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## ON THE ABSTRACT EVOLUTION EQUATION OF PARABOLIC TYPE

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### 1. Introduction

The object of the present paper is to prove the existence and uniqueness of the solution of the initial value problem for the abstract evolution equation of parabolic type

$$(E) \quad du/dt + A(t)u = f(t), \quad 0 < t \leq T$$

in a Banach space  $X$  by constructing the evolution operator to (E). Here  $u = u(\cdot)$  and  $f(\cdot)$  are functions on  $[0, T]$  to  $X$  and  $A(\cdot)$  is one on  $[0, T]$  to the set of linear operators, and 'of parabolic type' means that each  $-A(t)$  is the infinitesimal generator of an 'analytic' semi-group.

Several papers have already been published concerning this problem, for instance, [1], [2] and [3]. The present author has also shown in [4] that

$$\|A(t)(\lambda - A(t))^{-1}dA(t)^{-1}/dt\| \leq N/|\lambda|^\rho, \quad \rho \in (0, 1] \quad (1.1)$$

is one of sufficient conditions for the integrability of (E). In this article we will give a new sufficient condition weaker than (1.1). That is the estimation

$$\left. \begin{aligned} & \|\partial(\lambda - A(t))^{-1}/\partial t\| \leq N/|\lambda|^\rho, \quad \rho \in (0, 1] \\ & \|A(t)(\lambda - A(t))^{-1}dA(t)^{-1}/dt - A(s)(\lambda - A(s))^{-1}dA(s)^{-1}/ds\| \\ & \leq K \sum_{i=1}^k |\lambda|^{\alpha_i} |t-s|^{\beta_i}, \quad (\alpha_i, \beta_i) \in \{(\alpha, \beta); -1 \leq \alpha < \beta \leq 1\}, \end{aligned} \right\} \quad (1.2)$$

which is obviously weaker than (1.1) in view of

$$\partial(\lambda - A(t))^{-1}/\partial t = -A(t)(\lambda - A(t))^{-1}(dA(t)^{-1}/dt)A(t)(\lambda - A(t))^{-1}.$$

We also note that (1.2) is weaker than

$$\left. \begin{aligned} & \|\partial(\lambda - A(t))^{-1}/\partial t\| \leq N/|\lambda|^\rho, \quad \rho \in (0, 1] \\ & \|dA(t)^{-1}/dt - dA(s)^{-1}/ds\| \leq K|t-s|^\beta, \quad \beta \in (0, 1] \end{aligned} \right\} \quad (1.3)$$

which is made in [2], as is easily seen from

$$\begin{aligned} & A(t)(\lambda - A(t))^{-1}dA(t)^{-1}/dt - A(s)(\lambda - A(s))^{-1}dA(s)^{-1}/ds \\ &= \lambda \int_s^t (\partial(\lambda - A(\tau))^{-1}/\partial\tau) d\tau (dA(t)^{-1}/dt) \\ & \quad + A(s)(\lambda - A(s))^{-1}(dA(t)^{-1}/dt - dA(s)^{-1}/ds). \end{aligned}$$

As the previous paper [4] we follow the idea of [1] of establishing various estimates under the additional assumption that  $A(t)$  is bounded and then approximating  $A(t)$  by a sequence of bounded operators.

The author thanks Prof. M. Watanabe and Prof. H. Tanabe with all his heart. Prof. M. Watanabe informed him that (1.1) implies the condition

$$\|A(t)^\delta dA(t)^{-1}/dt\| \leq N, \delta \in (0, 1] \quad (1.4)$$

in [3] because of the equality

$$A(t)^\delta dA(t)^{-1}/dt = \frac{1}{2\pi i} \int_\Gamma \lambda^{\delta-1} A(t)(\lambda - A(t))^{-1} (dA(t)^{-1}/dt) d\lambda$$

with  $\delta \in (0, \rho)$ . Thus (1.1) and (1.4) are essentially equivalent.

