



Title	On the abstract evolution equation of parabolic type
Author(s)	Yagi, Atsushi
Citation	Osaka Journal of Mathematics. 1977, 14(3), p. 557-568
Version Type	VoR
URL	https://doi.org/10.18910/5486
rights	
Note	

The University of Osaka Institutional Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

The University of Osaka

ON THE ABSTRACT EVOLUTION EQUATION OF PARABOLIC TYPE

ATSUSHI YAGI

(Received July 15, 1976)

1. Introduction

The object of the present paper is to prove the existence and uniqueness of the solution of the initial value problem for the abstract evolution equation of parabolic type

$$(E) \quad du/dt + A(t)u = f(t), \quad 0 < t \leq T$$

in a Banach space X by constructing the evolution operator to (E). Here $u = u(\cdot)$ and $f(\cdot)$ are functions on $[0, T]$ to X and $A(\cdot)$ is one on $[0, T]$ to the set of linear operators, and 'of parabolic type' means that each $-A(t)$ is the infinitesimal generator of an 'analytic' semi-group.

Several papers have already been published concerning this problem, for instance, [1], [2] and [3]. The present author has also shown in [4] that

$$\|A(t)(\lambda - A(t))^{-1}dA(t)^{-1}/dt\| \leq N/|\lambda|^\rho, \quad \rho \in (0, 1] \quad (1.1)$$

is one of sufficient conditions for the integrability of (E). In this article we will give a new sufficient condition weaker than (1.1). That is the estimation

$$\left. \begin{aligned} &\|\partial(\lambda - A(t))^{-1}/\partial t\| \leq N/|\lambda|^\rho, \quad \rho \in (0, 1] \\ &\|A(t)(\lambda - A(t))^{-1}dA(t)^{-1}/dt - A(s)(\lambda - A(s))^{-1}dA(s)^{-1}/ds\| \\ &\leq K \sum_{i=1}^k |\lambda|^{\alpha_i} |t-s|^{\beta_i}, \quad (\alpha_i, \beta_i) \in \{(\alpha, \beta); -1 \leq \alpha < \beta \leq 1\}, \end{aligned} \right\} \quad (1.2)$$

which is obviously weaker than (1.1) in view of

$$\partial(\lambda - A(t))^{-1}/\partial t = -A(t)(\lambda - A(t))^{-1}(dA(t)^{-1}/dt)A(t)(\lambda - A(t))^{-1}.$$

We also note that (1.2) is weaker than

$$\left. \begin{aligned} &\|\partial(\lambda - A(t))^{-1}/\partial t\| \leq N/|\lambda|^\rho, \quad \rho \in (0, 1] \\ &\|dA(t)^{-1}/dt - dA(s)^{-1}/ds\| \leq K|t-s|^\beta, \quad \beta \in (0, 1] \end{aligned} \right\} \quad (1.3)$$

which is made in [2], as is easily seen from

$$\begin{aligned}
& A(t)(\lambda - A(t))^{-1}dA(t)^{-1}/dt - A(s)(\lambda - A(s))^{-1}dA(s)^{-1}/ds \\
&= \lambda \int_s^t (\partial(\lambda - A(\tau))^{-1}/\partial\tau) d\tau (dA(t)^{-1}/dt) \\
&+ A(s)(\lambda - A(s))^{-1}(dA(t)^{-1}/dt - dA(s)^{-1}/ds).
\end{aligned}$$

As the previous paper [4] we follow the idea of [1] of establishing various estimates under the additional assumption that $A(t)$ is bounded and then approximating $A(t)$ by a sequence of bounded operators.

The author thanks Prof. M. Watanabe and Prof. H. Tanabe with all his heart. Prof. M. Watanabe informed him that (1.1) implies the condition

$$\|A(t)^\delta dA(t)^{-1}/dt\| \leq N, \delta \in (0, 1] \quad (1.4)$$

in [3] because of the equality

$$A(t)^\delta dA(t)^{-1}/dt = \frac{1}{2\pi i} \int_\Gamma \lambda^{\delta-1} A(t)(\lambda - A(t))^{-1} (dA(t)^{-1}/dt) d\lambda$$

with $\delta \in (0, \rho)$. Thus (1.1) and (1.4) are essentially equivalent.

