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SPECTRA AND DYNAMICS OF BOUNDED PSEUDO-DIFFERENTIAL OPERATORS

MAN WAH WONG

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1. Introduction

Let $S^0$ be the set of all $C^\infty$ functions $\sigma$ on $\mathbb{R}^n$ such that, for each multi-index $\alpha$, there exists a positive constant $C_\alpha$ for which

$$|(D^\alpha \sigma)(\xi)| \leq C_\alpha (1 + |\xi|)^{-|\alpha|}, \quad \xi \in \mathbb{R}^n.$$ 

We call any function $\sigma$ in $S^0$ a symbol. Let $\sigma \in S^0$. Then we define the pseudo-differential operator $T_\sigma$ on the Schwartz space $S$ by

$$\text{(1.1)} \quad (T_\sigma \varphi)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \sigma(\xi) \hat{\varphi}(\xi) d\xi, \quad x \in \mathbb{R}^n,$$

where

$$\text{(1.2)} \quad \hat{\varphi}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \varphi(x) dx, \quad \xi \in \mathbb{R}^n,$$

for all functions $\varphi$ in $S$. It is obvious that $T_\sigma$ maps $S$ into $S$. That $T_\sigma$ can be extended to a bounded linear operator $T_{\sigma p}$ from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$, $1 < p < \infty$, is well-known and, in fact, follows from more general $L^p$-boundedness results given in, e.g., Theorem 4.1 in Chapter 2 and the bibliographical notes on page 411 of the book [2] by Kumano-go or Theorem 9.7 in Chapter 9 of the book [7] by Wong. If we let $K$ be the tempered distribution on $\mathbb{R}^n$ given by $\hat{K} = \sigma$, where the Fourier transform $\hat{K}$ of $K$ is taken in the distribution sense, then, by Lemma 9.12 in Chapter 9 of the book [7] by Wong, $K$ is a function on $\mathbb{R}^n - \{0\}$ such that, for each sufficiently large positive integer $N$, there is a positive constant $C_N$ for which

$$|K(x)| \leq C_N |x|^{-N}, \quad x \neq 0.$$ 

Moreover, for each $x$ in $\mathbb{R}^n$ and each $\varphi$ in $S$ vanishing in a neighbourhood of $x$,

$$\text{(1.3)} \quad (T_\sigma \varphi)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} K(x - y) \varphi(y) dy.$$ 

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Thus, the pseudo-differential operator $T_\sigma$ can be considered as a singular integral operator with convolution kernel $K$.

The aim of this paper is to compute the spectrum and essential spectrum of the bounded linear operator $T_{\sigma p} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$.

We first recall in Section 2 the spectrum and essential spectrum of a bounded linear operator $A$ from a complex Banach space $X$ into $X$. In Section 3, the spectrum and essential spectrum of the bounded linear operator $T_{\sigma p} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ are computed. Results related to the "spectral invariance", i.e., Theorem 3.2 in this paper, can be found in, e.g., the paper [3] by Leopold and Triebel or the paper [9] by Wong. An application to the asymptotic stability of the zero solution of a semilinear evolution equation modelled by a bounded pseudo-differential operator on $L^p(\mathbb{R}^n)$ is given in Section 4.

2. The Spectrum and Essential Spectrum

Let $A$ be a bounded linear operator from a complex Banach space $X$ into $X$. We denote the norm in $X$ by $|| \cdot ||$. Let $\rho(A)$ be the resolvent set of $A$ defined by

$$\rho(A) = \{ \lambda \in \mathbb{C} : A - \lambda I \text{ is one to one and onto} \},$$

where $\mathbb{C}$ is the set of all complex numbers and $I$ is the identity operator from $X$ into $X$. The spectrum $\Sigma(A)$ of $A$ is defined to be the set complement of $\rho(A)$ in $\mathbb{C}$. We denote the adjoint of $A$ by $A^t$, the range of $A$ by $R(A)$, and the null spaces of $A$ and $A^t$ by $N(A)$ and $N(A^t)$ respectively. We call $A$ a Fredholm operator if $R(A)$ is a closed subspace of $X$, and the dimensions $\alpha(A)$ and $\beta(A)$ of $N(A)$ and $N(A^t)$ respectively are finite. For any Fredholm operator $A$, we define the index $i(A)$ of $A$ by

$$i(A) = \alpha(A) - \beta(A).$$

Let $\Phi(A)$ be the set defined by

$$\Phi(A) = \{ \lambda \in \mathbb{C} : A - \lambda I \text{ is Fredholm with zero index} \}.$$

Then the essential spectrum $\Sigma_e(A)$ of $A$, defined by Schechter in [4], is the set complement of $\Phi(A)$ in $\mathbb{C}$. It can be shown that a complex number $\lambda$ is in the essential spectrum $\Sigma_e(A)$ if there exists a sequence $\{u_k\}$ of elements in $X$ such that $||u_k|| = 1, k = 1, 2, \cdots, (A - \lambda I)u_k \to 0$ in $X$ as $k \to \infty$ and $\{u_k\}$ has no convergent subsequence in $X$.