## 2. Construction of the evolution operator

**Theorem 1.** *Let  $\{A(t)\}_{0 \leq t \leq T}$  be a family of densely defined, closed linear operators acting in the Banach space  $X$ . Suppose that  $\{A(t)\}_{0 \leq t \leq T}$  satisfies the following conditions:*

(I) *For each  $t \in [0, T]$  the resolvent set of  $A(t)$  contains a fixed closed angular domain*

$$\Sigma = \{\lambda \in \mathbf{C}; \arg \lambda \in (-\theta, \theta)\}, \theta \in (0, \pi/2).$$

*For any  $t \in [0, T]$  and  $\lambda \in \Sigma$  the resolvent of  $A(t)$  satisfies the inequality*

$$\|(\lambda - A(t))^{-1}\| \leq N_1/(|\lambda| + 1)$$

*with some constant  $N_1$  independent of  $t$  and  $\lambda$ .*

(II)  *$A(t)^{-1}$  (and therefore  $(\lambda - A(t))^{-1}$  also) is strongly continuously differentiable in  $t$ , and the derivative  $\partial(\lambda - A(t))^{-1}/\partial t$  satisfies*

$$\|\partial(\lambda - A(t))^{-1}/\partial t\| \leq N_2/|\lambda|^\rho$$

*for any  $t \in [0, T]$  and  $\lambda \in \Sigma$  with some constants  $N_2$  and  $\rho \in (0, 1]$  independent of  $t$  and  $\lambda$ .*

(III) *The estimation*

$$\begin{aligned} & \|A(t)(\lambda - A(t))^{-1}dA(t)^{-1}/dt - A(s)(\lambda - A(s))^{-1}dA(s)^{-1}/ds\| \\ & \leq N_3 \sum_{i=1}^k |\lambda|^{\alpha_i} |t-s|^\beta, \end{aligned}$$

holds for any  $(t, s) \in [0, T]^2$  and  $\lambda \in \Sigma$  with some constant  $N_3$  and some non-empty finite family  $\{(\alpha_i, \beta_i)\}_{1 \leq i \leq k}$  of the elements of  $\{(\alpha, \beta); -1 \leq \alpha < \beta \leq 1\}$ ,  $N_3$  and the family being independent of  $(t, s)$  and  $\lambda$ .

Then there exists a family  $\{U(t, s)\}_{0 \leq s < t \leq T}$  of bounded operators on  $X$  having the following properties:

- 1)  $U(t, s)$  is strongly continuous in  $(t, s)$ .
- 2)  $U(t, r)U(r, s) = U(t, s)$  for  $0 \leq s \leq r \leq t \leq T$ , and  $U(s, s) = I$ .
- 3) For any  $t > s$ ,  $R(U(t, s)) \subset D(A(t))$ . The operator  $A(t)U(t, s)$  defined for  $0 \leq s < t \leq T$  is strongly continuous in  $(t, s)$  and is estimated by

$$\|A(t)U(t, s)\| \leq C_1(t-s)^{-1}$$

with some constant  $C_1$  determined by  $N_1, N_2, N_3, \theta, \rho, \{(\alpha_i, \beta_i)\}_{1 \leq i \leq k}, T$  and  $\sup_{0 \leq t \leq T} \|dA(t)^{-1}/dt\|$ .

- 4)  $U(t, s)$  is strongly differentiable in  $t \in (s, T]$ , and

$$\partial U(t, s)/\partial t = -A(t)U(t, s).$$

- 5) For any  $t > s$  and  $u_0 \in D(A(s))$ ,

$$\lim_{h \rightarrow 0} h^{-1} \{U(t, s+h) - U(t, s)\}u_0 = U(t, s)A(s)u_0.$$

The proof consists of three steps.

The first step. Assuming in addition to (I), (II) and (III) that  $A(t)$  is a bounded operator for each  $t \in [0, T]$ , we will establish various estimates concerning the evolution operator  $U(t, s)$  using only the constants  $N_1, N_2, N_3, \theta, \rho, \{(\alpha_i, \beta_i)\}_{1 \leq i \leq k}, T$  and  $\sup_{0 \leq t \leq T} \|dA(t)^{-1}/dt\|$  entering in the assumptions (I), (II) and (III).

Let  $\{U(t, s)\}_{0 \leq s < t \leq T}$  be the evolution operator to  $\{A(t)\}_{0 \leq t \leq T}$ . Evidently  $\{U(t, s)\}_{0 \leq s < t \leq T}$  has the properties 1)~5).  $U(t, s)$  is connected with the analytic semi-group  $\exp(-(t-s)A(t))$  generated by  $-A(t)$  as follows

$$\begin{aligned} & U(t, s) - \exp(-(t-s)A(t)) \\ & = - \int_s^t \frac{\partial}{\partial \tau} \{U(t, \tau) \exp(-(\tau-s)A(\tau))\} d\tau = \int_s^t U(t, \tau) P(\tau, s) d\tau, \quad (2.1) \end{aligned}$$

where

$$P(t, s) = -(\partial/\partial t + \partial/\partial s) \exp(-(t-s)A(t)). \tag{2.2}$$

