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SPECTRA AND DYNAMICS OF BOUNDED PSEUDO-DIFFERENTIAL OPERATORS

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1. Introduction

Let S^0 be the set of all C^∞ functions σ on $\mathbb{R}^n$ such that, for each multi-index α, there exists a positive constant C_α for which

$$|(D^\alpha\sigma)(\xi)| \le C_\alpha(1+|\xi|)^{-|\alpha|}, \quad \xi \in \mathbb{R}^n.$$

We call any function σ in S^0 a symbol. Let $\sigma \in S^0$. Then we define the pseudo-differential operator T_σ on the Schwartz space $\mathcal{S}$ by

$$(1.1) \qquad (T_\sigma\varphi)(x) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\sigma(\xi)\hat{\varphi}(\xi)d\xi, \quad x \in \mathbb{R}^n,$$

where

$$(1.2) \qquad \hat{\varphi}(\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}\varphi(x)dx, \quad \xi \in \mathbb{R}^n,$$

for all functions φ in $\mathcal{S}$. It is obvious that T_σ maps $\mathcal{S}$ into $\mathcal{S}$. That T_σ can be extended to a bounded linear operator $T_{\sigma p}$ from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$, $1 < p < \infty$, is well-known and, in fact, follows from more general L^p-boundedness results given in, e.g., Theorem 4.1 in Chapter 2 and the bibliograhical notes on page 411 of the book [2] by Kumano-go or Theorem 9.7 in Chapter 9 of the book [7] by Wong. If we let K be the tempered distribution on $\mathbb{R}^n$ given by $\hat{K} = \sigma$, where the Fourier transform $\hat{K}$ of K is taken in the distribution sense, then, by Lemma 9.12 in Chapter 9 of the book [7] by Wong, K is a function on $\mathbb{R}^n - \{0\}$ such that, for each sufficiently large positive integer N, there is a positive constant C_N for which

$$|K(x)| \le C_N|x|^{-N}, \quad x \ne 0.$$

Moreover, for each x in $\mathbb{R}^n$ and each φ in $\mathcal{S}$ vanishing in a neighbourhood of x,

$$(T_\sigma\varphi)(x) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} K(x-y)\varphi(y)dy.$$

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Thus, the pseudo-differential operator T_σ can be considered as a singular integral operator with convolution kernel K.

The aim of this paper is to compute the spectrum and essential spectrum of the bounded linear operator $T_{\sigma p} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$.

We first recall in Section 2 the spectrum and essential spectrum of a bounded linear operator A from a complex Banach space X into X. In Section 3, the spectrum and essential spectrum of the bounded linear operator $T_{\sigma p} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ are computed. Results related to the "spectral invariance", i.e., Theorem 3.2 in this paper, can be found in, e.g., the paper [3] by Leopold and Triebel or the paper [9] by Wong. An application to the asymptotic stability of the zero solution of a semilinear evolution equation modelled by a bounded pseudo-differential operator on $L^p(\mathbb{R}^n)$ is given in Section 4.

2. The Spectrum and Essential Spectrum

Let A be a bounded linear operator from a complex Banach space X into X. We denote the norm in X by $\|\ \|$. Let $\rho(A)$ be the resolvent set of A defined by

$$\rho(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is one to one and onto}\},$$

where $\mathbb{C}$ is the set of all complex numbers and I is the identity operator from X into X. The spectrum $\Sigma(A)$ of A is defined to be the set complement of $\rho(A)$ in $\mathbb{C}$. We denote the adjoint of A by A^t, the range of A by $R(A)$, and the null spaces of A and A^t by $N(A)$ and $N(A^t)$ respectively. We call A a Fredholm operator if $R(A)$ is a closed subspace of X, and the dimensions $\alpha(A)$ and $\beta(A)$ of $N(A)$ and $N(A^t)$ respectively are finite. For any Fredholm operator A, we define the index $i(A)$ of A by

$$i(A) = \alpha(A) - \beta(A).$$

Let $\Phi(A)$ be the set defined by

$$\Phi(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is Fredholm with zero index}\}.$$

Then the essential spectrum $\Sigma_e(A)$ of A, defined by Schechter in [4], is the set complement of $\Phi(A)$ in $\mathbb{C}$. It can be shown that a complex number λ is in the essential spectrum $\Sigma_e(A)$ if there exists a sequence $\{u_k\}$ of elements in X such that $\|u_k\| = 1$, $k = 1, 2, \cdots$, $(A-\lambda I)u_k \to 0$ in X as $k \to \infty$ and $\{u_k\}$ has no convergent subsequence in X.