The notions and results in this section can be found in Section 4 of Chapter 1 of the book [6] by Schechter.

3. The Spectrum of $T_{\sigma p}$

We begin with the following result on the essential spectrum $\Sigma_e(T_{\sigma p})$ of $T_{\sigma p}$. 


Proposition 3.1. Let $\sigma \in S^0$ and let $\lambda$ be a complex number. If $\sigma(\xi)$ is not bounded away from $\lambda$ for all $\xi$ in $\mathbb{R}^n$, then $\lambda \in \Sigma_\epsilon(T_{\sigma^p})$.

Proof. Let $\{\xi_k\}$ be a sequence of elements in $\mathbb{R}^n$ such that $\sigma(\xi_k) \to \lambda$ as $k \to \infty$. Let $k_0$ be the smallest positive integer greater than $n/p$ and let $N$ be any fixed integer greater than $k_0 - n/p$. Let $\{\epsilon_k\}$ be a sequence of positive numbers such that

$$e^{\epsilon_k|x|} \to 0, \quad 1 < |\mu| < N,$$

as $k \to \infty$, where

$$\sigma^{(\mu)}(\xi) = (\partial^\mu \sigma)(\xi), \quad \xi \in \mathbb{R}^n.$$

Let $\psi \in C_c^\infty(\mathbb{R}^n)$ be such that $\psi(x) = 0$ for all $x$ in a neighbourhood of the origin and

$$||\psi||_p = 1.$$ 

For $k = 1, 2, \cdots$, let $\varphi_k$ be the function on $\mathbb{R}^n$ defined by

$$\varphi_k(x) = \epsilon_k^{n/p} \psi(\epsilon_k x) e^{i \xi_k \cdot x}, \quad x \in \mathbb{R}^n.$$ 

Then, for $k = 1, 2, \cdots, \varphi_k \in C_c^\infty(\mathbb{R}^n)$ and an easy computation gives

$$(T_{\sigma^p} - \lambda I)\varphi_k(x) = e^{i x \cdot \xi_k} \epsilon_k^{n/p} (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i x \cdot \zeta} \{\sigma(\epsilon_k \zeta + \xi_k) - \lambda\} \hat{\psi}(\zeta) d\zeta$$

for all $x$ in $\mathbb{R}^n$. By Taylor's formula with integral remainder, we get, for $k = 1, 2, \cdots$,

$$(T_{\sigma^p} - \lambda I)\varphi_k(x)$$

$$= e^{i x \cdot \xi_k} \epsilon_k^{n/p} (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i x \cdot \zeta} \left\{ \sum_{|\mu| < N} \frac{\sigma^{(\mu)}(\xi_k)}{\mu!} \epsilon_k^{|\mu|} \zeta^\mu - \lambda \right\} \hat{\psi}(\zeta) d\zeta$$

$$+ e^{i x \cdot \xi_k} \epsilon_k^{n/p} (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i x \cdot \zeta} R_k(\zeta) \hat{\psi}(\zeta) d\zeta, \quad x \in \mathbb{R}^n,$$

where

$$R_k(\zeta) = N \sum_{|\mu| = N} \frac{\epsilon_k^{N} \zeta^\mu}{\mu!} \int_0^1 (1 - \theta)^{N - 1} \sigma^{(\mu)}(\xi_k + \theta \epsilon_k \zeta) d\theta, \quad \zeta \in \mathbb{R}^n.$$
The first term on the right hand side of (3.4) is equal to $T_k(x)$, where

$$T_k(x) = e^{i x \cdot \xi_k} e^{n/\mu} \sigma(\xi_k) - \lambda \} \psi(\xi_k x) + e^{i x \cdot \xi_k} \sum_{1 < |\mu| < N} \frac{\epsilon_k^{n + |\mu|} \sigma(\mu)(\xi_k)(D^\mu \psi)(\xi_k x)}{\mu!}.$$  

