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# METRICAL THEORY FOR FAREY CONTINUED FRACTIONS

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## 1. Introduction

By making fundamental use of the Farey shift map and employing infinite (but $\sigma$-finite) measures together with the Chacon-Ornstein ergodic theorem it is possible to find new metrical results for continued fractions. Moreover this offers a unified approach to several existing theorems.

The application of ergodic theory to the study of continued fractions began with the Gauss transformation, $G$: $[0,1] \mapsto [0,1]$,

$$G(x)=\begin{cases} \frac{1}{x}-\left[\frac{1}{x}\right], & x\neq 0 \\ 0, & x=0, \end{cases}$$

which is ergodic with respect to the Gauss measure $\mu_g$, where

$$\mu_g(B)=\frac{1}{\log 2}\int_B \frac{1}{1+x}dx$$

for any Borel subset $B$ of $[0,1]$. H. Nakada [11] extended $G$ to the 2-dimensional case. Let $\tilde{G}$: $[0,1]\times[0,1] \mapsto [0,1]\times[0,1]$ be defined to be

$$\tilde{G}(x,y)=\left(G(x),\frac{1}{a_1+y}\right)$$

where $a_1=\left[\frac{1}{x}\right]$. The absolutely continuous invariant measure of $\tilde{G}$, $\tilde{\mu}_g$, is given by

$$d\tilde{\mu}_g=\frac{1}{\log 2}\cdot\frac{dxdy}{(1+xy)^2}.$$

Then the dynamical system $([0,1]\times[0,1],\mathscr{B}_2,\tilde{\mu}_g,\tilde{G})$ is the natural extension of $([0,1),\mathscr{B}_1,G)$ where $\mathscr{B}_n$ is the Borel algebra of $\boldsymbol{R}^n$. Hence $\tilde{G}$ is ergodic with respect to $\bar{\mu}_g$. Many metrical results for regular continued fractions can be proved using the ergodicity of $G$ or $\tilde{G}$. For example, [3] (W. Bosma *et al*) gave the distribution of the sequences of approximation constants $\{\theta_n\}$.

In this paper we focus on the convergents and the mediants of the Farey (or slow) continued fractions. Define $T$: $[0,1]\mapsto[0,1]$ by

$$T(x)=\begin{cases}\dfrac{x}{1-x}, & 0\le x\le\dfrac{1}{2},\\[2mm] \dfrac{1-x}{x}, & \dfrac{1}{2}\le x<1.\end{cases}$$

$T$ is called the Farey shift map (see [10]). $T$ preserves the measure $v$ given by

$$dv=\frac{1}{\log 2}\cdot\frac{dx}{x},$$

which is $\sigma$-finite but not a probability measure, and $T$ is ergodic with respect to this measere (see [12] or [10]). The natural extension of $T$, denoted by $\tilde{T}$, is the transformation on $[0,1]\times[0,1]$ given by

$$\tilde{T}(x,y)=\begin{cases}\left(\dfrac{x}{1-x},\dfrac{y}{1+y}\right), & 0\le x<\dfrac{1}{2}\\[2mm] \left(\dfrac{1-x}{x},\dfrac{1}{1+y}\right), & \dfrac{1}{2}\le x<1\end{cases}$$

The absolutely continuous invariant measure, $\tilde{v}$, of $\tilde{T}$ is determined by

$$d\tilde{v}=\frac{dx\,dy}{(x+y-xy)^2}\cdot\frac{1}{\log 2}.$$

The ergodicity of $\tilde{T}$ can be established from that of $\tilde{G}$ using an argument of [16] or by direct appeal to a general result given in [4]. Since the $v$ or $\tilde{v}$ are infinite ($\sigma$-finite, though), the Birkhoff Ergodic Theorem is not applicable for $T$ or $\tilde{T}$. To avoid this disadvatage, Ito considered another transformation $T_1$ induced by $T$, the invariant measure of which is a probability measure. By the ergodicity of $T_1$ and its natural extension, he obtained in [8] many metrical results related to convergents and nearest mediants. In this paper we consider $T$ and $\tilde{T}$ directly. We shall establish an ergodic theorem for $\tilde{T}$ though the Chacon-Ornstein ergodic theorem. In this way the results of [8] can be generalized for we can derive metrical results on Diophantine approximation by all the mediants not only the nearest ones. The results for nearest mediants become a special case. Any other metrical results obtained by applying the Birkhoff ergodic theorem for $G$ or

$\tilde{G}$ can also be obtained by using the ergodic theorem we build for $\tilde{T}$.

In Section 2, we recall some basic results about Farey and regular continued fractions and give some basic properties of $T$ and $\tilde{T}$. In Section 3, we establish an ergodic theorem for $T$ through the Chacon-Ornstein Ergodic Theorem. In Section 4, we apply the ergodic theorem established in Section 3 to prove some old and new metrical results both for the regular continued fractions and the Farey continued fractions.

## 2. Preliminaries

For an irrational $x \in [0,1]$ with regular continued fraction expansion

$$\cfrac{1}{a_1+\cfrac{1}{a_2+\ddots}}=[0;a_1,a_2,\cdots],$$

the $n$-th convergent is given by

$$\frac{p_n}{q_n}=\cfrac{1}{a_1+\cfrac{1}{a_2+\ddots+\cfrac{1}{a_n}}}=[0;a_1,a_2,\cdots,a_n].$$

The integers $p_n$, $q_n$ can be described inductively by setting

$$p_{-1}=1, \quad p_0=0, \quad q_{-1}=0, \quad q_0=1,$$

$$p_n=a_np_{n-1}+p_{n-2}, \quad q_n=a_nq_{n-1}+q_{n-2}.$$

We shall be concerned with a slower sequence $\{P_n/Q_n\}$ of approximations to $x$, corresponding to a branch of the Farey tree (see [7], [14] for details). For our present purpose it suffices to know that

$$P_n=kp_m+p_{m-1}, \quad Q_n=kq_m+q_{m-1},$$

where

$$n=a_0+a_1+\cdots a_m+k, \quad 0<k\leq a_{m+1}, \quad (a_0=0).$$

The sequence $\{p_n/q_n\}$ consists of the convergents of $x$, while $\{P_n/Q_n\}$ is the sequence of convergents and mediants.