Letting  $A(t)$  operate on (2.1), we get

$$W(t, s) = R_1(t, s) + R_2(t, s) + \int_s^t W(t, \tau)P(\tau, s)d\tau, \tag{2.3}$$

where

$$\begin{aligned} W(t, s) &= A(t)U(t, s) - A(t) \exp(-(t-s)A(t)), \\ R_1(t, s) &= \int_s^t \{A(t) \exp(-(t-\tau)A(t)) \\ &\quad - A(\tau) \exp(-(t-\tau)A(\tau))\} P(\tau, s)d\tau \end{aligned} \tag{2.4}$$

and

$$R_2(t, s) = \int_s^t A(\tau) \exp(-(t-\tau)A(\tau))P(\tau, s)d\tau. \tag{2.5}$$

Regarding (2.1) and (2.3) as the integral equations for  $U(t, s)$  and  $W(t, s)$  respectively, we will estimate their kernels. It is well known that

$$\exp(-(t-s)A(t)) = \frac{1}{2\pi i} \int_{\Gamma} e^{-\lambda(t-s)}(\lambda - A(t))^{-1}d\lambda \tag{2.6}$$

with some smooth contour  $\Gamma$  running in  $\Sigma$  from  $\infty e^{i\theta}$  to  $\infty e^{-i\theta}$ . From this formula we easily conclude

$$\|\exp(-(t-s)A(t))\| \leq N_4.$$

By the definition of  $P(t, s)$  and (2.6) we have

$$P(t, s) = \frac{-1}{2\pi i} \int_{\Gamma} e^{-\lambda(t-s)}(\partial(\lambda - A(t))^{-1}/\partial t)d\lambda, \tag{2.7}$$

and using the assumption (II), we get

$$\|P(t, s)\| \leq N_5(t-s)^{\rho-1}. \tag{2.8}$$

In view of (2.8) and

$$\|A(t) \exp(-(t-s)A(t)) - A(s) \exp(-(t-s)A(s))\| \leq N_6(t-s)^{\rho-1}$$

we have

$$\|R_1(t, s)\| \leq N_7(t-s)^{2\rho-1}.$$

The estimate of  $R_2(t, s)$  is rather complicated. We express  $R_2(t, s)$  as

$$R_2(t, s) = \sum_{j=1}^5 F_j(t, s), \tag{2.9}$$

where each summand on the right is defined below.

$$R_2(t, s) = \left(\int_r^t + \int_s^r\right) A(\tau) \exp(-(t-\tau)A(\tau))P(\tau, s)d\tau,$$

where  $r=(t+s)/2$ . Define

$$F_1(t, s) = \int_s^r A(\sigma) \exp(-(t-\sigma)A(\sigma))P(\sigma, s)d\sigma.$$

Since

$$\begin{aligned} & \partial(\lambda - A(t))^{-1}/\partial t \\ &= -A(t)(\lambda - A(t))^{-1}(dA(t)^{-1}/dt)A(t)(\lambda - A(t))^{-1} \\ &= \{1 - \lambda(\lambda - A(t))^{-1}\}(dA(t)^{-1}/dt)A(t)(\lambda - A(t))^{-1}, \end{aligned} \tag{2.10}$$

we can express  $P(t, s)$  in the form

$$\begin{aligned} P(t, s) &= -(dA(t)^{-1}/dt)A(t) \exp(-(t-s)A(t)) \\ &\quad - A(t)^{-1} \frac{1}{2\pi i} \int_{\Gamma} e^{-\lambda(t-s)} \lambda (\partial(\lambda - A(t))^{-1}/\partial t) d\lambda, \end{aligned}$$

hence

$$\begin{aligned} & \int_r^t A(\tau) \exp(-(t-\tau)A(\tau))P(\tau, s)d\tau \\ &= -\int_r^t A(\tau) \exp(-(t-\tau)A(\tau))(dA(\tau)^{-1}/d\tau)A(\tau) \exp(-(\tau-s)A(\tau))d\tau \\ &\quad - \int_r^t \exp(-(t-\tau)A(\tau)) \frac{1}{2\pi i} \int_{\Gamma} e^{-\lambda(\tau-s)} \lambda (\partial(\lambda - A(\tau))^{-1}/\partial \tau) d\lambda d\tau. \end{aligned}$$

Define

$$F_2(t, s) = -\int_r^t \exp(-(t-\tau)A(\tau)) \frac{1}{2\pi i} \int_{\Gamma} e^{-\lambda(\tau-s)} \lambda (\partial(\lambda - A(\tau))^{-1}/\partial \tau) d\lambda d\tau.$$