The second term on the right hand side of (3.4) is equal to $\sum_{|\mu| = N} T_k^\mu(x)$, where

$$T_k^\mu(x) = \frac{\epsilon_k^{n + N}}{\mu!(2\pi)^{n/2} N} \int_{\mathbb{R}^n} e^{i x \cdot \xi_k} I_k^\mu(\zeta) \zeta^\mu \hat{\psi}(\zeta) d\zeta$$

and

$$I_k^\mu(\zeta) = \int_0^1 (1 - \theta)^{N - 1} \sigma(\mu)(\xi_k + \theta \xi_k \zeta) d\theta.$$  

Since $\sigma \in S^0$, it follows from (3.7) and (3.8) that, for each multi-index $\mu$ with $|\mu| = N$, there exists a positive constant $C_\mu$ such that

$$|T_k^\mu(x)| \leq \frac{\epsilon_k^{n + N}}{\mu!(2\pi)^{n/2} N} C_\mu \int_{\mathbb{R}^n} |\zeta^\mu \hat{\psi}(\zeta)| d\zeta, \quad x \in \mathbb{R}^n.$$  

Hence, for each multi-index $\mu$ with $|\mu| = N$,

$$T_k^\mu \to 0$$

uniformly on $\mathbb{R}^n$ as $k \to \infty$. Let $\alpha$ be a multi-index with $|\alpha| = k_0$. Then, by (3.7), an integration by parts and Leibnitz' formula,

$$|x^\alpha T_k^\mu(x)| \leq \frac{\epsilon_k^{n + N - k_0}}{\mu!(2\pi)^{n/2} N} \int_{\mathbb{R}^n} |\partial_\zeta^{\alpha} \{I_k^\mu(\zeta) \zeta^\mu \hat{\psi}(\zeta)\}| d\zeta$$

$$= \frac{\epsilon_k^{n + N - k_0}}{\mu!(2\pi)^{n/2} N} \int_{\mathbb{R}^n} \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} |(\partial_\zeta^{\alpha - \gamma} I_k^\mu)(\zeta) \partial_\zeta^{\gamma} \{\zeta^\mu \hat{\psi}(\zeta)\}| d\zeta$$

for all $x \in \mathbb{R}^n$. Using (3.8) and the fact that $\sigma \in S^0$, we can find, for each multi-index $\mu$ with $|\mu| = N$ and each multi-index $\gamma$ with $\gamma \leq \mu$, a positive constant $C_{\mu, \gamma}$ such that

$$|(\partial_\zeta^{\gamma} I_k^\mu)(\zeta)| \leq C_{\mu, \gamma}, \quad k = 1, 2, \cdots,$$

for all $\zeta \in \mathbb{R}^n$. So, by (3.10) and (3.11), we get a positive constant $C_{\alpha, \mu}$ such that

$$|x^\alpha T_k^\mu(x)| \leq C_{\alpha, \mu} \epsilon_k^{n + N - k_0}, \quad k = 1, 2, \cdots,$$
for all \( x \) in \( \mathbb{R}^n \). Thus, using the elementary inequality on page 53 of the book [7] by Wong and (3.12), we get, for each multi-index \( \mu \) with \( |\mu| = N \), a positive constant \( C_{\mu k_0} \) such that

\[
|x|^{k_0}|T_k^\mu(x)| \leq C_{\mu k_0} \varepsilon_k^{n+N-k_0}, \quad k = 1, 2, \ldots,
\]

for all \( x \) in \( \mathbb{R}^n \). So, by (3.9), (3.13), the fact that \( k_0 p > n \) and the Lebesgue dominated convergence theorem,

\[
||T_k^\mu||_p \to 0
\]

as \( k \to \infty \) for each multi-index \( \mu \) with \( |\mu| = N \). Thus, by (3.4), (3.6), (3.7) and (3.14),

\[
||(T_{\sigma p} - \lambda I)\varphi_k||_p \to 0
\]

as \( k \to \infty \). Obviously, by (3.2) and (3.3), \( ||\varphi_k||_p = 1, k = 1, 2, \ldots, \) and hence it remains to prove that \( \{\varphi_k\} \) has no convergent subsequence in \( L^p(\mathbb{R}^n) \). But this follows from the argument on page 64 of the book [6] by Schechter. \( \square \)

The precise description of the spectrum of \( T_{\sigma p} \) is provided by the following theorem.

**Theorem 3.2.** Let \( \sigma \in S^0 \). Then

\[
\Sigma(T_{\sigma p}) = \Sigma_\sigma(T_{\sigma p}) = \{\sigma(\xi) : \xi \in \mathbb{R}^n\}^c,
\]

where \( \cdots^c \) denotes the closure in \( \mathbb{C} \) of the set \( \cdots \).

Theorem 3.2 is an immediate consequence of Proposition 3.1 and the following proposition.