The Farey shift map $T$: $[0,1]\mapsto[0,1]$, defined in the introduction, may be characterised as follows. For $x=[0;a_1,a_2,\cdots]$, we have

$$T(x)=\begin{cases}[0;a_1-1,a_2,\cdots], & a\geq 2\\ [0;a_2,a_3,\cdots], & a_1=1.\end{cases}$$

Moreover, for $x=[0;a_1,a_2,\cdots]$, $y=[0;b_1,b_2,\cdots]$ we have

$$\tilde{T}(x,y)=\begin{cases}([0;a_1-1,a_2,\cdots],[0;b_1+1,b_2,\cdots]) & a\geq 2\\ ([0;a_2,a_3,\cdots],[0;1,b_1,b_2,\cdots]), & a_1=1.\end{cases}$$

We make basic use of the numbers $X_n$, $Y_n$ defined by $\tilde{T}^n(x,1)=(X_n,Y_n)$. Note, in particular, that

$$Y_n=\begin{cases}[0;k+1,a_m,\cdots,a_1], & m\geq 1\\ [0;k+1], & m=0\end{cases}$$

where

$$n=a_0+a_1+\cdots+a_m+k,\quad 0\leq k<a_{m+1}.$$

It is well-known that for $\tilde{G}$, *defined in the introduction,*

$$\tilde{G}^m(x,0)=(x_m,y_m)=([0;a_{m+1},a_{m+2},\cdots],\ [0;a_m,\cdots,a_1]),$$

where $x_m=G^m x$, $y_m=q_{m-1}/q_m$.

Let us write

$$\Theta_n=Q_n^2\left|x-\frac{P_n}{Q_n}\right| \tag{1}$$

and

$$\theta_m=q_m^2\left|x-\frac{p_m}{q_m}\right|. \tag{2}$$

Then we have

$$\theta_m=x_m(1+x_m y_m)^{-1} \tag{3}$$

and this is contained in the following formula, given in [2],

$$\Theta_n=(1-kx_m)(k+y_m)(1+x_m y_m)^{-1}, \tag{4}$$

where

$$n=a_0+a_1+\cdots+a_m+k,\quad 0<k\leq a_{m+1}.$$

(4) allows discussion of $\Theta_n$ via $\tilde{G}$ but our strategy is to work directly with $\tilde{T}$ so we set about expressing the quantities $Q_{n-1}/Q_n$ and $\Theta_n$ in terms of $X_n$, $Y_n$.

**Lemma 1.** *For $n=1,2,\cdots$, we have*

$$Q_n/Q_{n+1}=(1+Y_{n-1})^{-1}=\max\{Y_n,1-Y_n\}.$$

Proof. We know that $Y_n$ equals $Y_{n-1}(1+Y_{n-1})^{-1}$ or $(1+Y_{n-1})^{-1}$ and that $0<Y_{n-1}\le 1$. Hence $(1+Y_{n-1})^{-1}$ equals $\max\{Y_n,1-Y_n\}$.

Next we use induction. When $n=1$, we have $Y_1=1/2$, $Q_1=1$. And we always have $Q_2=2$. Hence

$$Q_1/Q_2=1/2=Y_1=\max\{Y_1,1-Y_1\}.$$

Suppose for $l\le n$ we have

$$Q_l/Q_{l+1}=\max\{Y_l,1-Y_l\}.$$

Assume that $n=a_0+a_1+{}_2+\cdots+a_m+k$, $0\le k<a_{m+1}$. If $k=0$, then $Q_n=q_m$, $Q_{n+1}=1\cdot q_m+q_{m-1}$ and $Y_n=[0;1,a_m,a_{m-1},\cdots,a_1]\ge 1/2$. Hence $Q_n/Q_{n+1}=Y_n$. If $a_{m+1}\ge 2$, then $Q_{n+2}=2\cdot q_m+q_{m-1}=Q_n+Q_{n+1}$. Thus

$$Q_{n+1}/Q_{n+2}=(1+Q_n/Q_{n+1})^{-1}=(1+Y_n)^{-1}.$$

If $a_{m+1}=1$, then $Q_{n+1}=q_{m+1}$ and $Q_{n+2}=q_m+q_{m+1}$. Again we have $Q_{n+2}=Q_n+Q_{n+1}$ and again we get $Q_{n+1}/Q_{n+2}=(1+Y_n)^{-1}$.

When $0<k<a_{m+1}-1$, we have $Q_n=kq_m+q_{m-1}$, $Q_{n+1}=(k+1)q_m+q_{m-1}$ and $Q_{n+2}=(k+2)q_m+q_{m-1}$, and $Y_n$, which equals $[0;k+1]$ or $[0;k+1,a_m,\cdots,a_1]$, is at most $1/2$. Hence

$$\begin{aligned}Q_{n+1}/Q_{n+2}&=Q_{n+1}(2q_m+Q_n)^{-1}\\&=(2q_m/Q_{n+1}+(1-Y_n))^{-1}\\&=(2(k+1+y_m)^{-1}+(1-Y_n))^{-1}\\&=(2Y_n+1-Y_n)^{-1}=(1+Y_n)^{-1}.\end{aligned}$$

Lastly we consider $k=a_{m+1}-1>0$. We have $Q_n=kq_m+q_{m-1}=q_{m+1}-q_m$, $Q_{n+1}=q_{m+1}$ and $Q_{n+2}=q_{m+1}+q_m$. We also have $Y_n=y_{m+1}\le 1/2$. Hence, once more, $Q_{n+1}/Q_{n+2}=(1+Y_n)^{-1}$.