The notions and results in this section can be found in Section 4 of Chapter 1 of the book [6] by Schechter.

3. The Spectrum of $T_{\sigma p}$

We begin with the following result on the essential spectrum $\Sigma_e(T_{\sigma p})$ of $T_{\sigma p}$.

Proposition 3.1. *Let $\sigma \in S^0$ and let λ be a complex number. If $\sigma(\xi)$ is not bounded away from λ for all ξ in $\mathbb{R}^n$, then $\lambda \in \Sigma_e(T_{\sigma p})$.*

Proof. Let $\{\xi_k\}$ be a sequence of elements in $\mathbb{R}^n$ such that $\sigma(\xi_k) \to \lambda$ as $k \to \infty$. Let k_0 be the smallest positive integer greater than n/p and let N be any fixed integer greater than $k_0 - n/p$. Let $\{\varepsilon_k\}$ be a sequence of positive numbers such that

$$\varepsilon_k^{|\mu|}\sigma^{(\mu)}(\xi_k) \to 0, \quad 1 < |\mu| < N, \tag{3.1}$$

as $k \to \infty$, where

$$\sigma^{(\mu)}(\xi) = (\partial^\mu \sigma)(\xi), \quad \xi \in \mathbb{R}^n.$$

Let $\psi \in C_0^\infty(\mathbb{R}^n)$ be such that $\psi(x) = 0$ for all x in a neighbourhood of the origin and

$$||\psi||_p = 1. \tag{3.2}$$

For $k = 1, 2, \cdots$, let φ_k be the function on $\mathbb{R}^n$ defined by

$$\varphi_k(x) = \varepsilon_k^{n/p}\psi(\varepsilon_k x)e^{i\xi_k \cdot x}, \quad x \in \mathbb{R}^n. \tag{3.3}$$

Then, for $k = 1, 2, \cdots$, $\varphi_k \in C_0^\infty(\mathbb{R}^n)$ and an easy computation gives

$$((T_{\sigma p} - \lambda I)\varphi_k)(x) = e^{ix\cdot\xi_k}\varepsilon_k^{n/p}(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\varepsilon_k\zeta}\{\sigma(\varepsilon_k\zeta + \xi_k) - \lambda\}\hat{\psi}(\zeta)d\zeta$$

for all x in $\mathbb{R}^n$. By Taylor's formula with integral remainder, we get, for $k = 1, 2, \cdots$,

$$\begin{aligned} &((T_{\sigma p} - \lambda I)\varphi_k)(x) \\ &= e^{ix\cdot\xi_k}\varepsilon_k^{n/p}(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\varepsilon_k\zeta}\left\{\sum_{|\mu|<N}\frac{\sigma^{(\mu)}(\xi_k)}{\mu!}\varepsilon_k^{|\mu|}\zeta^\mu - \lambda\right\}\hat{\psi}(\zeta)d\zeta \\ &\quad + e^{ix\cdot\xi_k}\varepsilon_k^{n/p}(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\varepsilon_k\zeta}R_k(\zeta)\hat{\psi}(\zeta)d\zeta, \quad x \in \mathbb{R}^n, \end{aligned} \tag{3.4}$$

where

$$R_k(\zeta) = N\sum_{|\mu|=N}\frac{\varepsilon_k^N\zeta^\mu}{\mu!}\int_0^1 (1-\theta)^{N-1}\sigma^{(\mu)}(\xi_k + \theta\varepsilon_k\zeta)d\theta, \quad \zeta \in \mathbb{R}^n. \tag{3.5}$$

The first term on the right hand side of (3.4) is equal to $T_k(x)$, where

$$\begin{aligned}(3.6)\qquad T_k(x) &= e^{ix\cdot\xi_k}\varepsilon_k^{n/p}\{\sigma(\xi_k)-\lambda\}\psi(\varepsilon_k x)\\ &\quad + e^{ix\cdot\xi_k}\sum_{1<|\mu|<N}\frac{\varepsilon_k^{\frac{n}{p}+|\mu|}}{\mu!}\sigma^{(\mu)}(\xi_k)(D^\mu\psi)(\varepsilon_k x).\end{aligned}$$

