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WICK CALCULUS: A TIME-FREQUENCY APPROACH

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Abstract

Anti-Wick operators are a class of pseudodifferential-type operators that are also known under the names of Time-frequency localization operators or Gabor-Toeplitz operators or wave packets. We consider a general version of Wick operators, depending on different window functions, and we investigate the related calculus, obtaining a composition formula that extends the known results [1, 3, 5, 15, 16, 18].

1. Introduction

The purpose of this paper is to present a formula for the product of two Wick operators, defined in terms of different pairs of windows φ_1, φ_2. In principle, Wick operators can be converted to Weyl operators, and hence one may apply to them the standard symbolic calculus [14, 21]. It is natural, however, to consider the product in the Wick form, and try to compute directly the symbol in terms of the symbols of the factors; see in this direction [3, 5, 15, 16, 19]. Recently Ando and Morimoto [1] have given a full expansion for the Wick symbol of the product in the case when all the windows coincide with the Gaussian function.

We propose here a general formula. The expression is somewhat non-standard, because we write the product as a sum of anti-Wick operators corresponding to a sequence of different pairs of windows, with decreasing order. This seems to us the only possible expression of reasonable simplicity in the generic case.

In the remaining part of this Introduction we recall the definition of Wick operators and state the composition result. In Section 2 we summarize some concepts of time-frequency methods used in the proof. In Section 3 we introduce the classes of symbols we are arguing on. They are, essentially, those of Shubin [18], as generalized in [3]. Let us emphasize that other classes of symbols, under weaker assumptions on derivative estimates, would work as well. In Section 4 we prove the result. In Section 5 we give a composition formula for the particular case of Gaussian functions as a pair of windows and we recapture the results of Lerner [15, 16], Ando and Morimoto [1]. Section 6 is devoted to miscellaneous comments. Namely we show how to pass from a pair of windows to another and, finally, we construct a parametrix for the elliptic Wick operators by using our formula; a natural application, which we hope

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to detail in future papers, concerns regularity results in the frame of the modulation spaces [9, 12].

Before stating the precise definition, let us observe that Wick operators have been considered in the past under rather different points of view, and different names. They were introduced by Berezin [2] as a quantization procedure, and as an approximation of pseudodifferential operator s ("wave packets") by Cordoba and Fefferman [7, 11]. From the point of view of the time-frequency analysis, which we shall adopt in the following, they have been studied by Daubechies [8] and Ramanathan and Topiwala [17], and they are now extensively investigated as an important mathematical tool in signal analysis and other applications [22, 23, 10, 6], under the name of localization operators.

Let us begin by defining the operators of translation and modulation by

$$T_x f(t) = f(t-x) \quad \text{and} \quad M_\omega f(t) = e^{2\pi i \omega t} f(t)\,. \tag{1}$$

For a fixed non-zero $g \in \mathcal{S}(\mathbb{R}^d)$ the short-time Fourier transform of $f \in \mathcal{S}'(\mathbb{R}^d)$ with respect to the window g is given by

$$V_g f(x,\omega) = \langle f, M_\omega T_x g\rangle = \int_{\mathbb{R}^d} f(t)\,\overline{g(t-x)}\,e^{-2\pi i \omega t}\,dt\,. \tag{2}$$

Then the time-frequency localization operator $A_a^{\varphi_1,\varphi_2}$ with symbol a and windows φ_1, φ_2 is defined to be

$$A_a^{\varphi_1,\varphi_2} f(t) = \int_{\mathbb{R}^{2d}} a(x,\omega) V_{\varphi_1} f(x,\omega) M_\omega T_x \varphi_2(t)\,dx\,d\omega\,. \tag{3}$$

If $a \in \mathcal{S}'(\mathbb{R}^{2d})$ and $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$, then (3) is a well-defined continuous operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$. If $\varphi_1(t) = \varphi_2(t) = e^{-\pi t^2}$, then $A_a = A_a^{\varphi_1,\varphi_2}$ is the classical Anti-Wick operator and the mapping $a \to A_a^{\varphi_1,\varphi_2}$ is interpreted as a quantization rule [2, 18, 23].