**Lemma 2.** *For $n=2,3,\cdots$, we have*

$$\Theta_n=\begin{cases}(1-Y_n)(X_n+Y_n-X_nY_n)^{-1}, & Y_n<1/2,\\ X_nY_n(X_n+Y_n-X_nY_n)^{-1}, & Y_n>1/2,\end{cases}\tag{5}$$

$$\Theta_{n+1}=(1-X_n)(X_n+Y_n-X_nY_n)^{-1}.\tag{6}$$

Proof. For $n=a_0+a_1+a_2+\cdots+a_m\ge 2$, we have $X_n=x_m$ and $Y_n=(1+y_m)^{-1}>1/2$. An application of (3) gives

$$\Theta_n = \theta_m = X_n(1 + X_n(Y_n^{-1} - 1))^{-1} = X_n Y_n (X_n + Y_n - X_n Y_n)^{-1}.$$

For $n = a_1 + \cdots + a_m + k \geq 2$ where $0 < k < a_{m+1}$, we apply (4) after noting that

$$X_n = (x_m^{-1} - k)^{-1}, \quad Y_n = (k + 1 + y_m)^{-1} < 1/2.$$

This gives

$$\Theta_n = (1 - Y_n)(X_n + Y_n - X_n Y_n)^{-1}.$$

It is now easy to see that $Y_{n+1} > 1/2$ if and only if $X_n > 1/2$. When $X_n > 1/2$

$$(X_{n+1}, Y_{n+1}) = \tilde{T}(X_n, Y_n) = (X_n^{-1}(1 - X_n), (1 + Y_n)^{-1}).$$

In this case we see that

$$\begin{aligned}\Theta_{n+1} &= X_{n+1} Y_{n+1} (X_{n+1} + Y_{n+1} - X_{n+1} Y_{n+1})^{-1} \\ &= (1 - X_n)(X_n + Y_n - X_n Y_n)^{-1}\end{aligned}$$

For the case $X_n < 1/2$, we substitute $X_{n+1} = (1 - X_n)^{-1} X_n$, $Y_{n+1} = Y_n (1 + Y_n)^{-1}$ in the formula

$$\Theta_{n+1} = (1 - Y_{n+1})(X_{n+1} + Y_{n+1} - X_{n+1} Y_{n+1})^{-1}$$

to obtain the required result. ■

### 3. Frgodicity and ergodic theorem for $\tilde{T}$

We begin this section by showing that $\tilde{G}$ can be induced from $\tilde{T}$.

**Theorem 1.** *The dynamical system* $(\Omega, \mathscr{B}_2, \tilde{\mu}_g, \tilde{G})$ *is* (*isomorphic with*) *the system induced from* $(\Omega, \mathscr{B}_2, \tilde{\nu}, \tilde{T})$ *on the set* $E = \{(x,y) : y > 1/2\}$.

Proof. Recall that for $x = [0; a_1, a_2, \cdots]$, $y = [0; b_1, b_2, \cdots]$ we have

$$\tilde{T}(x,y) = \begin{cases} ([0; a_1 - 1, a_2, \cdots], [0; b_1 + 1, b_2, \cdots]), & a \geq 2, \\ ([0; a_2, a_3, \cdots], [0; 1, b_1, b_2, \cdots]), & a_1 = 1. \end{cases}$$

In particular the second coordinate of $\tilde{T}(x,y)$ is greater than $1/2$ if and only if $a_1 = 1$. Accordingly the induced map $\tilde{T}_E$ is given by

$$\tilde{T}_E(x,y) = (x^{-1} - a_1, (1 + (y^{-1} + a_1 - 1)^{-1})^{-1}).$$

Now consider the map $\phi : E \mapsto \Omega$ given by

$$\phi(x,y)=(x,y^{-1}-1).$$

The map transforms to $(x,y)\mapsto(x^{-1}-a_1,(y+a_1)^{-1})$ and the measure $dxdy(x+y-xy)^{-2}$ transforms to $dxdy(1+xy)^{-2}$. ■

By Theorem 1 and the ergodicity of $\tilde{G}$ together with a result of [16], we obtain the ergodicity of $\tilde{T}$. Theorem 1 also gives an abstract justification of the statement that any result derived from $G$ or $\tilde{G}$ can be obtained from $\tilde{T}$.

The map $\tilde{T}$ is ergodic and invertible and $\tilde{\nu}$ is non-atomic so it follows (see [5]) that $\tilde{T}$ is conservative. Therefore we may apply the Chacon-Ornstein theorem (cf. [13]) on the system $(\Omega,\mathscr{B}_2,\tilde{\nu},\tilde{T})$ (*we use* $\Omega$ to denote $[0,1]\times[0,1]$ for the remainder of the paper) to derive the following result.

**Theorem 2.** *For any* $f,g\in L^1(\Omega,\mathscr{B}_2,\tilde{\nu})$ *with* $\int g d\tilde{\nu}\neq 0$, *one has*

$$\lim_{n\to\infty}\frac{\sum_{k=0}^{n-1} f(\tilde{T}^k(x,y))}{\sum_{k=0}^{n-1} g(\tilde{T}^k(x,y))}=\frac{\int f d\tilde{\nu}}{\int g d\tilde{\nu}} \quad \text{a.e.}$$

Next we show that under Lipschitz conditions on $f$, $g$, the points $\tilde{T}^k(x,y)$ in Theorem 2 can be replaced by $(X_k,Y_k)=\tilde{T}^k(x,1)$.