Next, define  $F_3(t, s)$ ,  $F_4(t, s)$  and  $F_5(t, s)$  as follows

$$\begin{aligned} & -\int_r^t A(\tau) \exp(-(t-\tau)A(\tau))(dA(\tau)^{-1}/d\tau)A(\tau) \exp(-(\tau-s)A(\tau))d\tau \\ &= \int_r^t A(\tau) \exp(-(t-\tau)A(\tau))(dA(\tau)^{-1}/d\tau) \{A(t) \exp(-(\tau-s)A(t)) \\ &\quad - A(\tau) \exp(-(\tau-s)A(\tau))\} d\tau \\ &\quad + \int_r^t \{A(t) \exp(-(t-\tau)A(t))dA(t)^{-1}/dt \\ &\quad - A(\tau) \exp(-(t-\tau)A(\tau))dA(\tau)^{-1}/d\tau\} A(t) \exp(-(\tau-s)A(t))d\tau \\ &\quad - \int_r^t A(t) \exp(-(t-\tau)A(t))(dA(t)^{-1}/dt)A(t) \exp(-(\tau-s)A(t))d\tau \\ &= F_3(t, s) + F_4(t, s) + F_5(t, s). \end{aligned}$$

It is easily observed that

$$\|F_1(t, s)\| \leq N_8(t-s)^{\rho-1}, \|F_2(t, s)\| \leq N_9(t-s)^{\rho-1} \quad \text{and} \\ \|F_3(t, s)\| \leq N_{10}(t-s)^{\rho-1}.$$

By (III) we obtain

$$\|A(t) \exp(-(t-s)A(t))dA(t)^{-1}/dt - A(s) \exp(-(t-s)A(s))dA(s)^{-1}/ds\| \\ \leq N_{11} \sum_{i=1}^k (t-s)^{\beta_i - \alpha_i - 1},$$

hence

$$\|F_4(t, s)\| \leq N_{12}(t-s)^{\delta-1}, \quad \text{with } \delta = \min_{1 \leq i \leq k} \{\beta_i - \alpha_i\}.$$

As for  $F_5(t, s)$ , we show

$$F_5(t, s) = P(t, r) \exp(-(r-s)A(t)). \quad (2.11)$$

In fact

$$\exp(-(t+h-s)A(t+h)) - \exp(-(t-s)A(t)) \\ = \int_0^h \frac{\partial}{\partial \theta} \exp(-(t+\theta-s)A(t+h))d\theta \\ + \int_s^t \frac{\partial}{\partial \tau} \{\exp(-(t-\tau)A(t)) \exp(-(\tau-s)A(t+h))\}d\tau \\ = - \int_0^h A(t+h) \exp(-(t+\theta-s)A(t+h))d\theta \\ + \int_s^t A(t) \exp(-(t-\tau)A(t))(A(t+h)^{-1} \\ - A(t)^{-1})A(t+h) \exp(-(\tau-s)A(t+h))d\tau,$$

therefore we conclude

$$\frac{\partial}{\partial t} \exp(-(t-s)A(t)) = -A(t) \exp(-(t-s)A(t)) \\ + \int_s^t A(t) \exp(-(t-\tau)A(t))(dA(t)^{-1}/dt)A(t) \exp(-(\tau-s)A(t))d\tau$$

or

$$\int_s^t A(t) \exp(-(t-\tau)A(t))(dA(t)^{-1}/dt)A(t) \exp(-(\tau-s)A(t))d\tau \\ = -P(t, s).$$

(2.11) is obvious now. By (2.11)

$$\|F_5(t, s)\| \leq N_{13}(t-s)^{\rho-1}.$$

The second step. Let

$$A_n(t) = A(t)(1+n^{-1}A(t))^{-1}, \quad n=1, 2, \dots \quad (2.12)$$

be the Yosida approximation of  $A(t)$ . Since  $A_n(t)$  is bounded, we can define  $\{P_n(t, s)\}_{0 \leq s < t \leq T}$ ,  $\{R_{1,n}(t, s)\}_{0 \leq s < t \leq T}$  and  $\{R_{2,n}(t, s)\}_{0 \leq s < t \leq T}$  by (2.2), (2.4) and (2.5) respectively replacing  $A(t)$  by  $A_n(t)$ .

For the original family  $\{A(t)\}_{0 \leq t \leq T}$  we define  $\{P(t, s)\}_{0 \leq s < t \leq T}$  and  $\{R_1(t, s)\}_{0 \leq s < t \leq T}$  by (2.7) and (2.4) respectively and  $\{R_2(t, s)\}_{0 \leq s < t \leq T}$  by (2.9) noting that  $F_1(t, s), \dots, F_5(t, s)$  are all meaningful, if  $F_5(t, s)$  is defined by (2.11), even though  $A(t)$  is unbounded.

