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The Theory of Construction of Finite Semigroups III. Finite Unipotent Semigroups.

By Takayuki TAMURA

As defined in the previous paper [4], we mean by a unipotent semigroup a semigroup which has unique idempotent, and in particular by a z-semigroup a unipotent semigroup whose unique idempotent is a zero 0, i.e. $0x=x0=0$ for all x; and a unipotent semigroup which contains a non-trivial group as a proper subsemigroup is called a unipotent semigroup with group.

As a special case of [2], we see that the study of finite unipotent semigroups with group is reduced to that of finite z-semigroups. But, as far as finite z-semigroups are concerned, the complete theory has not yet been established, although it has been done partly in [1], [4]. In §1 we shall investigate the structure of finite z-semigroups by defining a new ordering, so that some results in the previous paper [1] will be explained more easily here. In §2, we shall construct finite z-semigroups by the decompositions of certain finite free z-semigroups, and finally in §3 we shall complete the construction theory of a unipotent semigroup with group to complement the results in the previous paper.

§1. Fundamental Properties of z-Semigroups.

S denotes a finite z-semigroup. It is easily shown that a subsemigroup of S is a finite z-semigroup and the homomorphic image of S is also a finite z-semigroup.

1. Partial Ordering. Let a and b be elements of S. The element a is called a multiple of the element b if one of the following four equalities holds:

(1.1) $a=bx$ for some $x\in S$.

(1.2) $a=yb$ for some $y\in S$.

(1.3) $a=zbu$ for some $z,\ u\in S$.

Lemma 1.1. *Every non-zero element of a finite z-semigroup S is never a multiple of itself.*

Proof. Suppose that a is a multiple of itself, and then consider $X=\{x\,;\, ax=a\}$, $Y=\{y\,;\, ya=a\}$,

$Z=\{z\,;\, zau=a$ for some $u\in S$ depending on $z\}$. X, Y, and Z are the subsemigroups which contain 0. Hence we get $a=0$, q. e. d.

We defined in [1] the left ordering $a\underset{l}{\gtrsim}b$ and the right ordering $a\underset{r}{\gtrsim}b$. Let us, here, introduce a new ordering $a\geqq b$ meaning that either $a=b$ or a is a multiple of b.

Lemma 1.2. *The ordering $a\geqq b$ is a partial ordering.*

Proof. The reflexive law is trivial. At first, we shall prove the transitive law. Suppose $a\geqq b$ and $b\geqq c$. The proof is obvious in the following cases: (1) $a=b$ or $b=c$ (2) $a\underset{l}{\gtrsim}b$ and $b\underset{l}{\gtrsim}c$ (3) $a\underset{r}{\gtrsim}b$ and $b\underset{r}{\gtrsim}c$. In the other cases, we get easily $a=zcu$ for suitable $z, u\in S$. Next, we shall prove the symmetric law. If $a\geqq b$ and $b\geqq a$, we see easily that a or b is a multiple of itself, leading to $a=b=0$ by Lemma 1.1, q.e.d.

Clearly the zero 0 is the greatest element of S with respect to the ordering $\geqq$. $a>b$ denotes that $a\geqq b$ and $a\neq b$.

Rewriting Lemma 1.1,

Lemma 1.1′. *If $a\neq 0$, then $ab>a$, $ba>a$ for any b, and $bac>a$ for any b, c.*

2. Primes and Height. According to [1], an element is minimal with respect to the right ordering if and only if it is minimal with respect to the left ordering, further if and only if it is minimal with respect to the ordering $\geqq$. A minimal element x is called a prime, which means that x cannot be divided into a product of elements. In a finite z-semigroup, there is a prime at least. $S-S^2$ is the set of all the primes of S.

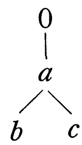
Lemma 1.3. *If a covers*[1] *b, then we have $a=bp$ or $a=pb$ for some prime p. But the converse is not true.*

Proof. We shall use Lemma 1.1′ in the proof. If $a=0$, then $b\neq 0$ and $bp>b$, $pb>b$, hence $bp=pb=0$ for any prime p. Hereafter assume $a\neq 0$. We have not $a=x'by'$ but $a=bp$ or $a=pb$ for some $p\neq 0$, because, if not so, $x'b\neq 0$, $b\neq 0$, and $a>x'b>b$, which implies that a would not cover b. If p is not a prime: $p=zu$, then $a=bzu$ or $a=zub$ where $bz\neq 0$, $ub\neq 0$. Henceforth we get $b<bz<a$ or $b<ub<a$, contradicting the assumption. The converse is not true as the following example shows. In the semigroup $\{0, a, b, c\}$ with multiplication

1) $a>b$ and $a>c>b$ for no c.