**Theorem 3.** *Suppose that* $f,g\in L^1(\Omega,\mathscr{B},\tilde{\nu})$ *satisfy*

$$|f(x,y)-f(x,y')|\leq L|y-y'|^\alpha$$

$$|g(x,y)-g(x,y')|\leq L|y-y'|^\alpha$$

*where* $L>0$, $\alpha>0$ *are constants. If* $\int g d\tilde{\nu}\neq 0$, *then for almost all* $(x,y)\in\Omega$ *one has*

$$\lim_{n\to\infty}\frac{\sum_{k=0}^{n-1} f(\tilde{T}^k(x,y))}{\sum_{k=0}^{n-1} g(\tilde{T}^k(x,y))}=\lim_{n\to\infty}\frac{\sum_{k=0}^{n-1} f(X_k,Y_k)}{\sum_{k=0}^{n-1} g(X_k,Y_k)}.$$

Proof. Let $Z_i$ be the second coordinate of $\tilde{T}^i(x,y)$, i.e. $\tilde{T}^i(x,y)=(X_i,Z_i)$. We claim that for almost all $x\in[0,1]$ and all $y\in(0,1]$, we have

$$\sum_{i=0}^{\infty}|Z_i-Y_i|^\alpha<+\infty.$$

In fact, for $x=[0;a_1,a_2,\cdots]$, $i=a_1+a_2+\cdots+a_m+k$, $0\leq k<a_{m+1}$, one has

$$Z_i=[0;k+1,a_m,\cdots,a_2,a_1-1+y^{-1}]$$

and

$$Y_i=[0;k+1,a_m,\cdots,a_2,a_1].$$

Hence we have

$$|Y_i-Z_i|\leq c/q_m^2$$

for some constant $c$, where $q_m$ is the denominator of the $m$-th convergent $\frac{p_m}{q_m}$ of the regular continued fraction expansion of $x$ (cf. [1] p.42]. Therefore we get that

$$\sum_{i=0}^{\infty}|Z_i-Y_i|^{\alpha}\leq\sum_{m=0}^{\infty}\sum_{k=1}^{a_{m+1}}\left(\frac{c}{q_m^2}\right)^{\alpha}=c^{\alpha}\sum_{m=0}^{\infty}\frac{a_{m+1}}{q_m^{2\alpha}}.$$

By induction we can see that

$$q_m\geq 2^{(m-1)/2}.$$

We need the following theorem (see [15]).

**Theorem A.** *Let* $F(n)>1$, *for* $n=1,2,\cdots$, *and suppose that* $\sum_{n=1}^{\infty}\frac{1}{F(n)}<\infty$. *Then the set*

$$A=\{x\in[0,1],\ a_k(x)>F(k)\ \textit{infinitely many times}\}$$

*has Lebesgue measure* 0.

Now we choose $F(n)=2^{n\alpha/2}$. By the above theorem we see that the set

$$E=\{x\in[0,1],\ a_k(x)>F(k)\ \textit{only finitely many times}\}$$

has Lebesgue measure 1. Hence for almost all $x\in[0,1]$,

$$\sum_{m=0}^{\infty}\frac{a_{m+1}}{q_m^{2\alpha}}\leq C(x)+\sum_{m=0}^{\infty}2^{\alpha(2-m)/2}<\infty.$$

Where $C(x)=\sum_{a_m>F(m)}\frac{a_{m+1}}{q_m^{2\alpha}}$. The required result follows easily when we bear in mind the fact that $\Sigma g(\tilde{T}^k(x,y))$ diverges almost everywhere because $\tilde{T}$ is conservative and ergodic. ∎

For some functions $f(x,y)$ though we do not have

$$(*) \qquad |f(x,y)-f(x,y')| \leq L|y-y'|^{\alpha}$$

for all $y,y' \in [0,1]$, it is still true that

$$|f(T^i(x,y))-f(X_i,Y_i)| \leq L|Z_i-Y_i|^{\alpha}, \quad \alpha>0,$$

for almost all $x \in [0,1]$ and $i$ large enough.

EXAMPLE. Let

$$f(x,y)=\begin{cases}\log(1-y) & y\in[0,1)\\ 0 & y=1\end{cases}$$

Then $f \in L^1(\Omega)$. We do not have $(*)$ for all $y,y' \in [0,1]$.

For $x=[0;a_1,a_2,\cdots]$, let

$$i=a_1+a_2+\cdots+a_m+k, \quad 0 \leq k < a_{m+1}$$

where $m \geq 2$. Then

$$\begin{aligned}&|f(T^i(x,y))-f(X_i,Y_i)|\\ &=|\log(1-Z_i)-\log(1-Y_i)|\\ &=\frac{1}{|1-\xi_i|}|Z_i-Y_i|\end{aligned}$$

where $\xi_i$ is in between $y_i$ and $Y_i$. It is easy to see that $1-\xi_i \geq 1/q_m$, i.e.

$$\frac{1}{1-\xi_i} \leq q_m \leq c|Y_i-Z_i|^{1/2}.$$

Therefore, $|f(T^i(x,y))-f(X_i,Y_i)| \leq c|Y_i-Z_i|^{1/2}$.

## 4. Applications

In this section we apply the ergodic theorems for $\tilde{T}$ to obtain metrical results for convergents and medians of regular continued fractions. For all the functions $f$, $g$ involved in this section it is valid to replace $\tilde{T}^k(x,y)$ by $(X_k, Y_k)$ as in Theorem 3 but omit the tedious verification.

For an irrational $x=[0;a_1,a_2,\cdots]$ we shall call

$$\frac{kp_n+p_{n-1}}{kq_n+q_{n-1}} \quad \text{and} \quad \frac{(a_{n+1}-k)p_n+p_{n-1}}{(a_{n+1}-k)q_n+q_{n-1}}$$

the $k$-th mediants of $x$, when $a_{n+1} \geq 2k$. We let $P_n^{(k)}/Q_n^{(k)}$ denote the sequence which consists of all convergents and $i$-th mediants of $x$ for all $i \leq k$. When $k=0$, we recover the convergents $\{p_m/q_m\}$ and, when $k=1$, we obtain the so-called nearest mediants of Ito, [8]. It is easy to see that the event "$P_n/Q_n$ appears as some $P_i^{(k)}/Q_i^{(k)}$" is characterised by $X_n > 1/(k+1)$ or $Y_n > 1/(k+2)$ while "$P_{n+1}/Q_{n+1}$ appears as some $P_i^{(k)}/Q_i^{(k)}$" is by $X_n > 1/(k+2)$ or $Y_n > 1/(k+1)$.