2. Construction of the evolution operator

Theorem 1. *Let $\{A(t)\}_{0 \leq t \leq T}$ be a family of densely defined, closed linear operators acting in the Banach space X . Suppose that $\{A(t)\}_{0 \leq t \leq T}$ satisfies the following conditions:*

(I) *For each $t \in [0, T]$ the resolvent set of $A(t)$ contains a fixed closed angular domain*

$$\Sigma = \{\lambda \in \mathbb{C}; \arg \lambda \in (-\theta, \theta)\}, \theta \in (0, \pi/2).$$

For any $t \in [0, T]$ and $\lambda \in \Sigma$ the resolvent of $A(t)$ satisfies the inequality

$$\|(\lambda - A(t))^{-1}\| \leq N_1/(|\lambda| + 1)$$

with some constant N_1 independent of t and λ .

(II) *$A(t)^{-1}$ (and therefore $(\lambda - A(t))^{-1}$ also) is strongly continuously differentiable in t , and the derivative $\partial(\lambda - A(t))^{-1}/\partial t$ satisfies*

$$\|\partial(\lambda - A(t))^{-1}/\partial t\| \leq N_2/|\lambda|^\rho$$

for any $t \in [0, T]$ and $\lambda \in \Sigma$ with some constants N_2 and $\rho \in (0, 1]$ independent of t and λ .

(III) The estimation

$$\begin{aligned} & \|A(t)(\lambda - A(t))^{-1}dA(t)^{-1}/dt - A(s)(\lambda - A(s))^{-1}dA(s)^{-1}/ds\| \\ & \leq N_3 \sum_{i=1}^k |\lambda|^{\alpha_i} |t-s|^{\beta_i}, \end{aligned}$$

holds for any $(t, s) \in [0, T]^2$ and $\lambda \in \Sigma$ with some constant N_3 and some non-empty finite family $\{(\alpha_i, \beta_i)\}_{1 \leq i \leq k}$ of the elements of $\{(\alpha, \beta); -1 \leq \alpha < \beta \leq 1\}$, N_3 and the family being independent of (t, s) and λ .

Then there exists a family $\{U(t, s)\}_{0 \leq s \leq t \leq T}$ of bounded operators on X having the following properties:

- 1) $U(t, s)$ is strongly continuous in (t, s) .
- 2) $U(t, r)U(r, s) = U(t, s)$ for $0 \leq s \leq r \leq t \leq T$, and $U(s, s) = I$.
- 3) For any $t > s$, $R(U(t, s)) \subset D(A(t))$. The operator $A(t)U(t, s)$ defined for $0 \leq s < t \leq T$ is strongly continuous in (t, s) and is estimated by

$$\|A(t)U(t, s)\| \leq C_1(t-s)^{-1}$$

with some constant C_1 determined by $N_1, N_2, N_3, \theta, \rho, \{(\alpha_i, \beta_i)\}_{1 \leq i \leq k}, T$ and $\sup_{0 \leq t \leq T} \|dA(t)^{-1}/dt\|$.

- 4) $U(t, s)$ is strongly differentiable in $t \in (s, T]$, and

$$\partial U(t, s)/\partial t = -A(t)U(t, s).$$

- 5) For any $t > s$ and $u_0 \in D(A(s))$,

$$\lim_{h \rightarrow 0} h^{-1} \{U(t, s+h) - U(t, s)\}u_0 = U(t, s)A(s)u_0.$$

The proof consists of three steps.

The first step. Assuming in addition to (I), (II) and (III) that $A(t)$ is a bounded operator for each $t \in [0, T]$, we will establish various estimates concerning the evolution operator $U(t, s)$ using only the constants $N_1, N_2, N_3, \theta, \rho, \{(\alpha_i, \beta_i)\}_{1 \leq i \leq k}, T$ and $\sup_{0 \leq t \leq T} \|dA(t)^{-1}/dt\|$ entering in the assumptions (I), (II) and (III).

Let $\{U(t, s)\}_{0 \leq s \leq t \leq T}$ be the evolution operator to $\{A(t)\}_{0 \leq t \leq T}$. Evidently $\{U(t, s)\}_{0 \leq s \leq t \leq T}$ has the properties 1)~5). $U(t, s)$ is connected with the analytic semi-group $\exp(-(t-s)A(t))$ generated by $-A(t)$ as follows

$$\begin{aligned} & U(t, s) - \exp(-(t-s)A(t)) \\ & = - \int_s^t \frac{\partial}{\partial \tau} \{U(t, \tau) \exp(-(\tau-s)A(\tau))\} d\tau = \int_s^t U(t, \tau) P(\tau, s) d\tau, \quad (2.1) \end{aligned}$$

where

$$P(t, s) = -(\partial/\partial t + \partial/\partial s) \exp(-(t-s)A(t)). \quad (2.2)$$