Often it is more convenient to interpret the definition of $A_a^{\varphi_1,\varphi_2}$ in a weak sense, then (3) can be recast as

$$\langle A_a^{\varphi_1,\varphi_2} f, g\rangle = \langle a V_{\varphi_1} f, V_{\varphi_2} g\rangle = \langle a, \overline{V_{\varphi_1} f}\, V_{\varphi_2} g\rangle, \qquad f, g \in \mathcal{S}(\mathbb{R}^d)\,. \tag{4}$$

In the sequel it will be useful to see (3) as *superposition* of rank one operators. Namely, let $Y = (y_1, y_2) \in \mathbb{R}^{2d}$ and consider the time-frequency shift operators given by

$$\Pi(Y) f = M_{y_2} T_{y_1} f$$

then we define

$$\Sigma_Y^{\varphi_1,\varphi_2} f(t) = (V_{\varphi_1} f)(Y)\Pi(Y)\varphi_2(t), \tag{5}$$

and (3) can be written as

$$A_a^{\varphi_1,\varphi_2} = \int_{\mathbb{R}^{2d}} a(Y)\Sigma_Y^{\varphi_1,\varphi_2}\, dY. \tag{6}$$

Our analysis of localization operators will heavily use the interplay between time-frequency methods and the Weyl calculus. Namely, the techniques used to prove our results will exploit the *representation of the localization operator $A_a^{\varphi_1,\varphi_2}$ as a Weyl transform*. Let $W(g, f)$ be the cross-Wigner distribution defined below (11). Then the Weyl transform L_σ of $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ is defined by

$$\langle L_\sigma f, g\rangle = \langle \sigma, W(g, f)\rangle, \qquad f, g \in \mathcal{S}(\mathbb{R}^d). \tag{7}$$

Every continuous operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ can be represented as a Weyl transform, and a calculation in [4, 11, 18] reveals that $A_a^{\varphi_1,\varphi_2} = L_{a*W(\varphi_2,\varphi_1)}$, so the (Weyl) symbol of $A_a^{\varphi_1,\varphi_2}$ is given by

$$\sigma = a * W(\varphi_2, \varphi_1)\,. \tag{8}$$

If we consider symbols in the classes of the subsequent Definition 3.1, our main result can be stated as follows

Theorem 1.1. *Let $\varphi_1, \varphi_2, \varphi_3, \varphi_4 \in \mathcal{S}(\mathbb{R}^d)$, $a \in S^{m_1}$, $b \in S^{m_2}$. Let N be a positive integer. If we define*

$$\Phi_\alpha = \frac{1}{(2\pi i)^{|\alpha_2|}} \sum_{\beta\le\alpha} \binom{\alpha}{\beta} (-1)^{|\beta_1|} \langle \varphi_3, t^{\alpha_1-\beta_1}\partial^{\alpha_2-\beta_2}\varphi_4\rangle t^{\beta_1}\partial^{\beta_2}\varphi_1, \tag{9}$$

where $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2) \in \mathbb{Z}_+^d \times \mathbb{Z}_+^d$, then we have the following composition formula

$$A_a^{\varphi_1,\varphi_2} A_b^{\varphi_3,\varphi_4} = \sum_{|\alpha|=0}^{N-1} \frac{(-1)^{|\alpha|}}{\alpha!} A_{a\partial^\alpha b}^{\Phi_\alpha,\varphi_2} + L_r, \tag{10}$$

where L_r is a Weyl operator with Weyl symbol $r \in S^{m_1+m_2-N}$.

An analogous formula could be proved by interchanging the rôle of the two symbols, then the symbol $a\partial^\alpha b$ is replaced by $(\partial^\alpha a)b$. More symmetric formulas with respect to a and b could be easily recovered as well.

NOTATION. We define $t^2 = t \cdot t$, for $t \in \mathbb{R}^d$, and $xy = x \cdot y$ is the scalar product on $\mathbb{R}^d$.

The Schwartz class is denoted by $\mathcal{S}(\mathbb{R}^d)$, the space of tempered distributions by $\mathcal{S}'(\mathbb{R}^d)$. We use the brackets $\langle f, g\rangle$ to denote the extension to $\mathcal{S}(\mathbb{R}^d)\times\mathcal{S}'(\mathbb{R}^d)$ of the inner product $\langle f, g\rangle = \int f(t)\overline{g(t)}\,dt$ on $L^2(\mathbb{R}^d)$. The Fourier transform is normalized to be $\hat{f}(\omega) = \mathcal{F}f(\omega) = \int f(t)e^{-2\pi i t\omega}\,dt$, the involution g^* is $g^*(t) = \overline{g(-t)}$ while the *reflection operator* is $\mathcal{I}g(x) = g(-x)$. Given two differentiable functions $a(x,\omega)$, $b(x,\omega)$ on $\mathbb{R}^{2d}$, we recall their Poisson brackets

$$\{a, b\} = \sum_{j=1}^{d}\left(\partial_{\omega_j}a\partial_{x_j}b - \partial_{x_j}a\partial_{\omega_j}b\right).$$