Let us write also

$$\Theta_n^{(k)} = (Q_n^{(k)})^2 |x - P_n^{(k)}/Q_n^{(k)}|.$$

Our main theorem can now be stated.

**Theorem 4.** *For almost all $x$ we have*

(i). *for* $k=0,1,2,\cdots$,

$$\lim_{n\to\infty} \frac{1}{n} \log Q_n^{(k)} = \frac{\pi^2}{12\log(2k+2)},$$

(ii). *for* $k=0,1,2,\cdots$,

$$\lim_{n\to\infty} \frac{1}{n} \log \left| x - \frac{P_n^{(k)}}{Q_n^{(k)}} \right| = -\frac{\pi^2}{6\log(2k+2)},$$

(iii). for $k=1,2,\cdots$,

$$\lim_{n\to\infty} \frac{1}{n} \#\{i:\ i \leq n,\ \Theta_i^{(k)} < z\}$$

$$= \frac{1}{\log(2k+2)} \begin{cases} z, & 0 \leq z < 1 \\ 1+\log z, & 1 \leq z < \dfrac{k+1}{2} \\ 2 - \dfrac{2z}{k+1} + \log\dfrac{2z^2}{k+1}, & \dfrac{k+1}{2} \leq z < k+1 \\ \log(2k+2), & k+1 < z \end{cases}$$

(iv). $$\lim_{n\to\infty} \frac{1}{\log Q_n} \#\{i:\ i \leq n,\ \Theta_i < z\} = \frac{12}{\pi^2} \begin{cases} z, & 0 < z \leq 1, \\ 1+\log z, & 1 < z. \end{cases}$$

REMARK. The case $k=0$ of (i) and (ii) are the basic results of Levy (see [1]), and the case $k=1$ of (i), (ii) and (iii) give results of Ito, [8]. The important result of Bosma *et al* in [3] corresponds to the case $k=0$ of (ii) and the proof which follows could be simplified to yield that special case. Nevertheless a suitable interpretation of the three terms corresponding to $1/2 \leq z < 1$ yields the appropriate distribution. We take the signed sum, i.e.

$$\lim_{n\to\infty} \frac{1}{n} \#\{i:\ i \leq n,\ \theta_n < z\} = \frac{1}{\log 2} \begin{cases} z, & 0 \leq z < \frac{1}{2} \\ 1 - z + \log(2z), & \frac{1}{2} \leq z < 1 \\ \log 2, & 1 \leq z \end{cases}$$

where

$$1 - z + \log(2z) = z - (1 + \log z) + [2 - 2z + \log(2z^2)].$$

Proof. For (i) we take

$$f(x,y) = \log(\max\{y, 1-y\}),$$

$$g(x,y) = \begin{cases} 1, & \text{when } x > (k+1)^{-1}, \text{or } y > (k+2)^{-1} \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\int f\,d\tilde{\nu} = -\frac{\pi^2}{12\log 2}, \quad \int g\,d\tilde{\nu} = \frac{\log(2k+2)}{\log 2}.$$

For $n = a_0 + a_1 + \cdots + a_m + k$, $0 \leq k < a_{m+1}$, we have, by Lemma 1,

$$\log \frac{Q_n}{Q_{n+1}} = f(X_n, Y_n)$$

while

$$\sum_{i=1}^{n} g(X_i, Y_i) = \#\{i:\ i \leq n,\ Q_i \text{ appears as some } Q_j^{(k)}\}.$$

Therefore

$$\sum_{i=1}^{n} f(X_i, Y_i) \Big/ \sum_{i=1}^{n} g(X_i, Y_i) = -s^{-1}(\log Q_{n+1} - \log Q_1).$$

where $s$ is determined by $Q_s^{(k)} \leq Q_n < Q_{s+1}^{(k)}$. By the ergodic theorem of the last section

$$\lim_{s\to\infty} \frac{\log Q_s^{(k)}}{s} = -\lim_{n\to\infty} \sum_{i=1}^{n} f(X_i, Y_i) \Big/ \sum_{i=1}^{n} g(X_i, Y_i) = -\int f\,d\tilde{\nu} \Big/ \int g\,d\tilde{\nu},$$

and the required result follows.

For (ii) noting that $\Theta_n^{(k)}=(Q_n^{(k)})^2|x-P_n^{(k)}/Q_n^{(k)}|$, by (i) it suffices to show that

$$\lim_{n\to\infty}\frac{\log\Theta_n^{(k)}}{n}=0 \quad \text{a.e.}$$

Remember that $\Theta_{n+1}$ is one of $\Theta_s^{(k)}$ if and only if $X_n>(k+2)^{-1}$ or $Y_n>(k+1)^{-1}$. Then when $\Theta_{n+1}$ is one of $\Theta_s^{(k)}$ we have

$$\Theta_{n+1}=\frac{1-X_n}{X_n+Y_n-X_nY_n}<k+1.$$

On the other hand,

$$\Theta_{n+1}\geq 1-X_n\geq 1-[0;1,a_{m+1},\cdots]>1/(a_{m+1}+1)\geq 1/2a_{m+1},$$

