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ON VARIATION BICOMPLEXES ASSOCIATED TO DIFFERENTIAL EQUATIONS*

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Introduction

This paper treats the Vinogradov's spectral sequence E of differential equations ([V1]) in the framework of Gelfand's formal differential geometry ([Ge]), and gives new interpretations of E generalizing formally the secondary characteristic classes and the Bott's vanishing theorems in the foliation theory.

We describe some part of E and the Lie algebra of the symmetries in a way useful for actual calculations. This description yields a quantitative formulation of the Noether theorem as a by-product.

The geometric language adopted here produces also a simple class of correspondences between the solutions of differential equations, which includes the usual Bäcklund transformations.

0.1. In Nice 1970, Gelfand introduced the idea of "formal differential geometry" and suggested its usefulness in investigations of problems involving jets ([Ge]). He and his collaborators realized this in their studies of the Hamiltonian struc-

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tures associated to Korteweg-de Vries equation ([GD]), the secondary characteristic classes of foliations ([FGG]) and combinatorial formulas for Pontrjagin classes ([GGL]).

The double complexes introduced in [FGG], called *the variation bicomplexes*, play important roles in their works explicitly or implicitly.

On the other hand, Vinogradov introduced in [V1] spectral sequences for differential equations and announced various interesting results. He uses however algebro-geometric and categorical language which seems to obscure the simple aspects of these spectral sequences, although his definition has the great advantage of being valid for very general classes of differential equations.

This paper constructs these naturally from the variation bicomplexes associated to differential equations.

This construction gives new interpretations of these spectral sequences not mentioned in [V1]. In fact, these permit one to generalize formally the basic tools of the quantitative studies of foliation, such as secondary characteristic classes and the Bott's vanishing theorems, etc.

However to make this generalization substantial, one must compute the spectral sequences, for which there are no systematic methods at presents. For example in the case of the integrability equation of G-structures, there arises the problem of computing various cohomologies of Gelfand-Fuks type not treated before.

The detailed contents are as follows.

0.2. For a differential equation R on a manifold M, the basic geometric object in this paper is its infinite prolongation $R_\infty \to M$ (§2.2) and a natural flat connection H_R on it (§2.4). It is a subbundle of TR_∞ of rank $n=\dim M$.

This pair (R_∞, H_R) is a good substitute for the solution space $\mathcal{S}ol(R)$: It provides us "differential calculus" on $\mathcal{S}ol(R)$ and also geometric intuition to the general notions about differential equations.

By splitting the de Rham complex on R_∞ with respect to H_R, one obtains the variation bicomplex $\Omega_R{}^{*,*}$ of R, where $\Omega_R^{p,q}=\Gamma(\wedge^p V_R{}^* \otimes \wedge^q H_R{}^*)$ (§2.5). The filtration $F^p=\oplus_{p'\geq p}\Omega_R^{p',*}$ generates the spectral sequence $E(R)=\{E_r^{p,q}(R), d_r\}$ (§2.7) mentioned above.

0.3. The E_1-, E_2- and E_∞-terms have various meanings (§4).

The complexe $\{E_1^{*,q}, d_1\}$ is related to the $H^q(M, \boldsymbol{R})$-valued "de Rham complex on $\mathcal{S}ol(R)$" (§4.2). In particular an element of $E_1^{0,q}$ gives us an $H^q(M, \boldsymbol{R})$-valued function on $\mathcal{S}ol(R)$, which may be called a characteristic class of solutions of R (§4.5). This unifies the concept of the conserved currents of evolution equations and that of the secondary characteristic classes of foliations. Further $E_1^{1,q}$ measures how a one-parameter family of solutions varies (§4.6).

There is a natural map $E_2^{p,q} \to H^p(\mathcal{S}ol(R), \boldsymbol{R}) \otimes H^q(M, \boldsymbol{R})$ (§4.2). In particular, $\omega \in E_2^{0,q}$ gives us a constant on each connected component of $\mathcal{Sol}(R)$ (§4.7). This generalizes the rigid characteristic classes of foliations.

Finally, E_∞-terms produce potential topological obstructions to the deformability of homotopy solutions to real ones (§4.4, 4.8). This is a formal generalization of the Bott's vanishing theorem [Bot] in the foliation theory.

0.4. The computation of $E(R)$ is generally difficult. This paper treats only the trivial equations $R=N$ (§5) and the "determined systems" (§7).

Section 5 computes it when R is trivial. The results coincide with those already announced by Vinogradov [V1]. As by-products, various useful facts follow about the Euler-Lagrange operators, which are however more or less well-known.

Section 7 rewrites $E_1^{1,n-1}$ ($n=\dim M$) as the solution space of a linear differential operator Φ_R^+, when R is a Cauchy-Kowalevsky system in a weak sense. This yields a new method of computing the conservation laws of wave equations, the Korteweg-de Vries equation and the BBM equation (§7.6).

When R is overdetermined, there are at present no systematic methods of computing $E_r^{p,q}$, nor of constructing nontrivial elements of $E_r^{p,q}$ (cf. §3.6).

0.5. The connection H_R allows us to define also the Lie algebra of "vector fields on $\mathcal{Sol}(R)$".

A vertical vector field X on $R_\infty \to M$ "preserves" the graphs of solutions if $[X, \Gamma H_R] \subset \Gamma H_R$. The space of all such X's forms a Lie algebra $\mathcal{L}(R)$, which is called the Lie algebra of *symmetries* of R (§6.1). An element of $\mathcal{L}(R)$ is sometimes called a Lie-Bäcklund transformation [AI].

Following [V1,2], section 6 expresses $\mathcal{L}(R)$ as the solution space of a linear differential operator Φ_R associated to R (§6.6). This expression is valid for any R in contrast to the similar one for $E_1^{1,n-1}$ mentioned in §0.4.

Then we get an effective method of computing the Lie algebra of contact transformations or bundle automorphisms leaving R invariant (§6.7).

0.6. When R is both an Euler-Lagrange equation and a Cauchy-Kowalevsky system in a weak sense, the expressions for $E_1^{1,n-1}$ and $\mathcal{L}(R)$ mentioned above coincide and give us a quantitative expression for the Noether Theorem:

$$E_1^{0,n-1} \subset E_1^{1,n-1} \simeq \mathcal{L}(R)$$

(Theorem 8.1).

By paraphrasing the result of [Mar], §8.2 characterizes in a practical way the image of $E_1^{0,n-1} \subset \mathcal{L}(R)$, whose elements are called the *Noether symmetries* of R. For $n=1$, this is given in [GD]. For general n, this is essentially obtained in [Mar].

0.7. Section 9 proposes a simple class of correspondences generalizing the usual Bäcklund transformations.

Further we remark that the Whalquist-Estabrook method of constructing Bäcklund transformations ([WE]) can be concisely summarized by the language developed in §1–2 (Remark 9.14).

0.8. Finally Section 10 gives remarks about motivation and terminologies of this paper and raises some problems.

0.9. General terminological and notational conventions are as follows.

When a set X has an equivalence relation on it, $[x]$ ($x\in X$) denotes the equivalence class represented by x.

The set of nonnegative integers and the field of real numbers are denoted respectively by $\boldsymbol{N}$ and $\boldsymbol{R}$. For $a<b$, $\boldsymbol{Z}(a,b)$ stands for the set of integers in $[a,b]$.

For $i\in\boldsymbol{Z}(1,n)$, (i) denotes the element of $\boldsymbol{N}^n$ which has 1 in the i-th component and zero in the others.

Let $I,J\in\boldsymbol{N}^n$ $(I=(i_1,\cdots,i_n), J=(j_1,\cdots,j_n))$. $I+J\in\boldsymbol{N}^n$ has i_k+j_k in the k-th component $(k\in\boldsymbol{Z}(1,n))$. $I\leqq J$ means $i_k\leqq j_k$ $(k\in\boldsymbol{Z}(1,n))$. When $I\geqq J$,

$$\binom{I}{J}=\binom{i_1}{j_1}\cdots\binom{i_n}{j_n}.$$

where $\binom{i}{j}=i!/j!(i-j)!$. For $I\in\boldsymbol{N}^n$ and a set of letters $x=(x^1,\cdots,x^n)$, x^I stands for $(x^1)^{i_1}\cdots(x^n)^{i_n}$. For $a,b\in\boldsymbol{Z}(1,n)$ with $a\leqq b$, $\boldsymbol{N}_{a,b}$ denotes the set of $I\in\boldsymbol{N}^n$ such that $i_k=0$ for $k\notin\boldsymbol{Z}(a,b)$. For $a>b$, $\boldsymbol{N}_{a,b}$ denotes $\{(0)\}$.

The dual space of a vector space V is denoted by V^*. If $(x^1,\cdots,x^n)$ is a basis of V, $\wedge^p V$ is also denoted by $\wedge^p[x^i]$. All the tensor products $\otimes$ are taken over $\boldsymbol{R}$ unless otherwise stated.

The word *smooth* will mean C^∞. Everything will be considered in the smooth category. Manifolds are always assumed to be connected and with countable basis. When M is a manifold, the algebra of functions on M (i.e. smooth maps: $M\to\boldsymbol{R}$), the tangent bundle, the cotangent bundle, the Lie algebra of vector fields and the de Rham complex are denoted respectively by $\mathcal{F}M$, TM, T^*M, $\mathcal{L}M$ and $\{\Omega^*M, d\}$.

$\boldsymbol{R}^n_x$ stands for $\boldsymbol{R}^n$ with the standard linear coordinate denoted by $x=(x^1,\cdots,x^n)$. $\mathcal{F}(x)$ stands for $\mathcal{F}\boldsymbol{R}^n_x$. The forms $dx^1\wedge\cdots\wedge dx^n$ and $(-1)^{i-1}dx^1\wedge\cdots\wedge\check{dx^i}\wedge\cdots\wedge dx^n$ are abriviated as dx and dx_i respectively.

Finally the decimal system is adopted for the reference numbers of paragraphs, theorems, etc.. The integral part denotes the section number and the fractional parts are arranged in each section according to their magnitude. For example Lemma 7.351 precedes Proposition 7.41 since 7.351<7.41.

1. Jet bundles

This section recalls some basic definitions and properties of the jet bundles, and fixes notations.

1.1. Finite jet bundles. Let $p: N \to M$ be a bundle, i.e., a surjective map everywhere of rank=dim M. The space of sections is denoted by ΓN.

Let $p_k: N_k \to M$ be the k-th jet bundle ($k \geqq 1$) (cf. [Bou]). Put $N_0 = N$. The *k-th* extension of $s \in \Gamma N$ will be denoted by s_k ($\in \Gamma N_k$).

If M is a point, we put $N_k = N$ for all k.

1.2. Infinite jet bundles. Denote the projective limits of $\{N_k\}$, $\{\Gamma N_k\}$ and $\{s_k\}$ ($s \in \Gamma N$) by N_∞, ΓN_∞ and s_∞ respectively. The projections $N_\infty \to N_k, \to M$ will be denoted by π_k ($k \geqq 0$), π respectively.

It is easy to show the following

Proposition 1.21. *$\pi_0: N_\infty \to N$ is an affine bundle. In particular, π_0 is a homotopy equivalence.*

The usual smooth objects can be naturally and easily generalized for this infinite-dimensional manifold N_∞. For example, a function on N_∞ is smooth if and only if it is in $\pi_k^* \mathcal{F} N_k$ for some $k \in \boldsymbol{N}$. Put $\mathcal{F} N_\infty = \bigcup \pi_k^* \mathcal{F} N_k$ and consider $\mathcal{F} N_k \subset \mathcal{F} N_\infty$.

For more details, see [BR]. The sections 1.4 and 1.5 express smooth objects using local coordinates on N_∞.

1.3. Standard local coordinates. Let $M = \boldsymbol{R}^n_x$, $N(n,m) = M \times \boldsymbol{R}^m_u$. Put $u^i_0 = u^i$ and define $u^i_I \in \mathcal{F} N(n,m)_\infty$ ($i \in \boldsymbol{Z}(1,m)$, $I \in \boldsymbol{N}^n - \{0\}$) by

$$s_\infty^* u^i_I = (\partial/\partial x)^I (u^i \circ s)$$

for $s \in \Gamma N$. Then $(x^i, u^j_I;\ i \in \boldsymbol{Z}(1,n), j \in \boldsymbol{Z}(1,m), I \in \boldsymbol{N}^n)$ is called *the standard coordinates on $N(n,m)_\infty$.*

Suppose $p: N \to M$ is a bundle. Each point y of N has a neighbourhood W such that $p|_W: W \to p(W)$ is isomorphic to $N(n,m) \to \boldsymbol{R}^n$ (n=dim M, m=dim N $-n$), whence there is a diffeomorphism $f: \pi_0^{-1} W \to N(n,m)_\infty$. Denote the pullbacks by f of x^i, s and u^j_I's by the same letters and put $U = \pi_0^{-1} W$. $(x^i, u^j_I; U)$ is called *a standard local chart* of N_∞. This will be used frequently throughout this paper.

Following [GM], we write $\mathcal{F}[x,u] = \mathcal{F}(x^i, u^j_I)$ and denote its element as $f[x,u]$.

1.4. Vector fields. The tangent bundle TN_∞ of N_∞ is by definition the projective limit of $\{\pi_k^* TN_k\}$. The space $\mathcal{L} N_\infty$ of the sections of TN_∞ is by definition the projective limit of $\{\Gamma_k\}$, where Γ_k is the injective limit of $\{\Gamma((\pi^l_k)^* TN_k);\ l \geqq k\}$. Here $\pi^l_k: N_l \to N_k$ is the natural projection.

$\mathcal{L} N_\infty$ acts on $\mathcal{F} N_\infty$ as derivations in the obvious way and hence carries a natural Lie algebra structure.

Let $(x^i, u_I^j; U)$ be a standard local chart of N_∞. Then an element of $\mathcal{L}U$, i.e., a vector field on U, is expressed uniquely as an infinite sum: $\sum f_i\partial/\partial x^i + \sum f_{j,I}\partial/\partial u_I^j$, where $f_i, f_{j,I} \in \mathcal{F}[x,u]$.

1.5. Differential forms. The bundle of p-forms $\wedge^p T^*N_\infty$ is the injective limit of $\{\pi_k^* \wedge^p T^*N_k\}$. The space $\Omega^p N_\infty$ of its sections, i.e., the space of p-forms, is the injective limit of $\{\Omega^p N_k\}$.

Let $(x^i, u_I^j; U)$ be a standard local chart of N_∞. Then $\Omega^p U$ has the expression:

$$\Omega^p U = \mathcal{F}U \otimes \wedge^p[du_I^j, dx^i] .$$

Hence, a p-form is a finite sum of such expression as

$$f[x, u]du_{I_1}^{j_1} \wedge \cdots \wedge du_{I_s}^{j_s} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_t}$$

$(s+t=p, f \in \mathcal{F}[x,u])$.

Note that there is a natural pairing $T^*N_\infty \times TN_\infty \to N_\infty \times \boldsymbol{R}$, where $T^*N_\infty = \wedge^1 T^*N_\infty$ is the cotangent bundle.

1.6. The connection H_N. Now we introduce a flat connection H_N on $N_\infty \to M$, which will play a fundamental role throughout this paper.

For each $y \in N_\infty$, define a linear subspace H_y of $T_y N_\infty$ as follows: Choose $s \in \Gamma N$ such that $y = s_\infty(z)$ $(z = \pi(y))$, which is possible obviously, and put

$$H_y = \operatorname{Im} d_z s_\infty ,$$

where $d_z s_\infty \colon T_z M \to T_y N_\infty$ is the differential.

Lemma. (i) *H_y does not depend on the choice of s.*
(ii) *$H_N = \cup H_y$ is a subbundle of TN_∞.*

Proof. (i) Let $(x^i, u_I^j; U)$ be a standard local chart. Choose $s \in \Gamma N$ such that $y = s_\infty(z)$. Then

$$\begin{aligned} X_i := d_z s_\infty(\partial/\partial x^i) &= \partial/\partial x^i + \sum(\partial/\partial x^i)_z(u_I^j \circ s_\infty)\partial/\partial u_I^j \\ &= \partial/\partial x^i + \sum u_{I+(i)}^j(s_\infty(z))\partial/\partial u_I^j \\ &= \partial/\partial x^i + \sum u_{I+(i)}^j(y)\partial/\partial u_I^j , \end{aligned}$$

for $i \in \boldsymbol{Z}(1,n)$. Thus X_i is independent of s and so is $H_y = \sum_{i=1,n} \boldsymbol{R}.X_i$.

(ii) Put

$$\partial_i = \partial/\partial x^i + \sum_{j,I} u_{I+(i)}^j \partial/\partial u_I^j . \tag{1.61}$$

This is a vector field on U and $\{\partial_1, \cdots, \partial_n\}$ spans $(H_N|_U)$ by (i). Hence H_N is a subbundle of TN_∞. Q.E.D.

The connection H_N lifts $X\in\mathcal{L}M$ up to $\tilde{X}\in\Gamma H_N\subset\mathcal{L}N_\infty$. Denote this map by τ: $\mathcal{L}M\to\mathcal{L}N_\infty$. Note that $\tilde{X}$ is uniquely characterized by

$$(1.62) \qquad \tilde{X}f\circ s_\infty = X(f\circ s_\infty) \quad \text{for all } s\in\Gamma N, f\in\mathcal{F}N_\infty .$$

For a standard local chart $(x^i, u^j_I; U)$, $\partial_i=(\partial/\partial x^i)^\sim$ is given by (1.61). These ∂_i's will appear frequently later.

1.7. Flatness of H_N. Now we prove $H=H_N$ is flat.

Let $H^\perp=\cup H_y^\perp\subset T^*N_\infty$ be the conormal bundle, where

$$H_y^\perp = \{\omega\in T_y^*N_\infty;\ \omega|H_y=0\} ,$$

for $y\in N_\infty$.

Let $(x^i, u^j_I; U)$ be a standard local chart. Put

$$(1.71) \qquad \delta u^j_I = du^j_I-\textstyle\sum_{i=1,n}u^j_{I+(i)}dx^i$$

$(j\in\boldsymbol{Z}(1,m),\ I\in\boldsymbol{N}^n)$.

Lemma 1.72. *$H^\perp|_U$ is spanned by $\{\delta u^j_I;\ j\in\boldsymbol{Z}(1,m),\ I\in\boldsymbol{N}^n\}$.*

Proof. Obviously $\{dx^i, \delta u^j_I\}$ is a frame of T^*U. Since $\delta u^j_I(\partial_i)=0$ ($i\in\boldsymbol{Z}(1,n)$) and $dx^i(\partial_j)=\delta_{ij}$, $H_y^\perp$ is spanned by $\{(\delta u^j_I)_y\}$. Q.E.D.

Proposition 1.73. *H is flat, i.e.,*

$$d\Gamma H^\perp\equiv 0 \qquad (\text{mod. } \Gamma H^\perp) .$$

Proof. It suffices to prove the assertion on a standard local chart $(x^i, u^j_I; U)$. We have

$$\begin{aligned} d\delta u^j_I &= -\textstyle\sum_i du^j_{I+(i)}\wedge dx^i \\ &\equiv -\textstyle\sum_{i,k}u^j_{I+(i)+(k)}dx^k\wedge dx^i \qquad (\text{mod. } \boldsymbol{R}.\ \delta u^j_{I+(i)}) \\ &= 0 . \end{aligned}$$

Hence by the previous lemma we have

$$d\Gamma H^\perp|_U\equiv 0 \qquad (\text{mod. } \Gamma H^\perp|_U) .$$

Q.E.D.

The flatness of H has another expression:

Proposition 1.74. *τ: $\mathcal{L}M\to\mathcal{L}N_\infty$ is a Lie algebra homomorphism.*

Proof. Let $X,\ Y\in\mathcal{L}M$, $s\in\Gamma N$, $f\in\mathcal{F}N_\infty$. Then by (1.62)

$$\begin{aligned} [\tilde{X}, \tilde{Y}]f\circ s_\infty &= (\tilde{X}\tilde{Y}f-\tilde{Y}\tilde{X}f)\circ s_\infty \\ &= (XY-YX)(f\circ s_\infty) \end{aligned}$$

$$= [X, Y]^{\sim} f \circ s_\infty .$$

Hence $[\tilde{X}\ \tilde{Y}]=[X, Y]^{\sim}$. Q.E.D.

Corollary 1.75. *The ∂_i's defined by* (1.61) *commute with each others.*

1.8. Flat sections. A section $s \in \Gamma N_\infty$ is called *flat* if $s(M)$ is tangent to H_N, i.e. $Ts(M) \subset H_N$.

Proposition 1.81. *For $s \in \Gamma N_\infty$, the following conditions are mutually equivalent.*

(i) *s is flat,*

(ii) $s^* \Gamma H_N^{\perp} = 0$,

(iii) $s = (\pi_0 \circ s)_\infty$.

Proof. (i)$\Leftrightarrow$(ii)$\Leftarrow$(iii) is obvious from the definitions.

Proof of (ii)$\Rightarrow$(iii). Let $(x^i, u_I^j; U)$ be a standard local chart. Then (ii) implies

$$s^* \delta u_I^j = ds_I^j - \sum_i s_{I+(i)}^j dx^i = 0 ,$$

where $s_I^j = u_I^j \circ s$. Hence

$$\partial s_I^j / \partial x^i = s_{I+(i)}^j ,$$

and we obtain

$$s_I^j = \partial^{|I|} s^j / \partial x^I ,$$

where $s^j = s_0^j$. Thus $(\pi_0 \circ s)_\infty = s_\infty$ on πU, whence (iii) follows. Q.E.D.