	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	a	0
c	0	0	0	a

whose diagram is



(isomorphic to 9_4 in [3]),

c is a prime, but bc does not cover b. q.e.d.

Let S be isomorphic to S' under f, and let x' and y' be the isomorphic images of x and y respectively. Then $x' > y'$ in S' if and only if $x > y$ in S. Meanwhile, if f is a homomorphism, then it holds that $x \geqq y$ in S implies $x' \geqq y'$ in S', and a homomorphic image of a prime is not necessarily a prime, q.e.d.

Consider a chain having x as its greatest element:

(1. 4) $x_1 < x_2 < \cdots < x_k = x$ where x_i covers $x_{i-1} (i=2, \cdots, k)$ and x_1 is a prime.

Then we say that x has a chain of length k. We term the height $H[x]$ of x as the maximum of lengths of chains which x has. The maximum of heights of elements of S is called the height $H[S]$ of S. Clearly $H[x]=1$ if and only if x is a prime.

Lemma 1. 4 is proved easily and Lemma 1. 5 is obtained in the same way as Lemma 9 and Corollary 6 in the previous paper [5].

Lemma 1. 4. (1. 5) $x < y$ *implies* $H[x] < H[y]$.

(1. 6) $H[a] = H[S]$ *if and only if* $a = 0$.

Lemma 1. 5. *Let* $l = H[a]$. *Then there is a chain*

$$x_1 < x_2 < \cdots < x_l = a,$$

where x_1 *is a prime,* x_i *covers* $x_{i-1} (i=2, 3, \cdots, l)$, *and* $H[x_i] = i \ (i=1, \cdots, l)$.

Now we get a theorem.

Theorem 1. 1. *Any element of a finite z-semigroup is factorized into a product of a finite number of primes. In detail, if x has a chain of length k, then x is expressed as a product of k primes. But the expression of factorization is not necessarily unique.*

Proof. Lemma 1. 3 makes us see easily that an element x having a chain (1. 4) is factorized into a product of k primes.

To verify the latter half, we give the examples as follow. Let us consider the semigroups 27_5, and 23_5 given in [3],

$$27_5 \quad \begin{array}{c|ccccc} & 0 & a & b & c & d \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 & a & a \\ c & 0 & 0 & a & b & b \\ d & 0 & 0 & a & b & b \end{array} \qquad 23_5 \quad \begin{array}{c|ccccc} & 0 & a & b & c & d \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 0 & a \\ c & 0 & 0 & 0 & 0 & a \\ d & 0 & 0 & a & a & b \end{array}$$

where c and d are primes and

$$a = d^3 = d^2c \qquad b = c^2 = cd = d^2 \qquad \text{in } 27_5$$
$$a = d^3 = cd \qquad \text{in } 23_5, \qquad \text{q.e.d.}$$

Lemma 1.6. *Let $h = H[S]$. Any product of h primes is* 0.

Proof. Let $p_1 \cdots p_h$ be a product of h primes. Without loss of generality, we may assume $p_1 \cdots p_h \neq 0$; and we get $p_1 < p_1p_2 < \cdots < p_1p_2 \cdots p_{h-1} < p_1p_2 \cdots p_h$ by Lemma 1.1′, so that

$$h = H[S] \geqq H[p_1p_2 \cdots p_h] \geqq h,$$

whence $H[p_1p_2 \cdots p_h] = h$ and so $p_1p_2 \cdots p_h = 0$ by (1.6) of Lemma 1.4.

Theorem 1.2. *For a finite z-semigroup S, there is a positive number h such that*

$$S^h = \{0\}, \quad \textit{and} \quad S^i \neq \{0\} \quad \textit{for} \quad 1 \leqq i < h.$$

This h is nothing but $H[S]$.

Proof. Let $h = H[S] = H[0]$. Any element y of S^h is expressed as $y = z_1z_2 \cdots z_h$. Since each z_i is divided into a product of primes, y is expressed as a product of h primes at least:

$$\begin{aligned} y &= p_1p_2 \cdots p_l & l &\geqq h \\ &= (p_1 \cdots p_h)(p_{h+1} \cdots p_l) = 0 & &\text{by Lemma 1.6.} \end{aligned}$$

Therefore $S^m = \{0\}$ for $m \geqq h$, and by Lemma 1.5 there is a chain

$$y_1 < y_2 < \cdots < y_{h-1} < y_h = 0$$

where $H[y_i] = i$, y_1 is a prime; and so y_i is divided into the product of i primes by Theorem 1.1. Hence $S^i \neq \{0\}$ if $1 \leqq i < h$. Thus this theorem has been proved, q. e. d.

If $ax = xa = 0$ for all $x \in S$, then a is called an annihilator of S. When $H[S] = h$, any product of $h-1$ primes is an annihilator. If $h \geqq 2$, then there is an annihilator beside 0, and it is covered by 0. $S^{h-1} - \{0\}$ is the set of all non-zero annihilators. If $h = 2$, then S is a semigroup defined as $xy = 0$ for all $x, y \in S$.