**Proposition 3.3.** Let \( \lambda \) be a complex number such that \( \sigma(\xi) \) is bounded away from \( \lambda \) for all \( \xi \) in \( \mathbb{R}^n \). Then \( \lambda \in \rho(T_{\sigma p}) \).

**Proof.** Let \( \tau \) be the function on \( \mathbb{R}^n \) defined by

\[
\tau(\xi) = \frac{1}{m(\xi)}, \quad \xi \in \mathbb{R}^n,
\]

where

\[
m(\xi) = \sigma(\xi) - \lambda, \quad \xi \in \mathbb{R}^n.
\]
Then, for all multi-indices \( \alpha \),

\[
(\partial^\alpha \tau)(\xi) = \sum C_{\alpha(1), \ldots, \alpha(k)} \frac{\partial^{\alpha(1)} m(\xi) \cdots \partial^{\alpha(k)} m(\xi)}{m(\xi)^{k+1}}, \quad \xi \in \mathbb{R}^n,
\]

where the summation is taken over all partitions \( \alpha^{(1)}, \ldots, \alpha^{(k)} \) of \( \alpha \) and \( C_{\alpha^{(1)}, \ldots, \alpha^{(k)}} \) is a constant depending on the partition \( \alpha^{(1)}, \ldots, \alpha^{(k)} \) of \( \alpha \). Since \( \sigma \in S^0 \), it follows from (3.15), (3.16) and (3.17) that there exist positive constants \( C_{\alpha^{(1)}}, \ldots, C_{\alpha^{(k)}} \) and \( C \) such that

\[
|[(\partial^\alpha \tau)(\xi)]| \leq \sum |C_{\alpha^{(1)}, \ldots, \alpha^{(k)}}| \frac{C_{\alpha^{(1)}} \cdots C_{\alpha^{(k)}} (1 + |\xi|)^{-|\alpha|}}{C^{k+1}}, \quad \xi \in \mathbb{R}^n.
\]

Thus, by (3.18), \( \tau \in S^0 \). So, by Theorem 9.7 in Chapter 9 of the book [7] by Wong, \( T_\tau \), initially defined on \( S \), can be extended to a unique bounded linear operator \( T_{\tau p} \) from \( L^p(\mathbb{R}^n) \) into \( L^p(\mathbb{R}^n) \). Now, let \( f \in L^p(\mathbb{R}^n) \). Then there exists a sequence \( \{\varphi_k\} \) of functions in \( S \) such that

\[
\varphi_k \rightarrow f \quad \text{in } L^p(\mathbb{R}^n) \text{ as } k \rightarrow \infty.
\]

Thus,

\[
T_\tau \varphi_k \rightarrow T_{\tau p} f
\]

in \( L^p(\mathbb{R}^n) \) as \( k \rightarrow \infty \). So,

\[
T_{\sigma - \lambda} T_\tau \varphi_k \rightarrow (T_{\sigma p} - \lambda I)T_{\tau p} f
\]

in \( L^p(\mathbb{R}^n) \) as \( k \rightarrow \infty \). But, by (3.15) and (3.16),

\[
\varphi_k = T_{\sigma - \lambda} T_\tau \varphi_k, \quad k = 1, 2, \ldots.
\]

Thus, by (3.19), (3.20), (3.21) and letting \( k \rightarrow \infty \), we get

\[
f = (T_{\sigma p} - \lambda I)T_{\tau p} f
\]

and hence

\[
(T_{\sigma p} - \lambda I)T_{\tau p} = I.
\]

Similarly,

\[
T_{\tau p}(T_{\sigma p} - \lambda I) = I.
\]

So, by (3.22) and (3.23), \( \lambda \in \rho(T_{\sigma p}) \) and the proof is complete. \( \square \)
Remark. It is important to note that if $\sigma \in S^0$, then the spectrum of $T_{\sigma p}$ is, in general, equal to $\{\sigma(\xi) : \xi \in \mathbb{R}^n\}_c$, but not $\{\sigma(\xi) : \xi \in \mathbb{R}^n\}$. To wit, let $\sigma$ be the function on $\mathbb{R}^2$ defined by

$$\sigma(\xi_1, \xi_2) = \frac{\xi_1 + i(\xi_1\xi_2 + 1)}{1 + \xi_1^2 + \xi_2^2}, \quad \xi_1, \xi_2 \in \mathbb{R}.$$ 

Then it is easy to prove that $\sigma \in S^0$. At any rate, it follows from Example 4.3, Example 4.4 and Exercise 4.1 in Chapter 4 of the book [7] by Wong. It is obvious that zero is not in the set $\{\sigma(\xi_1, \xi_2) : \xi_1, \xi_2 \in \mathbb{R}\}$ and $\sigma(1/k, -k) \to 0$ as $k \to \infty$. Thus, $0 \in \{\sigma(\xi_1, \xi_2) : \xi_1, \xi_2 \in \mathbb{R}\}_c$.