REMARK 1.82. In contrast to the case of the flat connections on finite-dimensional bundles, H_N has infinitely many flat sections passing through each point of N_∞.

2. Variation bicomplexes

This section introduces the variation bicomplex for each differential equation.

2.1. Differential equations. Let $p: N \rightarrow M$ be a bundle and put $n = \dim M$, $n+m = \dim N$.

A *differential equation of order* $\leqq k$ *on* N is a subbundle R of $N_k \rightarrow M$. Here "subbundle" just means that R is a closed submanifold of an open subset of N_k and $p_k|_R: R \rightarrow M$ is also a bundle.

An $s \in \Gamma N$ *is a solution of* R if and only if $s_k(M) \subset R$. The set of all the solutions of R will be denoted by $\mathcal{S}ol(R)$.

For convenience's sake the case $\dim M = 0$ is not excluded.

REMARK. This definition of R is general enough for actual applications. The more general one admitting singularities of R arouses unnecessary technical and terminological complication which seems to obscure the points.

2.2. Prolongation. We introduce here the infinite prolongation R_∞ of R which is so to speak the jet space of the solutions of R.

Let $R \subset N_k$ be a differential equation. By definition, R is a closed submanifold of an open set $U \subset N_k$. Let $\mathcal{I}_k \subset \mathcal{F}U$ be the defining ideal of R. Define $\mathcal{I}_\infty$ to be the ideal of $\mathcal{F}(\pi_k^{-1}U)$ generated by $\bigcup_{i \geqq k} \mathcal{I}_i$ where

$$\mathcal{I}_{i+1} = \{\tilde{X}f;\ X \in \mathcal{L}M,\ f \in \mathcal{I}_i\} \qquad (i \geqq k).$$

The zero set of $\mathcal{I}_\infty$ is called *the infinite prolongation of* R and will be denoted by R_∞.

Proposition 2.2. For $s \in \Gamma N$, s is a solution of R if and only if $s_\infty(M) \subset R_\infty$.

Proof. Since $\mathcal{I}_k \subset \mathcal{I}_\infty$, $s_\infty(M) \subset R_\infty$ implies s is a solution. Conversely suppose s is a solution. Since by (1.62) $s_\infty^* \mathcal{I}_i = 0$ implies $s_\infty^* \mathcal{I}_{i+1} = 0$, we have $s_\infty^* \mathcal{I}_i = 0$ $(^\forall i)$. Hence $s_\infty^* \mathcal{I}_\infty = 0$. Q.E.D.

REMARK 2.21. If $\pi | R_\infty$ is not surjective, R is called *incompatible*. Then Proposition 2.2 implies $\mathcal{S}ol(R) = \phi$.

REMARK 2.22. An $s \in \Gamma R_\infty$ will be called *a homotopy solution* of R. This is a suitable name since $R_\infty \to R$ is in many cases an affine bundle, and hence a homotopy equivalence (cf. Proposition 1.21).

REMARK 2.23. When R_∞ is finite-dimensional, R is called *maximally overdetermined* or *holonomic*. In this case the equation is essentially a system of ordinary differential equations, and hence $\mathcal{S}ol(R) \doteqdot$ a fiber of $R_\infty \to M$.

2.3. A basic assumption of R. We assume throughout this paper that R satisfies the following regularity:

Condition 2.3. *$R_\infty \to M$ is a subbundle of $N_\infty \to M$.*

Here "subbundle" means that $\pi_l R_\infty \to M$ is a subbundle of $N_l \to M$ $(^\forall l \geqq k)$.

Then, as in §1.2, we can define naturally the usual smooth objects on R_∞. Put $A(R) = \mathcal{F}R_\infty$, $A_l(R) = \mathcal{F}(\pi_l R_\infty)$ $(l \geqq k)$. By definition $A(R) = \cup A_l(R)$.

REMARK 2.31. Given a concrete R, it is usually easy to check this condition using standard local chart of N_∞ (see §3, §7.1). It is really satisfied by many differential equations in differential geometry and mathematical physics.

REMARK 2.32. The involutiveness of R implies the validity of Condition

2.3. Thus, the prolongation theorems [Kur, Mat] give a definite procedure to check Condition 2.3 for general R, although it seems rather complicated to carry it out actually.

2.4. The flat connection H_R. The bundle $R_\infty \to M$ inherits a flat connection from $N_\infty \to M$.

Put $H_R = H_N|_{R_\infty}$. Since $\Gamma H_N . \mathcal{J}_\infty \subset \mathcal{J}_\infty$, we have $H_R \subset TR_\infty$.

Proposition 2.41. *H_R is flat.*

Proof. Let i: $R_\infty \to N_\infty$ be the inclusion. Since $\Gamma H_R^\perp = i^* \Gamma H_N^\perp$, Proposition 1.73 implies

$$\begin{aligned} d\Gamma H_R^\perp &= i^* d\Gamma H_N^\perp \\ &\equiv 0 \qquad (\text{mod. } i^*\Gamma H_N^\perp = \Gamma H_R^\perp)\,. \end{aligned}$$

Q.E.D.

As for the flat sections of H_R, Proposition 1.81 implies

Proposition 2.42. *For $s \in \Gamma R_\infty$, the following conditions are mutually equivalent:*

(i) *s is flat,*

(ii) $s^* \Gamma H_R^\perp = 0$,

(iii) $s = (\pi_0 \circ s)_\infty$.

Thus $\mathcal{Sol}(R)$ can be identified with the set of flat sections of $R_\infty \to M$ with respect to H_R.

REMARK 2.43. The pair (R_∞, H_R) is a good substitute for $\mathcal{Sol}(R)$. It can express many important formal aspects of R concisely. For example, *an intermediate integral* is just a subbundle R' of R_∞ such that $H_R|_{R'} \subset TR'$.

2.5. The variation bicomplex. Let V_R be the subbundle of the vertical tangent vectors of $R_\infty \to M$. Then $TR_\infty \simeq V_R \oplus H_R$ and hence

$$\Omega^r R_\infty = \oplus_{p+q=r} \Omega_R^{p,q}$$

where $\Omega_R^{p,q} := \Gamma(\wedge^p V_R^* \otimes \wedge^q H_R^*)$. Since H_R is flat, d has only (1,0) and (0,1) components, which will be denoted by δ and $(-1)^p \partial$ on $\Omega_R^{p,q}$, respectively. $d^2 = 0$ implies then

$$\delta^2 = \partial^2 = \delta\partial - \partial\delta = 0\,.$$

Thus we obtained a bicomplex $\{\Omega_R^{*,*} = \oplus \Omega_R^{p,q}, \delta, \partial\}$, which is *the variation bicomplex of R.*

The flatness of an $s \in \Gamma R_\infty$ can be expressed using the variation bicomplex as follows:

Proposition 2.5. *For $s\in\Gamma R_\infty$ the following conditions are mutually equivalent.*

(i) *s is flat,*

(ii) $s^*\Omega_R^{+,*}=0$, *where* $\Omega_R^{+,*}=\oplus_{p>0}\Omega_R^{p,*}$,

(iii) $s^*\Omega_R^{1,q}=0$, *for some* $q\in \boldsymbol{Z}(0,n-1)$,

(iv) $s^*\Omega_R^{1,0}=0$,

(v) $ds^*f=s^*\partial f$, *for all* $f\in A(R)$.

Proof. Let $i: R_\infty\to N_\infty$ be the inclusion. Then $s\in\Gamma R_\infty$ is flat if and only if $i\circ s\in\Gamma N_\infty$ is flat. Moreover it is easy to see $i^*\Omega_N^{p,q}=\Omega_R^{p,q}$. Thus it suffices to prove the assertion when $R=N$.

Put $\Omega^{p,q}=\Omega_N^{p,q}$ for brevity. Since $\Gamma H_N^\perp=\Omega^{1,0}$, Proposition 1.81 implies (i)$\Leftrightarrow$(iv). Obviously (ii)$\Rightarrow$(iii), (iv)$\Rightarrow$(v). Thus it remains to show (iii)$\Rightarrow$(iv)$\Rightarrow$(ii) and (v)$\Rightarrow$(iv).

Obviously we may assume $N=M\times\boldsymbol{R}^m$, $M=\boldsymbol{R}^n$. Let (x^i, u_I^j) be the standard coordinates on N_∞.

Suppose (iii) holds, i.e., $s^*\Omega^{1,q}=0$ for some $q\in\boldsymbol{Z}(0, n-1)$. Put $s^*\delta u_I^j=\sum_{k=1,n}s_{I,k}^j dx^k$. Then

$$s^*(\delta u_I^j\wedge dx^1\wedge\cdots\wedge \check{dx^k}\wedge\cdots\wedge dx^{q+1})=0$$

implies $s_{I,k}^j=0$. Hence $s^*\delta u_I^j=0$ and (iv) follows.

Suppose (iv) holds, i.e., $s^*\Omega^{1,0}=0$. Then $s^*\delta u_I^j=0$, whence $s^*(\delta u_I^j\wedge\eta)=0$ ($^V\eta\in\Omega^*N_\infty$), i.e., (ii) is valid.

Finally, suppose (v) holds. Then

$$\begin{aligned} s^*\delta u_I^j &= s^*(du_I^j-\partial u_I^j)\\ &= ds^*u_I^j-s^*\partial u_I^j=0, \end{aligned}$$

whence $s^*\Omega^{1,0}=0$. Q.E.D.

2.6. Modifications of $\Omega_R^{*,*}$. According to the nature of problems concerned, it is necessary to modify the definition of $\Omega_R^{*,*}$.

First suppose a group G acts on R_∞ preserving H_R. Then the G-invariant elements of $\Omega_R^{*,*}$ form a subbicomplex of $\Omega_R^{*,*}$, which is denoted by $\Omega_{R;G}^{*,*}$.

Furthermore, suppose there is a G-invariant subalgebra B of $A(R)$. Suppose that the vector bundles H_R and V_R have systems of local frames whose transformation matrices have components in the sheaf of the germs of B. Then $\Omega_R^{*,*}$ has a subspace $\Omega_{R;B}^{*,*}$ consisting of those with coefficients in B with respect to these local frames. Put $\Omega_{R;G,B}^{*,*}=\Omega_{R;G}^{*,*}\cap\Omega_{R;B}^{*,*}$. When $\Omega^{*,*}=\Omega_{R;G,B}^{*,*}$ is a subbicomplex, B is called *admissible*. $\Omega^{*,*}$ is then called the *G-invariant* *variation bicomlex of R with coefficients in B.*

EXAMPLE 2.6. Let $M=\boldsymbol{R}_x^n$, $N=M\times\boldsymbol{R}_u^m$. Let (x^i, u_I^j) be the standard local coordinates of N_∞.

(i) The vector group $G=\boldsymbol{R}^n$ acts on N by the translation: $(x, (y, u))\mapsto(x+y, u)$ $(x, y\in\boldsymbol{R}^n, u\in\boldsymbol{R}^m)$, on ΓN by its action on graphs. and thus on N_∞. Then $\Omega_{R;G}^{*;*}$ consists of $\sum f\delta u_I^j\wedge\cdots\wedge dx^j\wedge\cdots$'s such that $f=f[u]$. We denote this bicomplex by $\Omega_{N(n,m),d.f.}^{*;*}$.

(ii) Let $G=(1)$. Put $B=\boldsymbol{R}[u_I^j]$. Then $\Omega_{R;G,B}^{*;*}=\Omega_{R;B}^{*;*}$ is a bicomplex. Its elements are written as $\sum f\delta u_I^j\wedge\cdots\wedge dx^i\wedge\cdots$, where f's are differential polynomials.

See also §3.2.

2.7. The spectral sequence. Let $\Omega^{*,*}=\Omega_{R;G,B}^{*;*}$. The filtration F on Ω^*R_∞ defined by

$$F^p\Omega^*R_\infty=\bigoplus_{p'\geq p}\Omega^{p',*}$$

is compatible with d and induces a spectral sequence $E(R,G,B)=\{E_r^{p,q}, d_r\}$. This is the one mentioned in the introduction and called *the G-invariant spectral sequence of the equation R with coefficients in B.*

REMARK. The spectral sequence associated to the other natural filtration reduces to the usual one for the fibering $R_\infty\to M$ when $G=(1)$, $B=A(R)$.

3. Examples

We consider various concrete differential equations and describe their variation bicomplexes using local coordinates. We also recall a few known results about their spectral sequences.

3.1. Trivial equations. First consider the trivial equation $R=N$, where $N=M\times\boldsymbol{R}^m\to M=\boldsymbol{R}^n$. Put $\Omega^{p,q}=\Omega_N^{p,q}$.

Let (x^i, u_I^j) be the standard coordinates. Define $\partial_i\in\mathcal{L}N_\infty$ and $\delta u_I^j\in\Omega^{1,0}$ by (1.61) and (1.71) respectively. Put $A=A(N)=\mathcal{F}[x,u]$. Then

$$\Omega^{p,q}=A\otimes\wedge^p[\delta u_I^j]\otimes\wedge^q[dx^i],$$

and the differentials are characterized by

$$\partial f=\sum\nolimits_{i=1,n}\partial_i f\otimes dx^i,$$
$$\delta f=\sum\nolimits_{I,j}\partial f/\partial u_I^j\otimes\delta u_I^j,$$

for $f\in A$.

The spectral sequence will be calculated in §5 (Theorem 5.1).

3.2. Riemannian metrics. Let M be a manifold and $N=S^2T^*M$, the symmetric product. Let $R\subset N$ be the subspace of the positive definite symmetric bilinear forms. Then $\mathcal{S}ol(R)=\Gamma R$ is just the space of the riemannian metrics on M.

The group $G=\mathrm{Diff}(M)$ acts on R in the obvoius way, which induces an action on R_∞ preserving H_R. Let B be the subalgebra of $A(R)$ generated locally by $\mathcal{F}M, (\partial/\partial x)^I g_{ij}, \det(g_{ij})^{-1}$. Let $\Omega^{*,*}=\Omega^{*,*}_{R,G,B}$, $E=E(R,G,B)$. The following theorem is suggested by Gelfand ([Ge]) as a specimen of topics in his "formal differential geometry".

Theorem ([Gi]). $E_1^{0,q}\simeq\mathrm{Pont}^q$ for $q\leqq n-1$,
$E_2^{0,q}\simeq\mathrm{Pont}^q$ for $q\leqq n$,

where Pont* *means the free graded commutative algebra generated by* p_i*'s* ($i\geqq 1$, $\deg p_i=4i$).

This is proved by Gilkey as a by-product of his characterization of the Pontrjagin forms.

Note that if $G=(1)$ and $B=A(R)$, then E is "trivial" (cf. Theorem 5.1).

3.3. Gelfand-Fuks cohomology. Let M be a manifold and put

$$N=\bigcup_{x\in M}\{(\omega_1,\cdots,\omega_n)\in(T_x^*M)^n;\ \omega_1,\cdots,\omega_n \text{ spanns } T_x^*M\}$$

($n=\dim M$), which is an open set of $\oplus_{i=1}^n T^*M$. Define $R\subset N_1$ by

$$R=\left\{(\omega)_1(x);\ \begin{matrix}x\in M \text{ and } \omega=(\omega_1,\cdots,\omega_n) \text{ is a local section}\\ \text{of } N \text{ near } x \text{ such that } d\omega_1=\cdots=d\omega_n=0\end{matrix}\right\}.$$

R_∞ may be considered as the infinite jet space of the local charts of M.

The group $G=\mathrm{Diff}(M)$ acts on R and hence on R_∞ preserving H_R. It is easy to identify the total complex of $\Omega^{*,*}_{R,G}$ with the standard cochain complex $C^*(W_n;\boldsymbol{R})$, where W_n is the topological Lie algebra of formal vector fields on n variables and $\boldsymbol{R}$ is the trivial W_n-module. Its cohomology $H^*(W_n;\boldsymbol{R})$ was calculated by Gelfand and Fuks and called by their names. They also determined $E=E(R,G)$ in their calculations:

Theorem 3.3. ([GF]).
(i) $E_1^{0,q}\simeq\wedge^q\boldsymbol{R}^n$,
$E_1^{p,q}=(0)$ *for* $q<n, p>0$.
(ii) *For* $r\geqq 2$,
$E_r^{p,q}=(0)$ *if* $q\neq n$ *and* $(p,q)\neq 0$,
$E_r^{p,n}\simeq H^{n+p}(W_n,\boldsymbol{R})$.

3.4. Wave equation. Let $M=\boldsymbol{R}^n$ ($n\geqq 2$), $N=M\times\boldsymbol{R}$. Let R be the wave equation, i.e.,

$$R=\{u_{2(1)}-\sum_{j=2,n}u_{2(j)}=0\}\subset N_2\,.$$

Then $\mathcal{J}_\infty$ (cf. §2.2) is generated by $\{u_{I+2(1)}-\sum_{j=2,n}u_{I+2(j)};\,I\in\boldsymbol{N}^n\}$. Thus R_∞ is a submanifold of N_∞ with

$$\{x^i,u_{j,J};\,i\in\boldsymbol{Z}(1,n),\ j=0,1,\ J\in\boldsymbol{N}_{2,n}\}$$

as a global coordinates, where $u_{j,J}=u_{j(1)+J}|_{R_\infty}$. In particular Condition 2.3 is satisfied.

A global frame of H_R^+ is given by $\{\delta u_{j,J};\,j=0,1,\,J\in\boldsymbol{N}_{2,n}\}$, where $\delta u_{i,J}=\delta u_{j(1)+J}|_{R_\infty}$. Using the above coordinates on R_∞,

$$\begin{aligned}\delta u_{0,J}&=du_{0,J}-u_{1,J}dx^1-\sum_{j=2,n}u_{0,J+(j)}dx^j\\ \delta u_{1,J}&=du_{1,J}-(\sum_{j=2,n}u_{0,J+2(j)})dx^1-\sum_{j=2,n}u_{1,J+(j)}dx^j\,.\end{aligned}$$

Then

$$\Omega_R^{p,q}=A(R)\otimes\wedge^p[\delta u_{j,J}]\otimes\wedge^q[dx^i]\,.$$

The lift ∂_i of $\partial/\partial x^i$ is characterized by $\langle\partial_i,dx^j\rangle=\delta_{i,j}$ and $\langle\partial_i,\delta u_{j,J}\rangle=0$, i.e.,

$$\begin{aligned}\partial_i x^j&=\delta_{i,j}\\ \partial_1 u_{0,J}&=u_{1,J}\\ \partial_1 u_{1,J}&=\sum_{j=2,n}u_{0,J+2(j)}\\ \partial_i u_{j,J}&=u_{j,J+(i)}\qquad\text{for } i\geqq 2\,.\end{aligned}$$

The differential δ is characterized by $x^i\mapsto 0$, $u_{j,J}\mapsto\delta u_{j,J}$ and ∂ by $x^i\mapsto dx^i$, $u_{j,J}\mapsto\sum_{i=1,n}\partial_i u_{j,J}dx^i$.

The $\Omega_R^{*,*}$ has a subbicomplex $\Omega_{R,B}^{*,*}$ with $B=\boldsymbol{R}[u_{j,J}]\subset A(R)$ (cf. Example 2.6). Its $E_1^{0,n-1}$ and $E_1^{1,n-1}$ can be calculated using the result of §7 (cf. [T1]).

The following $(0,n-1)$-form represents a non-zero class in $E_1^{0,n-1}$:

(3.41) $\quad(u_{1,0}^2+\sum_{j=2,n}u_{0,(j)}^2)dx_1-2\sum_{j=2,n}u_{1,0}u_{0,(j)}dx_j\,.$

See Remark 4.51(ii).

3.5. Korteweg-de Vries equation. Let $N=M\times\boldsymbol{R}\to M=\boldsymbol{R}^2$ and $(x^k,u_{i,j})$ the standard coordinates on N_∞, where $u_{i,j}=u_{(i,j)}$. Let R be the Korteweg-de Vries equation, i.e.,

$$R=\{K:=u_{3,0}+uu_{1,0}+u_{0,1}=0\}\subset N_3\,.$$

Let $\tilde{\partial}_i$ be the lift of $\partial/\partial x^i$ $(i=1,2)$ on N_∞. Then $\mathcal{J}_\infty$ is generated by $\{\tilde{\partial}^I K;\,I\in\boldsymbol{N}^2\}$. Since $\{x^i,u_{j,0};\,i=1,2,\,j\in\boldsymbol{N}\}\cup\{\tilde{\partial}^I K;\,I\in\boldsymbol{N}^2\}$ is a global chart on N_∞ (cf. Proposition 7.12), R_∞ is a submanifold with (x^i, u_j) as a global chart, where $u_j = u_{j,0}|_{R_\infty}$. Hence Condition 2.3 is satisfied.

Let $\partial_i = \tilde{\partial}_i|_{R_\infty}$. This is tangent to R_∞ and characterized by $\partial_i x^j = \delta_{i,j}$, $\partial_1 u_j = u_{j+1}$, and $\partial_2 u_j = -\partial_1^j(uu_1 + u_3)$.

Then $H_R^\perp$ is spanned by $\{\delta u_j; j \in \boldsymbol{N}\}$, where

$$\delta u_j := \delta u_{j,0}|_{R_\infty} = du_j - \textstyle\sum_{i=1,2} \partial_i u_j dx^i ,$$

and thus

$$\Omega_R^{p,q} = A(R) \otimes \wedge^p[\delta u_j] \otimes \wedge^q[dx^i] .$$

The differential δ is determined by $x^i \mapsto 0$, $u_i \mapsto \delta u_j$ and ∂ by $x^i \mapsto dx^i$, $u_j \mapsto \sum_i \partial_i u_j dx^i$.