where $m$ is determined by $Q_n\leq q_m<Q_{n+1}$. Therefore,

$$-\frac{\log a_{m+1}+\log 2}{s}<\frac{\Theta_s^{(k)}}{s}<\frac{\log(k+1)}{s}.$$

Noting that $m\leq s\leq(2k+1)m$ we obtain

$$\frac{\log a_{m+1}}{s}\leq\frac{\log a_{m+1}}{m}=\left(\frac{a_{m+1}}{m^\alpha}\right)^{1/\alpha}\cdot\frac{\log a_{m+1}}{a_{m+1}^{1/\alpha}}\to 0 \quad \text{a.e.}$$

by Theorem A, where $\alpha>1$ is a constant. Therefore

$$\lim_{n\to\infty}\frac{\log\Theta_n^{(k)}}{n}=0 \quad \text{a.e.}$$

To prove (iii) and (iv) we also consider $\Theta_{n+1}$ instead of $\Theta_n$. Let

$$E_{z,t}=\{(x,y)\in\Omega:\frac{1-x}{x+y-xy}<z,\ x>\frac{t}{1+t}\}$$
$$\cup\{(x,y)\in\Omega:\frac{1-x}{x+y-xy}<z,\ y>t\}=E_1\cup E_2$$

where $0<z$, $0\leq t\leq 1/2$ and

$$f(x,y)=\begin{cases}1, & (x,y)\in E_{z,t}\\ 0, & \text{othewise.}\end{cases}$$

We recast the inequality

$$(1-x)(x+y-xy)^{-1}<z$$

in the form

$$x>(1-zy)(1+z(1-y))^{-1}=u(y), \text{ say,}$$

and note that $(1-zy)(1+z(1-y))^{-1}>t/(1+t)$ if and only if $y>z^{-1}-t$. Therefore

$$E_1=\begin{cases} \{x>u(y),\quad 0<y<1\}, & 0<z<\dfrac{1}{1+t} \\ \{x>u(y),\quad 0<y<\dfrac{1}{z}-t\}\cup\{x>\dfrac{t}{1+t},\quad \dfrac{1}{z}-t<y<1\}, & \dfrac{1}{1+t}<z<\dfrac{1}{t} \\ \{x>\dfrac{t}{1+t},\quad 0<y<1\}, & \dfrac{1}{t}<z. \end{cases}$$

Suppose first that $0<z<1/(1+t)$. Then $E_{z,t}=E_1=\{x>u(y),\quad 0>y>1\}$. Hence

$$\log 2\int f\,d\tilde{v}=\int_0^1 dy\int_{u(y)}^1 (x+y-xy)^{-2}dx=z.$$

Now consider the case $1/(1+t)\leq z<1$. Note that $t\leq 1-t<z^{-1}-t$. Then again $E_{z,t}=\{x>u(y),\ 0<y<1\}$ and $\log 2\int f\,d\tilde{v}=z$.

When $1<z<1/2t$ we also have $z^{-1}-t>t$. Remember that $u(y)>0$ when $y>z^{-1}$. Thus

$$\begin{aligned} E_{z,t}&=\{(x,y)\in\Omega:\ x>u(y),\ 0<y<1\}\\ &=\{x>u(y),\ 0<y<z^{-1}\}\cup\{x>0,\ z^{-1}<y<1\}, \end{aligned}$$

and

$$\log 2\int f\,d\tilde{v}=1+\log z.$$

If $1/2t<z<1/t$, then $0<z^{-1}-t<t$ and

$$\begin{aligned} E_{z,t}=\{x>u(y),\ 0<y<\frac{1}{z}-t\}&\cup\{x>\frac{t}{1+t},\ \frac{1}{z}-t<y<t\}\\ &\cup\{x>u(y),\ t<y<\frac{1}{z}\}\cup\{x>0,\ \frac{1}{z}<y<1\} \end{aligned}$$

Calculate that

$$\log 2\int f\,d\tilde{\nu}=2-2tz+\log(2tz^2).$$

Lastly, when $z>1/t$ we have

$$E_{z,t}=\{x>t/(1+t),\ 0<y<t\}\cup\{x>0,\ t<y<1\}$$

and $\log 2\int f\,d\tilde{\nu}=\log 2-\log t$.

It is now easy to piece together the result of (iii) by taking $t=(k+1)^{-1}$ and

$$g(x,y)=\begin{cases}1, & \text{when } x>(k+2)^{-1} \text{ or } y>(k+1)^{-1}\\ 0, & \text{otherwise.}\end{cases}$$

To obtain (iv) we set $t=0$ and replace $g$ by the function $\log(\max\{y,1-y\})$ used as $f$ in proving (i). This completes the proof. ■

REMARK. Part (iii) of the theorem (in some sense a limiting case as $k\to\infty$) shows that $\frac{1}{n}\#\{i\colon\ i\le n,\ \Theta_n<z\}\to 0$ as $n\to\infty$ and hence that $\{\Theta_n\}$ does not have a distribution function. We can obtain some more information about $\Theta_n$. We have

For almost all $x\in[0,1]$ and any $\varepsilon>0$,

(i). $\lim\limits_{n\to\infty}\dfrac{\Theta_n}{m^{1+\varepsilon}}=0$, where $q_m\le Q_n<q_{m+1}$;

(ii). $\lim\limits_{n\to\infty}\dfrac{1}{n}\sum\limits_{i=1}^{n}\Theta_i=\infty$; *and*

(iii). $\lim\limits_{n\to\infty}\dfrac{1}{n^{2+\varepsilon}}\sum\limits_{i=1}^{n}\Theta_i=0.$

In fact, for $n=a_0+a_1+\cdots+a_m+k$, where $0\le k<a_{m+1}$, by Lemma 2

$$\Theta_{n+1}<Y_n=[k+1;a_m,\cdots,a_1]<k+2. \tag{$*$}$$

Then (i) follows from Theorem A. Using ($*$) we can get the the following estimation:

$$\frac{1}{16}\left[\sum_{i=1}^{m}a_i^2+k^2-n\right]-(m+1)<\sum_{i=1}^{n}\Theta_i<\sum_{i=1}^{m}a_i^2+k^2. \tag{$**$}$$

Then follow (ii) and (iii).