Letting $A(t)$ operate on (2.1), we get

$$W(t, s) = R_1(t, s) + R_2(t, s) + \int_s^t W(t, \tau) P(\tau, s) d\tau, \quad (2.3)$$

where

$$\begin{aligned} W(t, s) &= A(t)U(t, s) - A(t) \exp(-(t-s)A(t)), \\ R_1(t, s) &= \int_s^t \{A(t) \exp(-(t-\tau)A(t)) \\ &\quad - A(\tau) \exp(-(t-\tau)A(\tau))\} P(\tau, s) d\tau \end{aligned} \quad (2.4)$$

and

$$R_2(t, s) = \int_s^t A(\tau) \exp(-(t-\tau)A(\tau)) P(\tau, s) d\tau. \quad (2.5)$$

Regarding (2.1) and (2.3) as the integral equations for $U(t, s)$ and $W(t, s)$ respectively, we will estimate their kernels. It is well known that

$$\exp(-(t-s)A(t)) = \frac{1}{2\pi i} \int_{\Gamma} e^{-\lambda(t-s)} (\lambda - A(t))^{-1} d\lambda \quad (2.6)$$

with some smooth contour Γ running in Σ from $\infty e^{i\theta}$ to $\infty e^{-i\theta}$. From this formula we easily conclude

$$\|\exp(-(t-s)A(t))\| \leq N_4.$$

By the definition of $P(t, s)$ and (2.6) we have

$$P(t, s) = \frac{-1}{2\pi i} \int_{\Gamma} e^{-\lambda(t-s)} (\partial(\lambda - A(t))^{-1}/\partial t) d\lambda, \quad (2.7)$$

and using the assumption (II), we get

$$\|P(t, s)\| \leq N_5(t-s)^{\rho-1}. \quad (2.8)$$

In view of (2.8) and

$$\|A(t) \exp(-(t-s)A(t)) - A(s) \exp(-(t-s)A(s))\| \leq N_6(t-s)^{\rho-1}$$

we have

$$\|R_1(t, s)\| \leq N_7(t-s)^{2\rho-1}.$$

The estimate of $R_2(t, s)$ is rather complicated. We express $R_2(t, s)$ as

$$R_2(t, s) = \sum_{j=1}^5 F_j(t, s), \quad (2.9)$$

where each summand on the right is defined below.

$$R_2(t, s) = \left(\int_r^t + \int_s^r \right) A(\tau) \exp(-(t-\tau)A(\tau)) P(\tau, s) d\tau,$$

where $r=(t+s)/2$. Define

$$F_1(t, s) = \int_s^r A(\sigma) \exp(-(t-\sigma)A(\sigma)) P(\sigma, s) d\sigma.$$

Since

$$\begin{aligned} & \partial(\lambda - A(t))^{-1}/\partial t \\ &= -A(t)(\lambda - A(t))^{-1}(dA(t)^{-1}/dt)A(t)(\lambda - A(t))^{-1} \\ &= \{1 - \lambda(\lambda - A(t))^{-1}\}(dA(t)^{-1}/dt)A(t)(\lambda - A(t))^{-1}, \end{aligned} \quad (2.10)$$

we can express $P(t, s)$ in the form

$$\begin{aligned} P(t, s) &= -(dA(t)^{-1}/dt)A(t) \exp(-(t-s)A(t)) \\ &\quad - A(t)^{-1} \frac{1}{2\pi i} \int_{\Gamma} e^{-\lambda(t-s)} \lambda (\partial(\lambda - A(t))^{-1}/\partial t) d\lambda, \end{aligned}$$

hence

$$\begin{aligned} & \int_r^t A(\tau) \exp(-(t-\tau)A(\tau)) P(\tau, s) d\tau \\ &= - \int_r^t A(\tau) \exp(-(t-\tau)A(\tau)) (dA(\tau)^{-1}/d\tau) A(\tau) \exp(-(\tau-s)A(\tau)) d\tau \\ &\quad - \int_r^t \exp(-(t-\tau)A(\tau)) \frac{1}{2\pi i} \int_{\Gamma} e^{-\lambda(\tau-s)} \lambda (\partial(\lambda - A(\tau))^{-1}/\partial \tau) d\lambda d\tau. \end{aligned}$$

Define

$$F_2(t, s) = - \int_r^t \exp(-(t-\tau)A(\tau)) \frac{1}{2\pi i} \int_{\Gamma} e^{-\lambda(\tau-s)} \lambda (\partial(\lambda - A(\tau))^{-1}/\partial \tau) d\lambda d\tau.$$