The vector group $G = \boldsymbol{R}^2$ acts on N_∞ as in Example 2.6(i). Obviously R_∞ is G-invariant. The subbicomplex $\Omega_{R;G}^{*,*}$ is described as

$$\Omega_{R;G}^{p,q} = \mathcal{F}(u_k) \otimes \wedge^p[\delta u_j] \otimes \wedge^q[dx^i] .$$

Note that if dx^2 is put to zero in $\Omega_{R;G}^{*,*}$, then we obtain $\Omega_{N(1,1),\mathrm{d.f.}}^{*,*}$ (cf. Example 2.6(i)). Moreover, ∂_2 acts on $\Omega_{N(1,1),\mathrm{d.f.}}^{*,*}$ as a derivation. Thus $\Omega_{R;G}^{*,*}$ can be replaced without losing any information by the pair $(\Omega_{N(1,1),\mathrm{d.f.}}^{*,*}, \partial_2)$, where only ∂_2 depends on R. This is used in [GD], where the bicomplex $\Omega_{R;G}^{*,*}$ is implicitly used.

$E_1^{0,1}(R)$ is essentially calculated in [MGK, KMGZ]. See also §7.62 and Example 6.71.

3.6. Integrability of connections. Connections on the line bundle $M = \boldsymbol{R}_x^n \to L = \boldsymbol{R}_{x'}^{n-1}$ $(x' = (x^2, \cdots, x^n))$ is given uniquely by a 1-form $\omega = dx^1 - \sum_{i=2,n} u^i(x) dx^i$. Its integrability condition is $d\omega \equiv 0$ (mod. ω), i.e.,

$$(3.61) \qquad K_{ij} := u_j^i - u_i^j + u^j u_1^i - u^i u_1^j = 0, \ \ i, j \in \boldsymbol{Z}(2, n) ,$$

where $u_j^i = \partial u^i / \partial x^j$.

Let R be the submanifold of N_1 defined by (3.61). Here $N = M \times \boldsymbol{R}_u^{n-1}$ $(u = (u^2, \cdots, u^n))$, (x^i, u_j^i) is the standard coordinate and $u_j^i = u_{(j)}^i$.

It is easy to see that R_∞ is a submanifold of N_∞ with $\{x^i, u_J^j;\ i \in \boldsymbol{Z}(1, n),\ j \in \boldsymbol{Z}(2, n),\ J \in \boldsymbol{N}_{1,j}\}$ as a global chart. Here $u_J^j = u_J^j | R_\infty$.

The variation bicomplex is described as

$$\Omega^{p,q} = A(R) \otimes \wedge^p[\delta u_J^j] \otimes \wedge^q[dx^i] ,$$
$$\partial f = \textstyle\sum_{i=1,n} \partial_i f dx^i ,$$

where

$$\partial_i u_J^j = \begin{cases} u_{J+(i)}^j & \text{for } i \leqq j , \\ \partial^J(u_j^i + u^j u_1^i - u^i u_1^j) & \text{for } j < i \end{cases}$$

($j\in \boldsymbol{Z}(2,n)$). Note that the right hand side for the case $j<i$ makes sense since $J\in \boldsymbol{N}_{1,j}\subset \boldsymbol{N}_{1,i}$.

It is easy to see that

$$\omega = \textstyle\sum_{i,j\in \boldsymbol{Z}(2,n)}\sum_{k\in \boldsymbol{Z}(1,n)} u^i_1 u^j_{(1)+(k)} dx^i\wedge dx^j\wedge dx^k$$

is ∂-closed in $\Omega^{0,3}$. When $n\geqq 4$, it can be proved that $[\omega]$ is non-zero in $E_1^{0,3}$. This corresponds to the Godbillon-Vey class of foliations ([GV]). See Remark 4.51(iii).

Note that this equation is typical among the overdetermined systems that arise in the problems of integrability in defferential geometry.

3.7. Foliations. Let M be a manifold of dimension n. Let $N=Gr_pM$ be the Grassmann bundle of p-planes, where $p\in \boldsymbol{Z}(1,n-1)$. Define $R\subset N_1$ to be the equation of integrability of p-plane fields on M. $\mathcal{S}ol(R)$ is just the space of foliations of codimension q on M ($q=n-p$).

The space R_∞ can be expressed as follows. Let $G(n)$ be the group of the infinite jets of germs of diffeomorphisms of $\boldsymbol{R}^n$ preserving the origin, and $G(n,p)$ the subgroup consisting of those preserving the foliation $x^i=$constant ($i\in \boldsymbol{Z}(p+1,n)$). Let PM be the infinite frame bundle of M. This is a principal $G(n)$-bundle and identical with the R_∞ of §3.3. Then $R_\infty=PM/G(n,p)$.

The group $G=\mathrm{Diff}(M)$ acts on R in the obvious way. Put $\Omega^{*,*}=\Omega^{*,*}_{R;G}$, $E=E(R,G)$.

Since PM is G-homogeneous, we can describe the total complex $\Omega^*=\{\Omega^{*,*}, \delta\pm\partial\}$ algebraically. Let $L_{n,p}$ be the subalgebra of W_n (cf. §3.3) spanned by $\{\sum_{i=1,p} f_i(x)\partial/\partial x^i+\sum_{j=1,q} g_j(y)\partial/\partial y^j\}$, where $y^i=x^{p+i}$ ($i\in \boldsymbol{Z}(1,q)$) and f_i, g_j are formal power series without the constant terms. Then $\Omega^*\simeq C^*(W_n, L_{n,p}; \boldsymbol{R})$, where the right hand side is the relative continuous cohomology (cf. [HS]), a subcomplex of $C^*(W_n; \boldsymbol{R})$. The filtration of Ω^* corresponds to the restriction of that on $C^*(W_n; \boldsymbol{R})$ induced by the subalgebra $\sum_{i=1,n}\boldsymbol{R}.\partial/\partial x^i$.

3.8. Complex structures. Let M be an oriented manifold of dimension $2n$. Let N be the bundle of complex structures on the tangent spaces compatible with the orientation. ΓN is the space of almost complex structures on M. The integrability condition is given by a subbundle $R\subset N_1$. $\mathcal{S}ol(R)$ is the space of complex structures on M.

The space R_∞ can be described as follows. Let $G^+(2n)$ and $G^{\boldsymbol{C}}(n)$ be the subgroups of $G(2n)$ consisting of orientation preserving ones and bihomolomorphic ones on $\boldsymbol{C}^n=\boldsymbol{R}^{2n}$ respectively. The infinite jet space P^+M of local charts compatible with the orientation is a principal $G^+(2n)$-bundle. Then $R_\infty=P^+M/G^{\boldsymbol{C}}(n)$.

The group $G=\mathrm{Diff}^+(M)$ of the orientation preserving diffeomorphisms acts on R in the obvious way. Put $\Omega^{*,*}=\Omega^{*,*}_{R;G}$, $E=E(R,G)$.

As in §3.7, $\Omega^*=\{\Omega^{*,*}, \delta\pm\partial\}$ is isomorphic to the relative Gelfand-Fuks cohomology $C^*(W_{2n}, L_n^{\boldsymbol{C}}; \boldsymbol{R})$, where $L_n^{\boldsymbol{C}}$ is the subalgebra of holomorphic vector fields on $\boldsymbol{C}^n=\boldsymbol{R}^{2n}$ vanishing at the origin.

REMARK 3.81. Neither $H^*(W_n, L_{n,p}; \boldsymbol{R})$ nor $H^*(W_{2n}, L_n^{\boldsymbol{C}}; \boldsymbol{R})$ seems to have been calculated completely. However it is not difficult to show $H^i(W_2, L_1^{\boldsymbol{C}}; \boldsymbol{R}) \simeq \boldsymbol{R}$ $(i=0)$, $\boldsymbol{R}^2$ $(i=6)$, (0) $(i\neq 0, 6)$.

See Remark 4.82 and (10.35).

4. Interpretations of the spectral sequences

Fix $N\to M$ and $R\subset N_k$ satisfying Condition 2.3. Put $\Omega^{*,*}=\Omega_R^{*,*}$, $E=E(R)$. This section gives some interpretations of the E_1-, E_2- and E_∞- terms. Roughly, they give us formal differential calculus on the space $\mathcal{Sol}(R)$.

4.1. Solution manifolds. A map σ from a manifold X to ΓN is called *smooth* if the map $\tilde{\sigma}\colon X\times M\to N$ defined by $\tilde{\sigma}(x,y)=\sigma(x)(y)$ $(x\in X, y\in M)$ is smooth.

A map $\sigma\colon X\to\mathcal{Sol}(R)$ is called *smooth* if its composition with the inclusion $\mathcal{Sol}(R)\to\Gamma N$ is smooth. A smooth map $\sigma\colon X\to\mathcal{Sol}(R)$ is called *a solution manifold.*

A solution manifold $\sigma\colon X\to\mathcal{Sol}(R)$ induces a smooth map $\tilde{\sigma}_\infty\colon X\times M\to R_\infty$ defined by $\tilde{\sigma}_\infty(x,y)=(\sigma(x))_\infty(y)$ $(x\in X, y\in M)$. Let $\sigma^*\colon\Omega^*R_\infty\to\Omega^*(X\times M)$ be the induced map. $\Omega^*(X\times M)$ has an obvious underlying bicomplex structure $\{\Omega^{*,*}(X\times M), d_X, d_M\}$.

Proposition 4.11. *σ^* is a bicomplex map.*

Proof. Let $(x,y)\in X\times M$, $z=\sigma(x,y)$. Let $\tau\colon T_xX\oplus T_yM\to T_zR=V_z\oplus H_z$ $(V=V_R, H=H_R)$ be the differential of $\tilde{\sigma}_\infty$ at (x,y).

We have $\tau T_yM=H_z$ by definition. On the other hand $\tau T_xX\subset V_z$ since $\pi_\infty\circ\tilde{\sigma}_\infty$ is the projection $X\times M\to M$. Thus $\tau^*\Omega^{p,q}\subset\Omega^{p,q}(X\times M)$. Q.E.D.

We call σ^* *the characteristic map of* the solution manifold $\sigma\colon X\to\mathcal{Sol}(R)$.

4.2. Characteristic maps. Let $\{'E_r^{p,q}, 'd_r\}$ be the spectral sequence induced by the filtration $'F^p=\oplus_{p'\geq p}\Omega^{p',*}(X\times M)$. Recall that

$$'E_1^{p,q}\simeq\Omega^pX\otimes H^q(M, \boldsymbol{R}),$$
$$'E_2^{p,q}\simeq H^p(X, \boldsymbol{R})\otimes H^q(M, \boldsymbol{R}),$$

and $\{'E_1^{*,q}, 'd_1\}$ is isomorphic to the $H^q(M, \boldsymbol{R})$-valued de Rham complex on X.

Since σ^* preserves the filtrations, we obtain a spectral sequence homomorphism $\{\sigma_r\}\colon\{E_r^{p,q}\}\to\{'E_r^{p,q}\}$. In particular, σ induces $\sigma_1\colon E_1^{p,q}\to\Omega^pX\otimes H^q(M, \boldsymbol{R})$, $\sigma_2\colon E_2^{p,q}\to H^p(X, \boldsymbol{R})\otimes H^q(M, \boldsymbol{R})$, and a homomorphism from $\{E_1^{*,q},$

$d_1\}$ to the $H^q(M, \boldsymbol{R})$-valued de Rham complex of X.

We call these also *the characteristic maps of* σ.

Example 4.21. Let R be as in §3.7.
We can show that there is a natural spectral sequence map:

$$E(W_q, o_q) \to E(R)$$

(cf. [T3]). Here o_q is the subalgebra of orthogonal vector fields and $E(W_q, o_q)$ is the o_q-basic spectral sequence for the Weil algebra of W_q: For example

$$E_1^{i,j}(W_q, o_q) = H^j(W_q, o_q; S^i W_q') ,$$

where W_q' is the dual W_q-module of W_q and $S^i W_q'$ its i-th symmetric product. Thus the Gelfand-Fuks cohomologies $\{H^j(W_q, o_q; S^i W_q')\}$ give us "characteristic classes" for families of foliations on a fixed manifold. These cohomologies are not yet calculated except for $i=0$ ([GF]) and $i=1$ ([GFF]).

4.3. Homotopy invariance. Let $\sigma: X \to \mathcal{Sol}(R)$ be a solution manifold and $f: Y \to X$ a smooth map. Then $\sigma\circ f: Y \to \mathcal{Sol}(R)$ is a solution manifold. Obviously we have

Proposition 4.31.
$$(\sigma\circ f)^* = (f\times \mathrm{id}_M)^*\circ\sigma^* ,$$
$$(\sigma\circ f)_2 = (f^*\otimes \mathrm{id})\circ\sigma_2 .$$

Let $\sigma', \sigma'': X \to \mathcal{Sol}(R)$ be solution manifolds. σ' and σ'' are called *homotopic* if there is a solution manifold $\sigma: X\times(0,3) \to \mathcal{Sol}(R)$ such that $\sigma|_{X\times\{1\}}=\sigma'$, $\sigma|_{X\times\{2\}}=\sigma''$. The above proposition implies

Corollary 4.32. *Suppose* $\sigma', \sigma'': X \to \mathcal{Sol}(R)$ *are homotopic. Then* $\sigma_r'=\sigma_r''$ $(r\geqq 2)$.

Corollary 4.33. *Suppose a solution manifold* $\sigma: X \to \mathcal{Sol}(R)$ *induces a nonzero map* σ_2, *then* σ *is not homotopic to a constant map.*

Thus $E_2^{p,q}$ detects nonzero elements of $[X, \mathcal{Sol}(R)]$, the space of homotopy classes of solution manifolds. In this way $E_2^{*,*}$ provides a potential tool for studying the homotopy structure of $\mathcal{Sol}(R)$.

4.4. Topological obstructions. The space $E_\infty^{*,*}$ also provides a useful tool for the study of R.

Let $\sigma: X \to \Gamma R_\infty$ be smooth, i.e., the associated map $\tilde{\sigma}: X\times M \to R_\infty$ is smooth. When can we deform $\tilde{\sigma}$ to the $\tilde{\tau}_\infty$ of a solution manifold $\tau: X \to \mathcal{Sol}(R)$?

Suppose $\tilde{\sigma}$ can be deformed to a $\tilde{\tau}_\infty$. Since $\tau^* F^{p+1}\Omega^*=0$ $(p=\dim X)$, we have $\tau^* F^{p+1} H^*(R_\infty, \boldsymbol{R})=0$, where

$$F^{p+1}H^*(R_\infty, \boldsymbol{R}) = \mathrm{Im}(H^*(F^{p+1}\Omega^*)\to H^*\Omega^*) .$$

Since $\tau^*=\bar{\sigma}^*: H^*(R_\infty, \boldsymbol{R})\to H^*(X, \boldsymbol{R})\otimes H^*(M, \boldsymbol{R})$, we obtain the following

Proposition 4.4. *Suppose a smooth map $\sigma: X\to \Gamma R_\infty$ induces a nonzero map $\sigma^*: F^{p+1}H^*(R_\infty, \boldsymbol{R})\to H^*(X, \boldsymbol{R})\otimes H^*(M, \boldsymbol{R})$ ($p=\dim X$). Then σ can not be deformed to a solution manifold.*

Thus $\oplus_{p'>p}E_\infty^{p',*}\simeq F^{p+1}H^*(R_\infty, \boldsymbol{R})$ gives us potential obstructions to the deformability of a homotopy solution manifold of dimension p to a solution manifold. (See §4.8).

4.5. Characteristic classes. From now, we specialize to the case $p=0$, and relate the notions of §4.2–4 with the usual ones.

Consider $s\in \mathcal{S}ol(R)$ as a solution manifold $\bar{s}$: $\{s\}\to \mathcal{S}ol(R)$. By §4.2, $\bar{s}$ induces $\bar{s}_1: E_1^{0,q}\to H^q(M, \boldsymbol{R})$. This map is also given simply as follows. Since $E_1^{0,q}=H^q(\{\Omega^{0,*}, \partial\})$, $E_1^{0,q}\ni\omega$ is represented by a ∂-closed $\omega'\in\Omega^{0,q}$. Then $\bar{s}_1\omega=[s_\infty^*\omega']$. Note that by Proposition 2.5 $ds_\infty^*\omega'=s_\infty^*\partial\omega'=0$. Thus $\omega\in E_1^{0,q}$ defines a map $\underline{\omega}$: $\mathcal{S}ol(R)\to H^q(M, \boldsymbol{R})$. We call $\underline{\omega}(s)$ *the ω-characteristic class of* $s\in \mathcal{S}ol(R)$.

REMARK 4.51. The elements of $E_1^{0,*}$ appeared mainly in three different contexts:

(i) The Pontrjagin classes of Riemannian metrics (cf. §3.2).

(ii) When R is an ordinary differential equation, $E_1^{0,0}$ is the space of *constants of motions*. When R is an evolution equation, $E_1^{0,n-1}$ ($n=\dim M$) is the space of *conserved currents* of R.

Suppose for example $M=\boldsymbol{R}^n$, $N=M\times \boldsymbol{R}^m$ and $R\subset N_k$ satisfies Condition 2.3. Then an element of $E_1^{0,n-1}$ is represented by a ∂-closed $\omega\in\Omega^{0,n-1}$, which we write as $\omega=\sum_{i=1,n}J_i dx_i$ ($J_i\in A(R)$). Then ∂-closedness is equivalent to $\partial_1 J_1+\cdots+\partial_n J_n=0$. Thus, if $u(x)=(u^i(x))\in \mathcal{S}ol(R)$, then $J_i[u]:=u^*J_i\in\mathcal{F}M$ satisfies, by (1.61),

$$\partial J_1[u]/\partial x^1 = -(\partial J_2[u]/\partial x^2+\cdots+\partial J_n[u]/\partial x^n) .$$

Thus if $J_i[u]$'s decay rapidly at infinity, the integral $\int_{\boldsymbol{R}^{n-1}} J_1[u]dx^2\cdots dx^n$ does not depend on x^1, i.e., it is a conserved quantity. For example, the class represented by (3.41) corresponds to the conservation of energy for the wave equation.

(iii) By Example 4.21, when R is as in §3.7, there is a map: $H^*(W_q, o_q; \boldsymbol{R})\to E_1^{0,*}$. These are the secondary characteristic classes of foliations on M. See also §3.6 for the particular case $q=1$.

REMARK 4.52. Nontriviality of $\underline{\omega}$'s is difficult to show in general. [F1] announces a program of showing this, and [St] says that Petro proved it by a different method.

4.6. Deformation of solutions. Let $\sigma: (-\varepsilon, \varepsilon) \to \mathcal{S}ol(R)$ $(\varepsilon > 0)$ be a solution manifold, which is also called *a deformation of* $s = \sigma(0) \in \mathcal{S}ol(R)$.

Let $\eta \in E_1^{1,q}$. By §4.2, we have an element $\sigma_1(\eta) \in \Omega^1(-\varepsilon, \varepsilon) \otimes H^q(M, \boldsymbol{R})$. Define $\underline{\eta}(\sigma) = (\partial/\partial t \lrcorner \sigma_1(\eta))|_{t=0} \in H^q(M, \boldsymbol{R})$, where t is the standard coordinate on $(-\varepsilon, \varepsilon)$. We call $\underline{\eta}(\sigma)$ *the* η*-characteristic class of the deformation* σ *of* s.

Let $\omega \in E_1^{0,q}$. Then

Proposition 4.6. $\partial/\partial t(\underline{\omega}(\sigma(t)))|_{t=0} = \underline{d_1\omega}(\sigma)$.

Proof. By Proposition 4.31, $\underline{\omega}(\sigma(t)) = \sigma_1(\omega)(t)$, where $\sigma_1(\omega) \in \Omega^0(-\varepsilon, \varepsilon) \otimes H^q(M, \boldsymbol{R})$ is regarded as a $H^q(M, \boldsymbol{R})$-valued function on $(-\varepsilon, \varepsilon)$. Since d_1 corresponds to $d_t = d_{(-\varepsilon, \varepsilon)}$, we have

$$\begin{aligned} \underline{d_1\omega}(\sigma) &= \partial/\partial t \lrcorner \sigma_1(d_1\omega)|_{t=0} \\ &= \partial/\partial t \lrcorner d_t \sigma_1(\omega)|_{t=0} \\ &= (\partial/\partial t)\sigma_1(\omega)|_{t=0} \\ &= (\partial/\partial t)\underline{\omega}(\sigma(t))|_{t=0}\,. \end{aligned}$$

Q.E.D.

Thus the differential $d_1: E_1^{0,q} \to E_1^{1,q}$ describes the "universal differentiation" of the characteristic classes of solutions.

EXAMPLE 4.61. By Example 4.21, $H^*(W_q, o_q; W_q')$ gives us characteristic classes of variations of foliations (cf. [GFF, He]). See also 10.35 (ii).

4.7. Rigid classes. Let $\zeta \in E_2^{0,q} = \mathrm{Ker}(d_1: E_1^{0,q} \to E_1^{1,q})$. By Proposition 4.6, $t \mapsto \underline{\zeta}(\sigma(t))$ is constant for any solution manifold $\sigma: (-\varepsilon, \varepsilon) \to \mathcal{S}ol(R)$, and so ζ may be called *a rigid characteristic class.*

REMARK 4.71. Rigid characteristic classes have appeared in foliation theory by the same name and in [AI] as "weak Lagrangeans" of a differential equations. Note however that when R is the Euler-Lagrange equation of a Lagrangean L homogeneous in u_i^j's, then the class $[L|_{R_\infty} dx]$ is necessarily zero by virtue of the Galindo-Martinez formula (5.43).