4. A Semilinear Evolution Equation

In this section, we consider the dynamical system modelled by the semilinear evolution pseudo-differential equation

$$u'(t) + (T_{\sigma p} - \lambda I)\{u(t)\} = f\{u(t)\}, \quad t \geq 0,$$

where $\sigma \in S^0$, $u$ is a function from $[0, \infty)$ into $L^p(\mathbb{R}^n)$, $\lambda$ is a real constant and $f$ is a continuous mapping from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ such that

$$||f(u)||_p = o(||u||_p)$$

as $u \to 0$ in $L^p(\mathbb{R}^n)$. In (4.1), the derivative $u'(t)$, at any time $t$, is understood to be the strong limit in $L^p(\mathbb{R}^n)$, if it exists, of the difference quotient

$$\frac{u(t + h) - u(t)}{h}$$

as $h \to 0$. It is clear that, in view of (4.2), $u(t) \equiv 0$ is an equilibrium solution of (4.1). The equilibrium solution $u(t) \equiv 0$ of (4.1) is said to be asymptotically stable if, for any positive number $\varepsilon$, there exists a positive constant $\delta$ such that any solution $u(t), t \geq 0$, of (4.1) with $||u(0)||_p < \delta$ satisfies

$$||u(t)||_p < \varepsilon, \quad t \geq 0,$$

and

$$\lim_{t \to \infty} ||u(t)||_p = 0.$$

**Theorem 4.1.** Let $\sigma \in S^0$ and $f$ be a continuous mapping from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ satisfying (4.2). Then, for any real number $\lambda$ with $\lambda < \mu$, where

$$\mu = \inf_{\xi \in \mathbb{R}^n} \{\text{Re} \sigma(\xi)\},$$
the equilibrium solution \( u(t) \equiv 0 \) of (4.1) is asymptotically stable.

Proof. By Theorem 3.2, the spectrum of \( T_{\sigma_p} \) is equal to

\[ \{ \sigma(\xi) : \xi \in \mathbb{R}^n \}^c. \]

Hence, by Theorem 3.8 in Chapter 3 of the book [5] by Schechter, i.e., the spectral mapping theorem, the spectrum of the bounded linear operator \( e^{-T_{\sigma_p}t} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n), t \geq 0 \), is equal to

\[ \{ e^{-\sigma(\xi)t} : \xi \in \mathbb{R}^n \}^c. \]

Thus, the spectral radius of \( e^{-T_{\sigma_p}t} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n), t \geq 0 \), is equal to \( e^{-\mu t} \). Therefore, by a standard result in the theory of semigroups, i.e., Theorem 1.22 on page 15 of the book [1] by Davies,

\[ \lim_{t \to \infty} \frac{1}{t} \ln ||e^{-T_{\sigma_p}t}||_* = -\mu, \]

where || ||_* is the norm in the Banach algebra of all bounded linear operators from \( L^p(\mathbb{R}^n) \) into \( L^p(\mathbb{R}^n) \). Hence, for any number \( \varepsilon \) in \( (0, \mu - \lambda) \), we can find a positive number \( t_\varepsilon \) such that

\[ \frac{1}{t} \ln ||e^{-T_{\sigma_p}t}||_* < -\mu + \varepsilon, \quad t > t_\varepsilon. \]

Thus,

\[ ||e^{-T_{\sigma_p}t}||_* < e^{-(\mu - \varepsilon)t}, \quad t > t_\varepsilon, \]

and consequently,

\[ ||e^{-(T_{\sigma_p} - \lambda I)t}||_* < e^{-(\mu - \lambda - \varepsilon)t}, \quad t > t_\varepsilon. \]

Therefore there exists a positive constant \( M \) such that

\[ ||e^{-(T_{\sigma_p} - \lambda I)t}||_* < Me^{-(\mu - \lambda - \varepsilon)t}, \quad t \geq 0. \]

So, by Theorem 2.1 in the paper [8] by Wong, the equilibrium solution \( u(t) \equiv 0 \) of (4.1) is asymptotically stable.

Remark. The technique used in the proof of Theorem 4.1 is similar to that used in the proof of Theorem 1.1 in the paper [10] by Wong.

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References


Department of Mathematics
and Statistics
York University
4700 Keele Street
Toronto, Ontario M3J 1P3
CANADA

e-mail: mwwong@mathstat.yorku.ca