Next, define $F_3(t, s)$, $F_4(t, s)$ and $F_5(t, s)$ as follows

$$\begin{aligned} & - \int_r^t A(\tau) \exp(-(t-\tau)A(\tau)) (dA(\tau)^{-1}/d\tau) A(\tau) \exp(-(\tau-s)A(\tau)) d\tau \\ &= \int_r^t A(\tau) \exp(-(t-\tau)A(\tau)) (dA(\tau)^{-1}/d\tau) \{A(t) \exp(-(\tau-s)A(t)) \\ &\quad - A(\tau) \exp(-(\tau-s)A(\tau))\} d\tau \\ &\quad + \int_r^t \{A(t) \exp(-(t-\tau)A(t)) dA(t)^{-1}/dt \\ &\quad - A(\tau) \exp(-(t-\tau)A(\tau)) dA(\tau)^{-1}/d\tau\} A(t) \exp(-(\tau-s)A(t)) d\tau \\ &\quad - \int_r^t A(t) \exp(-(t-\tau)A(t)) (dA(t)^{-1}/dt) A(t) \exp(-(\tau-s)A(t)) d\tau \\ &= F_3(t, s) + F_4(t, s) + F_5(t, s). \end{aligned}$$

It is easily observed that

$$\begin{aligned} \|F_1(t, s)\| &\leq N_8(t-s)^{\rho-1}, \|F_2(t, s)\| \leq N_9(t-s)^{\rho-1} \quad \text{and} \\ \|F_3(t, s)\| &\leq N_{10}(t-s)^{\rho-1}. \end{aligned}$$

By (III) we obtain

$$\begin{aligned} &\|A(t) \exp(-(t-s)A(t))dA(t)^{-1}/dt - A(s) \exp(-(t-s)A(s))dA(s)^{-1}/ds\| \\ &\leq N_{11} \sum_{i=1}^k (t-s)^{\beta_i - \alpha_i - 1}, \end{aligned}$$

hence

$$\|F_4(t, s)\| \leq N_{12}(t-s)^{\delta-1}, \quad \text{with} \quad \delta = \min_{1 \leq i \leq k} \{\beta_i - \alpha_i\}.$$

As for $F_5(t, s)$, we show

$$F_5(t, s) = P(t, r) \exp(-(r-s)A(t)). \quad (2.11)$$

In fact

$$\begin{aligned} &\exp(-(t+h-s)A(t+h)) - \exp(-(t-s)A(t)) \\ &= \int_0^h \frac{\partial}{\partial \theta} \exp(-(t+\theta-s)A(t+h))d\theta \\ &\quad + \int_s^t \frac{\partial}{\partial \tau} \{\exp(-(t-\tau)A(t)) \exp(-(\tau-s)A(t+h))\}d\tau \\ &= - \int_0^h A(t+h) \exp(-(t+\theta-s)A(t+h))d\theta \\ &\quad + \int_s^t A(t) \exp(-(t-\tau)A(t))(A(t+h)^{-1} \\ &\quad \quad - A(t)^{-1})A(t+h) \exp(-(\tau-s)A(t+h))d\tau, \end{aligned}$$

therefore we conclude

$$\begin{aligned} \frac{\partial}{\partial t} \exp(-(t-s)A(t)) &= -A(t) \exp(-(t-s)A(t)) \\ &\quad + \int_s^t A(t) \exp(-(t-\tau)A(t))(dA(t)^{-1}/dt)A(t) \exp(-(\tau-s)A(t))d\tau \end{aligned}$$

or

$$\begin{aligned} &\int_s^t A(t) \exp(-(t-\tau)A(t))(dA(t)^{-1}/dt)A(t) \exp(-(\tau-s)A(t))d\tau \\ &= -P(t, s). \end{aligned}$$

(2.11) is obvious now. By (2.11)

$$\|F_5(t, s)\| \leq N_{13}(t-s)^{\rho-1}.$$

The second step. Let

$$A_n(t) = A(t)(1+n^{-1}A(t))^{-1}, \quad n=1, 2, \dots \quad (2.12)$$

be the Yosida approximation of $A(t)$. Since $A_n(t)$ is bounded, we can define $\{P_n(t, s)\}_{0 \leq s < t \leq T}$, $\{R_{1,n}(t, s)\}_{0 \leq s < t \leq T}$ and $\{R_{2,n}(t, s)\}_{0 \leq s < t \leq T}$ by (2.2), (2.4) and (2.5) respectively replacing $A(t)$ by $A_n(t)$.

For the original family $\{A(t)\}_{0 \leq t \leq T}$ we define $\{P(t, s)\}_{0 \leq s < t \leq T}$ and $\{R_1(t, s)\}_{0 \leq s < t \leq T}$ by (2.7) and (2.4) respectively and $\{R_2(t, s)\}_{0 \leq s < t \leq T}$ by (2.9) noting that $F_1(t, s), \dots, F_5(t, s)$ are all meaningful, if $F_5(t, s)$ is defined by (2.11), even though $A(t)$ is unbounded.