REMARK 4.72. Even if $\zeta \in E_1^{0,q}$ is not rigid, it can be constant for deformations of solutions. In fact the variability of non-rigid classes is difficult to show in general. In foliation theory, [F2] proposes a program of showing this.

REMARK 4.73. Each $\xi \in H^q(R_\infty, \boldsymbol{R})$ defines $\underline{\xi}: \mathcal{S}ol(R) \to H^q(M, \boldsymbol{R})$ by $\underline{\xi}(s) = s_\infty^* \xi$. Let $\lambda: H^q(R_\infty, \boldsymbol{R}) \to E_\infty^{0,q}$ be the projection, and $\mu: E_\infty^{0,q} \hookrightarrow E_1^{0,q}$. Then

$\underline{\xi}=\underline{\mu\lambda(\xi)}$ obviously and $\underline{\xi}$ is rigid. In foliation theory, elements in $\mathrm{Im}(\mu\lambda)$ are often called the primary characteristic classes, whereas those not in $\mathrm{Im}(\mu\lambda)$ the secondary characteristic classes. Note that $\mu\lambda$ is neither injective nor surjective generally. $\mathrm{Ker}\,\mu\lambda=F^1H^q(R_\infty, \boldsymbol{R})$ has an important meaning as is shown by Proposition 4.4. See also the next paragraph.

4.8. Bott's vanishing theorem. Because of its importance, we rewrite Proposition 4.4 for the case $p=0$.

Proposition 4.8. *Let $\omega\in E_\infty^{p,q}$ $(p>0)$ be represented by $\omega'\in F^pH^{p+q}(R_\infty, \boldsymbol{R})$. Suppose a homotopy solution $f\in\Gamma R_\infty$ satisfies $f^*\omega'\neq 0$. Then there is no solution g such that g_∞ is homotopic to f.*

EXAMPLE 4.81. Let R be as in §3.7., i.e., the integrability equation of plane fields of codimension q. Let $F\to N$ be the tautological vector bundle: $F_x(x\in N)$ is the p-plane in $T_{px}M$ corresponding to x. Put $\nu=p^*TM/F$, the universal normal bundle. Let $\nu_\infty=\pi_0{}^*\nu$ and denote by p_i the *i-th* Pontrjagin class of ν_∞. Then it is easy to see

$$p_i\in F^1H^{4i}(R_\infty, \boldsymbol{R}) \qquad \text{for } 4i>2q\,.$$

Let $f\in\Gamma N$, i.e., a plane field. Since $R_\infty\to R$ and $R\to N$ are both affine bundles, there is a lifting $\tilde{f}\in\Gamma R_\infty$ such that $\pi_0\tilde{f}=f$. Obviously $\tilde{f}^*p_i$ is the *i-th* Pontrjagin class of the normal bundle of f. Thus if $p_i(f)\neq 0$ for some $i>q/2$, then f is not homotopic to an integrable one, i.e., a foliation. This is just the Bott's vanishing theorem.

REMARK 4.82. The Bott's vanishing theorem and its variants ([Bot]) are the only cases where some nonzero elements of $E_\infty^{+,*}=\oplus_{p>0}E_\infty^{p,*}$ were constructed.

It seems interesting to calculate $E_\infty^{+,*}$ for the R of §3.8, i.e., the integrability equation of almost complex structures. In contrast to the case of foliations, we cannot get universal obstruction by taking $G=\mathrm{Diff}(M)$. In fact, with this G, we have $E_\infty^{+,*}\simeq H^+(W_{2n}, L_n^C; \boldsymbol{R})$. But it is not difficult to see that $f^*E_\infty^{+,*}(f\in\Gamma R_\infty)$ can be expressed by Chern classes of f. Then they must be identically zero, because of the independency of the Chern classes of complex manifolds.

However for $G=(1)$, $E_\infty^{+,*}$ may not be zero. Note that in this case $E_\infty^{+,*}$ of an M can differ from that of another M.

5. Trivial equations

We calculate the spectral sequence of the trivial equation $R=N$. Most of the results of this section have been obtained by Vinogradov in [V1] where he relates $\{E_r\}$ with the Spencer cohomologies. We give here a self-contained proof which uses only a simple fact about the Koszul complexes. The results produce some useful facts about the Euler-Lagrange operators, which are now more or less well-known.

In this section $R=N\to M$ will be fixed and $\Omega^{*,*}=\Omega_R^{*,*}$, $E=E(R)$. Put $n=\dim M$, $m=\dim N-n$.

5.1. Statement of theorems.

Theorem 5.1. *Suppose* $n\geqq 1$.

(i) *For* $r\geqq 1$,

$$E_r^{0,q}\simeq H^q(N,\boldsymbol{R}) \quad \text{if } q\leqq n-1,$$
$$E_r^{p,q}\simeq(0) \quad \text{if } p>0,\ q\neq n.$$

(ii) *For* $r\geqq 2$,

$$E_r^{p,n}\simeq H^{n+p}(N,\boldsymbol{R}).$$

We specialize to the case $N=N(n,m)$.

Let $V=\boldsymbol{R}^n$ and S_*V the symmetric algebra. The abelian Lie algebra V acts on S_*V, on $\boldsymbol{R}^m\otimes S_*V$ and then on $W_p:=\wedge^p(\boldsymbol{R}^m\otimes S_*V)$. This induces an action of the algebra S_*V on W_p. Put $B_p=W_p/V.W_p$. Note that $B_1\simeq\boldsymbol{R}^m$.

Theorem 5.1 can be refined for $N=N(n,m)$ as follows. See §3.1 for the notations.

Theorem 5.11. *Suppose* $M=\boldsymbol{R}^n$, $R=N=M\times\boldsymbol{R}^m$. *Then*

(i) *For* $r\geqq 1$, $E_r^{0,0}=\boldsymbol{R}$, $E_r^{p,q}\simeq(0)$ $(q\neq n,\ (p,q)\neq(0,0))$.

(ii) *For* $r\geqq 2$, $E_r^{p,q}=(0)$ *if* $(p,q)\neq(0,0)$.

(iii) $E_1^{0,n}\simeq A/\partial_1A+\cdots+\partial_nA$,

(iv) $E_1^{p,n}\simeq A\otimes B_p$, *for* $p>0$.

(v) *The map* $\alpha\colon A^m\to E_1^{1,n}$ *defined by* $\alpha(f_1,\cdots,f_m)=[\sum f_i\delta u^i\wedge dx]$ *induces the isomorphism of* (iv) $(p=1)$.

Corollary 5.12. *An element* $\sum f_{i,I}\delta u_I^i\wedge dx$ *of* $\Omega^{1,n}$ *is uniquely expressed as* $\sum f_i\delta u^i\wedge dx$ *modulo* $\partial\Omega^{1,n-1}$, *where* $f_i=\sum_{I\in N^n}(-\partial)^I f_{i,I}$.

Proof of Corollary 5.12. Uniqueness follows from (v) of the theorem. It suffices to show then that

$$\sum f_{i,I}\delta u_I^i\wedge dx-\sum f_i\delta u^i\wedge dx\in\partial\Omega^{1,n-1}.$$

This follows from

Lemma 5.121. *For* $f\in A$,

$$\begin{aligned}&(f\delta u^i_{\alpha_1\cdots\alpha_k}-(-\partial_{\alpha_1})\cdots(-\partial_{\alpha_k})f\delta u^i)\wedge dx\\ &=\partial\{\textstyle\sum_{j=1,k}(-\partial_{\alpha_1})\cdots(-\partial_{\alpha_{j-1}})f\delta u^i_{\alpha_{j+1}\cdots\alpha_k}\wedge dx_{\alpha_j}\},\end{aligned}$$

where $\alpha_1,\cdots,\alpha_k\in\boldsymbol{Z}(1,n)$, $i\in\boldsymbol{Z}(1,m)$ *and* $\delta u^i_{\alpha_1\cdots\alpha_k}:=\delta u_I^i$ $(I=(\alpha_1)+\cdots+(\alpha_k))$.

This can be easily verified, whence the Corollary 5.12. Q.E.D.

5.2. Proof of Theorem 5.1 assuming Theorem 5.11. Since E converges to $H^*(N_\infty, \boldsymbol{R})$, which is equal to $H^*(N, \boldsymbol{R})$ by Proposition 1.21, it suffices to show $E_1^{p,q}=0$ for $p>0$ and $q\leqq n-1$.

Fix $p>0$ and put $C^q=\Omega^{p,q}$. Let $\underline{C}^*$ be the sheaf of complexes on N_∞ corresponding to $C^*=\oplus C^q$.

Let $\hat{C}^*$ be the presheaf of complexes over N defined by $U\mapsto\Gamma(\pi_0^{-1}U, \underline{C}^*)$. Obviously $\hat{C}^*$ is a fine sheaf on N and hence we have a spectral sequence $'E$ converging to $H^*\Gamma(N, \hat{C}^*)=H^*C^*$ such that

$$'E_2^{p,q}\simeq\check{H}^p(N, \mathcal{H}^q\hat{C}^*)$$

([God]). Here $\mathcal{H}^q\hat{C}^*$ denotes the cohomology sheaf of $\hat{C}^*$. But by Theorem 5.11, $\mathcal{H}^q\hat{C}^*=(0)$ if $q\leqq n-1$. Hence $'E_2^{p,q}=0$ if $q\leqq n-1$. Thus $H^qC^*=0$ ($q\leqq n-1$). Q.E.D.

5.3. Proof of Theorem 5.11. Before proving Theorem 5.11, We cite a simple result on the cohomologies of the abelian Lie algebras.

Let $V=\boldsymbol{R}^n$ be the abelian Lie algebra and W a V-module. Let $\{C^*(V; W), d\}$ be the standard cochain complex: $C^p(V, W)=\wedge^pV^*\otimes W$ and

$$d\omega(X_1, \cdots, X_{p+1}) = \textstyle\sum_{i=1,p+1}(-1)^{i+1}X_i\omega(X_1, \cdots, \check{X}_i, \cdots, X_{p+1})$$

for $\omega\in C^p(V;W)$, $X_i\in V$. This is isomorphic to the usual Koszul complex associated to the V-module W. Note that $H^n(V; W)=W/V.W$ by definition.

Let S_*V be the V-module of §5.1. Then it is not difficult to see

Proposition 5.31. $H^pC^*(V; S_*V)\simeq\begin{cases}0 & \textit{for } p\neq n\\ \boldsymbol{R} & \textit{for } p=n\,.\end{cases}$

(cf. [Ko]).

Let W be a V-module. W is then also an S_*V-module. Proposition 5.31 implies

Corollary 5.32. *Suppose a V-module W is free as an S_*V-module. Then, for $p\neq n$, $H^pC^*(V; W)=0$.*

Hence in particular

Corollary 5.33. *Suppose a V-module W is, as an S_*V-module, a direct summand of a free module, then*

$$H^pC^*(V; W)\simeq\begin{cases}0 & \textit{for } p\neq n\\ W/V.W & \textit{for } p=n\,.\end{cases}$$

Proof of Theorem 5.11. The notations are as in §3.1.

The assertion (iii) is obvious.

Since N_∞ is contractible, it suffices for the proof of (i), (ii) and (iv) to show, for $p>0$, that $E_1^{p,q}=(0)$ $(q\leqq n-1)$ and $E_1^{p,n}\simeq A\otimes B_p$.

Fix $p>0$ and denote by $\{C^*, d\}$ the complex $\{E_0^{p,*}, d_0\}$. Let F^k be the subspace of C^* spanned over A by

$$\{\delta u_{I(1)}^{i(1)}\wedge\cdots\wedge\delta u_{I(p)}^{i(p)}\wedge dx^{j(1)}\wedge\cdots\wedge dx^{j(q)};\ \textstyle\sum_{j=1,p}|I(j)|\leqq q-k\}\ .$$

Then $\{F^k\}$ is a decreasing filtration on C^* compatible with d. Let $\{\bar{E}_r^{s,t}, \bar{d}_r\}$ be the associated spectral sequence.

Obviously, the complex $\{\bar{E}_0, \bar{d}_0\}$ is isomorphic to $A\otimes\bar{C}^*$, where $\bar{C}^*$ is the subcomplex of C^* defined by

$$\bar{C}^* = \wedge^p[\delta u_I^j]\otimes\wedge^*[dx^i]\ .$$

Lemma 5.34. *$H^i\bar{C}^*\simeq 0$ $(i\neq n)$, B_p $(i=n)$.*

Proof. Obviously $\bar{C}^*=C^*(V; W_p)$, where W_p is the V-module introduced in §5.1. Since W_p is a direct summand of the free S_*V-module $\otimes^p(\boldsymbol{R}^m\otimes S_*V)$, Corollary 5.33 implies the assertions. Q.E.D.

Thus $\bar{E}_1^i=0$ $(i\neq n)$, $\bar{E}_1^n=A\otimes B_p$. Hence the spectral sequence degenerates: $E_\infty^i\simeq\bar{E}_1^i$, and so $E_1^{p,q}\simeq\bar{E}_1^q\simeq(0)$ $(q\leqq n-1)$, $E_1^{p,n}\simeq\bar{E}_1^n\simeq A\otimes B_p$. Hence (i), (ii) and (iv) are proved.

The assertion (v) is now easily verified. Q.E.D.

5.4. Kernels of Euler-Lagrange operators. We give now some applications of Theorem 5.11.

Let $M=\boldsymbol{R}^n$, $N=M\times\boldsymbol{R}^m$. The notations are as in §3.1.

Define $\delta_j\colon A\to A$ $(j\in\boldsymbol{Z}(1, m))$ by

$$\delta_j f = \textstyle\sum_{I\in N^n}(-\partial)^I\partial f/\partial u_I^j\ .$$

Often δ_j is also denoted by $\delta/\delta u^j$.

By Lemma 5.121, we have

Lemma 5.41. *For $f\in A$,*

$$\delta(fdx) = \textstyle\sum_{j=1,m}\delta_j f\,\delta u^j\wedge dx+\partial\omega_f\ .$$

Here

$$(5.411)\quad \omega_f = \textstyle\sum_{j,I,k}((-\partial)^J\partial f/\partial u_I^j)\delta u^j_{i(k+1)\cdots i(l)}\wedge dx_{i(k)}\ ,$$

where $j\in\boldsymbol{Z}(1, m)$, $I\in\boldsymbol{N}^n$, $l=|I|$, $k\in\boldsymbol{Z}(1, l)$, $J=\sum_{a=1,k-1}(i(a))$ and an expression of I as $I=(i(1))+\cdots+(i(l))$ is fixed.

By Corollary 5.12, $\delta\partial=\partial\delta$ implies

$$\delta_j\partial_i f = 0,\qquad \text{for}\ \ i\in\boldsymbol{Z}(1, n), j\in\boldsymbol{Z}(1, m),\ f\in A\ .$$

Thus together with (iii) and (v) of Theorem 5.11, we obtain the following commutative diagram:

$$\begin{array}{ccc} \tilde{A} & \xrightarrow[\simeq]{\beta} & E_1^{0,n} \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle d_1} \\ A^m & \xrightarrow[\simeq]{\alpha} & E_1^{1,n} \end{array}$$

where $\tilde{A}=A/\partial_1 A+\cdots+\partial_n A$, $\beta([f])=[fdx]$ and $\delta([f])=(\delta_1 f, \cdots, \delta_m f)$.

Since, for $n\geqq 1$, $\operatorname{Ker}(d_1: E_1^{0,n}\to E_1^{1,n})=E_2^{0,n}=(0)$ by (i) of Theorem 5.11, so δ is injective, i.e.,

Proposition 5.42. *For $f\in A$, $fdx\in \operatorname{Im}\partial$ if and only if $\delta_j f=0$ ($j\in \boldsymbol{Z}(1, m)$).*

This also follows from the following useful variant of the Taylor formula proved in [GM]:

$$(5.43) \qquad f[x, u] = f[x, 0]+\sum_{I\in N^n}\sum_j \partial^I\Big(u^j\int_0^1(\delta_{j,I} f)\,[x, tu]dt\Big)$$

where

$$(5.44) \qquad \delta_{j,I} f = \sum_{J\geq 0}\binom{I+J}{I}(-\partial)^J\partial f/\partial u^j_{I+J}\,.$$

Note that $\delta_{j,0}=\delta_j$. Usually $\delta_{j,I}$ is denoted as $\delta/\delta u^j_I$.

5.5. Ranges of Euler-Lagrange operators. Define for $f\in A$, linear operators $D(u^j)f\colon A\to A$ by

$$(D(u^j)f)g = \sum_{I\in N^n}(\partial f/\partial u^j_I)\partial^I g, \ j\in \boldsymbol{Z}(1, m)\,.$$

Let $D(u^j)^+ f$ be its adjoint:

$$(D(u^j)^+ f)g = \sum_{I\in N^n}(-\partial)^I(g\partial f/\partial u^j_I)\,.$$

Then $E_1^{1,n}=0$ can be rewritten as follows:

Proposition 5.5. *For $f_j\in A$ ($j\in \boldsymbol{Z}(1, m)$), there exists an $f\in A$ such that $f_j=\delta_j f (j\in \boldsymbol{Z}(1, m))$ if and only if*

$$(5.51) \qquad D(u^j)f_i = D(u^i)^+ f_j \qquad \textit{for } i, j\in \boldsymbol{Z}(1, m)\,.$$

REMARK 5.52. The "only if" part was given in [Man].

Before the proof, we extend the operators δ_j, $D(u^i)f$ and $D(u^i)^+ f$ to $\Omega^{*,*}$. Since they are compositions of multiplication by elements of A, ∂_i and $\partial/\partial u^j_I$, it suffices to extend ∂_i and $\partial/\partial u^j_I$ from A to $\Omega^{*,*}$. They are extended to be the derivations characterized by $\partial_i dx^j=0$, $\partial_i\delta u^j_I=\delta u^j_{I+(i)}$, $(\partial/\partial u^j_I)dx^i=0$ and $(\partial/\partial u^j_I)\delta u^k_I=0$. Note that $\partial^I\delta u^j=\delta u^j_I$. We define then $\delta_j=\sum(-\partial)^I\,\partial/\partial u^j_I$, $D(u^i)f=\sum\partial f/\partial u^i_J\partial^J$, etc..

Lemma 5.53.

(i) $[\delta_j, \delta] = 0$,

(ii) $\delta_i \partial \omega = 0$ *for* $\omega \in \Omega^{p,n-1}$.

Proof. (i). It suffices to show $[\partial/\partial u_I^j, \delta]=0$ and $[\partial_i, \delta]=0$. The former is obvious. The latter follows immediately from

$$(5.54) \qquad [\partial/\partial u_I^j, \partial_k] = \begin{cases} \partial/\partial u_{I-(k)}^j & (I \geqq (k)) \\ 0 & (\text{otherwise}). \end{cases}$$

(ii) It suffices to show $\delta_j(\partial_i \eta \wedge dx)=0$ for $j \in \boldsymbol{Z}(1, m)$, $i \in \boldsymbol{Z}(1, n)$ and $\eta \in \Omega^{*,0}$.

$$\begin{aligned} &\delta_j(\partial_i \eta \wedge dx) \\ &= \sum (-\partial)^I \partial/\partial u_I^j(\partial_i \eta \wedge dx) \\ &= \sum (-\partial)^I (\partial/\partial u_I^j) \partial_i \eta \wedge dx \\ &= \{\textstyle\sum_I (-\partial)^I \partial_i (\partial/\partial u_I^j) \eta + \sum_I (-\partial)^{I+(i)} (\partial/\partial u_I^j) \eta\} \wedge dx \\ &= 0, \end{aligned}$$

where we used again (5.54). Q.E.D.

Proof of Proposition 5.5. Suppose $f_j = \delta_j f$ $(j \in \boldsymbol{Z}(1, m))$ for some $f \in A$. Note that if $\phi_j = \sum f_{j,I} \partial^I$ $(j \in \boldsymbol{Z}(1, m), f_{j,I} \in A)$ satisfies $\sum_{j=1,m} \phi_j(\delta u^j \wedge dx) = 0$, then $\phi_j = 0$ $(j \in \boldsymbol{Z}(1, m))$. Thus it suffices to show

$$(5.55) \quad \textstyle\sum_{i=1,m} (D(u^j)^+ f_i)(\delta u^i \wedge dx) = \sum_{i=1,m} (D(u^i) f_j)(\delta u^i \wedge dx).$$

The left hand side is

$$\begin{aligned} &\textstyle\sum_{i,I} (-\partial)^I (\partial f_i / \partial u_I^j \delta u^i \wedge dx) \\ &= \textstyle\sum_I (-\partial)^I (\partial/\partial u_I^j) (\sum_i f_i \delta u^i \wedge dx) \\ &= \delta_j(\delta(f dx) + \partial \omega_f) && (\text{for some } \omega_f \in \Omega^{1,n-1}) \\ &= \delta_j \delta f dx && (\text{by Lemma 5.53 (ii)}) \\ &= \delta \delta_j (f dx) && (\text{by Lemma 5.53 (i)}) \\ &= \delta(\delta_j f) dx \\ &= \delta(f_j dx) \\ &= \textstyle\sum_{i,I} \partial f_j / \partial u_I^i \delta u_I^i \wedge dx, \end{aligned}$$

which is the right hand side of (5.55). (This proof is a direct generalization of the one given in [GMS] for $n=1$.)

