequences are not "pure" nearest mediants. We shall consider the sequences

$$\left\{\frac{p_n + p_{n-1}}{q_n + q_{n-1}}, a_{n+1} \geq 2\right\} \quad \text{and} \quad \left\{\frac{(a_{n+1} - 1)p_n + p_{n-1}}{(a_{n+1} - 1)q_n + q_{n-1}}, a_{n+1} \geq 2\right\}$$

which are characterised by $1/3 < Y_n < 1/2$ and $X_n > 1/2, Y_n < 1/2$ respectively. In general we use $\{a_n^{(k)}/b_n^{(k)}\}$ and $\{c_n^{(k)}/d_n^{(k)}\}$ to denote the "pure" $k$-th mediant sequences for each of the two directions

$$\left\{\frac{kp_n + p_{n-1}}{kq_n + q_{n-1}}, a_{n+1} \geq 2k\right\} \quad \text{and} \quad \left\{\frac{(a_{n+1} - k)p_n + p_{n-1}}{(a_{n+1} - k)q_n + q_{n-1}}, a_{n+1} \geq 2k\right\}$$

respectively, where $k \leq 1$. It is not hard to see that $\{a_n^{(k)}/b_n^{(k)}\}$ is the subsequence of $\{P_n/Q_n\}$ determined by $X_n < 1/k$ and $(k+2)^{-1} < Y_n < (k+1)^{-1}$ while $\{c_n^{(k)}/d_n^{(k)}\}$ determined by $(k+1)^{-1} < X_n < k^{-1}$ and $Y_n < (k+1)^{-1}$. Define

$$\sigma_n^{(k)} = b_n^{(k)} - a_n^{(k)},$$

and

$$\rho_n^{(k)} = d_n^{(k)} - c_n^{(k)}.$$
Proof. For (i) we take \( f \) to be the same function as in the proof of Theorem 4(i). We let

\[
g_1(x,y) = \begin{cases} 
1, & 0 < x < k^{-1}, (k + 2)^{-1} < y < (k + 1)^{-1} \\
0, & \text{otherwise}
\end{cases}
\]

for the first one and

\[
g_2(x,y) = \begin{cases} 
1, & (k + 1)^{-1} < y < k^{-1}, 0 < x < (k + 1)^{-1} \\
0, & \text{otherwise}.
\end{cases}
\]

Then

\[
\int g_1 d\bar{v} = \int g_2 d\bar{v} = \frac{\log(2k + 1) - \log(2k)}{\log 2}.
\]

Therefore we get (i).

(ii) can be proved by a similar argument as the proof of Theorem 4(ii).

As for (iii), we take \( \sigma_n^{(k)} \) as an example. Since we are concerned \( X_n < 1/k \) and \( (k + 2)^{-1} < Y_n < (k + 1)^{-1} \), we have

\[
\Theta_n = (1 - Y_n)(X_n + Y_n - X_n Y_n)^{-1}
\]

by Lemma 2. It is easy to see that for the \( \Theta_n \) in consideration we have

\[
k/2 < \Theta_n < k + 1.
\]

Let

\[
f(x,y) = \begin{cases} 
1, & \frac{1 - y}{x + y - xy} < z, x < \frac{1}{k}, k + 2 < y < \frac{1}{k + 1} \\
0, & \text{otherwise}.
\end{cases}
\]
Then \( \Sigma_n^r = f(X_n, Y_i) \) counts the number of \( \Theta_i, i \leq n \) appears as some \( \sigma^q_i \) and \( \prec z \). The non-zero regions of \( f \) with respect to different values of \( z \) are as follows:

\[
\begin{align*}
\left\{ \frac{1-zx}{1+z-zx} < y < \frac{1}{k+1}, \quad \frac{k-z}{kz} < x < \frac{1}{k} \right\}, & \text{ when } \frac{k^2+k}{2k+1} < z \leq k^2 + k, \\
\left\{ \frac{1-zx}{1+z-zx} < y < \frac{1}{k+1}, \quad \frac{k-z}{kz} < x < \frac{k+1-z}{(k+1)z} \right\} & \cup \left\{ \frac{1}{k+2} < y < \frac{1}{k+1}, \quad \frac{k+1-z}{(k+1)z} < x < \frac{1}{k} \right\}, & \text{ when } \frac{k^2+k}{2k+1} < z \leq k, \\
\left\{ \frac{1-zx}{1+z-zx} < y < \frac{1}{k+1}, \quad 0 < x \leq \frac{k+1-z}{(k+1)z} \right\} & \cup \left\{ \frac{1}{k+2} < y < \frac{1}{k+1}, \quad \frac{k+1-z}{(k+1)z} < x < \frac{1}{k} \right\}, & \text{ when } k < z \leq k+1.
\end{align*}
\]

The proof is completed by calculating several integrals and taking \( g \) as \( g_1 \) in the proof of (i). 

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References

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