The chief aim of this step is to show that for each $t > s$, $\{P_n(t, s)\}_{n=1,2,\dots}$, $\{R_{1,n}(t, s)\}_{n=1,2,\dots}$ and $\{R_{2,n}(t, s)\}_{n=1,2,\dots}$ are, roughly speaking, boundedly convergent to $P(t, s)$, $R_1(t, s)$ and $R_2(t, s)$, respectively, which implies the convergence of the solutions of the equations obtained by substituting $A_n(t)$ for $A(t)$ in (2.1) and (2.3).

In order to establish the uniform boundedness with respect to n , using the first step, we must show that $\{A_n(t)\}_{0 \leq t \leq T}$ satisfies (I), (II) and (III) with some constants independent of n .

Since $\lambda \in \Sigma$ implies $n\lambda(n-\lambda)^{-1} \in \Sigma$, it is easily seen that the resolvent set of $A_n(t)$ contains Σ and for $\lambda \in \Sigma$ the resolvent is given by

$$(\lambda - A_n(t))^{-1} = (n - \lambda)^{-1}(n + A(t))(n\lambda(n - \lambda)^{-1} - A(t))^{-1} \quad (2.13)$$

or

$$= -(n - \lambda)^{-1} + n^2(n - \lambda)^{-2}(n\lambda(n - \lambda)^{-1} - A(t))^{-1}. \quad (2.14)$$

On the other hand we have the inequality

$$|\lambda - n| \geq \sin(\theta/2)(|\lambda| + n), \quad \text{for } \lambda \in \Sigma. \quad (2.15)$$

(2.14) together with (2.15) gives

$$\|(\lambda - A_n(t))^{-1}\| \leq M_1/(|\lambda| + 1) \quad (2.16)$$

for $\lambda \in \Sigma$ with some constant M_1 determined by N_1 and θ alone. If we take $\lambda = 0$ in (2.14), then

$$A_n(t)^{-1} = n^{-1} + A(t)^{-1},$$

hence $A_n(t)^{-1}$ is strongly differentiable in t and

$$dA_n(t)^{-1}/dt = dA(t)^{-1}/dt. \quad (2.17)$$

By (2.12), (2.13) and (2.17) and with the aid of (2.10), we get for $\lambda \in \Sigma$

$$\partial(\lambda - A_n(t))^{-1}/\partial t = n^2(n - \lambda)^{-2}\partial(n\lambda(n - \lambda)^{-1} - A(t))^{-1}/\partial t,$$

hence

$$\|\partial(\lambda - A_n(t))^{-1}/\partial t\| \leq M_2/|\lambda|^p$$

with some constant M_2 determined only by N_2 , θ and ρ . Similarly by (2.12),

(2.13) and (2.17) we get for $\lambda \in \Sigma$

$$\begin{aligned} & A_n(t)(\lambda - A_n(t))^{-1} dA_n(t)^{-1} / dt \\ &= n(n - \lambda)^{-1} A(t)(n\lambda(n - \lambda)^{-1} - A(t))^{-1} dA(t)^{-1} / dt, \end{aligned}$$

hence

$$\begin{aligned} & \|A_n(t)(\lambda - A_n(t))^{-1} dA_n(t)^{-1} / dt - A_n(s)(\lambda - A_n(s))^{-1} dA_n(s)^{-1} / ds\| \\ & \leq N_3 \sum_{i=1}^k |n(n - \lambda)^{-1}|^{1+\alpha_i} |\lambda|^{\alpha_i} |t - s|^{\beta_i} \\ & \leq M_3 \sum_{i=1}^k |\lambda|^{\alpha_i} |t - s|^{\beta_i} \end{aligned}$$

with some constant M_3 determined by N_3 , θ and $\{\alpha_i\}_{1 \leq i \leq k}$ alone. Thus (I), (II) and (III) are fulfilled with the same constants except M_1 , M_2 and M_3 .

