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## SPECTRA AND DYNAMICS OF BOUNDED PSEUDO-DIFFERENTIAL OPERATORS

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### 1. Introduction

Let  $S^0$  be the set of all  $C^\infty$  functions  $\sigma$  on  $\mathbb{R}^n$  such that, for each multi-index  $\alpha$ , there exists a positive constant  $C_\alpha$  for which

$$|(D^\alpha \sigma)(\xi)| \leq C_\alpha (1 + |\xi|)^{-|\alpha|}, \quad \xi \in \mathbb{R}^n.$$

We call any function  $\sigma$  in  $S^0$  a symbol. Let  $\sigma \in S^0$ . Then we define the pseudo-differential operator  $T_\sigma$  on the Schwartz space  $\mathcal{S}$  by

$$(1.1) \quad (T_\sigma \varphi)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \sigma(\xi) \hat{\varphi}(\xi) d\xi, \quad x \in \mathbb{R}^n,$$

where

$$(1.2) \quad \hat{\varphi}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} \varphi(x) dx, \quad \xi \in \mathbb{R}^n,$$

for all functions  $\varphi$  in  $\mathcal{S}$ . It is obvious that  $T_\sigma$  maps  $\mathcal{S}$  into  $\mathcal{S}$ . That  $T_\sigma$  can be extended to a bounded linear operator  $T_{\sigma p}$  from  $L^p(\mathbb{R}^n)$  into  $L^p(\mathbb{R}^n)$ ,  $1 < p < \infty$ , is well-known and, in fact, follows from more general  $L^p$ -boundedness results given in, e.g., Theorem 4.1 in Chapter 2 and the bibliographical notes on page 411 of the book [2] by Kumano-go or Theorem 9.7 in Chapter 9 of the book [7] by Wong. If we let  $K$  be the tempered distribution on  $\mathbb{R}^n$  given by  $\hat{K} = \sigma$ , where the Fourier transform  $\hat{K}$  of  $K$  is taken in the distribution sense, then, by Lemma 9.12 in Chapter 9 of the book [7] by Wong,  $K$  is a function on  $\mathbb{R}^n - \{0\}$  such that, for each sufficiently large positive integer  $N$ , there is a positive constant  $C_N$  for which

$$|K(x)| \leq C_N |x|^{-N}, \quad x \neq 0.$$

Moreover, for each  $x$  in  $\mathbb{R}^n$  and each  $\varphi$  in  $\mathcal{S}$  vanishing in a neighbourhood of  $x$ ,

$$(T_\sigma \varphi)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} K(x - y) \varphi(y) dy.$$

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Thus, the pseudo-differential operator  $T_\sigma$  can be considered as a singular integral operator with convolution kernel  $K$ .

The aim of this paper is to compute the spectrum and essential spectrum of the bounded linear operator  $T_{\sigma p} : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ .

We first recall in Section 2 the spectrum and essential spectrum of a bounded linear operator  $A$  from a complex Banach space  $X$  into  $X$ . In Section 3, the spectrum and essential spectrum of the bounded linear operator  $T_{\sigma p} : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$  are computed. Results related to the "spectral invariance", i.e., Theorem 3.2 in this paper, can be found in, e.g., the paper [3] by Leopold and Triebel or the paper [9] by Wong. An application to the asymptotic stability of the zero solution of a semilinear evolution equation modelled by a bounded pseudo-differential operator on  $L^p(\mathbb{R}^n)$  is given in Section 4.

## 2. The Spectrum and Essential Spectrum

Let  $A$  be a bounded linear operator from a complex Banach space  $X$  into  $X$ . We denote the norm in  $X$  by  $\| \cdot \|$ . Let  $\rho(A)$  be the resolvent set of  $A$  defined by

$$\rho(A) = \{ \lambda \in \mathbb{C} : A - \lambda I \text{ is one to one and onto} \},$$