3. Depth. By the way we shall define a term "depth". By the "depth $D[x]$ of an element x" we mean the maximum of lengths d of chains combining x with 0:

$$x = x_1 < x_2 < \cdots < x_d = 0,$$

x_i convering x_{i-1} $(i=2, \cdots, d)$. The maximum of depths of elements of S is called the depth $D[S]$ of S. Of course $D[x]=1$ if and only if $x=0$, and $D[S]=H[S]$.

We get easily the following lemmas

Lemma 1.7. *$x<y$ implies $D[x]>D[y]$.*
Analogously to Lemma 1.5,

Lemma 1.8. *Let $k=D[a]$. Then there is a chain*

$$a = y_k < y_{k-1} < \cdots < y_2 < y_1 = 0$$

where y_{i-1} covers y_i $(i=2, \cdots, k)$ and $D[y_i]=i$ $(i=1, \cdots, k)$.

Lemma 1.9. *Let $p_1, p_2, \cdots, p_m$ be all the primes of S, and let $x \neq 0$. $D[x]=i$ if and only if*

$$\text{(1.7)} \qquad \underset{1 \leq j \leq m}{\text{Max.}} \{D[xp_j], D[p_jx]\} = i-1.$$

§ 2. Decompositions of the Free z-Semigroup.

1. The number of primes of S is called the basic breadth of S. The height and the basic breadth of S will play an important rôle in the construction of a finite z-semigroup S. As the simplest case, we found the structure of S of basic breadth 1 in the paper [1]: If and only if a finite z-semigroup S has a unique prime, S is a finite power z-semigroup; and this is characterized by the condition that S is a chain with respect to the ordering $\geqq$.

In order to research construction of a finite z-semigroup S when the height h and the basic breadth m of S are given, it is convenient to obtain S as a homomorphic image of certain free z-semigroup generated by m elements. It is for this reason that the free z-semigroup is finite as we shall see. Hereafter suppose $h>2$, $m>1$.

Here are given $m+1$ elements $p_1, \cdots, p_m, 0$. Let F consist of 0 and all the expressions

$$p_{i_1} p_{i_2} \cdots p_{i_\nu}$$

where $1 \leqq \nu < h$ and each i_j is one of $1, \cdots, m$ $(j=1, \cdots, \nu)$ and the multiplication in F is defined as follows.

$$x0 = 0x = 0 \qquad \text{for all } x \in F, \text{ and}$$

for $x = p_{i_1} \cdots p_{i_\nu}$, $y = p_{j_1} \cdots p_{j_\mu}$,

$$xy = \begin{cases} p_{i_1} \cdots p_{i_\nu} p_{j_1} \cdots p_{j_\mu} & \nu+\mu < h \\ 0 & \nu+\mu \geqq h. \end{cases}$$

we see easily that F is a z-semigroup of height h with primes $p_1, \cdots, p_m$, and the number of elements of F is

$$\sum_{i=0}^{h-1} m^i = \frac{m^h-1}{m-1}$$

in which m^i is the number of elements of height i. F is called the free z-semigroup of height h and basic breadth m. Any z-semigroup S of height h and of basic breadth m is obtained as a homomorphic image of F. Hence we have

Lemma 2.1. *Let n be the order of S.*

$$m+h-1 \leqq n \leqq \frac{m^h-1}{m-1}.$$

2. In order to construct all S of height $h>1$ and of basic breadth $m>1$, we must find all the decompositions $F = \sum_{\alpha \in S} C_\alpha$ ($C_\alpha \cap C_\beta = \phi$, $\alpha \neq \beta$) of F satisfying the following conditions:

(2.1) For α and a prime p_i, there are β and γ (depending on p_i) such that $x, y \in C_\alpha$ imply $xp_i, yp_i \in C_\beta$, $p_i x, p_i y \in C_\gamma$.

(2.2) Each prime element of F composes a class of only itself:

$$C_{\alpha_i} = \{p_i\} \qquad (i = 1, \cdots, m).$$

(2.3) There are h classes $C_{\beta_1}, C_{\beta_2}, \cdots, C_{\beta_h}$ such that $\{p_1\} = C_{\beta_1}$, and C_{β_j} contains a multiple of an element $C_{\beta_{j-1}}$ ($j=2, \cdots, h$).