The chief aim of this step is to show that for each  $t > s$ ,  $\{P_n(t, s)\}_{n=1,2,\dots}$ ,  $\{R_{1,n}(t, s)\}_{n=1,2,\dots}$  and  $\{R_{2,n}(t, s)\}_{n=1,2,\dots}$  are, roughly speaking, boundedly convergent to  $P(t, s)$ ,  $R_1(t, s)$  and  $R_2(t, s)$ , respectively, which implies the convergence of the solutions of the equations obtained by substituting  $A_n(t)$  for  $A(t)$  in (2.1) and (2.3).

In order to establish the uniform boundedness with respect to  $n$ , using the first step, we must show that  $\{A_n(t)\}_{0 \leq t \leq T}$  satisfies (I), (II) and (III) with some constants independent of  $n$ .

Since  $\lambda \in \Sigma$  implies  $n\lambda(n-\lambda)^{-1} \in \Sigma$ , it is easily seen that the resolvent set of  $A_n(t)$  contains  $\Sigma$  and for  $\lambda \in \Sigma$  the resolvent is given by

$$(\lambda - A_n(t))^{-1} = (n - \lambda)^{-1}(n + A(t))(n\lambda(n - \lambda)^{-1} - A(t))^{-1} \tag{2.13}$$

or

$$= -(n - \lambda)^{-1} + n^2(n - \lambda)^{-2}(n\lambda(n - \lambda)^{-1} - A(t))^{-1}. \tag{2.14}$$

On the other hand we have the inequality

$$|\lambda - n| \geq \sin(\theta/2)(|\lambda| + n), \quad \text{for } \lambda \in \Sigma. \tag{2.15}$$

(2.14) together with (2.15) gives

$$\|(\lambda - A_n(t))^{-1}\| \leq M_1/(|\lambda| + 1) \tag{2.16}$$

for  $\lambda \in \Sigma$  with some constant  $M_1$  determined by  $N_1$  and  $\theta$  alone. If we take  $\lambda = 0$  in (2.14), then

$$A_n(t)^{-1} = n^{-1} + A(t)^{-1},$$

hence  $A_n(t)^{-1}$  is strongly differentiable in  $t$  and

$$dA_n(t)^{-1}/dt = dA(t)^{-1}/dt. \tag{2.17}$$

By (2.12), (2.13) and (2.17) and with the aid of (2.10), we get for  $\lambda \in \Sigma$

$$\partial(\lambda - A_n(t))^{-1}/\partial t = n^2(n - \lambda)^{-2}\partial(n\lambda(n - \lambda)^{-1} - A(t))^{-1}/\partial t,$$

hence

$$\|\partial(\lambda - A_n(t))^{-1}/\partial t\| \leq M_2/|\lambda|^p$$

with some constant  $M_2$  determined only by  $N_2$ ,  $\theta$  and  $\rho$ . Similarly by (2.12),



(2.13) and (2.17) we get for  $\lambda \in \Sigma$

$$\begin{aligned} & A_n(t)(\lambda - A_n(t))^{-1}dA_n(t)^{-1}/dt \\ &= n(n - \lambda)^{-1}A(t)(n\lambda(n - \lambda)^{-1} - A(t))^{-1}dA(t)^{-1}/dt, \end{aligned}$$

hence

$$\begin{aligned} & \|A_n(t)(\lambda - A_n(t))^{-1}dA_n(t)^{-1}/dt - A_n(s)(\lambda - A_n(s))^{-1}dA_n(s)^{-1}/ds\| \\ & \leq N_3 \sum_{i=1}^k |n(n - \lambda)^{-1}|^{1+\alpha_i} |\lambda|^{\alpha_i} |t - s|^{\beta_i} \\ & \leq M_3 \sum_{i=1}^k |\lambda|^{\alpha_i} |t - s|^{\beta_i} \end{aligned}$$

with some constant  $M_3$  determined by  $N_3, \theta$  and  $\{\alpha_i\}_{1 \leq i \leq k}$  alone. Thus (I), (II) and (III) are fulfilled with the same constants except  $M_1, M_2$  and  $M_3$ .