The second term on the right hand side of (3.4) is equal to $\sum_{|\mu|=N}T_k^\mu(x)$, where

$$(3.7)\qquad T_k^\mu(x)=\frac{\varepsilon_k^{\frac{n}{p}+N}e^{ix\cdot\xi_k}}{\mu!(2\pi)^{n/2}}N\int_{\mathbb{R}^n}e^{ix\cdot\varepsilon_k\zeta}I_k^\mu(\zeta)\zeta^\mu\hat{\psi}(\zeta)d\zeta$$

and

$$(3.8)\qquad I_k^\mu(\zeta)=\int_0^1(1-\theta)^{N-1}\sigma^{(\mu)}(\xi_k+\theta\varepsilon_k\zeta)d\theta.$$

Since $\sigma\in S^0$, it follows from (3.7) and (3.8) that, for each multi-index μ with $|\mu|=N$, there exists a positive constant C_μ such that

$$|T_k^\mu(x)|\le\frac{\varepsilon_k^{\frac{n}{p}+N}}{\mu!(2\pi)^{n/2}}NC_\mu\int_{\mathbb{R}^n}|\zeta^\mu\hat{\psi}(\zeta)|d\zeta,\quad x\in\mathbb{R}^n.$$

Hence, for each multi-index μ with $|\mu|=N$,

$$(3.9)\qquad T_k^\mu\to 0$$

uniformly on $\mathbb{R}^n$ as $k\to\infty$. Let α be a multi-index with $|\alpha|=k_0$. Then, by (3.7), an integration by parts and Leibnitz' formula,

$$\begin{aligned}(3.10)\qquad |x^\alpha T_k^\mu(x)| &\le \frac{\varepsilon_k^{\frac{n}{p}+N-k_0}}{\mu!(2\pi)^{n/2}}N\int_{\mathbb{R}^n}|\partial_\zeta^\alpha\{I_k^\mu(\zeta)\zeta^\mu\hat{\psi}(\zeta)\}|d\zeta\\ &= \frac{\varepsilon_k^{\frac{n}{p}+N-k_0}}{\mu!(2\pi)^{n/2}}N\int_{\mathbb{R}^n}\sum_{\gamma\le\alpha}\binom{\alpha}{\gamma}|(\partial^\gamma I_k^\mu)(\zeta)\partial_\zeta^{\alpha-\gamma}\{\zeta^\mu\hat{\psi}(\zeta)\}|d\zeta\end{aligned}$$

for all x in $\mathbb{R}^n$. Using (3.8) and the fact that $\sigma\in S^0$, we can find, for each multi-index μ with $|\mu|=N$ and each multi-index γ with $\gamma\le\mu$, a positive constant $C_{\mu\gamma}$ such that

$$(3.11)\qquad |(\partial^\gamma I_k^\mu)(\zeta)|\le C_{\mu\gamma},\quad k=1,2,\cdots,$$

for all ζ in $\mathbb{R}^n$. So, by (3.10) and (3.11), we get a positive constant $C_{\alpha\mu}$ such that

$$(3.12)\qquad |x^\alpha T_k^\mu(x)|\le C_{\alpha\mu}\varepsilon_k^{\frac{n}{p}+N-k_0},\quad k=1,2,\cdots,$$

for all x in $\mathbb{R}^n$. Thus, using the elementary inequality on page 53 of the book [7] by Wong and (3.12), we get, for each multi-index μ with $|\mu| = N$, a positive constant $C_{\mu k_0}$ such that

$$|x|^{k_0}|T_k^{\mu}(x)| \leq C_{\mu k_0}\varepsilon_k^{\frac{n}{p}+N-k_0}, \quad k = 1, 2, \cdots, \tag{3.13}$$

for all x in $\mathbb{R}^n$. So, by (3.9), (3.13), the fact that $k_0 p > n$ and the Lebesgue dominated convergence theorem,

$$||T_k^{\mu}||_p \to 0 \tag{3.14}$$

as $k \to \infty$ for each multi-index μ with $|\mu| = N$. Thus, by (3.4), (3.6), (3.7) and (3.14),

$$||(T_{\sigma p} - \lambda I)\varphi_k||_p \to 0$$

as $k \to \infty$. Obviously, by (3.2) and (3.3), $||\varphi_k||_p = 1$, $k = 1, 2, \cdots$, and hence it remains to prove that $\{\varphi_k\}$ has no convergent subsequence in $L^p(\mathbb{R}^n)$. But this follows from the argument on page 64 of the book [6] by Schechter. □

The precise description of the spectrum of $T_{\sigma p}$ is provided by the following theorem.