where  $\mathbb{C}$  is the set of all complex numbers and  $I$  is the identity operator from  $X$  into  $X$ . The spectrum  $\Sigma(A)$  of  $A$  is defined to be the set complement of  $\rho(A)$  in  $\mathbb{C}$ . We denote the adjoint of  $A$  by  $A^t$ , the range of  $A$  by  $R(A)$ , and the null spaces of  $A$  and  $A^t$  by  $N(A)$  and  $N(A^t)$  respectively. We call  $A$  a Fredholm operator if  $R(A)$  is a closed subspace of  $X$ , and the dimensions  $\alpha(A)$  and  $\beta(A)$  of  $N(A)$  and  $N(A^t)$  respectively are finite. For any Fredholm operator  $A$ , we define the index  $i(A)$  of  $A$  by

$$i(A) = \alpha(A) - \beta(A).$$

Let  $\Phi(A)$  be the set defined by

$$\Phi(A) = \{ \lambda \in \mathbb{C} : A - \lambda I \text{ is Fredholm with zero index} \}.$$

Then the essential spectrum  $\Sigma_e(A)$  of  $A$ , defined by Schechter in [4], is the set complement of  $\Phi(A)$  in  $\mathbb{C}$ . It can be shown that a complex number  $\lambda$  is in the essential spectrum  $\Sigma_e(A)$  if there exists a sequence  $\{u_k\}$  of elements in  $X$  such that  $\|u_k\| = 1$ ,  $k = 1, 2, \dots$ ,  $(A - \lambda I)u_k \rightarrow 0$  in  $X$  as  $k \rightarrow \infty$  and  $\{u_k\}$  has no convergent subsequence in  $X$ .

The notions and results in this section can be found in Section 4 of Chapter 1 of the book [6] by Schechter.

## 3. The Spectrum of $T_{\sigma p}$

We begin with the following result on the essential spectrum  $\Sigma_e(T_{\sigma p})$  of  $T_{\sigma p}$ .

**Proposition 3.1.** *Let  $\sigma \in S^0$  and let  $\lambda$  be a complex number. If  $\sigma(\xi)$  is not bounded away from  $\lambda$  for all  $\xi$  in  $\mathbb{R}^n$ , then  $\lambda \in \Sigma_\varepsilon(T_{\sigma p})$ .*

*Proof.* Let  $\{\xi_k\}$  be a sequence of elements in  $\mathbb{R}^n$  such that  $\sigma(\xi_k) \rightarrow \lambda$  as  $k \rightarrow \infty$ . Let  $k_0$  be the smallest positive integer greater than  $n/p$  and let  $N$  be any fixed integer greater than  $k_0 - n/p$ . Let  $\{\varepsilon_k\}$  be a sequence of positive numbers such that

$$(3.1) \quad \varepsilon_k^{|\mu|} \sigma^{(\mu)}(\xi_k) \rightarrow 0, \quad 1 < |\mu| < N,$$

as  $k \rightarrow \infty$ , where

$$\sigma^{(\mu)}(\xi) = (\partial^\mu \sigma)(\xi), \quad \xi \in \mathbb{R}^n.$$

Let  $\psi \in C_0^\infty(\mathbb{R}^n)$  be such that  $\psi(x) = 0$  for all  $x$  in a neighbourhood of the origin and

$$(3.2) \quad \|\psi\|_p = 1.$$

For  $k = 1, 2, \dots$ , let  $\varphi_k$  be the function on  $\mathbb{R}^n$  defined by

$$(3.3) \quad \varphi_k(x) = \varepsilon_k^{n/p} \psi(\varepsilon_k x) e^{i\xi_k \cdot x}, \quad x \in \mathbb{R}^n.$$

Then, for  $k = 1, 2, \dots$ ,  $\varphi_k \in C_0^\infty(\mathbb{R}^n)$  and an easy computation gives

$$((T_{\sigma p} - \lambda I)\varphi_k)(x) = e^{ix \cdot \xi_k} \varepsilon_k^{n/p} (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot \varepsilon_k \zeta} \{\sigma(\varepsilon_k \zeta + \xi_k) - \lambda\} \hat{\psi}(\zeta) d\zeta$$

for all  $x$  in  $\mathbb{R}^n$ . By Taylor's formula with integral remainder, we get, for  $k = 1, 2, \dots$ ,