We have deduced, therefore, that

$$\begin{aligned} & \|\exp(-(t-s)A_n(t))\| \leq M_4, \quad \|P_n(t, s)\| \leq M_5(t-s)^{\rho-1}, \\ & \|R_{1,n}(t, s)\| \leq M_6(t-s)^{2\rho-1} \quad \text{and} \\ & \|R_{2,n}(t, s)\| \leq M_7\{(t-s)^{\rho-1} + (t-s)^{\delta-1}\} \quad \text{with } \delta = \min_{1 \leq i \leq k} \{\beta_i - \alpha_i\}. \end{aligned}$$

Next, we establish the convergence. The strong convergence of  $\{\exp(-(t-s)A_n(t))\}_{n=1,2,\dots}$ ,  $\{P_n(t, s)\}_{n=1,2,\dots}$  and  $\{R_{1,n}(t, s)\}_{n=1,2,\dots}$  to  $\exp(-(t-s)A(t))$ ,  $P(t, s)$  and  $R_1(t, s)$  respectively follows from that of  $\{(\lambda - A_n(t))^{-1}\}_{n=1,2,\dots}$  to  $(\lambda - A(t))^{-1}$  for each  $\lambda \in \Sigma$  together with (2.16) and the repeated use of the theorem of bounded convergence. The strong convergence of  $\{R_{2,n}(t, s)\}_{n=1,2,\dots}$  to  $R_2(t, s)$  follows from that of each family  $\{F_{j,n}(t, s)\}_{n=1,2,\dots}$  to  $F_j(t, s)$ .

Now, it would be natural to expect that the desired evolution operator  $U(t, s)$  is the solution of the integral equation

$$\begin{aligned} & U(t, s) = \exp(-(t-s)A(t)) + \int_s^t U(t, \tau)P(\tau, s)d\tau, \quad 0 \leq s < t \leq T \\ & U(s, s) = I. \end{aligned} \tag{2.18}$$

In order to derive the desired properties of  $U(t, s)$  we need the operator valued function  $W(t, s)$  which is the solution of the integral equation

$$W(t, s) = R_1(t, s) + R_2(t, s) + \int_s^t W(t, \tau)P(\tau, s)d\tau, \quad 0 \leq s < t \leq T. \tag{2.19}$$

Since the inhomogeneous terms and kernel of (2.18) and (2.19) are strongly continuous and satisfy

$$\begin{aligned} & \|\exp(-(t-s)A(t))\| \leq M_8, \quad \|P(t, s)\| \leq M_9(t-s)^{\rho-1}, \\ & \|R_1(t, s)\| \leq M_{10}(t-s)^{2\rho-1} \quad \text{and} \\ & \|R_2(t, s)\| \leq M_{11}\{(t-s)^{\rho-1} + (t-s)^{\delta-1}\}, \end{aligned}$$

the solutions of (2.18) and (2.19) exist and are unique. As is easily seen,  $U(t, s)$  and  $W(t, s)$  are the strong limits of  $U_n(t, s)$  and  $W_n(t, s)$ .

The final step. We prove  $U(t, s)$  satisfies 1)~5). 1) is easily seen and 2) is a direct consequence of that for  $U_n(t, s)$ . The strong convergence of  $W_n(t, s)$  to  $W(t, s)$  together with

$$\|W_n(t, s)\| \leq M_{12} \{(t-s)^{\rho-1} + (t-s)^{\delta-1}\}$$

implies 3) and 4). These arguments are similar to the proof of Theorem 1 of [4], and the detail is omitted. For the proof of 5) we use the following lemma (see P. 124 in [6]).

**Lemma.** Let  $\{V(t, s)\}_{0 \leq s < t \leq T}$  be the family of bounded operators on  $X$  constructed by

$$\begin{aligned} V(t, s) &= \exp(-(t-s)A(s)) + Z(t, s) \\ Z(t, s) &= \int_s^t Q(t, \tau) \exp(-(\tau-s)A(s)) d\tau, \end{aligned}$$

where  $\{Q(t, s)\}_{0 \leq s < t \leq T}$  is the solution of

$$\begin{aligned} Q(t, s) &= Q_1(t, s) + \int_s^t Q(t, \tau) Q_1(\tau, s) d\tau, \\ Q_1(t, s) &= (\partial/\partial t + \partial/\partial s) \exp(-(t-s)A(s)). \end{aligned}$$

Then  $\{V(t, s)\}_{0 \leq s < t \leq T}$  has 1) and 5).

The existence of  $Q(t, s)$  (hence  $Z(t, s)$  and  $V(t, s)$ ) is proved by the analogous estimation

$$\|Q_1(t, s)\| \leq M_{12}(t-s)^{\rho-1}$$

to (2.8). The proof of Lemma is not difficult, so we omit it.