Conversely suppose (5.51) holds. Then

$$\delta(\textstyle\sum_j f_j \delta u^j \wedge dx)$$

$$
\begin{aligned}
&= \textstyle\sum_j \partial f_j/\partial u^i_I \delta u^i_I \wedge \delta u^j \wedge dx \\
&= \textstyle\sum_{i,j} (D(u^i) f_j \delta u^i) \wedge \delta u^j \wedge dx \\
&\equiv \textstyle\sum_{i,j} \delta u^i \wedge (D(u^i)^+ f_j) \delta u^j \wedge dx \qquad (\text{mod. } \partial\Omega^{2,n-1}) \\
&= \textstyle\sum_{i,j} \delta u^i \wedge D(u^j) f_i \delta u^j \wedge dx \qquad (\text{by } (5.51)) \\
&= -\textstyle\sum_{i,j} (D(u^j) f_i) \delta u^j \wedge \delta u^i \wedge dx \\
&= -\delta(\textstyle\sum_i f_i \delta u^i \wedge dx) .
\end{aligned}
$$

Hence $\delta(\sum_j f_j \delta u^j \wedge dx) \in \partial\Omega^{2,n-1}$. Since $E_2^{1,n}=0$, there are $f \in A$ and $\omega \in \Omega^{1,n-1}$ such that

$$\sum f_j \delta u^j \wedge dx = \delta f \wedge dx + \partial\omega .$$

Then Corollary 5.12 and Lemma 5.41 show $f_j = \delta_j f$ ($j \in \boldsymbol{Z}(1,m)$). Q.E.D.

REMARK 5.56. There are various conditions for f_j's to be written as $\delta_j f$ ($f \in A$). The following follows immediately from (5.43).

Proposition 5.561 ([GM]). *For $f_j \in A$ ($j \in \boldsymbol{Z}(1,m)$), put*

$$Z = \textstyle\sum_{j=1,m} u^j \displaystyle\int_0^1 f_j[x, su]ds .$$

Then $f_j = \delta_j f$ ($j \in \boldsymbol{Z}(1,m)$) for some $f \in A$ if and only if $f_j = \delta_j Z$ ($j \in \boldsymbol{Z}(1,m)$).

EXAMPLE 5.57. Let $n=2, m=1$. Suppose $f = u_{0,1} + u u_{1,0} + u_{3,0}$ ($u_{i,j} = u_{(i,j)}$). Then

$$
\begin{aligned}
D(u)f &= \partial_2 + u\partial_1 + u_{1,0} + \partial_1^3 \\
D(u)^+ f &= -\partial_2 - u\partial_1 - \partial_1^3 .
\end{aligned}
$$

Thus $f \notin \operatorname{Im} \delta/\delta u$. However by introducing a potential $w = \int u dx^1$, we have

$$f[u] = g[w] = w_{1,1} + w_{1,0} w_{2,0} + w_{4,0} .$$

Then

$$D(w)g = D(w)^+ g = \partial_1\partial_2 + w_{1,0}\partial_1^2 + u_{2,0}\partial_1 + \partial_1^4$$

and in fact

$$g = (1/6)\delta/\delta w(w(3w_{1,1} + 2w_{1,0}w_{2,0} + 3w_{4,0})) .$$

5.6. Parameters. In actual situations, some of the independent variables often behave as parameters. Correspondingly then we must modify the variation bicomplex.

Specifically, let $p: N \to M$ be a bundle and P a manifold, which is the parameter space. Put $\bar{p}: \bar{N} = N \times P \to \bar{M} = M \times P$ and let $q: M \times P \to M$ be the natural projection. Let $R \subset \bar{N}_1$ be the image of $\alpha: q^* N_1 \to \bar{N}_1$ defined by $\alpha(y, s_1(x)) = (q^*s)_1(x,y)$ ($x \in M$, $y \in P$, $s \in \Gamma N$), where $q^*s \in \Gamma\bar{N}$ is defined by $(q^*s)(x,y) = s(x)$.

Let $\Omega^{*,*}$ be the variation bicomplex of R. This includes δ- and ∂-closed ideal $\mathcal{J}$ generated by $\pi'^*(\oplus_{i>0}\Omega^i P)$, where π' is the composition of $R_\infty \to \bar{M} \to P$. We put $\Omega_N^{*,*}\langle P\rangle = \Omega^{*,*}/\mathcal{J}$ and call it *the variation bicomplex of N with the parameter space P*.

EXAMPLE 5.61. Suppose $M=\boldsymbol{R}_x^n$, $N=M\times\boldsymbol{R}_u^m$ and $P=\boldsymbol{R}_y^{m'}$. Then $\Omega^{*,*}=\Omega_N^{*,*}\langle P\rangle$ is described as follows:

$$\Omega^{p,q} = \bar{A}\otimes\wedge^p[\delta u_I^j; I\in N^n]\otimes\wedge^q[dx^i]$$

where $\bar{A}=\mathcal{F}[x,y,u]$, and

$$\delta f = \sum \partial f/\partial u_I^j \delta u_I^j$$
$$\partial f = \sum_{i=1,n}\partial_i f dx^i$$

$(\partial_i = \partial/\partial x^i + \sum_{j,I} u_{I+(i)}^j \partial/\partial u_I^j$ for $i\in\boldsymbol{Z}(1,n))$. Thus

$$\Omega^{*,*} = \Omega_N^{*,*}\otimes_{\mathcal{F}(x,u)}\mathcal{F}(x,y,u)\,.$$

Let E be the spectral sequence of $\Omega_N^{*,*}\langle P\rangle$. We can show just as in the case of Theorem 5.1 the following

Theorem 5.62.

(i) *For* $r\geqq 1$,

$E_r^{0,q}\simeq H^q(N, \boldsymbol{R})\otimes\mathcal{F}P$ *if* $q\leqq n-1$,

$E_r^{p,q}=(0)$ *if* $p>0$, $q\neq n$.

(ii) *For* $r\geqq 2$,

$E_r^{p,n}\simeq H^{p+n}(N, \boldsymbol{R})\otimes\mathcal{F}P$.

Corresponding to Theorem 5.11, we have, using the notations of Example 5.61, the following

Theorem 5.63. *Suppose* $M=\boldsymbol{R}_x^n$, $N=M\times\boldsymbol{R}_u^m$ *and* $P=\boldsymbol{R}_y^{m'}$. *Then*

(i) *For* $r\geqq 1$,

$$E_r^{p,q}\simeq\begin{cases}\mathcal{F}(y)\,, & \text{for } (p,q)=(0,0)\,,\\ (0)\,, & \text{for } (p,q)\neq(0,0),\ q\neq n\,.\end{cases}$$

(ii) *For* $r\geqq 2$, $E_r^{p,q}\simeq(0)$ *if* $(p,q)\neq(0,0)$.

(iii) $E_1^{0,n}\simeq\bar{A}/\partial_1\bar{A}+\cdots+\partial_n\bar{A}$.

(iv) *The correspondence* $(f_1,\cdots,f_m)\mapsto[f_i\delta u^i\wedge dx]$ *defines an isomorphism* $\bar{A}^m\to E_1^{1,n}$.

This enables us to generalize Proposition 5.5 immediately. Define $D(u^j)$, $D(u^j)^+$ and δ_j as in §5.5. Then

Proposition 5.64. *For* $f_j\in\bar{A}\,(j\in Z(1,m))$, *there exists* $f\in\bar{A}$ *such that* $f_j=$

$\delta_j f$ $(j \in Z(1,m))$ *if and only if*

$$D(u^j)f_i = D(u^i)^+ f_j \qquad (i, j \in Z(1, m)).$$

6. Symmetries

Let $N \to M$ be a bundle and $R \subset N_k$ a differential equation satisfying Condition 2.3.

This section defines symmetries of R, which might be considered as "vector fields" on $\mathcal{Sol}(R)$, and expresses the space $\mathcal{L}(R)$ of symmetries as the kernel of a certain linear differential operator.

The notations are as in §2. Put $H = H_R$, $\Omega^{*,*} = \Omega_R^{*,*}$.

6.1. Definition of symmetries. Let $\mathcal{L}R_\infty = \Gamma TR_\infty$ be the Lie algebra of the smooth vector fields on R_∞. Define its subalgebras by

$$\mathcal{L}(H) = \{X \in \mathcal{L}R_\infty;\ [X, \Gamma H] \subset \Gamma H\},$$
$$\mathcal{L}^V(H) = \mathcal{L}(H) \cap \Gamma V_R.$$

Since ΓH is an ideal of $\mathcal{L}(H)$, we obtain a Lie algebra

$$\mathcal{L}(R) = \mathcal{L}(H)/\Gamma H.$$

The elements of $\mathcal{L}(R)$ are called *the symmetries of R*.

Obviously we have

$$\mathcal{L}(H) \simeq \mathcal{L}^V(H) \oplus \Gamma H, \tag{6.11}$$

whence, as Lie algebras,

$$\mathcal{L}(R) \simeq \mathcal{L}^V(H). \tag{6.12}$$

REMARK 6.13. To see that the above definition is appropriate, consider the situation of foliations. Let $(M, \mathcal{F})$ be a foliated manifold and $T\mathcal{F} \subset T$ the leaf tangent bundle. Then the Lie algebra of the vector fields preserving leaves are $\mathcal{L}(\mathcal{F}) := \{X \in \mathcal{L}M;\ [X, \Gamma T\mathcal{F}] \subset \Gamma T\mathcal{F}\}$. Its ideal $\Gamma T\mathcal{F}$ consists of those vector fields fixing the leaves. Thus the quotient Lie algebra $\mathcal{L}(\mathcal{F})/\Gamma T\mathcal{F}$ is the Lie algebra of infinitesimal transformations on the space of the leaves of $\mathcal{F}$. Since H corresponds to $T\mathcal{F}$, we see that the definition of $\mathcal{L}(R)$ above is natural.

REMARK 6.14. The equality (6.12) is quite useful. This enable us to discard the horizontal part of symmetries, making the situation very simple, for example in the Noether theorem. This was first remarked explicitly in [R].

REMARK 6.15. $\mathcal{L}(R)$ is generally very small. The soliton equations have the remarkable property that $\mathcal{L}(R)$'s have infinite dimension.

REMARK 6.16. Whereas $\mathcal{L}(R)$ is useful to obtain solutions from one known solution, ΓH sometimes helps us to construct solutions. In fact the Cauchy characteristics of R can be considered as a subbundle $C(H)$ of H such that $[\Gamma C(H), \Gamma C(H)]\subset\Gamma C(H)$ and that through each point $p\in R_\infty$ passes unique submanifold of dimension=rank $C(H)$. Thus, once we have constructed an integrable submanifold X of H whose tangent space spanns $H|_X$ modulo $C(H)$, there is a unique maximal integrable submanifold $\tilde{X}$, i.e., a solution of R, containing X. We note that the second geometric condition for $C(H)$ can be rewritten infinitesimally.

6.2. Computation of $\mathcal{L}(N)$. We determine $\mathcal{L}(R)$ for $R=N=M\times\boldsymbol{R}^m\to M=\boldsymbol{R}^n$. The notations are as in §3.1.

Let $X\in\Gamma V$. Then obviously $X\in\mathcal{L}^V(H)$ is equivalent to $[X, \partial_i]\in\Gamma H$ $(i\in\boldsymbol{Z}(1,n))$. Put $X=\sum X_{j,I}\partial/\partial u_I^j$ $(X_{j,I}\in A)$. Then

$$[X, \partial_i] = \sum (X_{j,I+(i)}-\partial_i X_{j,I})\partial/\partial u_I^j\,.$$

Thus, $[X,\partial_i]\in\Gamma H$ if and only if

$$X_{j,I+(i)} = \partial_i X_{j,I} \quad \text{for} \quad j\in\boldsymbol{Z}(1,m),\ I\in\boldsymbol{N}^n\,.$$

Hence $X\in\mathcal{L}^V(H)$ if and only if

$$X = \sum \partial^I X_j\partial/\partial u_I^j\,,$$

where $X_j=X_{j,0}$. By (6.12), we have proved

Proposition 6.21. *For $N=M\times\boldsymbol{R}^m\to M=\boldsymbol{R}^n$, the map $(f_j)_{j\in\boldsymbol{Z}(1,m)}\mapsto[\sum\partial^I f_j\partial/\partial u_I^j]$ defines an isomorphism $A^m\simeq\mathcal{L}(N)$.*

The above argument shows also the following

Proposition 6.22. *For general $N\to M$, $X\in\Gamma V$ is in $\mathcal{L}^V(H)$ if and only if $[X, \tilde{Y}]=0$ $({}^\forall Y\in\mathcal{L}M)$.*

Proof. Since the problem is local, we may assume $N=M\times\boldsymbol{R}^m\to M=\boldsymbol{R}^n$. Then as is shown in the derivation of Proposition 6.21, we have $X\in\mathcal{L}^V(H)$ if and only if $[X, \partial_i]=0$ $(i\in\boldsymbol{Z}(1,n))$. But $[X,\sum f_i\partial_i]=\sum f_i[X, \partial_i]$ for $f_i\in\mathcal{F}M$. Thus $[X,\partial_i]=0$ $({}^\forall i)$ if and only if $[X, \tilde{Y}]=0$ $({}^\forall Y\in\mathcal{L}M)$. Q.E.D.

6.3. Cartan formulas. We recall some of the Cartan fromulas on R_∞ and prove two basic facts about the subalgebra $\mathcal{L}^V(H_R)$ of $\mathcal{L}R_\infty$.

First, note that as in the finite-dimensional case, $\omega\in\Omega^p=\Omega^p R_\infty$ can be regarded as an alternating map from $\mathcal{L}R_\infty\times\cdots\times\mathcal{L}R_\infty$ (p-times) to A, multilinear over A. The exterior differentiation can be expressed as

$$\begin{aligned} d\omega(X_1,\cdots,X_{p+1}) &= \sum(-1)^{i+1}X_i\omega(X_1,\cdots,\check{X}_i,\cdots,X_{p+1}) \\ &\quad+\sum(-1)^{i+j}\omega([X_i,X_j],X_1,\cdots,\check{X}_i,\cdots,\check{X}_j,\cdots,X_{p+1}) \end{aligned}$$

for $\omega\in\Omega^p$, $X_i\in\mathcal{L}R_\infty$.

For $\omega\in\Omega^p$ and $X\in\mathcal{L}R_\infty$, define $i(X)\omega\in\Omega^{p-1}$ by

$$i(X)\omega(X_1,\cdots,X_{p-1}) = \omega(X,X_1,\cdots,X_{p-1})$$

$(X_i\in\mathcal{L}R_\infty)$.

Define $L_X=i(X)d+di(X)$ $(X\in\mathcal{L}R_\infty)$. L_X is a derivation of Ω^* commuting with d and coincides with X on $A=\Omega^0$.

Lemma 6.31. *If $X\in\mathcal{L}^V(H)$, then $L_X=i(X)\delta+\delta i(X)$, and $i(X)\partial=\partial i(X)$.*

Proof. Since, for $\omega\in\Omega^{p,q}$, $L_X\omega=(i(X)\delta+\delta i(X)+(-1)^p\{i(X)\partial-\partial i(X)\})\omega$, $L_X\omega$ has the (p,q) component $(i(X)\delta+\delta i(X))\omega$ and the $(p-1,q+1)$ component $(-1)^p\{i(X)\partial-\partial i(X)\}\omega$. It suffices to show that $L_X\omega$ does not have the $(p-1,q+1)$ component. Since L_X is a derivation and $\Omega^{*,*}$ is generated locally by $\Omega^{1,0}$ and $\Omega^{0,1}$, it suffices to show (i) $L_X\Omega^{0,1}\subset\Omega^{0,1}$ and (ii) $L_X\Omega^{1,0}\subset\Omega^{1,0}$. (i) is trivial. Now let $\omega\in\Omega^{1,0}$ and $Y\in\Gamma H$.
Then

$$\begin{aligned}(L_X\omega)(Y) &= (i(X)d\omega+di(X)\omega)(Y)\\ &= d\omega(X,Y)+Y\omega(X)\\ &= X\omega(Y)-Y\omega(X)-\omega([X,Y])+Y\omega(X)=0,\end{aligned}$$

since $[X,Y]\in\Gamma H$. Thus $L_X\omega\in\Omega^{1,0}$. Q.E.D.

Lemma 6.32. *If $Y\in\mathcal{L}M$, then $i(\tilde{Y})\delta+\delta i(\tilde{Y})=0$ and $L_{\tilde{Y}}=(-1)^p\{i(\tilde{Y})\partial-\partial i(\tilde{Y})\}$ on $\Omega^{p,q}$.*

Proof. First note that $[\tilde{Y},\Gamma V]\subset\Gamma V$. In fact, for $f\in\mathcal{F}M$ and $X\in\Gamma V$, $[\tilde{Y},X]f=\tilde{Y}Xf-X\tilde{Y}f=-X(Yf)=0$. Thus $[\tilde{Y},X]\in\Gamma V$. Then the same argument as in the proof of the previous lemma shows $L_{\tilde{Y}}\Omega^{p,q}\subset\Omega^{p,q}$, which implies the assertion immediately. Q.E.D.

We can show the following weaker version of Proposition 6.22.

Proposition 6.33. *For $X\in\mathcal{L}^V(H)$ and $Y\in\mathcal{L}M$, $[X,\tilde{Y}]=0$.*

Proof. For $f\in A$,

$$\begin{aligned}X\tilde{Y}f &= i(X)\delta i(\tilde{Y})\partial f\\ &= -i(X)i(\tilde{Y})\delta\partial f \qquad \text{(by Lemma 6.32)}\\ &= i(\tilde{Y})i(X)\partial\delta f\\ &= i(\tilde{Y})\partial i(X)\delta f \qquad \text{(by Lemma 6.31)}\\ &= \tilde{Y}Xf.\end{aligned}$$

Q.E.D.

Proposition 6.34. *For $X\in\mathcal{L}^V(H)$, $[L_X,\partial]=[L_X,\delta]=0$, i.e., L_X: $\Omega^{*,*}\to$*

$\Omega^{*,*}$ *is a bicomplex homomorphism.*

Proof. Obvious from Lemma 6.31. Q.E.D.

REMARK 6.35. This proposition implies that the Lie algebra $\mathcal{L}(R)$ acts on the spectral sequence $E(R)$. This action is explicitly calculated in [Kh] for some examples.

6.4. Bundle automorphisms. Call an element of $\mathcal{L}(R)$ *integrable* if it is represented by an integrable vector field X on R_∞. Here a vector field is called integrable if it generates a local one-parameter group of local diffeomorphisms of R_∞.

In general, elements of $\mathcal{L}(N)$ is not integrable. But there are integrable ones in $\mathcal{L}(N)$ which are represented by the liftings of vector fields on $\mathcal{L}N$ and (if $m=1$) of the contact vector fields on N_1.

First we consider $\mathcal{L}N$ in this paragraph.

Lemma 6.41. *For $X\in\mathcal{L}N$, there is a unique $\tilde{X}\in\mathcal{L}(H)$ such that $\tilde{X}=X$ on $A_0=\mathcal{F}N$.*

Proof. First we prove the uniqueness. Let $Y\in\mathcal{L}(H)$ satisfy $Y.A_0=0$. Let $(x^i, u^j_I; U)$ be a standard local chart (cf §1.3). Then by (6.11) we can write $Y|_U=\sum Y_i\partial_i+Y'$, where $Y_i\in\mathcal{F}[x, u]$, $Y'\in\mathcal{L}^V(H|_U)$. But $Y.A_0=0$ implies $Y_i=0$ and $Y'u^i=0$. Then Proposition 6.21 implies $Y'|_U=0$. Hence $Y=0$.

By virtue of the uniqueness, we may assume $M=\boldsymbol{R}^n$, $N=M\times\boldsymbol{R}^m$ in constructing $\tilde{X}$. Write $X=\sum X_i\partial/\partial x^i+\sum Y_j\partial/\partial u^j$ $(X_i, Y_j\in\mathcal{F}N)$. Define then

$$\tilde{X}=\sum X_i\partial_i+\sum\partial^I(Y_j-X_iu^j_{(i)})\partial/\partial u^j_I\,. \tag{6.411}$$

Then obviously $\tilde{X}\in\mathcal{L}(H)$ and

$$\tilde{X}x^i=X_i\,,$$
$$\tilde{X}u^j=\sum X_iu^j_{(i)}+Y_j-\sum X_iu^j_{(i)}=Y_j\,.$$

Hence $\tilde{X}=X$ on A_0. Q.E.D.

REMARK 6.42. $\tilde{X}$ can be constructed more geometrically as follows: let ϕ_t be the local one parameter diffeomorphism group generated by X. Then roughly ϕ_t acts on ΓN and hence on N_∞. The induced vector fields is $\tilde{X}$.

Define α: $\mathcal{L}N\to\mathcal{L}(N)$ by $\alpha(X)=[\tilde{X}]$.

Lemma 6.43. α: $\mathcal{L}N\to\mathcal{L}(N)$ *is a Lie algebra homomorphism.*

Proof. Let $X, Y\in\mathcal{L}N$. Since $[\tilde{X},\tilde{Y}]=[X,Y]^\sim$ on A_0 and $[\tilde{X},\tilde{Y}]\in\mathcal{L}(H_N)$, the uniqueness implies $[\tilde{X},\tilde{Y}]=[X,Y]^\sim$. Q.E.D.

Suppose $N=M\times \boldsymbol{R}^m\to M=\boldsymbol{R}^n$. Then by Proposition 6.21 there is an isomorphism $\gamma: \mathcal{L}(N)\to A^m$. (6.411) implies then

$$(6.44)\qquad \gamma\alpha(\sum X_i\partial/\partial x^i+\sum Y_j\partial/\partial u^j) = (Y_j-\textstyle\sum_{i=1,n}u^j_{(i)}X_i)$$

$(X_i, Y_j\in \mathcal{F}N)$.