3. Find a system $\{a_1, \cdots, a_\nu\}$ of mutually incomparable[2] non-prime elements of F for which there is an element b at least such that $H[b] = h-1$ and $a_i \not\leqq b$ for all i. Every ideal I_1 is determined by the system $\{a_1, \cdots, a_\nu\}$ as $I_1 = \sum P(a_i)$ where $P(a_i) = \{x;\, x \geqq a_i\}$ and so $a_1, \cdots, a_\nu$ are minimal in I_1. Let us define $I_2, \cdots, I_h$ by induction. Assuming that $I_1, I_2, \cdots, I_{j-1}$ are already obtained, I_j denotes the set of all the elements x of $F - \sum_{k<j} I_k$ which fulfil

2) If neither $x>y$, nor $x=y$ nor $x<y$, we say that x and y are incomparable.

$$xp_i \in \sum_{k<j} I_k \quad \text{and} \quad p_i x \in \sum_{k<j} I_k \quad \text{for all primes } p_i\,.$$

$I_2, \cdots, I_h$ are non-empty as we shall show.

Lemma 2. 2. *$H[x]=h-j$ implies $x \in \sum_{\nu=1}^{j+1} I_\nu$.* $(j=0, 1, \cdots, h-1)$

Proof. The lemma is true for $j=0$, because $H[x]=h$ implies $x=0 \in I_1$. Assuming that it holds for $j \leqq \lambda-1$, we shall prove that $H[x]=h-\lambda$ implies $x \in \sum_{\nu=1}^{\lambda+1} I_\nu$. By Lemma 1. 4,

$H[xp_i]>H[x]=h-\lambda$ i.e. $H[xp_i] \geqq h-(\lambda-1)$ for all primes p_i from which we get $xp_i \in \sum_{\nu=1}^{\lambda} I_\nu$ for all p_i by the assumption of induction. Similarly $p_i x \in \sum_{\nu=1}^{\lambda} I_\nu$ for all p_i. Henceforth $x \in \sum_{\nu=1}^{\lambda+1} I_\nu$ by the definition of I_j.

Lemma 2. 3. *I_j $(j=2, \cdots, h)$ are not empty.*

Proof. Any element b of height $h-1$ outside the ideal I_1 is contained in I_2. By Lemma 1. 5, we have a chain

$$b_1 < b_2 < \cdots < b_{h-2} < b_{h-1} = b$$

where b_{i+1} covers b_i, $H[b_i]=i$ $(i=1, \cdots, h-1)$ and b_1 is a prime. We shall prove $b_{h-j} \in I_{j+1}$ $(j=2, \cdots, h-1)$. Since we have known $b_{h-j} \in \sum_{\nu=1}^{j+1} I_\nu$ $(j=2, \cdots, h-1)$ because of Lemma 2. 2, we may show

$$b_{h-j} \bar{\in} \sum_{\nu=1}^{j} I_\nu \quad (j=2, \cdots, h-1) \tag{2. 4}$$

Assuming truth of (2. 4) for $j \leqq \lambda-1$, we have $b_{h-\lambda} \bar{\in} \sum_{\nu=1}^{\lambda} I_\nu$ because $b_{h-(\lambda-1)}=b_{h-\lambda} p_i$ or $p_i b_{h-\lambda}$ for certain p_i, q. e. d.

Thus we have obtained a sequence

$$I_1,\ I_2,\ \cdots,\ I_j,\ \cdots,\ I_h$$

which will be called a principal sequence.

4. First, let $C_0=I_1$, and decompose all the elements of I_2 into arbitrary number of classes:

$$I_2=\sum_{\alpha=1}^{\mu_2} C_\alpha\,, \quad C_\alpha \frown C_\beta = \phi \quad (\alpha \neq \beta)$$

without condition except one (2. 2); thus we have a partition of $I_1 \cup I_2$

$$I_1 \cup I_2 = \sum_{\alpha=0}^{\mu_2} C_\alpha\,.$$

Let $x \overset{2}{\sim} y$ mean that the elements x and y of $I_1 \cup I_2$ belong to a certain class C_α. Reminding us of Theorem 1.1 and the definition of I_2, we may see immediately that $\overset{2}{\sim}$ is a congruence relation in $I_1 \cup I_2$.

By induction we shall define the relation $x \overset{j}{\sim} y$ $(2 < j \leq h)$ in the subsemigroup $\sum_{\nu=1}^{j} I_\nu$ of F. Assuming that the relation $x \overset{j-1}{\sim} y$ in $\sum_{\nu=1}^{j-1} I_\nu$ is defined and that $\sum_{\nu=1}^{j-1} I_\nu = \sum_{\alpha=0}^{\mu_{j-1}} C_\alpha$, a relation $x \overset{j}{\sim} y$ in $\sum_{\nu=1}^{j} I_\nu$, which gives a decomposition $\sum_{\nu=1}^{j} I_\nu = \sum_{\alpha=0}^{\mu_j} C_\alpha$, is defined such that

(2.5) If $x, y \in \sum_{\nu=1}^{j-1} I_\nu$, then $x \overset{j}{\sim} y$ is equivalent to $x \overset{j-1}{\sim} y$.