It follows from 4) of  $U(t, s)$  and 5) of  $V(t, s)$  that  $V(t, \tau)U(\tau, s)$  is strongly differentiable in  $\tau \in (s, t)$  and  $\partial V(t, \tau)U(\tau, s)/\partial \tau$  vanishes identically. Hence  $V(t, \tau)U(\tau, s)$  is constant in  $\tau \in (s, t)$ . Letting  $\tau \rightarrow s$  and  $\tau \rightarrow t$ , we get

$$V(t, s) = U(t, s), \quad 0 \leq s \leq t \leq T.$$

REMARK. If the following condition:

$$\begin{aligned} \text{(IV)} \quad & \|dA(t)^{-1}/dt A(t)(\lambda - A(t))^{-1} - dA(s)^{-1}/ds A(s)(\lambda - A(s))^{-1}\| \\ & \leq N_4 \sum_{j=1}^l |\lambda|^{\alpha_j} |t-s|^{\beta_j} \end{aligned}$$

holds for any  $(t, s) \in [0, T]^2$  and  $\lambda \in \Sigma$  with some constant  $N_4$  and some non-

empty finite family  $\{(\alpha_j, \beta_j)\}_{1 \leq j \leq l}$  of the elements of  $\{(\alpha, \beta); -1 \leq \alpha < \beta \leq 1\}, N_4$  and the family being independent of  $(t, s)$  and  $\lambda$ .

is added to (I), (II) and (III), then 5) is strengthened as follows:

6) For any  $t > s$  the bounded extension  $\overline{U(t, s)A(s)}$  of  $U(t, s)A(s)$  exists and is strongly continuous in  $0 \leq s < t \leq T$ . There exists some constant  $C_2$  such that

$$\|\overline{U(t, s)A(s)}\| \leq C_2(t-s)^{-1}.$$

7)  $U(t, s)$  is strongly differentiable in  $s \in [0, t)$ , and

$$\partial U(t, s)/\partial s = \overline{U(t, s)A(s)}.$$

### 3. The initial value problem for (E)

**Theorem 2.** Let  $\{A(t)\}_{0 \leq t \leq T}$  be a family of densely defined, closed linear operators in  $X$  satisfying (I), (II) and (III), and  $\{U(t, s)\}_{0 \leq s \leq t \leq T}$  be its evolution operator. Then the following statements hold:

i) If  $f$  is continuous, then any strict solution  $u \in C([0, T]; X) \cap C^1((0, T]; X)$  of (E) on  $[0, T]$  with its initial value  $u_0 \in X$  can be expressed in the form

$$u(t) = U(t, 0)u_0 + \int_0^t U(t, \tau)f(\tau)d\tau. \tag{3.1}$$

ii) If  $f$  is Hölder continuous and  $u_0$  is an arbitrary element of  $X$ , then the function  $u$  defined by (3.1) is the strict solution of (E) on  $[0, T]$ .

The proof of i) and ii) is almost the same as the proof of 4) and 5) in [4], therefore, it is omitted. But we will give, here, another proof of i) without using the differentiability in  $s$  of  $U(t, s)$ .

For any  $\varepsilon > 0$ ,  $U_n(t, \tau)u(\tau)$  is continuously differentiable in  $\tau \in [\varepsilon, t]$ , and

$$\begin{aligned} \partial U_n(t, \tau)u(\tau)/\partial \tau &= U_n(t, \tau)f(\tau) \\ &\quad + U_n(t, \tau)\{(1+n^{-1}A(\tau))^{-1}-I\}A(\tau)u(\tau). \end{aligned} \tag{3.2}$$

Integrating (3.2) on  $[\varepsilon, t]$ , we have

$$\begin{aligned} u(t) - U_n(t, \varepsilon)u(\varepsilon) &= \int_\varepsilon^t U_n(t, \tau)f(\tau)d\tau \\ &\quad + \int_\varepsilon^t U_n(t, \tau)\{(1+n^{-1}A(\tau))^{-1}-I\}A(\tau)u(\tau)d\tau. \end{aligned} \tag{3.3}$$

Since  $U_n(t, s)$  is uniformly bounded, the second integral of (3.3) vanishes when  $n$  tends to infinity. Thus we have

$$u(t) - U(t, \varepsilon)u(\varepsilon) = \int_\varepsilon^t U(t, \tau)f(\tau)d\tau.$$

Letting  $\varepsilon$  tend to zero, we obtain the desired equality (3.1).