We have deduced, therefore, that

$$\begin{aligned} & \|\exp(-(t-s)A_n(t))\| \leq M_4, \quad \|P_n(t, s)\| \leq M_5(t-s)^{p-1}, \\ & \|R_{1,n}(t, s)\| \leq M_6(t-s)^{2p-1} \quad \text{and} \\ & \|R_{2,n}(t, s)\| \leq M_7\{(t-s)^{p-1} + (t-s)^{\delta-1}\} \quad \text{with} \quad \delta = \min_{1 \leq i \leq k} \{\beta_i - \alpha_i\}. \end{aligned}$$

Next, we establish the convergence. The strong convergence of $\{\exp(-(t-s)A_n(t))\}_{n=1,2,\dots}$, $\{P_n(t, s)\}_{n=1,2,\dots}$ and $\{R_{1,n}(t, s)\}_{n=1,2,\dots}$ to $\exp(-(t-s)A(t))$, $P(t, s)$ and $R_1(t, s)$ respectively follows from that of $\{(\lambda - A_n(t))^{-1}\}_{n=1,2,\dots}$ to $(\lambda - A(t))^{-1}$ for each $\lambda \in \Sigma$ together with (2.16) and the repeated use of the theorem of bounded convergence. The strong convergence of $\{R_{2,n}(t, s)\}_{n=1,2,\dots}$ to $R_2(t, s)$ follows from that of each family $\{F_{j,n}(t, s)\}_{n=1,2,\dots}$ to $F_j(t, s)$.

Now, it would be natural to expect that the desired evolution operator $U(t, s)$ is the solution of the integral equation

$$\begin{aligned} U(t, s) &= \exp(-(t-s)A(t)) + \int_s^t U(t, \tau)P(\tau, s)d\tau, \quad 0 \leq s < t \leq T \\ U(s, s) &= I. \end{aligned} \quad (2.18)$$

In order to derive the desired properties of $U(t, s)$ we need the operator valued function $W(t, s)$ which is the solution of the integral equation

$$W(t, s) = R_1(t, s) + R_2(t, s) + \int_s^t W(t, \tau)P(\tau, s)d\tau, \quad 0 \leq s < t \leq T. \quad (2.19)$$

Since the inhomogeneous terms and kernel of (2.18) and (2.19) are strongly continuous and satisfy

$$\begin{aligned} & \|\exp(-(t-s)A(t))\| \leq M_8, \quad \|P(t, s)\| \leq M_9(t-s)^{p-1}, \\ & \|R_1(t, s)\| \leq M_{10}(t-s)^{2p-1} \quad \text{and} \\ & \|R_2(t, s)\| \leq M_{11}\{(t-s)^{p-1} + (t-s)^{\delta-1}\}, \end{aligned}$$

the solutions of (2.18) and (2.19) exist and are unique. As is easily seen, $U(t, s)$ and $W(t, s)$ are the strong limits of $U_n(t, s)$ and $W_n(t, s)$.

The final step. We prove $U(t, s)$ satisfies 1)~5). 1) is easily seen and 2) is a direct consequence of that for $U_n(t, s)$. The strong convergence of $W_n(t, s)$ to $W(t, s)$ together with

$$\|W_n(t, s)\| \leq M_{12} \{(t-s)^{\rho-1} + (t-s)^{\delta-1}\}$$

implies 3) and 4). These arguments are similar to the proof of Theorem 1 of [4], and the detail is omitted. For the proof of 5) we use the following lemma (see P. 124 in [6]).

Lemma. Let $\{V(t, s)\}_{0 \leq s \leq t \leq T}$ be the family of bounded operators on X constructed by

$$\begin{aligned} V(t, s) &= \exp(-(t-s)A(s)) + Z(t, s) \\ Z(t, s) &= \int_s^t Q(t, \tau) \exp(-(\tau-s)A(s)) d\tau, \end{aligned}$$

where $\{Q(t, s)\}_{0 \leq s \leq t \leq T}$ is the solution of

$$\begin{aligned} Q(t, s) &= Q_1(t, s) + \int_s^t Q(t, \tau) Q_1(\tau, s) d\tau, \\ Q_1(t, s) &= (\partial/\partial t + \partial/\partial s) \exp(-(t-s)A(s)). \end{aligned}$$

Then $\{V(t, s)\}_{0 \leq s \leq t \leq T}$ has 1) and 5).

The existence of $Q(t, s)$ (hence $Z(t, s)$ and $V(t, s)$) is proved by the analogous estimation

$$\|Q_1(t, s)\| \leq M_{12}(t-s)^{\rho-1}$$

to (2.8). The proof of Lemma is not difficult, so we omit it.