Next let us compare Theorem 4 (iv) with some results of P. Erdös [6] and

J. Blom [2]. Let

$$\Theta\left(\frac{p}{q},x\right)=q^2\left|x-\frac{p}{q}\right|.$$

Define

$$U(x,z,n)=\#\{(p,q)\in \boldsymbol{Z}\times \boldsymbol{N}\colon (p,q)=1,\ \Theta\left(\frac{p}{q},x\right)\le z,\ q\le n\},$$

$$U_1(x,z,n)=\#\{(p,q)\in \boldsymbol{Z}\times \boldsymbol{N}\colon (p,q)=1,\ \Theta\left(\frac{p}{q},x\right)\le z,\ q\le n,\ \frac{p}{q} \text{ is a convergent of } x\}$$

and

$$U_2(x,z,n)=\#\{(p,q)\in \boldsymbol{Z}\times \boldsymbol{N}\colon (p,q)=1,\ \Theta\left(\frac{p}{q},x\right)\le z,\ q\le n\},\ \frac{p}{q} \text{ is a best approximant of } x\}.$$

By best approximant we mean that if there is a fraction $\frac{a}{b}$ different from $\frac{p}{q}$ such that

$$\left|\frac{a}{b}-x\right|\le\left|\frac{p}{q}-x\right|$$

then $b>q$. Erdös [6]) proved that for any $z\ge 0$

$$\lim_{n\to\infty}\frac{U(x,z,n)}{n}=\frac{12}{\pi^2}z \quad \text{a.e.}$$

Blom [2] gave that

$$\lim_{n\to\infty}\frac{U_1(x,z,n)}{n}=\frac{12}{\pi^2}(f)z \quad \text{a.e.}$$

and

$$\lim_{n\to\infty}\frac{U_2(x,z,n)}{n}=\frac{12}{\pi^2}(f(z)+h(z)) \quad \text{a.e.}$$

where

$$f(z)=\begin{cases} z & 0\le z\le \frac{1}{2} \\ 1-z+\log(2z) & \frac{1}{2}\le z\le 1 \\ \log 2 & z\ge 1, \end{cases}$$

$$h(z)=\begin{cases} 0 & 0\le z\le \frac{1}{2} \\ z-\frac{1}{2}-\frac{1}{2}\log(2z) & \frac{1}{2}\le z\le 1 \\ \frac{1}{2}+\frac{1}{2}\log(2z) & z\ge 1. \end{cases}$$

When $z\le 1$ Theorem 4(iii) corresponds to the result of Erdös [6]. [8] and [9] also gave new proofs in this case. When $z>1$ this result takes a different form. This fact tells us that, for $z>1$, there is no result for convergents and mediants analogous to the theorem of Legendre for $z=1/2$ or the theorem of Fatou and Koksma for $z=1$ (see [9]).

The result of Blom [2] can also be proved by Theorem 2 or 3. In fact for irrational $x\in[0,1)$, a best approximant is an element of $\left\{\frac{P_n}{Q_n}\right\}$ characterized by

$$Y_n>\frac{1}{2} \text{ or } Y_n^{-1}>X_n^{-1}+1, \quad n\ge 2.$$

Thus we can prove these results by choosing appropriate functions $f$ and $g$.

Jager [9] considered the two sequences

$$\left\{\frac{p_n+p_{n-1}}{q_n+q_{n-1}}\right\}, \quad \left\{\frac{(a_{n+1}-1)p_n+p_{n-1}}{(a_{n+1}-1)q_n+q_{n-1}}\right\}$$

separately and obtained some metrical results. If $P_N/Q_N$ appears as $(p_n+p_{n-1})/(q_n+q_{n-1})$ then we have

$$X_N=[0;\,a_{n+1}-1,a_{n+2},\cdots], \quad Y_N=[0;2,a_n,\cdots,a_1] \quad \text{when } a_{n+1}\ge 2,$$

or

$$X_N=[0;a_{n+2},\cdots], \quad Y_N=[0;1,1,a_n,\cdots,a_1] \quad \text{when } a_{n+1}=1.$$

Thus the first sequece is characterised by $1/3<Y_n<2/3$. The second one is more complicated. If $a_{n+1}\ge 2$, then $((a_{n+1}-1)p_n+p_{n-1})/((a_{n+1}-1)q_n+q_{n-1})$ corresponds

$$X_N=[0;1,a_{n+2},\cdots]\in(1/2,1), \quad Y_N=[0;a_{n+1},\cdots,a_1]\in(0,1/2).$$

When $a_{n+1}=1$ we get $p_{n-1}/q_{n-1}$ which corresponds to

$$\begin{cases} X_N=[0;a_n,1,a_{n+2},\cdots]\in\left(\dfrac{1}{a_n+1},\dfrac{2}{2a_n+1}\right), \\ Y_N=[0,1,a_{n-1},\cdots,a_1]\in(1/2,1). \end{cases}$$

Hence the second one is characterised by

$$X_n\in(1/2,1),\ Y_n\in(0,1/2)\quad\text{or}\quad X_n\in\bigcup_{i=1}^{\infty}\left(\frac{1}{i+1},\frac{2}{2i+1}\right),\ Y_n\in(1/2,1).$$