6.5. Contact transformations. Suppose now $m=1$. N_1 has a canonical contact structure defined by the line subbundle $L\subset T^*N_1$ which is locally generated by $du-\sum_{i=1,n}u_i dx^i$ $(u_i=u_{(i)})$. $X\in\mathcal{L}N_1$ is called *contact* if $L_X\Gamma L\subset\Gamma L$. A diffeomorphism ϕ of N_1 is called *a contact transformation* if $\phi^*L=L$. The space of all the contact vector fields obviously form a Lie subalgebra of $\mathcal{L}N_1$, which is denoted by $\mathcal{L}^{ct}N_1$.

Lemma 6.51. *For $X\in\mathcal{L}^{ct}N_1$, there is a unique $\tilde{X}\in\mathcal{L}(H_N)$ such that* $\tilde{X}=X$ on $A_1=\mathcal{F}N_1$.

Proof. Uniqueness can be proved in the same way as in the proof of Lemma 6.41. So we may assume $M=\boldsymbol{R}^n$, $N=M\times\boldsymbol{R}$ in constructing $\tilde{X}$.

Express $X\in\mathcal{L}N_1$ as $X=\sum_{i=1,n}X_i\partial/\partial x^i+Y\partial/\partial u+\sum_{i=1,n}Z_i\partial/\partial u_i$ $(X_i, Y, Z_i\in A_1)$. Then a calculation shows

$$\begin{aligned} L_X(du-\textstyle\sum u_i dx^i)&\equiv\textstyle\sum_i(\partial W/\partial x^i+u_i\partial W/\partial u-Z_i)dx^i\\ &\quad+\textstyle\sum_i(\partial W/\partial u_i+X_i)du_i\end{aligned}$$

modulo $du-\sum_i u_i dx^i$, where $W=Y-\sum_{j=1,n}u_jX_j$. Thus $X\in\mathcal{L}^{ct}N_1$ if and only if

$$(6.511)\qquad \begin{aligned} Z_i &= \partial W/\partial x^i+u_i\partial W/\partial u\,,\\ X_i &= -\partial W/\partial u_i\,,\end{aligned}$$

hold.

Define for $X\in\mathcal{L}^{ct}N_1$

$$\tilde{X}=\textstyle\sum_i X_i\partial_i+\sum(\partial^I W)\partial/\partial u_I\,.$$

Then

$$\begin{aligned} \tilde{X}x^i &= X_i = Xx^i\,,\\ \tilde{X}u &= \textstyle\sum X_i u_i+W = Y = Xu\,,\\ \tilde{X}u_i &= \textstyle\sum_j X_j u_{ji}+\partial_i W\\ &= \textstyle\sum_j X_j u_{ji}+\partial W/\partial x^i+u_i\partial W/\partial u+\sum_j u_{ij}\partial W/\partial u_j\\ &= Z_i \qquad\qquad \text{(by (6.511))}\\ &= Xu_i\,.\end{aligned}$$

Thus $\tilde{X}=X$ on A_1. Q.E.D.

Corollary 6.52. *In the case $N=M\times\boldsymbol{R}$, $M=\boldsymbol{R}^n$, the map μ: $\mathcal{L}^{ct}N_1\rightarrow A_1$ defined by $\mu(X)=W$ $(X=\sum X_i\partial/\partial x^i+Y\partial/\partial u+\sum Z_i\partial/\partial u_i$, $W=Y-\sum u_iX_i)$ is an isomorphism.*

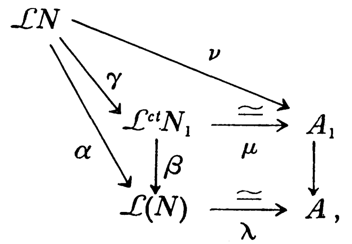
Proof. By (6.511) and $Y=W+\sum_{j=1,n}u_jX_j$, we can reconstruct X from W, i.e., μ is surjective. The injectivity is obvious. Q.E.D.

Define β: $\mathcal{L}^{ct}N_1\rightarrow\mathcal{L}(N)$ by $\beta(X)=[\tilde{X}]$. Then the following can be proved just as Lemma 6.43.

Lemma 6.53. *β is a Lie algebra homomorphism.*

In the case $M=\boldsymbol{R}^n$, $N=M\times\boldsymbol{R}$, the relation between λ: $\mathcal{L}(N)\rightarrow A$ of Proposition 6.21, α of Lemma 6.43, and β, μ of this section is as follows.

Proposition 6.54. *The following diagram commutes*



where $\nu(\sum X_i\partial/\partial x^i+Y\partial/\partial u)=Y-\sum_{i=1,n}u_iX_i$ $(X_i, Y\in A_0)$,

$$\mu^{-1}W=\textstyle\sum_i(-\partial W/\partial u_i)\partial/\partial x^i+(W-\sum_i u_i\partial W/\partial u_i)\partial/\partial u$$
$$+\textstyle\sum_i(\partial W/\partial x^i+u_i\partial W/\partial u)\partial/\partial u_i \qquad (W\in A_1)$$

and $\gamma=\mu^{-1}\nu$.

6.6. Description of $\mathcal{L}(R)$. Let $M=\boldsymbol{R}^n$, $N=M\times\boldsymbol{R}^m$. The notations are as in §3.1. Suppose $R=\{F_1=\cdots=F_k=0\}$ $(F_j\in A)$ satisfies Condition 2.3.

Define Φ: $A^m\rightarrow A^k$ by $(\Phi f)_i=\sum_j(D(u^j)F_i)f_j$ $(i\in\boldsymbol{Z}(1,k))$. Since $\Phi(\mathcal{J}_\infty^m)\subset\mathcal{J}_\infty^k$ ($\mathcal{J}_\infty$: the defining ideal of R_∞), Φ induces Φ_R: $A(R)^m\rightarrow A(R)^k$.

Theorem 6.6. *The correspondence $A^m\rightarrow\Gamma(TN_\infty|R_\infty)$ defined by $\alpha(f_1,\cdots,f_m)=\sum(\partial^If_j)\partial/\partial u_I^j|_{R_\infty}$ $(f_i\in A)$ induces an isomorphism* Ker $\Phi_R\simeq\mathcal{L}(R)$.

Remark. This is announced in [V2]. See also [IS1,2].

Before the proof, we need two lemmas.

Fix $X\in\mathcal{L}^V(H)$. Let $v_I^i=u_I^i|_{R_\infty}$. Put $f_i=Xv^i\in A(R)$. Choose $\tilde{f}_i\in A$ such that $\tilde{f}_i|_{R_\infty}=f_i$, which is always possible. By Proposition 6.21, $\tilde{X}:=\sum\partial^I\tilde{f}_j\partial/\partial u_I^j\in\mathcal{L}^V(H_N)$.

Lemma 6.61. *$\tilde{X}$ is tangent to R_∞ and its restriction on R_∞ coincides with X.*

Proof. It suffices to show that, for all $y\in R_\infty$, $\tilde{X}_y=X_y$. This will follow from $\tilde{X}_y u_I^i=X_y u_I^i$ ($^\forall i, I$). Since X_y is tangent to R_∞, $X_y u_I^i=X_y v_I^i=(Xv_I^i)(y)$. Thus we have only to show $p(\tilde{X}u_I^i)=Xv_I^i$, where $p\colon A\to A(R)$ is the projection.

Denote $\partial_i|_{R_\infty}$ also by ∂_i. Then ∂_i commutes with p. Thus $p(\tilde{X}u_I^i)=p(\tilde{X}\partial^I u^i)$, which is by Proposition 6.22 equal to $\partial^I(p\tilde{X}u^i)=\partial^I(p\tilde{f}_i)=\partial^I(f_i)=\partial^I Xv^i$, which is by Proposition 6.33 equal to $X\partial^I v^i=Xv_I^i$. Hence $p(\tilde{X}u_I^i)=Xv_I^i$.

Q.E.D.

Lemma 6.62. *Let* $g=(g_i)\in A^m$. *Then* $X_g:=\sum\partial^I g_i\partial/\partial u_I^i$ *is tangent to* R_∞ *if and only if* $\Phi(g)\in\mathcal{J}_\infty^k$.

Proof. Since $\mathcal{J}_\infty$ is generated by $\{\partial^I F_j; j\in\boldsymbol{Z}(1,k), I\in\boldsymbol{N}^n\}$, X_g is tangent to R_∞ if and only if $X_g\partial^I F_j\in\mathcal{J}_\infty$ for all I, j. But $X_g\partial^I F_j=\partial^I X_g F_j$ by Proposition 6.33. Since $\mathcal{J}_\infty$ is ∂_i-closed, we see X_g is tangent to R_∞ if and only if $X_g F_j\in\mathcal{J}_\infty$. The assertion follows finally from

$$X_g F_j = \sum\partial^I g_i\partial F_j/\partial u_I^i = (\Phi g)_j\,.$$

Q.E.D.

Proof of Theorem 6.6. Define $\alpha\colon \operatorname{Ker}\Phi\to\mathcal{L}(R)$ by $\alpha(g)=X_g|R_\infty$. This is well-defined by Lemma 6.62. By Lemma 6.61, α is surjective. Suppose $\alpha(g)=0$, i.e., $X_g|_{R_\infty}=0$. This implies $g_j=X_g u^j$ ($j\in\boldsymbol{Z}(1,m)$) is zero on R_∞, i.e., $g_j\in\mathcal{J}_\infty$. Hence $\operatorname{Ker}\alpha\subset\mathcal{J}_\infty^m$. Since obviously $\mathcal{J}_\infty^m\subset\operatorname{Ker}\alpha$, we have $\operatorname{Ker}\alpha=\mathcal{J}_\infty^m$. The assertion then follows from $\operatorname{Ker}\Phi_R=\operatorname{Ker}\Phi/\mathcal{J}_\infty^m$. Q.E.D.

6.7. Contact transformations of R. Theorem 6.6 gives a method of computing the Lie algebras of vector fields on N and if $m=1$ of contact vector fields on N_1 which leave R invariant.

Specifically let R be as in §6.6 and put

$$\mathcal{L}_0(R) = \{[\tilde{X}];\, X\in\mathcal{L}N,\ \tilde{X}\text{ is tangent to } R_\infty\}\ ,$$

and for $m=1$

$$\mathcal{L}^{ct}(R) = \{[\tilde{X}];\, X\in\mathcal{L}^{ct}N_1,\ \tilde{X}\text{ is tangent to } R_\infty\}\ .$$

Put further

$$\bar{A}_0 = \{(f_1,\cdots,f_m)|_{R_\infty}; f_j = Y_j-\textstyle\sum_{i=1,n}X_i u_i^j\ (Y_j, X_i\in A_0(N))\}\ .$$

Then

Theorem 6.7.

(i) $\mathcal{L}_0(R)\simeq\bar{A}_0\cap\operatorname{Ker}\Phi_R$,

(ii) if $m = 1$, $\mathcal{L}^{ct}(R)\simeq A_1\cap\operatorname{Ker}\Phi_R$.

Proof. (i) Let $X\in\mathcal{L}_0(R)$. By definition $X=[\tilde{Y}]$ for some $Y\in\mathcal{L}N$.

Write $Y=\sum X_i \partial/\partial x^i+\sum Y_j \partial/\partial u^j$ and put $f_j=Y_j-\sum_{i=1,n} u_i^j X_i$ ($j\in \boldsymbol{Z}(1, m)$). Since $X_f=\tilde{Y}-\sum X_i\partial_i$ is also tangent to R_∞, Lemma 6.62 implies $\Phi(f_j)\in\mathcal{J}_\infty^k$. Thus $(f_j)|_{R_\infty}\in \bar{A}_0\cap \operatorname{Ker}\Phi_R$. This correspondence $X\mapsto(f_j)|_{R_\infty}$ is obviously bijective.

(ii) can be proved similarly. Q.E.D.

EXAMPLE 6.71. Let R be as in §3.5, i.e., the Korteweg-de Vries equation. The notations are as in §3.5.

Here we have

$$\Phi f=(\partial_2+u\partial_1+\partial_1{}^3+u_1)f, \qquad f\in A$$

and

$$A_1=\mathcal{F}(x^1, x^2, u, u_1, u_3)\subset A\ .$$

A little calculation shows that $\operatorname{Ker}\Phi\cap A_1$ is spanned by $\{-u_1, -(u_3+uu_1), 1-x^2u_1, -x^1u_1+3x^2(u_3+uu_1)-2u\}$. These correspond to $X_1=\partial/\partial x^1$, $X_2=\partial/\partial x^2$, $X_3=x^2\partial/\partial x^1+\partial/\partial u$ and $X_4=x^1\partial/\partial x^1+3x^2\partial/\partial x^2-2u\partial/\partial u\in\mathcal{L}N$ respectively. Their finite forms are

$$\begin{aligned}
\phi_{1,t}(x^1, x^2, u) &= (x^1+t, x^2, u)\ ,\\
\phi_{2,t}(x^1, x^2, u) &= (x^1, x^2+t, u)\ ,\\
\phi_{3,t}(x^1, x^2, u) &= (x^1+tx^2, x^2, u+t)\ ,\\
\phi_{4,t}(x^1, x^2, u) &= (e^t x^1, e^{3t}x^2, e^{-2t}u)
\end{aligned}$$

($t\in\boldsymbol{R}$). Their commutators are $[X_1,X_2]=[X_1,X_3]=0$, $[X_1,X_4]=[X_2,X_3]=X_1$, $[X_2,X_4]=3X_2$ and $[X_3,X_4]=-2X_3$. In short

Proposition 6.711. *The correspondence*

$$-\sum_{i=1}^4 a_i X_i \mapsto \begin{pmatrix} a_4 & a_3 & 0 & a_1\\ 0 & 3a_4 & 0 & a_2\\ 0 & 0 & -2a_4 & a_3\\ 0 & 0 & 0 & 0\end{pmatrix}$$

is a Lie algebra injection of $\mathcal{L}^{ct}(R)=\mathcal{L}_0(R)$ *into* $\mathfrak{gl}_4\boldsymbol{R}$.

We note that this result is already in [Kum].

REMARK 6.72. Elements of $\mathcal{L}_0(R)$ and $\mathcal{L}^{ct}(R)$ are integrable by definition. When $R=N$, these are the only integrable ones, which is a classical result by Bäcklund ([Bä]). This is valid for equations provided they satisfy certain conditions easy to check. However for general R, there can exist integrable ones not in $\mathcal{L}_0(R)$ nor in $\mathcal{L}^{ct}(R)$.

7. Cauchy-Kowalevsky systems

This section gives a useful description of $E_1^{0,n-1}$ and $E_1^{1,n-1}$ of R which is Cauchy-Kowalevsky system in a weak sense (pseudo *CK* system in short).

In this section $M=\boldsymbol{R}^n$, $N=M\times\boldsymbol{R}^m$ and the notations of §3.1 are used. Put $A=A(N)$, $A_k=A_k(N)$ (cf. §2.3).

7.1. Pseudo *CK* systems. $R\subset N_k$ is called *a pseudo CK system* if

$$R=\{K_1=\cdots=K_m=0\}\ ,$$

where $K_j=u^j_{k_j(1)}-F_j\in A_k$ with $k_j>0$ and

$$\text{(7.11)}\qquad \partial F_j/\partial u^i_I=0,\quad \text{for all } i\in\boldsymbol{Z}(1,m),\ I\in N(i)\ ,$$

where $N(i)=\{I\in\boldsymbol{N}^n;\ i_1\geqq k_i(I=(i_1,\cdots,i_n))\}$.

Obviously we have

Proposition 7.12. *If $R=\{K_1=\cdots=K_m=0\}$ is a pseudo CK system, then N_∞ admits the following system of global coordinates: $\{\partial^I K_j;\ j\in\boldsymbol{Z}(1,m),\ I\in\boldsymbol{N}^n\}\cup\{x^i,\ u^j_I;\ i\in\boldsymbol{Z}(1,n),\ j\in\boldsymbol{Z}(1,m),\ I=(i_1,\cdots,i_n)\in\boldsymbol{N}^n$ with $i_1\leqq k_j-1\}$.*

This implies

Corollary 7.13. *A pseudo CK system satisfies Condition* 2.3.

REMARK 7.14. If $k_i=k$ and $K_i\in\mathcal{F}N_k$ $(i\in\boldsymbol{Z}(1,m))$, R is a Cauchy-Kowalevsky system in the usual sense.

REMARK 7.15. In [Mar], a pseudo *CK* system is called *a normal system.*

7.2. Statement of theorems. Define linear differential operator $\Phi^+: A^m\to A^m$ by

$$(\Phi^+f)_i=\textstyle\sum_j(D(u^i)^+K_j)f_j\qquad \text{for } i\in\boldsymbol{Z}(1,m),\ f_j\in A\ .$$

Here $D(u^i)^+K_j$ is the operator defined in §5.5. Since $\Phi^+(\mathcal{J}^m_\infty)\subset\mathcal{J}^m_\infty$, Φ^+ in-induces Φ^+_R: $A(R)^m\to A(R)^m$.

Theorem 7.21. *Suppose $R=\{K_1=\cdots=K_m=0\}$ is a pseudo CK system. Then*

(i) $E_1^{1,n-1}(R)\simeq\operatorname{Ker}\Phi^+_R$,

(ii) if $n\geqq 2$, $E_1^{0,0}(R)\simeq\boldsymbol{R}$, $E_1^{p,q}(R)=(0)$ $(q\leqq n-2,\ (p,q)\neq(0,0))$.

Corollary 7.22. *Suppose R is a pseudo CK system.*

(i) *If $n\geqq 2$, then $E_2^{0,n-1}(R)=0$, i.e., d_1: $E_1^{0,n-1}(R)\to E_1^{1,n-1}(R)$ is injective.*

(ii) *If $n=1$, then $E_2^{0,0}(R)=\boldsymbol{R}$, i.e,*

$$0\to\boldsymbol{R}\to E_1^{0,0}(R)\to E_1^{1,0}(R)$$

is exact.

(iii) $E_2^{1,n-1}(R)=(0)$.

Proof. Put $E_r^{p,q}=E_r^{p,q}(R)$. By Theorem 7.21(ii), we have $E_2^{0,n-1}\simeq E_\infty^{0,n-1}\simeq H^{n-1}(R_\infty, \boldsymbol{R})$, which yields (i) and (ii), since R_∞ is contractible by Proposition 7.12. Furthur $E_2^{1,n-1}\simeq E_\infty^{1,n-1}\subset\bigoplus_{i+j=n}E_\infty^{i,j}\simeq H^n(R_\infty, \boldsymbol{R})=(0)$, whence (iii). Q.E.D.

REMARK 7.23. Thus we can compute the space of conserved currents by solving the linear differential equation $\Phi^+f=0$ ($f\in A^m$). Some criterions for $f\in\operatorname{Ker}\Phi_R^+$ to correspond to an element of $E_1^{0,n-1}$ will be given in §7.5–6.

REMARK 7.24. Theorem 7.21 is stated in [V1]. For $m=n=1$, it is already in [GD].

7.3. Proof of Theorem 7.21. There are two proofs—the intrinsic and the extrinsic ones. The latter uses coordinates on N_∞ and makes clear the meaning of Φ_R^+, but is a little long. So we give the intrinsic proof using coordinates of R_∞.

7.31. Proposition 7.12 gives the following coordinate on R_∞: $\{x^1,\cdots,x^n\}\cup\{v_{j,J}^i; i\in\boldsymbol{Z}(1,m), j\in\boldsymbol{Z}(0,k_j-1), J\in\boldsymbol{N}_{2,n}\}$, where $v_{j,J}^i=u_{j(1)+J}^i|_{R_\infty}$. Put $v_j^i=v_{j,0}^i$. Then

$$A(R)=\mathcal{F}(x^i, v_{k,J}^i)\,.$$

The derivations ∂_j of $A(R)$ is characterized by

$$\begin{aligned}
&\partial_i x^j=\delta_{i,j}\,,\\
&\partial_i v_{k,J}^j=v_{k,J+(i)}^j,\quad i\geqq 2\,,\\
&\partial_1 v_{k,J}^j=\begin{cases} v_{k+1,J}^j, & k\leqq k_j-2\,,\\ \partial^J\bar{F}^j, & k=k_j-1\,.\end{cases}
\end{aligned}$$

Here $\bar{F}^j=F^j|_{R_\infty}$.

Put $t=x^1$, $x'=(x^2,\cdots,x^n)$, $\Omega^{*,*}=\Omega_R^{*,*}$, $E_r^{p,q}=E_r^{p,q}(R)$.

7.32. Fix $p\geqq 0$ and let $'F$ be the filtration of $\Omega^{p,*}$ defined by $'F^0=\Omega^{p,*}$, $'F^1=\Omega^{p,*}\wedge dt$, $'F^2=(0)$. Obviously $'F$ is compatible with ∂. Let $\{{}^pE_r^{s,t}, d\}$ be the induced spectral sequence which converges to $E_1^{p,*}$.

It is easy to see the following

Lemma 7.321.

(i) ${}^pE_0^{0,t}\cong A(R)\otimes\wedge^p[\delta v_{j,J}^i]\otimes\wedge^t[dx'], d_0=\sum_{i=2,n}(\partial_i\otimes 1+1\otimes\partial_i)\otimes dx^i$,

(ii) $\{{}^pE_0^{1,*}, d_0\}\simeq\{{}^pE_0^{0,*}, d_0\}$ *as complexes,*

(iii) ${}^pE_0^{s,*}=(0)$, *for* $s\geqq 2$.

(See §6.3 for the definition of the ∂_i's action on $\wedge^p[\delta v_{j,J}^i]$.)