If $Z = (z_1, z_2)\in\mathbb{R}^{2d}$ and $Y = (y_1, y_2)\in\mathbb{R}^{2d}$, we set

$$[Y, Z] = z_1y_2 - y_1z_2.$$

Thus $[\cdot,\cdot]$ is a non-degenerate, antisymmetric bilinear form, called the *symplectic form* [20]. Let X_j the jth coordinate of $X\in\mathbb{R}^{2d}$, then H_{X_j} denotes the Hamilton vector field of X_j, that is, for $1\le j\le d$, $H_{X_j} = -\partial_{d+j}$ and $H_{X_{j+d}} = \partial_j$. and $M_\omega T_x = e^{2\pi i x\omega}T_xM_\omega$. We define $g^*(t) = \overline{g(-t)}$.

Throughout the paper, we shall use the notation $A\lesssim B$ to indicate $A\le cB$ for a suitable constant $c>0$, whereas $A\asymp B$ if $A\le cB$ and $B\le kA$, for suitable $c, k>0$.

2. Time-Frequency methods

First we summarize some concepts and tools of time-frequency analysis. Since these methods are now available in textbooks [11, 12], we shall omit the most part of the proofs.

2.1. Short-Time Fourier transform (STFT) and Wigner distribution.

The time-frequency representations needed for the Weyl calculus and for localization operators are the *short-time Fourier transform* and the *Wigner distribution*.

The short-time Fourier transform (STFT) of a distribution $f\in\mathcal{S}'(\mathbb{R}^d)$ with respect to a non-zero window $g\in\mathcal{S}(\mathbb{R}^d)$ is

$$V_gf(x,\omega) = \langle f, M_\omega T_xg\rangle = \int_{\mathbb{R}^d} f(t)\,\overline{g(t-x)}\,e^{-2\pi i\omega t}\,dt\,,$$

whereas the *cross-Wigner distribution* $W(f,g)$ of $f, g\in L^2(\mathbb{R}^d)$ is defined to be

$$W(f,g)(x,\omega) = \int f\left(x+\frac{t}{2}\right)\overline{g\left(x-\frac{t}{2}\right)}e^{-2\pi i\omega t}\,dt. \tag{11}$$

The quadratic expression $W(f,f)$ is usually called the Wigner distribution of f.

Both the STFT V_gf and the Wigner distribution $W(f,g)$ are defined on many pairs of Banach spaces. For instance, they both map $L^2(\mathbb{R}^d)\times L^2(\mathbb{R}^d)$ into $L^2(\mathbb{R}^{2d})$

and $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^{2d})$. Furthermore, they can be extended to a map from $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^{2d})$. Actually, the cross-Wigner distribution, is just an STFT in disguise [12, Lemma 4.3.1]; in fact, for all $f, g \in L^2(\mathbb{R}^d)$,

$$W(f, g)(x, \omega) = 2^d e^{4\pi i x\omega} V_{\mathcal{I}g} f(2x, 2\omega). \tag{12}$$

We first list some crucial properties of the STFT (for proofs, see [12, Ch. 3] and [13]).

Lemma 2.1. *Let $f, g, f_j, g_j \in L^2(\mathbb{R}^d)$, $j = 1, 2$, then we have*
(i) (*Switching f and g*),

$$(V_f g)(x, \xi) = e^{-2\pi i \xi x} \overline{(V_g f)(-x, -\xi)}. \tag{13}$$

(ii) (*Inversion formula*),

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} (V_g f)(x, \xi) M_\xi T_x h \, dx \, d\xi = \langle h, g \rangle f. \tag{14}$$

(iii) (*Orthogonality relations*),

$$\langle V_{g_1} f_1, V_{g_2} f_2 \rangle_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} = \langle f_1, f_2 \rangle_{L^2(\mathbb{R}^d)} \overline{\langle g_1, g_2 \rangle}_{L^2(\mathbb{R}^d)}. \tag{15}$$

(iv) (*STFT of time-frequency shifts*) *For $y, \xi \in \mathbb{R}^d$, we have*

$$V_g(M_\xi T_y f)(x, \omega) = e^{-2\pi i(\omega - \xi) y} (V_g f)(x - y, \omega - \xi), \tag{16}$$

$$V_{(M_\xi T_y g)}(M_\xi T_y f)(x, \omega) = e^{2\pi i(\xi x - \omega y)} (V_g f)(x, \omega). \tag{17}$$

Note that (16) and (17) can be read backwards and yield a formula for the $2d$-dimensional time-frequency shift $M_\zeta T_z(V_g f)$, $z, \zeta \in \mathbb{R}^{2d}$.