However, those two sequences are not "pure" nearest mediants. We shall consider the sequences

$$\left\{\frac{p_n+p_{n-1}}{q_n+q_{n-1}},a_{n+1}\geq 2\right\}\quad\text{and}\quad\left\{\frac{(a_{n+1}-1)p_n+p_{n-1}}{(a_{n+1}-1)q_n+q_{n-1}},a_{n+1}\geq 2\right\}$$

which are characterised by $1/3<Y_n<1/2$ and $X_n>1/2$, $Y_n<1/2$ respectively. In general we use $\{a_n^{(k)}/b_n^{(k)}\}$ and $\{c_n^{(k)}/d_n^{(k)}\}$ to denotes the "pure" $k$-th mediant sequences for each of the two directions

$$\left\{\frac{kp_n+p_{n-1}}{kq_n+q_{n-1}},a_{n+1}\geq 2k\right\}\quad\text{and}\quad\left\{\frac{(a_{n+1}-k)p_n+p_{n-1}}{(a_{n+1}-k)q_n+q_{n-1}},a_{n+1}\geq 2k\right\}$$

respectively, where $k\leq 1$. It is not hard to see that $\{a_n^{(k)}/b_n^{(k)}\}$ is the subsequence of $\{P_n/Q_n\}$ determined by $X_n<1/k$ and $(k+2)^{-1}<Y_n<(k+1)^{-1}$ while $\{c_n^{(k)}/d_n^{(k)}\}$ determined by $(k+1)^{-1}<X_n<k^{-1}$ and $Y_n<(k+1)^{-1}$. Define

$$\sigma_n^{(k)}=b_n^{(k)}|b_n^{(k)}x-a_n^{(k)}|$$

and

$$\rho_n^{(k)}=d_n^{(k)}|d_n^{(k)}x-c_n^{(k)}|.$$

**Theorem 5.** *For almost all $x\in[0,1]$ and $k=1,2,\cdots$, one has*

(i). $$\lim_{n\to\infty}\frac{\log b_n^{(k)}}{n}=\lim_{n\to\infty}\frac{\log d_n^{(k)}}{n}=\frac{\pi^2}{12(\log(2k+1)-\log(2k))},$$

(ii). $$\lim_{n\to\infty}\frac{1}{n}\log\left|x-\frac{a_n^{(k)}}{b_n^{(k)}}\right|=\lim_{n\to\infty}\frac{1}{n}\log\left|x-\frac{c_n^{(k)}}{d_n^{(k)}}\right|=-\frac{\pi^2}{6(\log(2k+1)-\log(2k))},$$

(iii). $$\lim_{n\to\infty}\frac{1}{n}\#\{\sigma_i^{(k)}<z,\ i\leq n\}=\lim_{n\to\infty}\frac{1}{n}\#\{\rho_n^{(k)}<z,\ i\leq n\}$$

$$=\frac{1}{\log(2k+1)-\log(2k)}\begin{cases}\frac{2z}{k}-1-\log\frac{2z}{k}, & \frac{k}{2}<z\le\frac{k^2+k}{2k+1}\\ \frac{z}{k^2+k}-\log\left(1+\frac{1}{2k+1}\right), & \frac{k^2+k}{2k+1}<z\le k\\ 1-\frac{z}{k+1}-\log\frac{2(k^2+k)}{(2k+1)z}, & k<z\le k+1\\ \log(2k+1)-\log(2k), & k+1<z.\end{cases}$$

Proof. For (i) we take $f$ to be the same function as in the proof of Theorem 4(i). We let

$$g_1(x,y)=\begin{cases}1, & 0<x<k^{-1},(k+2)^{-1}<y<(k+1)^{-1}\\ 0, & \text{otherwise}\end{cases}$$

for the first one and

$$g_2(x,y)=\begin{cases}1, & (k+1)^{-1}<y<k^{-1},0<x<(k+1)^{-1}\\ 0, & \text{otherwise.}\end{cases}$$

Then

$$\int g_1 d\tilde{v}=\int g_2 d\tilde{v}=\frac{\log(2k+1)-\log(2k)}{\log 2}.$$

Therefore we get (i).

(ii) can be proved by a similar argument as the proof of Theorem 4(ii).

As for (iii), we take $\sigma_n^{(k)}$ as an example. Since we are concerned $X_n<1/k$ and $(k+2)^{-1}<Y_n<(k+1)^{-1}$, we have

$$\Theta_n=(1-Y_n)(X_n+Y_n-X_nY_n)^{-1}$$

by Lemma 2. It is easy to see that for the $\Theta_n$ in consideration we have

$$k/2<\Theta_n<k+1.$$

Let

$$f(x,y)=\begin{cases}1, & \frac{1-y}{x+y-xy}<z, x<\frac{1}{k}, \frac{1}{k+2}<y\frac{1}{k+1}\\ 0, & \text{otherwise.}\end{cases}$$

Then $\Sigma_{i=1}^{n} f(X_i, Y_i)$ counts the number of $\Theta_i$, $i \le n$ appears as some $\sigma_j^{(k)}$ and $<z$. The non-zero regions of $f$ with respect to different values of $z$ are as follows:

$$\left\{\frac{1-zx}{1+z-zx}<y<\frac{1}{k+1},\ \frac{k-z}{kz}<x<\frac{1}{k}\right\},\ \text{when } \frac{k}{2}<z\le\frac{k^2+k}{2k+1},$$

$$\left\{\frac{1-zx}{1+z-zx}<y<\frac{1}{k+1},\ \frac{k-z}{kz}<x\le\frac{k+1-z}{(k+1)z}\right\}$$
$$\cup\left\{\frac{1}{k+2}<y<\frac{1}{k+1},\ \frac{k+1-z}{(k+1)z}<x<\frac{1}{k}\right\},\ \text{when } \frac{k^2+k}{2k+1}<z\le k,$$

$$\left\{\frac{1-zx}{1+z-zx}<y<\frac{1}{k+1},\ 0<x\le\frac{k+1-z}{(k+1)z}\right\}$$
$$\cup\left\{\frac{1}{k+2}<y<\frac{1}{k+1},\ \frac{k+1-z}{(k+1)z}<x<\frac{1}{k}\right\},\ \text{when } k<z\le k+1.$$

The proof is completed by calculating several integrals and taking $g$ as $g_1$ in the proof of (i). ■

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