Theorem 3.2. *Let $\sigma \in S^0$. Then*

$$\Sigma(T_{\sigma p}) = \Sigma_e(T_{\sigma p}) = \{\sigma(\xi) : \xi \in \mathbb{R}^n\}^c,$$

where $\{\cdots\}^c$ denotes the closure in $\mathbb{C}$ of the set $\{\cdots\}$.

Theorem 3.2 is an immediate consequence of Proposition 3.1 and the following proposition.

Proposition 3.3. *Let λ be a complex number such that $\sigma(\xi)$ is bounded away from λ for all ξ in $\mathbb{R}^n$. Then $\lambda \in \rho(T_{\sigma p})$.*

Proof. Let τ be the function on $\mathbb{R}^n$ defined by

$$\tau(\xi) = \frac{1}{m(\xi)}, \quad \xi \in \mathbb{R}^n, \tag{3.15}$$

where

$$m(\xi) = \sigma(\xi) - \lambda, \quad \xi \in \mathbb{R}^n. \tag{3.16}$$

Then, for all multi-indices α,

$$(3.17)\qquad (\partial^\alpha \tau)(\xi) = \sum C_{\alpha^{(1)},\cdots,\alpha^{(k)}} \frac{(\partial^{\alpha^{(1)}} m)(\xi)\cdots(\partial^{\alpha^{(k)}} m)(\xi)}{m(\xi)^{k+1}}, \quad \xi \in \mathbb{R}^n,$$

where the summation is taken over all partitions $\alpha^{(1)}, \cdots, \alpha^{(k)}$ of α and $C_{\alpha^{(1)},\cdots,\alpha^{(k)}}$ is a constant depending on the partition $\alpha^{(1)}, \cdots, \alpha^{(k)}$ of α. Since $\sigma \in S^0$, it follows from (3.15), (3.16) and (3.17) that there exist positive constants $C_{\alpha^{(1)}}, \cdots, C_{\alpha^{(k)}}$ and C such that

$$(3.18)\qquad |(\partial^\alpha \tau)(\xi)| \le \sum |C_{\alpha^{(1)},\cdots,\alpha^{(k)}}| \frac{C_{\alpha^{(1)}}\cdots C_{\alpha^{(k)}}(1+|\xi|)^{-|\alpha|}}{C^{k+1}}, \quad \xi \in \mathbb{R}^n.$$

Thus, by (3.18), $\tau \in S^0$. So, by Theorem 9.7 in Chapter 9 of the book [7] by Wong, T_τ, initially defined on $\mathcal{S}$, can be extended to a unique bounded linear operator $T_{\tau p}$ from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$. Now, let $f \in L^p(\mathbb{R}^n)$. Then there exists a sequence $\{\varphi_k\}$ of functions in $\mathcal{S}$ such that

$$(3.19)\qquad \varphi_k \to f$$

in $L^p(\mathbb{R}^n)$ as $k \to \infty$. Thus,

$$T_\tau \varphi_k \to T_{\tau p} f$$

in $L^p(\mathbb{R}^n)$ as $k \to \infty$. So,

$$(3.20)\qquad T_{\sigma-\lambda} T_\tau \varphi_k \to (T_{\sigma p} - \lambda I) T_{\tau p} f$$

in $L^p(\mathbb{R}^n)$ as $k \to \infty$. But, by (3.15) and (3.16),

$$(3.21)\qquad \varphi_k = T_{\sigma-\lambda} T_\tau \varphi_k, \quad k = 1, 2, \cdots.$$

Thus, by (3.19), (3.20), (3.21) and letting $k \to \infty$, we get

$$f = (T_{\sigma p} - \lambda I) T_{\tau p} f$$

and hence

$$(3.22)\qquad (T_{\sigma p} - \lambda I) T_{\tau p} = I.$$

Similarly,

$$(3.23)\qquad T_{\tau p}(T_{\sigma p} - \lambda I) = I.$$

So, by (3.22) and (3.23), $\lambda \in \rho(T_{\sigma p})$ and the proof is complete. □

REMARK. It is important to note that if $\sigma \in S^0$, then the spectrum of $T_{\sigma p}$ is, in general, equal to $\{\sigma(\xi) : \xi \in \mathbb{R}^n\}^c$, but not $\{\sigma(\xi) : \xi \in \mathbb{R}^n\}$. To wit, let σ be the function on $\mathbb{R}^2$ defined by

$$\sigma(\xi_1, \xi_2) = \frac{\xi_1 + i(\xi_1\xi_2 + 1)}{1 + \xi_1^2 + \xi_2^2}, \quad \xi_1, \xi_2 \in \mathbb{R}.$$

Then it is easy to prove that $\sigma \in S^0$. At any rate, it follows from Example 4.3, Example 4.4 and Exercise 4.1 in Chapter 4 of the book [7] by Wong. It is obvious that zero is not in the set $\{\sigma(\xi_1, \xi_2) : \ \xi_1, \xi_2 \in \mathbb{R}\}$ and $\sigma(1/k, -k) \to 0$ as $k \to \infty$. Thus, $0 \in \{\sigma(\xi_1, \xi_2) : \ \xi_1, \xi_2 \in \mathbb{R}\}^c$.