Given a non-zero window γ, we recall the adjoint operator of V_γ. Let F be a function on $\mathbb{R}^{2d}$, we define

$$V_\gamma^* F = \int_{\mathbb{R}^{2d}} F(x, \omega) M_\omega T_x \gamma \, dx \, d\omega \tag{18}$$

it is easy to see that, for $f \in L^2(\mathbb{R}^d)$,

$$\langle V_\gamma^* F, f \rangle = \langle F, V_\gamma f \rangle.$$

Let g_1, g_2, g_3 be non-zero windows in $\mathcal{S}(\mathbb{R}^d)$, and $f \in L^2(\mathbb{R}^d)$, from (14) and (18) we get

$$\langle g_2, g_3 \rangle V_{g_1} f = V_{g_1} V_{g_2}^* (V_{g_3} f). \tag{19}$$

The next lemma summarizes those properties of the Wigner distribution that we need in the sequel. For the proofs we refer again to [12, Ch. 4].

Lemma 2.2. (i) *Let* $\mathcal{T}_s F(x,t) = F(x+(t/2), x-(t/2))$ *be the symmetric coordinate transform and* $\mathcal{F}_2 F(x,\omega) = \int_{\mathbb{R}^d} F(x,t)e^{-2\pi i\omega t}\,dt$ *be the Fourier transform in the second variable, then*

$$W(f,g) = \mathcal{F}_2\mathcal{T}_s(f\otimes\overline{g}). \tag{20}$$

(ii) *Covariance of* $W(f,g)$*: For* $u,\eta\in\mathbb{R}^d$*, we have*

$$\begin{aligned} W(T_uM_\eta f, T_vM_\gamma g)(x,\omega) &= e^{\pi i(u+v)(\eta-\gamma)}e^{2\pi ix(\eta-\gamma)}e^{-2\pi i\omega(u-v)} \\ &\quad\times W(f,g)\Big(x-\frac{u+v}{2},\omega-\frac{\eta+\gamma}{2}\Big). \end{aligned} \tag{21}$$

(iii) $W(\widehat{f},\widehat{g})(x,\omega) = W(f,g)(-\omega,x)$.

Maybe the following lemma is well known, but since we have not found it in the literature we shall give the proof of the assumption. Another useful property of the cross-Wigner distribution is given by

Lemma 2.3. *Let* $f,g\in\mathcal{S}(\mathbb{R}^d)$*, then*

$$\int_{\mathbb{R}^{2d}} W(f,g)(x,\omega)\,dx\,d\omega = \langle f,g\rangle. \tag{22}$$

Proof. For the Wigner distribution the assumption holds true [12, Lemma 4.3.6]:

$$\int_{\mathbb{R}^{2d}} W(h,h)(x,\omega)\,dx\,d\omega = \langle h,h\rangle \quad \text{for all } h\in\mathcal{S}(\mathbb{R}^d).$$

Therefore relation (22) simply follows by polarization. □

We can now state a relation among cross-Wigner distributions that will play a central rôle in the proof of the main result.

Proposition 2.4. *Let* $Y=(y_1,y_2)\in\mathbb{R}^{2d}$*,* $Z=(z_1,z_2)\in\mathbb{R}^{2d}$*, then, for all* $f_1,f_2,g_1,g_2\in\mathcal{S}(\mathbb{R}^d)$*,*

$$2^{-d}\int_{\mathbb{R}^{2d}} W(f_1,g_1)\left(\frac{Y-Z}{2}\right)W(f_2,g_2)\left(-\frac{Y+Z}{2}\right)e^{2\pi i[Z,Y]}\,dZ = \langle f_1,g_2\rangle W(\mathcal{I}f_2,g_1)(Y). \tag{23}$$