$$(3.4) \quad \begin{aligned} & ((T_{\sigma p} - \lambda I)\varphi_k)(x) \\ &= e^{ix \cdot \xi_k} \varepsilon_k^{n/p} (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot \varepsilon_k \zeta} \left\{ \sum_{|\mu| < N} \frac{\sigma^{(\mu)}(\xi_k)}{\mu!} \varepsilon_k^{|\mu|} \zeta^\mu - \lambda \right\} \hat{\psi}(\zeta) d\zeta \\ & \quad + e^{ix \cdot \xi_k} \varepsilon_k^{n/p} (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot \varepsilon_k \zeta} R_k(\zeta) \hat{\psi}(\zeta) d\zeta, \quad x \in \mathbb{R}^n, \end{aligned}$$

where

$$(3.5) \quad R_k(\zeta) = N \sum_{|\mu|=N} \frac{\varepsilon_k^N \zeta^\mu}{\mu!} \int_0^1 (1-\theta)^{N-1} \sigma^{(\mu)}(\xi_k + \theta \varepsilon_k \zeta) d\theta, \quad \zeta \in \mathbb{R}^n.$$

The first term on the right hand side of (3.4) is equal to  $T_k(x)$ , where

$$(3.6) \quad T_k(x) = e^{ix \cdot \xi_k} \varepsilon_k^{n/p} \{ \sigma(\xi_k) - \lambda \} \psi(\varepsilon_k x) + e^{ix \cdot \xi_k} \sum_{1 < |\mu| < N} \frac{\varepsilon_k^{\frac{n}{p} + |\mu|}}{\mu!} \sigma^{(\mu)}(\xi_k) (D^\mu \psi)(\varepsilon_k x).$$

The second term on the right hand side of (3.4) is equal to  $\sum_{|\mu|=N} T_k^\mu(x)$ , where

$$(3.7) \quad T_k^\mu(x) = \frac{\varepsilon_k^{\frac{n}{p} + N}}{\mu! (2\pi)^{n/2}} N \int_{\mathbb{R}^n} e^{ix \cdot \varepsilon_k \zeta} I_k^\mu(\zeta) \zeta^\mu \hat{\psi}(\zeta) d\zeta$$

and

$$(3.8) \quad I_k^\mu(\zeta) = \int_0^1 (1 - \theta)^{N-1} \sigma^{(\mu)}(\xi_k + \theta \varepsilon_k \zeta) d\theta.$$

Since  $\sigma \in S^0$ , it follows from (3.7) and (3.8) that, for each multi-index  $\mu$  with  $|\mu| = N$ , there exists a positive constant  $C_\mu$  such that

$$|T_k^\mu(x)| \leq \frac{\varepsilon_k^{\frac{n}{p} + N}}{\mu! (2\pi)^{n/2}} N C_\mu \int_{\mathbb{R}^n} |\zeta^\mu \hat{\psi}(\zeta)| d\zeta, \quad x \in \mathbb{R}^n.$$

Hence, for each multi-index  $\mu$  with  $|\mu| = N$ ,

$$(3.9) \quad T_k^\mu \rightarrow 0$$

uniformly on  $\mathbb{R}^n$  as  $k \rightarrow \infty$ . Let  $\alpha$  be a multi-index with  $|\alpha| = k_0$ . Then, by (3.7), an integration by parts and Leibnitz' formula,

$$(3.10) \quad |x^\alpha T_k^\mu(x)| \leq \frac{\varepsilon_k^{\frac{n}{p} + N - k_0}}{\mu! (2\pi)^{n/2}} N \int_{\mathbb{R}^n} |\partial_\zeta^\alpha \{ I_k^\mu(\zeta) \zeta^\mu \hat{\psi}(\zeta) \}| d\zeta = \frac{\varepsilon_k^{\frac{n}{p} + N - k_0}}{\mu! (2\pi)^{n/2}} N \int_{\mathbb{R}^n} \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} |(\partial^\gamma I_k^\mu)(\zeta) \partial_\zeta^{\alpha-\gamma} \{ \zeta^\mu \hat{\psi}(\zeta) \}| d\zeta$$