### 4. Application

In this section we entirely follow the terminology and notations in §7 of [2], and some of them may be used without precise definitions. Let  $H$  and  $K$  be Hilbert spaces with the inner products  $(\cdot, \cdot)$  and  $((\cdot, \cdot))$  respectively such that  $K \subset H$  algebraically and topologically. Let  $\{a(t; \cdot, \cdot)\}_{0 \leq t \leq T}$  be a family of continuous sesquilinear forms on  $K \times K$ , and let  $\{V(t)\}_{0 \leq t \leq T}$  be a family of closed subspaces of  $K$ . We define the operator  $A(t)$  for each  $t \in [0, T]$  in the following manner:

$u \in V(t)$  belongs to  $D(A(t))$  and  $A(t)u = f \in H$  if and only if  $a(t; u, v) = (f, v)$  for any  $v \in V(t)$ .

**Theorem 3.** Assume (K. 1), (K. 2), (K. 3) and (K. 4) which are stated in [2] and, instead of (K. 5) and (K. 6), the following:

(K. 5)' There exist some constants  $M$  and  $\alpha \in (0, 1]$  such that for any  $(t, s) \in [0, T]^2$  and  $u \in K$

$$|(\dot{P}(t) - \dot{P}(s))u| \leq M |t - s|^\alpha \|u\|$$

and

$$|(\dot{Q}(t) - \dot{Q}(s))u| \leq M |t - s|^\alpha \|u\|.$$

(K. 6)'  $\lim_{h \rightarrow 0} \sup_{(u, v) \in B^2} |\dot{a}(t+h; u, v) - \dot{a}(t; u, v)| = 0,$

where  $B = \{u \in K; \|u\| \leq 1\}$ .

Then (I), (II) and (III) of  $\{-A(t)\}_{0 \leq t \leq T}$  hold with  $\rho = 1/2$  and  $\{(\alpha_i, \beta_i)\}_{1 \leq i \leq k} = \{(-1/2, 0), (0, \alpha), (1/2, 1)\}$ .

By Lemma 7.1 and Theorem 7.1 in [2], we have only to show (III).

Taking  $g = A(t)^*(\bar{\lambda} + A(t)^*)^{-1}g'$  in the formula (7.13) in [2], we have the expression

$$\begin{aligned} (A(t)(\lambda + A(t))^{-1}dA(t)^{-1}/dt f, g) &= -\dot{a}(t; A(t)^{-1}f, (\bar{\lambda} + A(t)^*)^{-1}g) \\ &+ (\dot{P}(t)A(t)^{-1}f, A(t)^*(\bar{\lambda} + A(t)^*)^{-1}g) + (f, \dot{Q}(t)(\bar{\lambda} + A(t)^*)^{-1}g) \\ &- a(t; A(t)^{-1}f, \dot{Q}(t)(\bar{\lambda} + A(t)^*)^{-1}g) - a(t; \dot{P}(t)A(t)^{-1}f, (\bar{\lambda} + A(t)^*)^{-1}g). \end{aligned}$$

From this we get for  $\lambda \in \Sigma$

$$\begin{aligned} &(\{A(t)(\lambda + A(t))^{-1}dA(t)^{-1}/dt - A(s)(\lambda + A(s))^{-1}dA(s)^{-1}/ds\} f, g) \\ &= I(t, s; f, g) + R(t, s; f, g), \end{aligned}$$

where

$$\begin{aligned} I(t, s; f, g) &= (\dot{P}(t)A(t)^{-1}f, A(t)^*(\bar{\lambda} + A(t)^*)^{-1}g) \\ &- (\dot{P}(s)A(s)^{-1}f, A(s)^*(\bar{\lambda} + A(s)^*)^{-1}g) \end{aligned}$$

and

$R(t, s; f, g)$  = the sum of the remaining terms.

By (7.15) and (7.16) in [2] we easily conclude

$$|R(t, s; f, g)| \leq M_1 |\lambda|^{-1/2} |f| |g|$$

with some constant  $M_1$ .

$$\begin{aligned} |I| \leq & |(\dot{P}(t) - \dot{P}(s))A(t)^{-1}f, A(t)^*(\bar{\lambda} + A(t)^*)^{-1}g| \\ & + |(\dot{P}(s)\{A(t)^{-1} - A(s)^{-1}\}f, A(t)^*(\bar{\lambda} + A(t)^*)^{-1}g)| \\ & + |(\dot{P}(s)A(s)^{-1}f, \bar{\lambda} \int_s^t (\partial(\bar{\lambda} + A(\tau)^*)^{-1}/\partial\tau)gd\tau)|, \end{aligned}$$

in view of (K. 5)' and (II) of  $\{-A(t)^*\}_{0 \leq t \leq T}$ , we finally conclude

$$|I(t, s; f, g)| \leq M_2 \{|t-s|^\alpha + |\lambda|^{1/2}|t-s|\} |f| |g|$$

with some constant  $M_2$ .

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