(2.6) If $x, y \in I_j$, then $xp_i \overset{j-1}{\sim} yp_i$ and $p_i x \overset{j-1}{\sim} p_i y$ for every prime p_i.

(2.7) If x or y is a prime, then $x \overset{j}{\sim} y$ means $x = y$.

It is easily proved that each relation $x \overset{j}{\sim} y$ is a congruence relation in $\sum_{\nu=1}^{j} I_\nu$ $(j = 2, \cdots, h)$.

At last the relation $x \overset{h}{\sim} y$ gives a decomposition of F:

(2.8) $$F = \sum_{\alpha=0}^{\mu_h} C_\alpha \quad \text{where} \quad C_\alpha \cap C_\beta = \phi \quad (\alpha \neq \beta).$$

As easily seen, the decomposition (2.8) fulfils the conditions (2.1), (2.2), (2.3). Denoting by S the factor semigroup of F due to (2.8), S is a z-semigroup of height h and of basic breadth m, and clearly the elements of I_j are associated with an element x of S such that $D[x] = j$.

5. Thus we establish the following theorem.

Theorem 2.1. *Let F be the free z-semigroup of height h and of basic breadth m with primes $p_1, \cdots, p_m$. Take a system $\{a_1, \cdots, a_\nu\}$ of mutually incomparable non-prime elements of F such that there is b, $a_i \nleqq b$, $H[b] = h-1$ for all i. I_1 is defined as $I_1 = \sum_{i=1}^{\nu} P(a_i)$. For a principal sequence $I_1, \cdots, I_h$, classify all the elements of I_2 such that* (2.2) *is satisfied: $I_2 = \sum_{\alpha=1}^{\mu_2} C_\alpha$, and then classify all the elements of $I_3, \cdots, I_h$ succcessively such that* (2.5), (2.6) *and* (2.7) *are fulfilled:*

$$I_j = \sum_{\alpha=\mu_{j-1}+1}^{\mu_j} C_\alpha \qquad (j = 3, \cdots, h).$$

The factor semigroup S of F due to the decomposition $F = \sum_{\alpha=0}^{\mu_h} C_\alpha$ is a z-semigroup of height h and of basic breadth m. Any S is obtained by the above process.

6. Finally we shall find a necessary and sufficient condition for two decompositions of F to give isomorphic factor semigroups. The height and the basic breadth are preserved under an isomorphism, and a prime is mapped to a prime. Let S and S' be the factor z-semigroups of h and of basic breadth m given by the decompositions (2. 8), (2. 9) of F respectively,

(2. 9) $$F=\sum_{\beta=0}^{\mu_h} C_\beta{}',$$

the relations of which are denoted by $\sim$ and $\approx$ respectively. If S is isomorphic onto S', then the isomorphism causes a permutation f between primes $p_1, \cdots, p_m$ of F. The image x' of $x=p_{i_1}\cdots p_{i_\lambda}$ of S into S' must be $x'=f(p_{i_1})\cdots f(p_{i_\lambda})$, and $p_{i_1}\cdots p_{i_\lambda}\sim p_{j_1}\cdots p_{j_\nu}$ corresponds to $f(p_{i_1})\cdots f(p_{i_\lambda})\approx f(p_{j_1})\cdots f(p_{j_\nu})$. Thus a class C_α is associated with a class $C_\beta{}'$ such that

(2. 10) *$p_{i_1}\cdots p_{i_\lambda}$ of C_α is mapped to $f(p_{i_1})\cdots f(p_{i_\lambda})$ of $C_\beta{}'$. $C_\beta{}'$ is denoted by $f(C_\alpha)$.*

Of course $C_a\to f(C_\alpha)$ is one to one. Hence we have $F=\sum_{\alpha=0}^{\mu_h} f(C_\alpha)$.

Conversely if, for the decompositions (2. 8) and (2. 9) of F, there is a permutation f of $\{p_1, \cdots, p_m\}$ such that every $C_\beta{}'$ is expressed as $f(C_\alpha)$ for some α in the meaning of (2. 10), then it is easily proved that S is isomorphic to S' under the correspondence $C_\alpha\to f(C_\alpha)$.

Theorem 2. 2. *The two decompositions* (2. 8) *and* (2. 9) *of F make the finite z-semigroups S and S' be isomorphic if and only if there is a permutation f between all the prime elements $p_1, \cdots, p_m$ such that the decomposition $\sum_{\beta=0}^{\mu_h} C_\beta{}'$ coincides with the decomposition $\sum_{\alpha=0}^{\mu_h} f(C_\alpha)$ where $f(C_\alpha)$ has the meaning of* (2. 10).

§ 3. Finite Unipotent Semigroup with Group.