for all  $x$  in  $\mathbb{R}^n$ . Using (3.8) and the fact that  $\sigma \in S^0$ , we can find, for each multi-index  $\mu$  with  $|\mu| = N$  and each multi-index  $\gamma$  with  $\gamma \leq \mu$ , a positive constant  $C_{\mu\gamma}$  such that

$$(3.11) \quad |(\partial^\gamma I_k^\mu)(\zeta)| \leq C_{\mu\gamma}, \quad k = 1, 2, \dots,$$

for all  $\zeta$  in  $\mathbb{R}^n$ . So, by (3.10) and (3.11), we get a positive constant  $C_{\alpha\mu}$  such that

$$(3.12) \quad |x^\alpha T_k^\mu(x)| \leq C_{\alpha\mu} \varepsilon_k^{\frac{n}{p} + N - k_0}, \quad k = 1, 2, \dots,$$

for all  $x$  in  $\mathbb{R}^n$ . Thus, using the elementary inequality on page 53 of the book [7] by Wong and (3.12), we get, for each multi-index  $\mu$  with  $|\mu| = N$ , a positive constant  $C_{\mu k_0}$  such that

$$(3.13) \quad |x|^{k_0} |T_k^\mu(x)| \leq C_{\mu k_0} \varepsilon_k^{\frac{n}{p} + N - k_0}, \quad k = 1, 2, \dots,$$

for all  $x$  in  $\mathbb{R}^n$ . So, by (3.9), (3.13), the fact that  $k_0 p > n$  and the Lebesgue dominated convergence theorem,

$$(3.14) \quad \|T_k^\mu\|_p \rightarrow 0$$

as  $k \rightarrow \infty$  for each multi-index  $\mu$  with  $|\mu| = N$ . Thus, by (3.4), (3.6), (3.7) and (3.14),

$$\|(T_{\sigma p} - \lambda I)\varphi_k\|_p \rightarrow 0$$

as  $k \rightarrow \infty$ . Obviously, by (3.2) and (3.3),  $\|\varphi_k\|_p = 1$ ,  $k = 1, 2, \dots$ , and hence it remains to prove that  $\{\varphi_k\}$  has no convergent subsequence in  $L^p(\mathbb{R}^n)$ . But this follows from the argument on page 64 of the book [6] by Schechter.  $\square$

The precise description of the spectrum of  $T_{\sigma p}$  is provided by the following theorem.

**Theorem 3.2.** *Let  $\sigma \in S^0$ . Then*

$$\Sigma(T_{\sigma p}) = \Sigma_e(T_{\sigma p}) = \{\sigma(\xi) : \xi \in \mathbb{R}^n\}^c,$$

where  $\{\dots\}^c$  denotes the closure in  $\mathbb{C}$  of the set  $\{\dots\}$ .

Theorem 3.2 is an immediate consequence of Proposition 3.1 and the following proposition.

**Proposition 3.3.** *Let  $\lambda$  be a complex number such that  $\sigma(\xi)$  is bounded away from  $\lambda$  for all  $\xi$  in  $\mathbb{R}^n$ . Then  $\lambda \in \rho(T_{\sigma p})$ .*

*Proof.* Let  $\tau$  be the function on  $\mathbb{R}^n$  defined by

$$(3.15) \quad \tau(\xi) = \frac{1}{m(\xi)}, \quad \xi \in \mathbb{R}^n,$$

where

$$(3.16) \quad m(\xi) = \sigma(\xi) - \lambda, \quad \xi \in \mathbb{R}^n.$$

Then, for all multi-indices  $\alpha$ ,

$$(3.17) \quad (\partial^\alpha \tau)(\xi) = \sum C_{\alpha^{(1)}, \dots, \alpha^{(k)}} \frac{(\partial^{\alpha^{(1)}} m)(\xi) \cdots (\partial^{\alpha^{(k)}} m)(\xi)}{m(\xi)^{k+1}}, \quad \xi \in \mathbb{R}^n,$$

where the summation is taken over all partitions  $\alpha^{(1)}, \dots, \alpha^{(k)}$  of  $\alpha$  and  $C_{\alpha^{(1)}, \dots, \alpha^{(k)}}$  is a constant depending on the partition  $\alpha^{(1)}, \dots, \alpha^{(k)}$  of  $\alpha$ . Since  $\sigma \in S^0$ , it follows from (3.15), (3.16) and (3.17) that there exist positive constants  $C_{\alpha^{(1)}}, \dots, C_{\alpha^{(k)}}$  and  $C$  such that

$$(3.18) \quad |(\partial^\alpha \tau)(\xi)| \leq \sum |C_{\alpha^{(1)}, \dots, \alpha^{(k)}}| \frac{C_{\alpha^{(1)}} \cdots C_{\alpha^{(k)}} (1 + |\xi|)^{-|\alpha|}}{C^{k+1}}, \quad \xi \in \mathbb{R}^n.$$