It follows from 4) of $U(t, s)$ and 5) of $V(t, s)$ that $V(t, \tau)U(\tau, s)$ is strongly differentiable in $\tau \in (s, t)$ and $\partial V(t, \tau)U(\tau, s)/\partial \tau$ vanishes identically. Hence $V(t, \tau)U(\tau, s)$ is constant in $\tau \in (s, t)$. Letting $\tau \rightarrow s$ and $\tau \rightarrow t$, we get

$$V(t, s) = U(t, s), \quad 0 \leq s \leq t \leq T.$$

REMARK. If the following condition:

$$\begin{aligned} \text{(IV)} \quad & \|dA(t)^{-1}/dt A(t)(\lambda - A(t))^{-1} - dA(s)^{-1}/ds A(s)(\lambda - A(s))^{-1}\| \\ & \leq N_4 \sum_{j=1}^l |\lambda|^{\alpha_j} |t-s|^{\beta_j} \end{aligned}$$

holds for any $(t, s) \in [0, T]^2$ and $\lambda \in \Sigma$ with some constant N_4 and some non-

empty finite family $\{(\alpha_j, \beta_j)\}_{1 \leq j \leq l}$ of the elements of $\{(\alpha, \beta); -1 \leq \alpha < \beta \leq 1\}, N_4$ and the family being independent of (t, s) and λ .

is added to (I), (II) and (III), then 5) is strengthened as follows:

6) For any $t > s$ the bounded extension $\overline{U(t, s)A(s)}$ of $U(t, s)A(s)$ exists and is strongly continuous in $0 \leq s < t \leq T$. There exists some constant C_2 such that

$$\|\overline{U(t, s)A(s)}\| \leq C_2(t-s)^{-1}.$$

7) $U(t, s)$ is strongly differentiable in $s \in [0, t]$, and

$$\partial U(t, s)/\partial s = \overline{U(t, s)A(s)}.$$

3. The initial value problem for (E)

Theorem 2. Let $\{A(t)\}_{0 \leq t \leq T}$ be a family of densely defined, closed linear operators in X satisfying (I), (II) and (III), and $\{U(t, s)\}_{0 \leq s \leq t \leq T}$ be its evolution operator. Then the following statements hold:

i) If f is continuous, then any strict solution $u \in C([0, T]; X) \cap C^1((0, T]; X)$ of (E) on $[0, T]$ with its initial value $u_0 \in X$ can be expressed in the form

$$u(t) = U(t, 0)u_0 + \int_0^t U(t, \tau)f(\tau)d\tau. \quad (3.1)$$

ii) If f is Hölder continuous and u_0 is an arbitrary element of X , then the function u defined by (3.1) is the strict solution of (E) on $[0, T]$.

The proof of i) and ii) is almost the same as the proof of 4) and 5) in [4], therefore, it is omitted. But we will give, here, another proof of i) without using the differentiability in s of $U(t, s)$.

For any $\varepsilon > 0$, $U_n(t, \tau)u(\tau)$ is continuously differentiable in $\tau \in [\varepsilon, t]$, and

$$\begin{aligned} \partial U_n(t, \tau)u(\tau)/\partial \tau &= U_n(t, \tau)f(\tau) \\ &\quad + U_n(t, \tau)\{(1+n^{-1}A(\tau))^{-1}-I\}A(\tau)u(\tau). \end{aligned} \quad (3.2)$$

Integrating (3.2) on $[\varepsilon, t]$, we have

$$\begin{aligned} u(t) - U_n(t, \varepsilon)u(\varepsilon) &= \int_\varepsilon^t U_n(t, \tau)f(\tau)d\tau \\ &\quad + \int_\varepsilon^t U_n(t, \tau)\{(1+n^{-1}A(\tau))^{-1}-I\}A(\tau)u(\tau)d\tau. \end{aligned} \quad (3.3)$$

Since $U_n(t, s)$ is uniformly bounded, the second integral of (3.3) vanishes when n tends to infinity. Thus we have

$$u(t) - U(t, \varepsilon)u(\varepsilon) = \int_\varepsilon^t U(t, \tau)f(\tau)d\tau.$$

Letting ε tend to zero, we obtain the desired equality (3.1).

4. Application

In this section we entirely follow the terminology and notations in §7 of [2], and some of them may be used without precise definitions. Let H and K be Hilbert spaces with the inner products (\cdot, \cdot) and $((\cdot, \cdot))$ respectively such that $K \subset H$ algebraically and topologically. Let $\{a(t; \cdot, \cdot)\}_{0 \leq t \leq T}$ be a family of continuous sesquilinear forms on $K \times K$, and let $\{V(t)\}_{0 \leq t \leq T}$ be a family of closed subspaces of K . We define the operator $A(t)$ for each $t \in [0, T]$ in the following manner:

$u \in V(t)$ belongs to $D(A(t))$ and $A(t)u = f \in H$ if and only if $a(t; u, v) = (f, v)$ for any $v \in V(t)$.