Proof. We translate the Wigner distribution into the corresponding STFT and we use the STFT properties to get the result. By means of (12),

$$
\begin{aligned}
W(f_1, g_1)\Big(\frac{Y-Z}{2}\Big) &= 2^d \exp\left\{4\pi i\Big(\frac{y_1 - z_1}{2}\Big)\Big(\frac{y_2 - z_2}{2}\Big)\right\} V_{\mathcal{I}g_1} f_1(Y - Z),\\
W(f_2, g_2)\Big(-\frac{Y+Z}{2}\Big) &= 2^d \exp\left\{4\pi i\Big(\frac{y_1 + z_1}{2}\Big)\Big(\frac{y_2 - z_2}{2}\Big)\right\} V_{\mathcal{I}g_2} f_2\big(-(Y+Z)\big).
\end{aligned}
$$

Making use of (13) and (16)

$$
\begin{aligned}
V_{\mathcal{I}g_2} f_2\big(-(Y+Z)\big) &= \overline{V_{f_2}(\mathcal{I}g_2)(Y+Z)}e^{-2\pi i(y_1+z_1)(y_2+z_2)}\\
&= \overline{V_{f_2}(\Pi(-Y)\mathcal{I}g_2)(Z)}e^{2\pi i(y_2+z_2)y_1}e^{-2\pi i(y_1+z_1)(y_2+z_2)}\\
&= \overline{V_{f_2}(\Pi(-Y)\mathcal{I}g_2)(Z)}e^{-2\pi i(z_1z_2+y_2z_1)}.
\end{aligned}
$$

Thanks to the previous computations we can translate the left-hand side of (23) in the following way

$$
\begin{aligned}
2^{-d}&\int_{\mathbb{R}^{2d}} W(f_1, g_1)\left(\frac{Y-Z}{2}\right) W(f_2, g_2)\left(-\frac{Y+Z}{2}\right) e^{2\pi i[Z,Y]}\, dZ\\
&= 2^d \int_{\mathbb{R}^{2d}} e^{2\pi i(y_1y_2+y_1z_2-2y_2z_1)} V_{\mathcal{I}g_1} f_1(Y-Z)\overline{V_{f_2}(\Pi(-Y)\mathcal{I}g_2)(Z)}\, dZ\\
&= 2^d e^{2\pi i y_1y_2} \int_{\mathbb{R}^{2d}} e^{2\pi i z_1(z_2-y_2)} V_{\mathcal{I}g_1} f_1(Y-Z)\\
&\qquad\qquad \times e^{2\pi i(y_1z_2-y_2z_1)}e^{-2\pi i z_1z_2}\overline{V_{f_2}(\Pi(-Y)\mathcal{I}g_2)(Z)}\, dZ\\
&= 2^d \int_{\mathbb{R}^{2d}} e^{-2\pi i z_1(y_2-z_2)} V_{\mathcal{I}g_1} f_1(Y-Z) V_{g_2}(\mathcal{I}(\Pi(Y)f_2))(Z)\, dZ
\end{aligned}
$$

where the last equality is obtained by the three following steps,
(i) Switching relation (13)

$$
e^{-2\pi i z_1z_2}\overline{V_{f_2}(\Pi(-Y)\mathcal{I}g_2)(Z)} = V_{(\Pi(-Y)\mathcal{I}g_2)} f_2(-Z)
$$

(ii) Since $\Pi(Y) = M_{y_2}T_{y_1} = e^{2\pi i y_1y_2}T_{y_1}M_{y_2}$ and by (17),

$$
\begin{aligned}
e^{2\pi i(y_1z_2-y_2z_1)} V_{(\Pi(-Y)\mathcal{I}g_2)} f_2(-Z) &= V_{(\Pi(Y)\Pi(-Y)\mathcal{I}g_2)}\big(\Pi(Y)f_2\big)(-Z)\\
&= e^{-2\pi i y_1y_2} V_{\mathcal{I}g_2}\big(\Pi(Y)f_2\big)(-Z)
\end{aligned}
$$

(iii) Observing that $T_{-z_1}\mathcal{I}g_2(t) = \mathcal{I}g_2(t+z_1) = g_2(-t-z_1)$, then

$$
\begin{aligned}
V_{\mathcal{I}g_2}\big(\Pi(Y)f_2\big)(-Z) &= \int_{\mathbb{R}^d} \big(\Pi(Y)f_2\big)(-t)e^{-2\pi i t z_2}\overline{g_2(t-z_1)}\, dt\\
&= \int_{\mathbb{R}^d} \big(\mathcal{I}(\Pi(Y)f_2)\big)(t)e^{-2\pi i t z_2}\overline{T_{z_1}g_2(t)}\, dt\\
&= V_{g_2}\big(\mathcal{I}(\Pi(Y)f_2)\big)(Z).
\end{aligned}
$$