Thus, by (3.18),  $\tau \in S^0$ . So, by Theorem 9.7 in Chapter 9 of the book [7] by Wong,  $T_\tau$ , initially defined on  $\mathcal{S}$ , can be extended to a unique bounded linear operator  $T_{\tau p}$  from  $L^p(\mathbb{R}^n)$  into  $L^p(\mathbb{R}^n)$ . Now, let  $f \in L^p(\mathbb{R}^n)$ . Then there exists a sequence  $\{\varphi_k\}$  of functions in  $\mathcal{S}$  such that

$$(3.19) \quad \varphi_k \rightarrow f$$

in  $L^p(\mathbb{R}^n)$  as  $k \rightarrow \infty$ . Thus,

$$T_\tau \varphi_k \rightarrow T_{\tau p} f$$

in  $L^p(\mathbb{R}^n)$  as  $k \rightarrow \infty$ . So,

$$(3.20) \quad T_{\sigma-\lambda} T_\tau \varphi_k \rightarrow (T_{\sigma p} - \lambda I) T_{\tau p} f$$

in  $L^p(\mathbb{R}^n)$  as  $k \rightarrow \infty$ . But, by (3.15) and (3.16),

$$(3.21) \quad \varphi_k = T_{\sigma-\lambda} T_\tau \varphi_k, \quad k = 1, 2, \dots$$

Thus, by (3.19), (3.20), (3.21) and letting  $k \rightarrow \infty$ , we get

$$f = (T_{\sigma p} - \lambda I) T_{\tau p} f$$

and hence

$$(3.22) \quad (T_{\sigma p} - \lambda I) T_{\tau p} = I.$$

Similarly,

$$(3.23) \quad T_{\tau p} (T_{\sigma p} - \lambda I) = I.$$

So, by (3.22) and (3.23),  $\lambda \in \rho(T_{\sigma p})$  and the proof is complete. □

REMARK. It is important to note that if  $\sigma \in S^0$ , then the spectrum of  $T_{\sigma p}$  is, in general, equal to  $\{\sigma(\xi) : \xi \in \mathbb{R}^n\}^c$ , but not  $\{\sigma(\xi) : \xi \in \mathbb{R}^n\}$ . To wit, let  $\sigma$  be the function on  $\mathbb{R}^2$  defined by

$$\sigma(\xi_1, \xi_2) = \frac{\xi_1 + i(\xi_1\xi_2 + 1)}{1 + \xi_1^2 + \xi_2^2}, \quad \xi_1, \xi_2 \in \mathbb{R}.$$

Then it is easy to prove that  $\sigma \in S^0$ . At any rate, it follows from Example 4.3, Example 4.4 and Exercise 4.1 in Chapter 4 of the book [7] by Wong. It is obvious that zero is not in the set  $\{\sigma(\xi_1, \xi_2) : \xi_1, \xi_2 \in \mathbb{R}\}$  and  $\sigma(1/k, -k) \rightarrow 0$  as  $k \rightarrow \infty$ . Thus,  $0 \in \{\sigma(\xi_1, \xi_2) : \xi_1, \xi_2 \in \mathbb{R}\}^c$ .