Theorem 3. Assume (K. 1), (K. 2), (K. 3) and (K. 4) which are stated in [2] and, instead of (K. 5) and (K. 6), the following:

(K. 5)' There exist some constants M and $\alpha \in (0, 1]$ such that for any $(t, s) \in [0, T]^2$ and $u \in K$

$$|(\dot{P}(t) - \dot{P}(s))u| \leq M |t - s|^\alpha \|u\|$$

and

$$|(\dot{Q}(t) - \dot{Q}(s))u| \leq M |t - s|^\alpha \|u\|.$$

$$(K. 6)' \quad \lim_{h \rightarrow 0} \sup_{(u, v) \in B^2} |\dot{a}(t+h; u, v) - \dot{a}(t; u, v)| = 0,$$

where $B = \{u \in K; \|u\| \leq 1\}$.

Then (I), (II) and (III) of $\{-A(t)\}_{0 \leq t \leq T}$ hold with $\rho = 1/2$ and $\{(\alpha_i, \beta_i)\}_{1 \leq i \leq k} = \{(-1/2, 0), (0, \alpha), (1/2, 1)\}$.

By Lemma 7.1 and Theorem 7.1 in [2], we have only to show (III).

Taking $g = A(t)^*(\bar{\lambda} + A(t)^*)^{-1}g'$ in the formula (7.13) in [2], we have the expression

$$\begin{aligned} (A(t)(\bar{\lambda} + A(t))^{-1}dA(t)^{-1}/dt f, g) &= -\dot{a}(t; A(t)^{-1}f, (\bar{\lambda} + A(t)^*)^{-1}g) \\ &\quad + (\dot{P}(t)A(t)^{-1}f, A(t)^*(\bar{\lambda} + A(t)^*)^{-1}g) + (f, \dot{Q}(t)(\bar{\lambda} + A(t)^*)^{-1}g) \\ &\quad - a(t; A(t)^{-1}f, \dot{Q}(t)(\bar{\lambda} + A(t)^*)^{-1}g) - a(t; \dot{P}(t)A(t)^{-1}f, (\bar{\lambda} + A(t)^*)^{-1}g). \end{aligned}$$

From this we get for $\lambda \in \Sigma$

$$\begin{aligned} &(\{A(t)(\bar{\lambda} + A(t))^{-1}dA(t)^{-1}/dt - A(s)(\bar{\lambda} + A(s))^{-1}dA(s)^{-1}/ds\} f, g) \\ &= I(t, s; f, g) + R(t, s; f, g), \end{aligned}$$

where

$$\begin{aligned} I(t, s; f, g) &= (\dot{P}(t)A(t)^{-1}f, A(t)^*(\bar{\lambda} + A(t)^*)^{-1}g) \\ &\quad - (\dot{P}(s)A(s)^{-1}f, A(s)^*(\bar{\lambda} + A(s)^*)^{-1}g) \end{aligned}$$

and

$R(t, s; f, g)$ = the sum of the remaining terms.

By (7.15) and (7.16) in [2] we easily conclude

$$|R(t, s; f, g)| \leq M_1 |\lambda|^{-1/2} |f| |g|$$

with some constant M_1 .

$$\begin{aligned} |I| \leq & |(\{\dot{P}(t) - \dot{P}(s)\} A(t)^{-1} f, A(t)^*(\bar{\lambda} + A(t)^*)^{-1} g)| \\ & + |(\dot{P}(s)\{A(t)^{-1} - A(s)^{-1}\} f, A(t)^*(\bar{\lambda} + A(t)^*)^{-1} g)| \\ & + |(\dot{P}(s)A(s)^{-1} f, \bar{\lambda} \int_s^t (\partial(\bar{\lambda} + A(\tau)^*)^{-1} / \partial \tau) g d\tau)|, \end{aligned}$$

in view of (K. 5)' and (II) of $\{-A(t)^*\}_{0 \leq t \leq T}$, we finally conclude

$$|I(t, s; f, g)| \leq M_2 \{|t-s|^{\alpha} + |\lambda|^{1/2} |t-s|\} |f| |g|$$

with some constant M_2 .

OSAKA UNIVERSITY

Bibliography

- [1] T. Kato: *Abstract evolution equations of parabolic type in Banach and Hilbert spaces*, Nagoya Math. J. **19** (1961), 93–125.
- [2] T. Kato and H. Tanabe: *On the abstract evolution equation*, Osaka Math. J. **14** (1962), 107–133.
- [3] H. Tanabe: *Note on singular perturbation for abstract differential equations*, Osaka J. Math. **1** (1964), 239–252.
- [4] A. Yagi: *On the abstract linear evolution equations in Banach spaces*, J. Math. Soc. Japan **28** (1976), 290–303.
- [5] S.G. Krein: *Linear differential equations in a Banach Space*, Moskow, 1967 (in Russian),
- [6] H. Tanabe: *Evolution equations*, Iwanami, Tokyo, 1975 (in Japanese).