Now, (19) can be rewritten in our case as follows

$$
\begin{aligned}
&2^d \int_{\mathbb{R}^{2d}} e^{-2\pi i z_1(y_2-z_2)} V_{\mathcal{I}g_1} f_1(Y-Z) V_{g_2}\big(\mathcal{I}(\Pi(Y)f_2)\big)(Z)\, dZ\\
&= 2^d \int_{\mathbb{R}^{2d}} V_{g_2}\big(\mathcal{I}(\Pi(Y)f_2)\big)(Z)\overline{V_{f_1}\Pi(Y)\mathcal{I}g_1}(Z)\, dZ\\
&= 2^d \big\langle V_{f_1}^* V_{g_2}\big(\mathcal{I}(\Pi(Y)f_2)\big), \Pi(Y)\mathcal{I}g_1\big\rangle\\
&= 2^d V_{\mathcal{I}g_1} V_{f_1}^* V_{g_2}\big(\mathcal{I}(\Pi(Y)f_2)\big)(Y) = 2^d \langle f_1, g_2\rangle V_{\mathcal{I}g_1}\mathcal{I}\big(\Pi(Y)f_2\big)(Y)\\
&= 2^d \langle f_1, g_2\rangle e^{4\pi i y_1 y_2} V_{\mathcal{I}g_1}\mathcal{I}f_2(2Y) = \langle f_1, g_2\rangle W(\mathcal{I}f_2, g_1)(Y),
\end{aligned}
$$

and we get the assumption. □

3. Symbol classes

Our classes of symbols, introduced by Shubin, Berezin and others authors (see e.g. [2, 18, 3]), will be defined in terms of general weight functions $\Lambda(Z)$ in $\mathbb{R}^{2d}$.

Weight functions [5, 3]. Let us denote $\langle Z\rangle = (1+|Z|^2)^{1/2}$, we consider a weight function $\Lambda(Z)$, continuous in $\mathbb{R}^{2d}$, for which there exists $\epsilon > 0$ such that

$$\langle Z\rangle^{\epsilon} \lesssim \Lambda(Z) \lesssim \langle Z\rangle, \qquad \text{for all } Z \in \mathbb{R}^{2d}. \tag{24}$$

Moreover, beside (24), we shall assume that Λ is *slowly varying*, i.e. there exists $\epsilon > 0$ such that

$$\Lambda(Z) \asymp \Lambda(X), \quad \text{for } |X-Z| \le \epsilon\Lambda(Z). \tag{25}$$

Let us observe that, starting from (24), (25), one can always find $\tilde{\Lambda}(Z) \in C^\infty(\mathbb{R}^{2d})$, with $\tilde{\Lambda}(Z) \asymp \Lambda(Z)$, satisfying (24), (25) and the additional property

$$|\partial^\gamma \tilde{\Lambda}(Z)| \lesssim \tilde{\Lambda}(Z)^{1-|\gamma|}. \tag{26}$$

We are therefore allowed to assume that (26) is also satisfied by Λ. From (25) it also easily follows that Λ is *temperate*, in the sense that

$$\Lambda(Z) \lesssim \Lambda(X)\langle Z-X\rangle. \tag{27}$$

Another property we shall require for Λ is the following. For all $t = (t_1, \dots, t_{2d}) \in \mathbb{R}^{2d}$, we have

$$\Lambda(tZ) \lesssim \Lambda(Z), \quad \text{with } tZ = (t_1 z_1, \dots, t_{2d} z_{2d}). \tag{28}$$

Combining (25) with (28) we obtain for $t' \in \mathbb{R}^{2d}$, $t'' \in \mathbb{R}^{2d}$

$$\Lambda(t'Z + t''X) \lesssim \Lambda(X)\langle Z-X\rangle. \tag{29}$$

Finally, from (27), it follows that for every $s \in \mathbb{R}$

$$\Lambda(Z)^s \lesssim \Lambda(X)^s \langle Z - X\rangle^{|s|}. \tag{30}$$

Symbols.