1. The theory of construction of a finite unipotent semigroup with group is included in the paper [2] and in Theorem 7 of [5] as a special case. Stating the result again

Theorem 3. 1. *Suppose that there are given a finite group G whose multiplication is denoted by $x\cdot y$, a finite z-semigroup Z whose multiplication is denoted by $x\times y$, and a mapping ψ of the set $\bar{Z}$ of all non-zero elements of Z into G such that*

$$x\in\bar{Z},\ y\in\bar{Z} \quad \textit{and} \quad x\times y\in\bar{Z} \quad \textit{imply} \quad \psi(x\times y)=\psi(x)\cdot\psi(y).$$

A multilplication xy *in* $S=G\cup\bar{Z}$ *is defined as*

$$(3.1)\qquad xy=\begin{cases} x\cdot y & \text{if } x\in G,\ y\in G \\ x\cdot\psi(y) & \text{if } x\in G,\ y\in\bar{Z} \\ \psi(x)\cdot y & \text{if } x\in\bar{Z},\ y\in G \\ \psi(x)\cdot\psi(y) & \text{if } x,\ y\in\bar{Z} \text{ and } x\times y=0 \\ x\times y & \text{if } x,\ y\in\bar{Z} \text{ and } x\times y\in\bar{Z}\,. \end{cases}$$

Then S *is a unipotent semigroup having* G *as the kernal (i.e. its greatest group and its least ideal), and the difference semigroup of* S *modulo* G *is isomorphic to* Z. S *is denoted by* $S=(G, Z, \psi)$. *Any unipotent semigroup with group is constructed in the above mentioned way.* $S_1=(G_1, Z_1, \psi_1)$ *is isomorphic onto* $S_2=(G_2, Z_2, \psi_2)$ *if and only if*

(3.2) G_1 *is isomorphic onto* G_2. *(The mapping of* G_1 *onto* G_2 *is denoted by* ξ.)

(3.3) Z_1 *is isomorphic onto* Z_2. *(The mapping of* Z_1 *onto* Z_2 *is denoted by* η.)

$$(3.4)\qquad \psi_1=\xi^{-1}\psi_2\eta\,.$$ [3]

We say that ψ_1 and ψ_2 are equivalent if there are ξ and η satisfying (3.4).

2. We shall add the way how ψ is found when a finite group G and a finite z-semigroup Z are given. A z-semigroup Z of height h and of basic breadth m is assigned as a decomposition of the free z-semigroup F,

$$F=\sum_{0\leqq\alpha\leqq\mu_n} C_\alpha$$

letting $Z=\{0, 1, 2, \cdots, \mu_1, \mu_1+1, \cdots, \mu_2, \mu_2+1, \cdots, \mu_{n-1}, \mu_{n-1}+1, \cdots, \mu_n\}$ and letting $p_1, \cdots, p_m$ be all the primes of Z. In order to determine ψ of $\bar{Z}$ into G, it is necessary and sufficient to give $\psi(p_1), \cdots, \psi(p_m)$ such that the following condition is satisfied.

(3.5) $0\neq p_{i_1}\cdots p_{i_\lambda}\sim p_{j_1}\cdots p_{j_\nu}\in C_\alpha$ *implies* $\psi(p_{i_1})\cdots\psi(p_{i_\lambda})=\psi(p_{j_1})\cdots\psi(p_{j_\nu})$.

Then ψ is completely determined in $\bar{Z}$ as

$$\psi(\alpha)=\psi(p_{i_1})\cdots\psi(p_{i_\lambda})\quad\text{for}\quad p_{i_1}\cdots p_{i_\lambda}\in C_\alpha,\quad \alpha\neq 0\,.$$

Now all the equalities of the kind of (3.5) are simplified, if possible, by cancellation law. The set of all the equalities thus obtained, if exist, is called the system of the primary equations for Z. In order to find ψ for the given Z and G, we must have all the systems of the solutions

3) $(\xi^{-1}\psi_2\eta)(x)=\xi^{-1}(\psi_2(\eta(x)))$ for $x\in\bar{Z}_1$.

$$(3.6)\qquad \begin{cases} \psi_1(p_1), \cdots, \psi_1(p_m), \\ \cdots\cdots\cdots\cdots \\ \psi_s(p_1), \cdots, \psi_s(p_m) \end{cases}$$

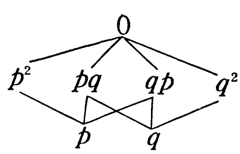
of the system of the primary equations for Z, where ψ_i and ψ_j $(i \neq j)$ are not equivalent. (cf. (3.4)) We shall call (3.6) a complete system of solutions. Of course the system of the primary equations has always one system of solutions at least, e.g.

$$\psi(p_1) = \psi(p_2) = \cdots = \psi(p_m) = e. \quad \text{(the unit of } G)$$

If and only if Z satisfies the condition that any non-zero element is uniquely factorized into the product of primes, then there is no primary equation for Z, that is to say, we can arbitrarily choose $\psi(p_1), \cdots, \psi(p_m)$. Even then a complete system of solutions is considered. Such a z-semigroup Z is called a uniquely factorizable z-semigroup.