Obviously then we have (cf. Example 5.61)

Lemma 7.322. *$\{{}^pE_0^{0,*}, d_0\}$ is isomorphic to the complex $\{\bar{E}_0^{p,*}, d_0\}$ of the variation bicomplex of $\bar{N}=\bar{M}\times \boldsymbol{R}^{\bar{m}}\to\bar{M}=\boldsymbol{R}^{n-1}\times\boldsymbol{R}_t$ ($\bar{m}=\sum_{j=1,n}k_j$) with $\boldsymbol{R}_t$ as the parameter space.*

7.33. Proof of Theorem 7.21 (ii) for $p=0$. Lemma 7.322 and Theorem 5.63 yield the following commutative diagram:

$$\begin{array}{ccc} {}^0E_1^{0,0} & \xrightarrow[\simeq]{\alpha} & \mathcal{F}(t) \\ \downarrow d_1 & & \downarrow \partial/\partial t \\ {}^0E_1^{1,0} & \xrightarrow[\simeq]{\beta} & \mathcal{F}(t) \end{array}$$

where $\alpha^{-1}(f)=[f]$, $\beta^{-1}(f)=[fdt]$ ($f\in\mathcal{F}(t)\subset A$). Thus we have ${}^0E_2^{0,0}\simeq\boldsymbol{R}$, ${}^0E_2^{1,0}\simeq(0)$. This finishes the proof when $n=2$.

Suppose $n\geqq 3$. Lemmas 7.321, 7.322 and Theorem 5.63 yield

$${}^0E_1^{v,w}=(0) \qquad (1\leqq w\leqq n-2)\,.$$

Hence

$${}^0E_\infty^{v,w}=\begin{cases}\boldsymbol{R} & \text{for } (v,w)=(0,0)\,,\\ (0) & \text{for } v+w\in\boldsymbol{Z}(1,n-2)\,.\end{cases}$$

This proves (ii) for $p=0$. Q.E.D.

7.34. Proof of Theorem 7.21 (ii) for $p>0$. By Lemmas 7.321, 7.322 and Theorem 5.63, we have for all v

$${}^pE_1^{v,w}=(0) \qquad \text{for } w\leqq n-2\,,$$

whence

$${}^pE_\infty^{v,w}=(0) \qquad \text{for } v+w\leqq n-2\,.$$

This proves (ii) for $p>0$.

7.35. Proof of Theorem 7.21 (i).

Denote the elements of $A(R)^{\bar{m}}$ ($\bar{m}=\sum_{j=1,m}k_j$) as $f=(f_{j,k})$ with $j\in\boldsymbol{Z}(1,m)$, $k\in\boldsymbol{Z}(0,k_j-1)$.

Define a linear endomorphism Ψ of $A(R)^{\bar{m}}$ by

$$\Psi(f_{j,k})=(g_{j,k})\,,$$

where, for $i\in\boldsymbol{Z}(1,m)$,

$$g_{i,0}=\partial_1 f_{i,0}+\textstyle\sum_{a=1,m}(\mathcal{D}_{i,0}^{\dagger}\bar{F}_a)f_{a,k_a-1}\,,$$

and, for $j\in\boldsymbol{Z}(1,k_i-1)$,

$$g_{i,j} = \partial_1 f_{i,j} + f_{i,j-1} + \textstyle\sum_{a=1,m} (\mathcal{D}^+_{i,j}\bar{F}_a) f_{a,k_a-1} \, .$$

Here $\mathcal{D}^+_{i,j}G\colon A(R) \to A(R)$ ($i \in \boldsymbol{Z}(1,m)$, $j \in \boldsymbol{Z}(0,k_i-1)$, $G \in A(R)$) is defined by putting for $f \in A(R)$

$$(\mathcal{D}^+_{i,j}G)f = \textstyle\sum_{J \in N_{2,n}} (-\partial)^J (f \partial G / \partial v^i_{j,J}) \, .$$

Define $\alpha\colon A(R)^{\bar m} \to {}^1E_1^{0,n-1}$ and $\beta\colon A(R)^{\bar m} \to {}^1E_1^{1,n-1}$ respectively by

$$\alpha(f_{i,j}) = [\textstyle\sum f_{i,j} \delta v^i_j \wedge dx'] \, ,$$
$$\beta(f_{i,j}) = [\textstyle\sum f_{i,j} \delta v^i_j \wedge dx] \, .$$

Then by Theorem 5.63 (iv) and Lemma 7.322, α and β are isomorphisms.

Lemma 7.351. *The following commutes:*

$$\begin{array}{ccc} A(R)^{\bar m} & \xrightarrow[\simeq]{\alpha} & {}^1E_1^{0,n-1} \\ \big\downarrow \Psi & & \big\downarrow d_1 \\ A(R)^{\bar m} & \xrightarrow[\simeq]{\beta} & {}^1E_1^{1,n-1} \end{array}$$

Proof. By definition, $d_1\alpha(f_{i,j})$ is represented by

$$\begin{aligned} &\textstyle\sum_{j=1,m} \partial_1 (\sum_{k=0,k_j-1} f_{j,k} \delta v^j_k) \wedge dx \\ &= \textstyle\sum_{j=1,m} \{ \sum_{k=0,k_j-1} \partial_1 f_{j,k} \delta v^j_k + \sum_{k=0,k_j-2} f_{j,k} \delta v^j_{k+1} \\ &\quad + \textstyle\sum_{r,s,J} f_{j,k_j-1} \partial \bar{F}_j / \partial v^r_{s,J} \delta v^r_{s,J} \wedge dx \} \, . \end{aligned}$$

Since the third term represents $[(\mathcal{D}^+_{r,s}\bar{F}_j) f_{j,k_j-1} \delta v^r_s \wedge dx]$ it follows $d_1\alpha(f_{i,j})$ is represented by $\beta\Psi(f_{i,j})$. Q.E.D.

Lemma 7.352. *The map $\gamma\colon A(R)^{\bar m} \to A(R)^m$ defined by $\gamma(f_{i,j}) = (f_{i,k_i-1})_{i \in \boldsymbol{Z}(1,m)}$ induces an isomorphism*

$$\operatorname{Ker}\Psi \to \operatorname{Ker}\Phi^+_R \, .$$

Proof. By induction, we can easily show that $\Psi(f_{i,j}) = 0$ if and only if

$$\begin{aligned} &f_{i,j} = \textstyle\sum_a (D^+_{i,j+1}K_a)\phi_a \quad \text{for } j \in \boldsymbol{Z}(0,k_i-2) \, , \\ &\textstyle\sum_a (D^+_{i,0}K_a)\phi_a = 0 \, , \end{aligned} \tag{7.353}$$

($\phi_a \colon = f_{a,k_a-1}$) for $i \in \boldsymbol{Z}(1,m)$. Here

$$D^+_{i,j}K_a \colon = (-\partial_1)^{k_i-j} \delta_{i,a} - \textstyle\sum_{k \in \boldsymbol{Z}(0,k_i-1-j)} (-\partial_1)^k \circ \mathcal{D}^+_{i,k+j}\bar{F}_a \, ,$$

$\delta_{i,a}$ being the Kronecker's delta. Note that $(\sum_a (D^+_{i,0}K_a)\phi_a) = \Phi^+_R(\phi_a)$.

Thus γ induces $\bar{\gamma}\colon \operatorname{Ker}\Psi \to \operatorname{Ker}\Phi^+_R$. (7.353) shows γ is bijective. Q.E.D.

Hence ${}^1E_2^{0,n-1} \simeq \operatorname{Ker}\Phi^+_R$. Since ${}^1E_2^{s,t} = 0$ ($t \in \boldsymbol{Z}(0,n-2)$),

$$
(7.354)\qquad \begin{aligned} E_1^{1,n-1} &\simeq \oplus_{s+t=n-1}{}^1E_\infty^{s,t} \\ &= {}^1E_\infty^{0,n-1} \\ &\simeq {}^1E_2^{0,n-1} \simeq \operatorname{Ker}\Phi_R^+ . \end{aligned}
$$

This completes the proof of Theorem 7.21. Q.E.D.

7.4. First description of $E_1^{0,n-1} \subset \operatorname{Ker}\Phi_R^+$. Because of its importance in applications, this and the next paragraphs give two explicit descriptions of the composition λ of $E_1^{0,n-1} \to E_1^{1,n-1} \to \operatorname{Ker}\Phi_R^+$, $\operatorname{Im}\lambda$ and λ^{-1}.

The first one uses the facts in §7.35.

Define δ_j^i: $A(R) \to A(R)$ ($i \in \boldsymbol{Z}(1,m)$, $j \in \boldsymbol{Z}(0,k_i-1)$) by

$$\delta_j^i f = \textstyle\sum_{J\in N_{2,n}} (-\partial)^J \partial f/\partial v_{j,J}^i .$$

Define for $f,g \in A(R)$,

$$
\begin{aligned}
(E_{i,j}f)g &= \textstyle\sum_{J\in N_{2,n}} \partial f/\partial v_{j,J}^i \partial^J g , \\
(E_{i,j}^+ f)g &= \textstyle\sum_{J\in N_{2,n}} (-\partial)^J (g\partial f/\partial v_{j,J}^i) .
\end{aligned}
$$

Then

Proposition 7.41.

(i) *Suppose* $\sum_{i=1,n} J_i dx_i$ ($J_i \in A(R)$) *represents* $\omega \in E_1^{0,n-1}$. *Then* $\lambda\omega = (\phi_1, \cdots, \phi_m)$, *where* $\phi_i = \delta_{k_i-1}^i J_1$.

(ii) *For* $\phi = (\phi_1, \cdots, \phi_m) \in \operatorname{Ker}\Phi_R^+$, *define* $f_{i,j} \in A(R)$ ($i \in \boldsymbol{Z}(1,m)$, $j \in \boldsymbol{Z}(0,k_i-2)$) *by* (7.353) *and* $f_{i,k_i-1} = \phi_i$. *Then* $\phi \in \operatorname{Im}\lambda$ *if and only if*

$$E_{i,j} f_{i',j'} = E_{i',j'}^+ f_{i,j}$$

for all i, i', j, j'.

Proof. (i) Obviously under the isomorphism $E_1^{1,n-1} \simeq {}^1E_2^{0,n-1}$ of (7.354), $d_1\omega$ corresponds to

$$
\begin{aligned}
[\delta J_1 \wedge dx'] &= [\textstyle\sum_J \partial J_1/\partial v_{j,J}^i \delta v_{j,J}^i \wedge dx'] \\
&= [\textstyle\sum (\delta_j^i J_1) \delta v_j^i \wedge dx'] ,
\end{aligned}
$$

which immediately implies the assertion.

(ii) Let ${}^0E$ and ${}^1E$ be the spectral sequences introduced in §7.32. Let $\{\delta_r^{p,q}\}$: ${}^0E \to {}^1E$ be the homomorphism induced by δ: $\{\Omega^{0,*}, \partial\} \to \{\Omega^{1,*}, \partial\}$. Consider the following commutative diagram:

$$
\begin{array}{ccccccc}
0 \longrightarrow & {}^0E_2^{0,n-1} & \longrightarrow & {}^0E_1^{0,n-1} & \xrightarrow{d_1} & {}^0E_1^{1,n-1} \\
 & \downarrow \delta_2^{0,n-1} & & \downarrow \delta_1^{0,n-1} & & \downarrow \delta_1^{1,n-1} \\
0 \longrightarrow & {}^1E_2^{0,n-1} & \longrightarrow & {}^1E_1^{0,n-1} & \xrightarrow{d_1} & {}^1E_1^{1,n-1} \\
 & \wr\downarrow a & & \wr\downarrow \alpha^{-1} & & \wr\downarrow \beta^{-1} \\
0 \longrightarrow & \operatorname{Ker}\Psi & \longrightarrow & A(R)^{\overline{m}} & \xrightarrow{\Psi} & A(R)^{\overline{m}} ,
\end{array}
$$

where the horizontal maps are exact, α and β are as in Lemma 7.351. Note that $\delta_1^{i,n-1}$ $(i=0,1)$ are described by the commutative diagram:

$$\begin{array}{ccc} {}^0E_1^{i,n-1} & \xrightarrow{\simeq} & \bar{E}_1^{0,n-1} \\ \Big\downarrow \delta_1^{i,n-1} & & \Big\downarrow d_1 \\ {}^1E_1^{i,n-1} & \xrightarrow{\simeq} & \bar{E}_1^{1,n-1}, \end{array}$$

where $\bar{E}$ is the spectral sequence of Lemma 7.322. In particular $\delta_1^{1,n-1}$ is injective by Theorem 5.63 (ii) for $n\geqq 2$.

By Proposition 5.64, there is a $J_1\in A(R)$ such that $\delta_1^{0,n-1}$ maps $\omega=[J_1dx']\in {}^0E_1^{0,n-1}$ to $\eta=[\sum f_{i,j}\delta v_j^i\wedge dx']\in {}^1E_1^{0,n-1}$. This completes the proof for $n=1$. Suppose now $n\geqq 2$. By Lemma 7.352 and the assumption that $\phi\in\operatorname{Ker}\Phi_R^+$, we have

$$d_1\eta=\beta\Psi\alpha^{-1}\eta=0\,.$$

Hence

$$\begin{aligned}\delta_1^{1,n-1}d_1\omega &= d_1\delta_1^{0,n-1}\omega \\ &= d_1\eta=0\,.\end{aligned}$$

Thus $d_1\omega=0$ and we have

$$(f_{i,j})\in\operatorname{Im} a\delta_2^{0,n-1}\,.$$

This implies the assertion since there is the following commutative diagram:

$$\begin{array}{ccccccc} E_1^{0,n-1} & \xrightarrow[\simeq]{} & {}^0E_2^{0,n-1} & & & & \\ \Big\downarrow d_1 & & \Big\downarrow \delta_2^{0,n-1} & & & & \\ E_1^{1,n-1} & \xrightarrow[\simeq]{c} & {}^1E_2^{0,n-1} & \xrightarrow[\simeq]{a} & \operatorname{Ker}\Psi & \xrightarrow[\simeq]{b} & \operatorname{Ker}\Phi_R^+ \end{array}$$

with $\lambda=bacd_1$ and $b(f_{i,j})=\phi$. Q.E.D.

7.5. Second description of $E_1^{0,n-1}\subset\operatorname{Ker}\Phi_R^+$. This is essentially due to Martinez [Mar].

Consider $A(R)$ as a subalgebra of A by the substitutions: $v_{j,J}^i=u_{j(1)+J}^i(i\in \boldsymbol{Z}(1,m), j\in\boldsymbol{Z}(0,k_i-1), J\in\boldsymbol{N}_{2,n})$. Then Proposition 7.12 implies $A=A(R)\oplus\mathcal{J}_\infty$. Denote the projection $A\to A(R)$ by $f\mapsto\bar{f}$.

To avoid confusion denote by $\bar{\partial}_i$ the derivation ∂_i of $A(R)$. Note that $\partial_i\,|\,A(R)=\bar{\partial}_i$ $(i\geqq 2)$.

Theorem 7.5. *Let* $\phi=(\phi_1,\cdots,\phi_m)\in A(R)^m$. *Then* $\phi\in\operatorname{Im}\lambda$ *if and only if* (7.51) $\delta_j(\sum_{i=1,m}\phi_iK_i)=0$ $(j\in\boldsymbol{Z}(1,m))$, *where* δ_j *is defined in* §5.4.

Remark. This a rephrase of the content of [R].

This implies then by Proposition 5.42 and (5.43) the following

Corollary 7.52. *Let* $\phi=(\phi_1,\cdots,\phi_m)\in\operatorname{Im}\lambda$. *Then* $\lambda^{-1}\phi$ *is represented by the restriction on* R_∞ *of such* $\omega\in\Omega_N^{0,n-1}$ *as* $\partial\omega=\kappa dx$ $(\kappa=\sum_{i=1,m}\phi_i K_i)$. *In particular it is represented by* $\sum_i J_i|_{R_\infty}dx_i$ *where* $J_1=\tilde{J}_1+\int_0^{x^1}\kappa[x,0]dx^1$, *and* $J_i=\tilde{J}_i$ $(i\geqq 2)$. *Here*

$$\tilde{J}_i=\sum_j\sum_{I\in N_{1,i}}\partial^I\Big(u^j\int_0^1\delta_{j,I+(i)}\kappa[x,su]ds\Big).$$

Consider $\Omega^{*,*}\subset\Omega_N^{*,*}$ by the maps $A(R)\subset A$, $\delta v^i_{j,J}\mapsto\delta u^i_{j(1)+J}$. Denote the differentials of $\Omega^{*,*}$ by $\bar{\delta}$ and $\bar{\partial}$.

Proof of Theorem 7.5. Let $\phi\in A(R)^m$. Suppose $\phi=\lambda\omega$ $({}^{\exists}\omega\in E_1^{0,n-1})$. Represent ω by $\varpi=\sum_i J_i dx_i$ $(J_i\in A(R)\subset A)$. Then $\bar{\partial}\varpi=0$, i.e.,

$$f:=\partial_1J_1+\cdots+\partial_nJ_n\in\mathcal{J}_\infty. \tag{7.53}$$

Since $\partial_iJ_i=\bar{\partial}_iJ_i\in A(R)$ $(i\geqq 2)$, f is the $\mathcal{J}_\infty$-component of ∂_1J_1, i.e.,

$$f=\sum_{I\in N_{2,n}}\partial J_1/\partial v^i_{k_i-1,I}\partial^IK_i\,,$$

By an obvious ananolgue of Lemma 5.121,

$$f-\sum_i(\delta^i_{k_i-1}J_1)K_i\in\partial_2\mathcal{J}_\infty+\cdots+\partial_n\mathcal{J}_\infty. \tag{7.54}$$

By Proposition 7.41, $\phi_i=\delta^i_{k_i-1}J_1$. Hence by (7.53) and (7.54)

$$\sum_{i=1,m}\phi_iK_i\in\partial_1A+\cdots+\partial_nA\,,$$

which is equivalent to (7.51) by Proposition 5.42.

Suppose now conversely $\phi\in A(R)^m$ satisfies (7.51). By Proposition 5.42, there is an $\omega=\sum_{i=1,n}J_idx_i\in\Omega_N^{0,n-1}$ such that $\partial\omega=\sum_j\phi_jK_j$. Put $\varpi=\sum_{i=1,n}\bar{J}_idx_i$. Then ϖ is $\bar{\partial}$-closed. By (7.54)

$$\sum_{j=1,m}\psi_jK_j-\sum_{i=1\,n}\partial_i\bar{J}_i\in\partial_2\mathcal{J}_\infty+\cdots+\partial_n\mathcal{J}_\infty.$$

where $\psi=(\psi_1,\cdots,\psi_m)=\lambda([\varpi])$. Since $J_i-\bar{J}_i\in\mathcal{J}_\infty$, we have

$$\sum_{j=1,m}(\phi_j-\psi_j)K_j\in\partial_1\mathcal{J}_\infty+\cdots+\partial_n\mathcal{J}_\infty. \tag{7.55}$$

Lemma 7.56 ([Mar]). *If* $a=(a_i)\in A(R)^m$ *satisfies* $\sum_{i=1,m}a_iK_i\in\partial_1\mathcal{J}_\infty+\cdots+\partial_n\mathcal{J}_\infty$, *then* $a=0$.

Proof. By assumption, there are $f_{i,j,I}\in A$ such that

$$\begin{aligned}\sum_ia_iK_i&=\sum_i\partial_i\Big(\sum_{j=1,m}\sum_{I\in N^n}f_{i,j,I}\partial^IK_j\Big)\\&=\sum_{i,j,I}(\partial_if_{i,j,I}\partial^IK_j+f_{i,j,I}\partial^{I+(i)}K_j)\,.\end{aligned} \tag{7.57}$$

By Proposition 7.12 we can consider $\{x,\ v^i_{j,J},\ \partial^IK_j\}$ as a system of global cordinates on N_∞. Defferentiate (7.57) with respect to $\partial^I K_j$ and put $\partial^I K_j=0$ $({}^\forall j, I)$. Then

$$a_i = b_{i,0}, \quad \text{for } i\in \boldsymbol{Z}(1,m),$$
$$b_{i,I} = 0, \quad \text{for } i\in \boldsymbol{Z}(1,m), I\in \boldsymbol{N}^n-\{0\},$$

where

$$b_{i,I} = \textstyle\sum_{j=1,n}(\bar\partial_j \bar f_{j,i,I}+\bar f_{j,i,I-(j)}).$$

Here $\bar f_{k,i,I}=0$ if some component of I is negative.

Then

$$\begin{aligned} a_i &= b_{i,0} \\ &= \textstyle\sum_{I\in N^n}(-\bar\partial)^I b_{i,I} \\ &= \textstyle\sum_{i,j,I}(-\bar\partial)^I(\bar\partial_j \bar f_{j,i,I}+\bar f_{j,i\ I-(j)}) \\ &= \textstyle\sum_{i,j,I}((-\bar\partial)^I\bar\partial_j \bar f_{j,i,I}+(-\bar\partial)^{I+(j)}\bar f_{j\ i,I}) = 0. \end{aligned}$$

Q.E.D.

Thus (7.55) implies $\phi_j=\psi_j$. In particular $\phi=(\phi_i)=\lambda[\varpi]$, which completes the proof of Theorem 7.5.