4. A Semilinear Evolution Equation

In this section, we consider the dynamical system modelled by the semilinear evolution pseudo-differential equation

$$u'(t) + (T_{\sigma p} - \lambda I)\{u(t)\} = f\{u(t)\}, \quad t \geq 0, \tag{4.1}$$

where $\sigma \in S^0$, u is a function from $[0, \infty)$ into $L^p(\mathbb{R}^n)$, λ is a real constant and f is a continuous mapping from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ such that

$$||f(u)||_p = o(||u||_p) \tag{4.2}$$

as $u \to 0$ in $L^p(\mathbb{R}^n)$. In (4.1), the derivative $u'(t)$, at any time t, is understood to be the strong limit in $L^p(\mathbb{R}^n)$, if it exists, of the difference quotient

$$\frac{u(t+h) - u(t)}{h}$$

as $h \to 0$. It is clear that, in view of (4.2), $u(t) \equiv 0$ is an equilibrium solution of (4.1). The equilibrium solution $u(t) \equiv 0$ of (4.1) is said to be asymptotically stable if, for any positive number ε, there exists a positive constant δ such that any solution $u(t)$, $t \geq 0$, of (4.1) with $||u(0)||_p < \delta$ satisfies

$$||u(t)||_p < \varepsilon, \quad t \geq 0,$$

and

$$\lim_{t \to \infty} ||u(t)||_p = 0.$$

Theorem 4.1. *Let $\sigma \in S^0$ and f be a continuous mapping from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ satisfying* (4.2). *Then, for any real number λ with $\lambda < \mu$, where*

$$\mu = \inf_{\xi \in \mathbb{R}^n} \{\text{Re } \sigma(\xi)\},$$

the equilibrium solution $u(t) \equiv 0$ *of* (4.1) *is asymptotically stable.*

Proof. By Theorem 3.2, the spectrum of $T_{\sigma p}$ is equal to

$$\{\sigma(\xi) : \xi \in \mathbb{R}^n\}^c.$$

Hence, by Theorem 3.8 in Chapter 3 of the book [5] by Schechter, i.e., the spectral mapping theorem, the spectrum of the bounded linear operator $e^{-T_{\sigma p}t} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, $t \geq 0$, is equal to

$$\{e^{-\sigma(\xi)t} : \xi \in \mathbb{R}^n\}^c.$$

Thus, the spectral radius of $e^{-T_{\sigma p}t} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, $t \geq 0$, is equal to $e^{-\mu t}$. Therefore, by a standard result in the theory of semigroups, i.e., Theorem 1.22 on page 15 of the book [1] by Davies,

$$\lim_{t\to\infty} \frac{1}{t} \ln ||e^{-T_{\sigma p}t}||_* = -\mu,$$

where $|| \ ||_*$ is the norm in the Banach algebra of all bounded linear operators from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$. Hence, for any number ε in $(0, \mu - \lambda)$, we can find a positive number t_ε such that

$$\frac{1}{t} \ln ||e^{-T_{\sigma p}t}||_* < -\mu + \varepsilon, \quad t > t_\varepsilon.$$

Thus,

$$||e^{-T_{\sigma p}t}||_* < e^{-(\mu-\varepsilon)t}, \quad t > t_\varepsilon,$$

and consequently,

$$||e^{-(T_{\sigma p}-\lambda I)t}||_* < e^{-(\mu-\lambda-\varepsilon)t}, \quad t > t_\varepsilon.$$

Therefore there exists a positive constant M such that

$$||e^{-(T_{\sigma p}-\lambda I)t}||_* < Me^{-(\mu-\lambda-\varepsilon)t}, \quad t \geq 0.$$

So, by Theorem 2.1 in the paper [8] by Wong, the equilibrium solution $u(t) \equiv 0$ of (4.1) is asymptotically stable. □

REMARK. The technique used in the proof of Theorem 4.1 is similar to that used in the proof of Theorem 1.1 in the paper [10] by Wong.

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