§ 4. Examples.

1. We shall construct all z-semigroups of height 3 and of basic breadth 2. The free z-semigroup of $\{0, p, q, p^2, pq, qp, q^2\}$ has a diagram



where the ordering is defined in § 1. We can find all the principal sequences except equivalent ones as the following table shows.

Table 1. Principal Sequences

No. of Principal Sequence	I_1	I_2	I_3
1	(0)	(p^2, pq, qp, q^2)	(p, q)
2	$(0, p^2)$	(pq, qp, q^2)	(p, q)
3	$(0, pq)$	(p^2, qp, q^2)	(p, q)
4	$(0, p^2, pq)$	(qp, q^2)	(p, q)
5	$(0, p^2, qp)$	(pq, q^2)	(p, q)
6	$(0, p^2, q^2)$	(pq, qp)	(p, q)
7	$(0, pq, qp)$	(p^2, q^2)	(p, q)
8	$(0, p^2, pq, qp)$	(q^2)	(p, q)
9	$(0, p^2, pq, q^2)$	(qp)	(p, q)

We obtain all the decompositions (2.8) of F.

Table 2. Decompositions of F

No. of Principal Sequence	*No. of Decomposition*	I_1	I_2	I_3	*Remark*
1	1	(0)	(p^2), (pq), (qp), (q^2)	(p), (q)	○ *
	2	(0)	(p^2, pq), (qp), (q^2)	(p), (q)	
	3	(0)	(p^2, qp), (pq), (q^2)	(p), (q)	
	4	(0)	(p^2, q^2), (pq), (qp)	(p), (q)	○ *
	5	(0)	(pq, qp), (p^2), (q^2)	(p), (q)	× ○
	6	(0)	(p^2, pq), (qp, q^2)	(p), (q)	○
	7	(0)	(p^2, qp), (pq, q^2)	(p), (q)	○
	8	(0)	(p^2, q^2), (pq, qp)	(p), (q)	× ○
	9	(0)	(p^2, pq, qp), (q^2)	(p), (q)	×
	10	(0)	(p^2, pq, q^2), (qp)	(p), (q)	
	11	(0)	(p^2, pq, qp, q^2)	(p), (q)	× ○
2	12	$(0, p^2)$	(pq), (qp), (q^2)	(p), (q)	
	13	$(0, p^2)$	(pq, qp), (q^2)	(p), (q)	×
	14	$(0, p^2)$	(pq, q^2), (qp)	(p), (q)	
	15	$(0, p^2)$	(qp, q^2), (pq)	(p), (q)	
	16	$(0, p^2)$	(pq, qp, q^2)	(p), (q)	×
3	17	$(0, pq)$	(p^2), (qp), (q^2)	(p), (q)	
	18	$(0, pq)$	(p^2, qp), (q^2)	(p), (q)	
	19	$(0, pq)$	(p^2, q^2), (qp)	(p), (q)	
	20	$(0, pq)$	(qp, q^2), (p^2)	(p), (q)	
	21	$(0, pq)$	(p^2, qp, q^2)	(p), (q)	
4	22	$(0, p^2, pq)$	(qp), (q^2)	(p), (q)	
	23	$(0, p^2, pq)$	(qp, q^2)	(p), (q)	
5	24	$(0, p^2, qp)$	(pq), (q^2)	(p), (q)	
	25	$(0, p^2, qp)$	(pq, q^2)	(p), (q)	
6	26	$(0, p^2, q^2)$	(pq), (qp)	(p), (q)	○ *
	27	$(0, p^2, q^2)$	(pq, qp)	(p), (q)	× ○
7	28	$(0, pq, qp)$	(p^2), (q^2)	(p), (q)	× ○
	29	$(0, pq, qp)$	(p^2, q^2)	(p), (q)	× ○
8	30	$(0, p^2, pq, qp)$	(q^2)	(p), (q)	×
9	31	$(0, p^2, pq, q^2)$	(qp)	(p), (q)	

These are not isomorphic each other. In the above table, the mark $\times$ means "commutative", $*$ does "self-dual", and the semigroup marked $\bigcirc$ has an automorphism except an identical mapping. Among the above, the semigroups of order at most 5 are seen in [3].

2. The z-semigroups 1, 12, 17, 22, 24, 26, 28, 30 and 31 are uniquely factorizable. We shall show the primary equations for others in the following table.

Table 3.