#### 4. A Semilinear Evolution Equation

In this section, we consider the dynamical system modelled by the semilinear evolution pseudo-differential equation

$$(4.1) \quad u'(t) + (T_{\sigma p} - \lambda I)\{u(t)\} = f\{u(t)\}, \quad t \geq 0,$$

where  $\sigma \in S^0$ ,  $u$  is a function from  $[0, \infty)$  into  $L^p(\mathbb{R}^n)$ ,  $\lambda$  is a real constant and  $f$  is a continuous mapping from  $L^p(\mathbb{R}^n)$  into  $L^p(\mathbb{R}^n)$  such that

$$(4.2) \quad \|f(u)\|_p = o(\|u\|_p)$$

as  $u \rightarrow 0$  in  $L^p(\mathbb{R}^n)$ . In (4.1), the derivative  $u'(t)$ , at any time  $t$ , is understood to be the strong limit in  $L^p(\mathbb{R}^n)$ , if it exists, of the difference quotient

$$\frac{u(t+h) - u(t)}{h}$$

as  $h \rightarrow 0$ . It is clear that, in view of (4.2),  $u(t) \equiv 0$  is an equilibrium solution of (4.1). The equilibrium solution  $u(t) \equiv 0$  of (4.1) is said to be asymptotically stable if, for any positive number  $\varepsilon$ , there exists a positive constant  $\delta$  such that any solution  $u(t)$ ,  $t \geq 0$ , of (4.1) with  $\|u(0)\|_p < \delta$  satisfies

$$\|u(t)\|_p < \varepsilon, \quad t \geq 0,$$

and

$$\lim_{t \rightarrow \infty} \|u(t)\|_p = 0.$$

**Theorem 4.1.** *Let  $\sigma \in S^0$  and  $f$  be a continuous mapping from  $L^p(\mathbb{R}^n)$  into  $L^p(\mathbb{R}^n)$  satisfying (4.2). Then, for any real number  $\lambda$  with  $\lambda < \mu$ , where*

$$\mu = \inf_{\xi \in \mathbb{R}^n} \{\operatorname{Re} \sigma(\xi)\},$$



the equilibrium solution  $u(t) \equiv 0$  of (4.1) is asymptotically stable.

**Proof.** By Theorem 3.2, the spectrum of  $T_{\sigma p}$  is equal to

$$\{\sigma(\xi) : \xi \in \mathbb{R}^n\}^c.$$

Hence, by Theorem 3.8 in Chapter 3 of the book [5] by Schechter, i.e., the spectral mapping theorem, the spectrum of the bounded linear operator  $e^{-T_{\sigma p}t} : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ ,  $t \geq 0$ , is equal to

$$\{e^{-\sigma(\xi)t} : \xi \in \mathbb{R}^n\}^c.$$

Thus, the spectral radius of  $e^{-T_{\sigma p}t} : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ ,  $t \geq 0$ , is equal to  $e^{-\mu t}$ . Therefore, by a standard result in the theory of semigroups, i.e., Theorem 1.22 on page 15 of the book [1] by Davies,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln \|e^{-T_{\sigma p}t}\|_* = -\mu,$$

where  $\|\cdot\|_*$  is the norm in the Banach algebra of all bounded linear operators from  $L^p(\mathbb{R}^n)$  into  $L^p(\mathbb{R}^n)$ . Hence, for any number  $\varepsilon$  in  $(0, \mu - \lambda)$ , we can find a positive number  $t_\varepsilon$  such that

$$\frac{1}{t} \ln \|e^{-T_{\sigma p}t}\|_* < -\mu + \varepsilon, \quad t > t_\varepsilon.$$

Thus,

$$\|e^{-T_{\sigma p}t}\|_* < e^{-(\mu-\varepsilon)t}, \quad t > t_\varepsilon,$$

and consequently,

$$\|e^{-(T_{\sigma p}-\lambda I)t}\|_* < e^{-(\mu-\lambda-\varepsilon)t}, \quad t > t_\varepsilon.$$

Therefore there exists a positive constant  $M$  such that

$$\|e^{-(T_{\sigma p}-\lambda I)t}\|_* < M e^{-(\mu-\lambda-\varepsilon)t}, \quad t \geq 0.$$

So, by Theorem 2.1 in the paper [8] by Wong, the equilibrium solution  $u(t) \equiv 0$  of (4.1) is asymptotically stable.  $\square$

**REMARK.** The technique used in the proof of Theorem 4.1 is similar to that used in the proof of Theorem 1.1 in the paper [10] by Wong.

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