7.6. Applications

7.61. Wave equation. Let R be as in §3.4 and use the notations introduced there. R is defined by

$$K: = u_{2(1)}-\textstyle\sum_{i=2,n}u_{2(i)} = 0.$$

Then $\Phi_R^+: A(R)\to A(R)$ is

$$\Phi_R^+ f = (\partial_1^2-\textstyle\sum_{i=2,n}\partial_i^2)f.$$

By Theorem 7.21, $E_1^{1,n-1}(R)=\mathrm{Ker}\,\Phi_R^+$. It is shown in [T1] that if $n\geqq 3$ and f is a differential polynamial, then $\Phi_R^+ f=0$ if and only if $f=a+\sum_{i=0,1}\sum_{J\in N_{2,n}} a_{i,J}u_{i,J}$ where $a, a_{i,J}\in \boldsymbol{R}$. Proposition 7.41 then determines those conserved currents expressible by differential polynomials. See [AG,O] for other methods of computing $E_1^{0,n-1}(R)$. [AG] covers the case $n=2$. Both treat general linear differential equations.

7.62. Korteweg-de Vries equation. Let R be as in §3.5. Since $\Phi_R^+= D(u)^+K=-\partial_2-u\partial_1-\partial_1^3$,

$$E_1^{1,1}(R)\simeq\{f\in\mathcal{F}(x^i, u_j);\ (\partial_2+u\partial_1+\partial_1^3)f = 0\}.$$

Put $W_m=\mathrm{Ker}\,\Phi_R^+\cap\mathcal{F}(x^i, u_j; j\leqq m)$. It is easy to show $\dim W_{m+1}\leqq \dim W_m+1$. This implies immediately that *even in the space of C^∞ conservation laws, those found in* [MGK] *is independent and spanns $E_1^{0,1}(R)$.*

7.63. Benjamin-Bona-Mahony equation ([BBM]). Let $M=\boldsymbol{R}^2$, $N=M\times \boldsymbol{R}_u$ and $R=\{K=u_{2,1}-u_{0,1}+uu_{1,0}=0\}\subset N_3$, where $u_{i,j}=u_{(i,j)}$. Note that by a linear change of coordinates of $\boldsymbol{R}^2$, R can be transformed to a pseudo CK system and we can apply the results of this section to R. We have

$$D(u)^+K = \partial_1{}^2\partial_2-\partial_2+u\partial_1 .$$

In [T2] it is proved that $\operatorname{Ker}\Phi_R^+\simeq\boldsymbol{R}^3$ with 1, u, $u^2+2u_{1,1}$ as a basis. Moreover it is easily verified that $E_1^{0,1}\simeq E_1^{1,1}$. Thus the equation R has only three independent conservation laws.

8. Noether theorem

The results of §5–7 imply a quantitative version of the classical Noether theorem.

Let $M=\boldsymbol{R}^n$, $N=M\times\boldsymbol{R}^m$. The notations are as in §3.1 and §5.4.

8.1. Quantitative version.

Theorem 8.1. *Let R be the Euler-Lagrange equation with the Lagrangean $F\in\mathcal{F}N_\infty$, i.e., $R=\{\delta_1F=\cdots=\delta_mF=0\}$. Suppose R is a pseudo CK system. Then*

(i) $E_1^{1,n-1}(R)\simeq\mathcal{L}(R)$,

(ii) $d_1\colon E_1^{0,n-1}(R)\to E_1^{1,n-1}(R)$ *is injective for* $n\geqq 2$, *and*

$$0\to\boldsymbol{R}\to E_1^{0,0}(R)\to E_1^{1,0}(R)$$

is exact for $n=1$.

Recall that $E_1^{0,n-1}$ is the space of conserved currents of R (cf. Remark 4.5 (ii)).

Proof. (ii) is Corollary 7.22.

By Proposition 5.5, we have $\Phi=\Phi^+$, whence $\Phi_R=\Phi_R^+$. Then by Theorem 6.6 and Theorem 7.21

$$\begin{aligned} E_1^{1,n-1}(R) &\simeq \operatorname{Ker}\Phi_R^+ \\ &= \operatorname{Ker}\Phi_R \\ &\simeq \mathcal{L}(R) . \end{aligned}$$

Q.E.D.

8.2. Explicit version. Let $R=\{\delta_jF=0\}$ be as in Theorem 8.1, i.e.,

$$\delta_jF = u^j_{k\,j(1)}-F_j$$

with $F_j\in A$ satisfying (7.11). Put $E_r^{p,q}=E_r^{p,q}(R)$

Let $\mu\colon E_1^{0,n-1}\to\mathcal{L}(R)$ be the composition of $d_1\colon E_1^{0,n-1}\to E_1^{1,n-1}$ and the isomorphism of Theorem 8.1 (i). The elements of Im μ are called *the Noether symmetries of R.*

Indentify $\mathcal{L}(R)$ with Ker $\Phi_R \subset A(R)^m$ by Theorem 6.6, regard $A(R)\subset A$ as in §7.5 and use notations of §7.31. Then Theorem 7.5 and Corollary 7.52 implies the following direct and inverse Noetheor theorems.

Theorem 8.2.

(i) *μ maps $[\sum_i J_i dx_i]\in E_1^{0,n-1}$ to $\phi=(\phi_1,\cdots,\phi_m)\in A(R)^m$, with $\phi_i=\sum_{I\in N_{2,n}}(-\partial)^I\partial J_1/\partial v^i_{k_i-1,I}$.*

(ii) *$\phi=(\phi_1,\cdots,\phi_m)\in A(R)$ is a Noether symmetry if and only if*

$$\delta_i(\textstyle\sum_j \phi_j\delta_j F) = 0, \qquad i\in \boldsymbol{Z}(1,m)\,.$$

(iii) *If ϕ is a Noether symmetry, $\mu^{-1}\phi\in E_1^{0,n-1}$ is represented by $\sum_{i=1,n}J_i dx_i$* $(J_1=\tilde{J}_1+\int_0^{x^1}\kappa[x,0]dx^1, J_i=\tilde{J}_i\ (i\geqq 2))$, where

$$\tilde{J}_i = \textstyle\sum_{I\in N_{1,i}}\sum_{j=1,m}\partial^I(u^j\int_0^1\delta_{j,I+(i)}\kappa[x,su]ds)\,,$$

with $\delta_{j,I}$ defined by (5.44) *and $\kappa=\sum_{k=1,m}\phi_k\delta_k F$.*

REMARK 8.21. (i) and (ii) are given essentially in [Mar].

8.3. Classical version. Finally we write the condition of $\phi\in\mathcal{L}(N)$ to be tangent to R_∞ and induce a Noether symmetry.

Let R be as in §8.2. Indentify $\mathcal{L}(N)=A^m$ by Proposition 6.21.

Theorem 8.3. *Let $\phi=(\phi_1,\cdots,\phi_m)\in\mathcal{L}(N)$. Then ϕ is tangent to R_∞ and $[\phi\,|_{R_\infty}]\in\mathcal{L}(R)$ is a Noether symmetry if and only if*

$$L_\phi(Fdx)\in\partial\Omega_N^{0,n-1}$$

Proof. By definition

$$L_\phi(Fdx) = i(\phi)\delta(Fdx)\,.$$

By Lemma 5.41, there is an $\eta\in\Omega_N^{1,n-1}$ such that

$$\delta(Fdx) = \partial\eta+\delta_i F\delta u^i\wedge dx\,.$$

By Lemma 6.31, we have

$$\begin{aligned} L_\phi(Fdx) &= i(\phi)\,(\partial\eta+\delta_i F\delta u^i\wedge dx) \\ &= \partial i(\phi)\eta+\textstyle\sum_j i(\phi)\delta_j F\delta u^j\wedge dx\,. \end{aligned}$$

Since $\phi=\sum\partial^I\phi_j\partial/\partial u^j_I$, we obtain

$$L_\phi(Fdx)\equiv\textstyle\sum_j\phi_j\delta_j Fdx \quad (\text{mod. } \partial\Omega^{0,n-1})\,.$$

Thus $L_\phi(Fdx)\in\operatorname{Im}\partial$ if and only if

$$\textstyle\sum_j \phi_j \delta_j F dx \in \partial\Omega^{0,n-1},$$

which is equivalent $\phi\in\operatorname{Im}\mu$ by Theorem 8.2. Q.E.D.

This proof shows also the following

Corollary 8.31. *Suppose* $\phi=(\phi_1,\cdots,\phi_m)\in\mathcal{L}(N)$ *satisfies* $L_\phi(Fdx)=\partial\omega$ $({}^{\exists}\omega\in\Omega_N^{0,n-1})$. *Then* $\phi\,|_{R_\infty}$ *is a Noether symmetry and corresponds to*

$$[(\omega-i(\phi)\omega_F)\,|_{R_\infty}]\in E_1^{0,n-1}(R),$$

where $\omega_F\in\Omega_N^{1,n-1}$ *is defined by* (5.411).

9. Bäcklund correspondences

The flat connection H defines a class of correspondence between solutions of differential equations. This includes the usual Bäcklund transformation.

9.1. Focus our attention on the following aspect of the usual Bäcklund transformations: Let R and R' be two differential equations. A Bäcklund transformation T transforms solutions of R and R' to holonomic equations whose solutions are those of R' and R respectively (cf. Remark 2.23 for the meaning of "holonomic"). In other words solutions of R and R' allows us to obtain families of solutions of R' and R respectively by integrating ordinary differential equations. Thus T defines a point to finite-dimensional submanifold correspondence between $\mathcal{S}ol(R)$ and $\mathcal{S}ol(R')$.

This aspect, which seems to be essential, is not lost by the following generalization.

Let $p^i\colon N^i\to M$ $(i=1,2)$ be bundles and suppose $R^i\subset N^i_{k_i}$ $(i=1,2)$ satisfy Condition 2.3. We call a submanifold $R\subset N_k$ $(N=N^1\times_M N^2)$ a *Bäcklund map from* R^1 *to* R^2 if

$$\tilde{R}=R_\infty\cap(R^1_\infty\times_M R^2_\infty)$$

is a subbundle of $N_\infty=N^1_\infty\times_M N^2_\infty$ and $q^1\,|_{\tilde{R}}$ has finite-dimensional fibers, q^1 being the natural projection: $N_\infty\to N^1_\infty$. Here $N=N^1\times_M N^2=\{(y^1,y^2)\in N^1\times N^2;\ p^1y^1=p^2y^2\}$ is the fiber product. If R is also a Bäcklund map from R^2 to R^1, i.e., $q^2\,|_{\tilde{R}}\colon\tilde{R}\to R^2_\infty$ has finite-dimensional fibers, then R is called a *Bäcklund correspondence between* R^1 *and* R^2.

Let R be a Bäcklund map from R^1 to R^2. For $s^1\in\mathcal{S}ol(R^1)$, let $R(s^1)\subset R^2_\infty$ be the pull-back of the bundle $\tilde{R}\to R^1_\infty$ by s^1_∞:

$$R(s^1)=\{y^2\in R^2_\infty;\ (s^1_\infty(\pi^2(y^2)),y^2)\in\tilde{R}\},$$

where $\pi^2\colon N^2_\infty\to M$ is the projection. By definition, $R(s^1)\to M$ is a finite-dimensional bundle.

Proposition 9.11. *If R is a Bäcklund map from R^1 to R^2, then $R(s^1)$ is an intermediate integral of R^2 (cf. Remark 2.43), i.e., $TR(s^1)\supset H_{R^2}|_{R(s^1)}$.*

Proof. Let $y^2\in R(s^1)$. By definition $y=(y^1,y^2)\in\tilde{R}$, where $y^1=s^1_\infty(x)$, $x=\pi^2(y^2)$.

Let $X\in T_xM$. Let $\tilde{X}\in(H_N)_y$ and $\tilde{X}^i\in(H_{N^i})_{y^i}$ $(i=1,2)$ be its horizontal lifts. Obviously $\tilde{X}^i=q^i_*\tilde{X}$ $(i=1,2)$. Since H's are tangent to R_∞ and R^i_∞ $(i=1,2)$, we have $\tilde{X}\in T_yR_\infty$ and $\tilde{X}^i\in T_{y^i}R^i_\infty$ $(i=1,2)$. Hence

$$\tilde{X}=(\tilde{X}^1,\tilde{X}^2)\in T_y(R^1_\infty\times_M R^2_\infty)\cap T_yR_\infty=T_y\tilde{R}\,.$$

Moreover we have $\tilde{X}\in T_y((q^1)^{-1}s^1_\infty(M))$, since $\tilde{X}^1=s^1_{\infty*}X$. Hence

$$\tilde{X}^2\in q^2{}_*T_y((q^1)^{-1}s^1_\infty(M))=T_{y^2}R(s^1)\,.$$

Since $(H_{R^2})_{y^2}$ is spanned by $(\tilde{X}^2)'$s, the assertion follows. Q.E.D.

Thus, when we have a Bäcklund map R from R^1 to R^2 and a solution s^1 of R^1, we can obtain solutions of R^2 by solving the holonomic system $R(s^1)$, which can be reduced to ordinary differential equations.

Note that even if the fibers of $R\to R^1$ are not of finite dimension, Proposition 9.11 still holds. But we cannot obtain solutions of $R(s^1)$ by solving only ordinary differential equations. This shows that the finite-dimensionality of the fibers of $R\to R^1_\infty$ is the essential property of Bäcklund maps.

REMARK 9.12. Pommaret considers in [P] a similar situation as ours, which differs, however, from ours essentially in the following two points. First he starts from $R\subset N_k$ $(N=N^1\times_M N^2)$ and then takes up and solves the problem of finding the equations whose infinite prolongations are $q^i(R_\infty)$*'s* $(i=1,2)$. Thus he finds equations R^i*'s* for which a given R is a Bäcklund transformation. Secondly he does not pay attention to the condition of finite-dimensionality of $q^i|_{R_\infty}$ $(i=1,2)$, which is from our point of view the essential aspect of Bäcklund transformations.

EXAMPLE 9.13. Let $M=\boldsymbol{R}^2$, $N^i=M\times\boldsymbol{R}$ $(i=1,2)$. Denote by u^i the linear coordinate on the fibers of $N^i\to M$ $(i=1,2)$. Let $R^i\subset N^i_3$ $(i=1,2)$ be the modified KdV equations:

$$w_{0,1}-6w^2_{1,0}+w_{3,0}=0 \tag{9.131}$$

$(w=u^1,u^2)$. Define $R\subset N_1$ by

$$u^1_{1,0}+u^2_{1,0}+k^2-u^1+u^2=0 \tag{9.132}$$

where k is a constant. Then it is easy to see that

$$\mathcal{F}\tilde{R} \cong \mathcal{F}(x^1, x^2, u^1, [u^2])$$
$$\cong \mathcal{F}(x^1, x^2, [u^1], u^2),$$

where $[w]=\{w_{j,0}; j=0,1,2,\cdots\}$. In particular, the projections $\tilde{R}\to R^i_\infty$ $(i=1,2)$ have one dimensional fibers. Note that on $\tilde{R}$ we have automatically the equation

$$(9.133) \qquad u^1_{0,1}+u^2_{0,1} = 4(-k^2u^1_{1,0}+u^2_{1,0}+u^2_{1,0}X+u^2_{2,0}X)$$

$(X=u^1-u^2)$. Usually (9.133) is considered as the one half of the defining equations for the Bäcklund transformations but in fact (9.133) follows from (9.132) and (9.131) $(w=u^1, u^2)$.

REMARK 9.14. Given an equation R^1, it is important but difficult to construct Bäcklund correspondences between R^1 and other equations.

There are two methods known, Hirota's [Hi] and Whalquist-Estabrook's [WE]. The main part of the latter is to construct a Lie algebra $\mathfrak{g}$ of finite dimension and a $\mathfrak{g}$-valued 1-form $\omega\in\mathfrak{g}\otimes\Omega^{0,1}_{R^1}$ such that

$$\partial\omega-[\omega, \omega] = 0,$$

where $[\sum X_i\otimes\omega^i, \sum Y_j\otimes\eta^j]=\sum[X_i, Y_j]\otimes\omega^i\wedge\eta^j$ for $X_i, Y_j\in\mathfrak{g}$, $\omega^i, \eta^j\in\Omega^{0,1}_{R^1}$.

9.2. We compare our definition with that of Pirani and Robinson (cf. [PR, PRS]).

Their definition is in our terminologies as follows: Let $N^i\to M$ $(i=1,2)$ be bundles and ψ a cross-section of the bundle $\pi\colon N^1_k\times_M N^2_1\to N^1_k\times_M N^2$. Define $R^\psi=p^{-1}(\mathrm{Im}\,\psi)$, where $p\colon N_k=N^1_k\times_M N^2_k\to N^1_k\times_M N^2_1$ is the natural projection. Let $R^\psi_\infty\subset N^1_\infty\times_M N^2_\infty$ be the infinite prolongation of R^ψ. Obviously the projection $\pi_\infty\colon N^1_\infty\times_M N^2_\infty\to N^1_\infty\times_M N^2$ is a diffeomorphism on R^ψ_∞. Put $\tilde{R}^\psi=\pi_\infty(R^\psi_\infty)$. Then ψ is called *a Bäcklund map* if $\tilde{R}^\psi$ is in the infinite prolongation (in the obvious sense) of some $\tilde{R}\subset N^1_{k+1}\times_M N^2$. When the image of $\tilde{R}^\psi$ under the natural projection $N^1_\infty\times_M N^2\to N^1_\infty$ is in the infinite prolongation of some $R^1\subset N^1_{k+1}$, ψ is called *an ordinary Bäcklund map* for R^1. Finally, when the image of R^ψ_∞ under the natural projection $N_\infty\to N^2_\infty$ is in the infinite prolongation of some $R^2\subset N^2_{s+1}({}^{\mathcal{I}}s)$, ψ is called *a Bäcklund transformation between R^1 and R^2.*

Note that an ordinary Bäcklund map in their sense is also a Bäcklund map from R^1 to N^2 in our sense. In fact, by the definition of R^ψ_∞, it is obvious that the dimension of the fibers of $R^\psi_\infty\to N^1_\infty$ coincides with that of the fibers of $N^2\to M$.

However, a Bäcklund transformation ψ between R^1 and R^2 may not be a Bäcklund correspondence between R^1 and R^2 in our sense, since the finite-dimensionality of the fibers of $R^\psi_\infty\to R^2_\infty$ does not follow necessarily from their definition.

10. Concluding remarks

10.1. Motivations. The motivation of this paper is in Proposition 4.8, which came naturally from an effort to unite the two important aspects of differential forms: they define cohomology classes and differential systems.

This proposition aroused the problem of computing $E_r^{p,q}(R)$, which turned out to be connected with various problems such as characteristic classes of foliations (Remark 4.51 (iii)), symmetries (§6), Gelfand-Fuks cohomologies (§3.3) etc.. Thus the description of $E_1^{0,n-1}(R)$ for determined systems R (Theorem 7.21), which is useless viewed from the original motivation, is not insignificant and actually gives us a method of computing the conserved currents of various differential equations (§7.6).

However at present the knowledge about $E_r^{p,q}(R)$ is quite unsatisfactory, especially for overdetermined systems.

10.2. Languages. Manin says in [Man] there are three equivalent languages for the formal study of differential equations—classical, differential-algebraic, and geometric. This paper uses the geometric one in developing the basic notions and the classical one, which uses the standard coordinate on N_∞(§1.3), in the proofs and examples.

Note that the languages used in [Man] is "extrinsic", i.e., consider the pair $(N_\infty, \mathcal{J}_\infty)$ (§2.2) the basic object, whereas [V1] and this paper use intrinsic languages, i.e., the manifold R_∞ is regarded as the basic object (cf. Remark 2.43).

Further note that [V1] uses differential-algebraic, algebraic-geometric, and categorical terminologies to define various concepts including $E_r^{p,q}(R)$, whereas this paper uses only differential-geometric ones. In fact only one basic notion, the flat connection H_R (§2.4), is enough for theoretical development, with a little loss of generality of R however.

10.3. Problems. In spite of the simplicity of their definitions, the invariants H_R, $\Omega_R^{*,*}$ and $E_r^{*,*}(R)$ of R have many nontrivial problems related to them. Here are some, which seem to deserve further investigations.

10.31. Describe the theory of characteristics of R by H_R and $\Omega_R^{*,*}$ (cf. Remark 6.16 for the Cauchy characteristics).

10.32. When dim $M=2$, dim $N=3$, $R\subset N_2$, state the applicability to R of the Darboux's method of integration ([Gou]) in terms of $\Omega_R^{*,*}$. This seems to clarify the geometric background of this method and to enable us to generalize it.

10.33. Develop the theory of Bäcklund correspondences along the lines in §9. Unite it with the Whalquist-Estabrook method ([WE]), cf. Remark

9.14). This might give us a method of constructing elements of $E_r^{p,q}(R)$ (cf. [WSK]).

10.34. Calculate $E_1^{0,3}(R)$ for the Yang-Mills equation on $\boldsymbol{R}^4$. [GS] constructs 15 independent elements. Since R has soliton solutions, it can be infinite-dimensional.

10.35. (i) Compute $\oplus_{i>0} E_\infty^{i,6-i}(R)$ for the equation R of integrability of almost complex structures (§3.8) on $M=S^6$.

(ii) Is it possible to use $E_1^{p,q}(R)$ of §3.8 for the deformation theory of complex structures (cf. §4.6)?

(iii) Compute $H^*(W_{2n}, L_n^C; \boldsymbol{R})$ (cf. Remarks 3.81, 4.82). $F^{p+1}H^*$ might measures $\mathrm{Im}\,(\pi_p\, \mathcal{S}ol(R) \to \pi_p(\Gamma R) = \pi_p(\Gamma N))$. (cf. Proposition 4.4). Note that $\pi_p(\Gamma N) \otimes_{\boldsymbol{Z}} \boldsymbol{R}$ is "calculable" algebraically by the Sullivan theory ([Su]).

10.36. Compute $H^*(W_q, o_q; S^* W_q')$ (cf. §4.2).

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