Z	*the system of the primary equations*
2, 3, 6, 7, 9, 10, 11, 14, 15, 16, 18, 20, 21, 23, 25	$\psi(p)=\psi(q)$
4, 19, 29	$\psi(p)^2=\psi(q)^2$
5, 13, 27	$\psi(p)\psi(q)=\psi(q)\psi(p)$
8	$\psi(p)^2=\psi(q)^2$ $\psi(p)\psi(q)=\psi(q)\psi(p)$

Example 1. Let G be a group of order 3:

$$G: \quad \begin{array}{c|ccc} & a & b & c \\ \hline a & a & b & c \\ b & b & c & a \\ c & c & a & b \end{array}$$

For the uniquely factorizable Z of No. 31, reminding us of the automorphisms of G, we have a complete system of solutions:

$$\psi_1=\begin{pmatrix} p & q \\ a & a \end{pmatrix}, \quad \psi_2=\begin{pmatrix} p & q \\ a & b \end{pmatrix}, \quad \psi_3=\begin{pmatrix} p & q \\ b & b \end{pmatrix}, \quad \psi_4=\begin{pmatrix} p & q \\ b & c \end{pmatrix}.$$

Example 2. Let G be a symmetric group of degree 3

$$G: \quad \begin{array}{c|cccccc} & a & b & c & d & e & f \\ \hline a & a & b & c & d & e & f \\ b & b & c & a & e & f & d \\ c & c & a & b & f & d & e \\ d & d & f & e & a & c & b \\ e & e & d & f & b & a & c \\ f & f & e & d & c & b & a \end{array}$$

and let Z be of No. 8. From $\psi(p)^2=\psi(q)^2$, $\psi(p)\psi(q)=\psi(q)\psi(p)$, we get a complete system of solutions:

$$\psi_1=\begin{pmatrix} p & q \\ a & a \end{pmatrix}, \quad \psi_2=\begin{pmatrix} p & q \\ b & b \end{pmatrix}, \quad \psi_3=\begin{pmatrix} p & q \\ a & d \end{pmatrix}, \quad \psi_4=\begin{pmatrix} p & q \\ d & d \end{pmatrix}.$$

Example 3. Let us find all the unipotent semigroups $S=(G, Z, \psi)$

when G is a group of order 3 as given in Example 1, and Z is given as

$$Z: \quad \begin{array}{c|cccc} & 0 & d & e & f \\ \hline 0 & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 0 & 0 \\ e & 0 & 0 & 0 & 0 \\ f & 0 & 0 & d & 0 \end{array} \qquad \text{(No. 31 in Table 2)}$$

where e and f are the primes. According to Example 1, we have

$$\psi_1 = \begin{pmatrix} d & e & f \\ a & a & a \end{pmatrix}, \quad \psi_2 = \begin{pmatrix} d & e & f \\ b & a & b \end{pmatrix}, \quad \psi_3 = \begin{pmatrix} d & e & f \\ c & b & b \end{pmatrix}, \quad \psi_4 = \begin{pmatrix} d & e & f \\ a & b & c \end{pmatrix}.$$

By (3.1), we get S_i for ψ_i $(i=1, 2, 3, 4)$.

$$S_1 \quad \begin{array}{c|cccccc} & a & b & c & d & e & f \\ \hline a & a & b & c & a & a & a \\ b & b & c & a & b & b & b \\ c & c & a & b & c & c & c \\ d & a & b & c & a & a & a \\ e & a & b & c & a & a & a \\ f & a & b & c & a & d & a \end{array} \qquad S_2 \quad \begin{array}{c|cccccc} & a & b & c & d & e & f \\ \hline a & a & b & c & b & a & b \\ b & b & c & a & c & b & c \\ c & c & a & b & a & c & a \\ d & b & c & a & c & b & c \\ e & a & b & c & b & a & b \\ f & b & c & a & c & d & c \end{array} \qquad S_3 \quad \begin{array}{c|cccccc} & a & b & c & d & e & f \\ \hline a & a & b & c & c & b & b \\ b & b & c & a & a & c & c \\ c & c & a & b & b & a & a \\ d & c & a & b & b & a & a \\ e & b & c & a & a & c & c \\ f & b & c & a & a & d & c \end{array}$$

$$S_4 \quad \begin{array}{c|cccccc} & a & b & c & d & e & f \\ \hline a & a & b & c & a & b & c \\ b & b & c & a & b & c & a \\ c & c & a & b & c & a & b \\ d & a & b & c & a & b & c \\ e & b & c & a & b & c & a \\ f & c & a & b & c & d & b \end{array}$$

Although have $\psi'_2 = \begin{pmatrix} d & e & f \\ b & b & a \end{pmatrix}$, S'_2 which is obtained from ψ'_2 is anti-isomorphic to S_2.

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