DEFINITION 3.1. Let $m \in \mathbb{R}$, then the symbol class S^m_Λ, denoted for short S^m in the sequel, consists of the functions $a(Z) \in C^\infty(\mathbb{R}^{2d})$ which satisfy the estimates

$$|\partial^\gamma a(Z)| \lesssim \Lambda(Z)^{m-|\gamma|}.$$

Let us assume, without loss of generality, that $\Lambda(Z)$ satisfies (26); then we may write $\Lambda(Z) \in S^1$. We list in the following some basic propositions [18, 3].

Proposition 3.2. *We have* $S^{-\infty} = \bigcap_m S^m = \mathcal{S}(\mathbb{R}^{2d})$.

Proposition 3.3. (i) $S^m \subset S^{m'}$, *if* $m \le m'$. (ii) *If* $a \in S^m$, *and* $b \in S^{m'}$, *then* $ab \in S^{m+m'}$ *and* $a + b \in S^{\max\{m,m'\}}$. (iii) *If* $a \in S^m$, *then* $D^\alpha a \in S^{m-|\alpha|}$ *for all* α. (iv) *If* $a \in S^m$, *then* $T_W a(Z) = a(Z - W) \in S^m$ *for all* $W \in \mathbb{R}^{2d}$.

Let us observe that S^m is a Fréchet space with respect to the seminorms $|a|_{k,S^m} = \sup_{|\gamma|\le k} \sup_{Z\in\mathbb{R}^{2d}} \Lambda(Z)^{-m+|\gamma|}|\partial^\gamma a(Z)|$. The preceding Proposition 3.3 can be reconsidered in the corresponding topology; we have in particular continuity of the linear map $D^\alpha : S^m \to S^{m-|\alpha|}$.

DEFINITION 3.4. Let $a_j \in S^{m_j}$, $j = 1, 2, \ldots, m_j \to -\infty$ with $m_{j+1} \le m_j$ for all j, and let $a \in S^{m_1}$. We write $a \sim \sum_{j=1}^\infty a_j$ if for all integer $r \ge 2$ $a - \sum_{1\le j<r} a_j \in S^{m_r}$. We say also in this case that $\sum_{j=1}^\infty a_j$ is an *asymptotic expansion* for a.

Proposition 3.5. *Let* $a_j \in S^{m_j}$, $j = 1, 2, \ldots, m_j \to -\infty$ *with* $m_{j+1} \le m_j$ *for all* j. *Then there exists* $a \in S^{m_1}$ *such that* $a \sim \sum_{j=1}^\infty a_j$. *If another symbol* a' *has the same property, then* $a - a' \in \mathcal{S}(\mathbb{R}^{2d})$.

4. Proof of the main result

We begin with the following preliminary results. They can be seen as generalizations of [1, Lemmas 2.9, 2.10].

Lemma 4.1. *Let* $\varphi_1, \varphi_2 \in \mathcal{S}$,

(i) *consider the symbol* $p_Y(X) = T_Y W(\varphi_2, \varphi_1)(X)$, *then*

$$\Sigma_Y^{\varphi_1,\varphi_2} = L_{p_Y}, \tag{31}$$

where L_{p_Y} *is the Weyl operator with Weyl symbol* p_Y;

(ii) *consider the symbol*

$$p_{Y,Z}(X) = 2^{-d} e^{2\pi i[X-Z,X-Y]} W(\varphi_4, \mathcal{I}\varphi_1)\left(\frac{Y-Z}{2}\right) W(\varphi_2,\varphi_3)\left(X - \frac{Y+Z}{2}\right), \tag{32}$$

then

$$\Sigma_Y^{\varphi_1,\varphi_2}\Sigma_Z^{\varphi_3,\varphi_4} = L_{p_{Y,Z}}. \tag{33}$$

Proof. (i) Let $f, g \in \mathcal{S}$, we have

$$\langle \Sigma_Y^{\varphi_1,\varphi_2} f, g\rangle = V_{\varphi_1} f(Y)\overline{V_{\varphi_2} g(Y)}.$$

Therefore, by (4), the operator $\Sigma_Y^{\varphi_1,\varphi_2}$ can be seen as a localization operator with symbol the Dirac measure δ_Y. Its Weyl symbol is given by

$$p_y(X) = \delta_Y * W(\varphi_2,\varphi_1)(X) = W(\varphi_2,\varphi_1)(X-Y).$$

(ii) Let $f, g \in \mathcal{S}$, then

$$\begin{aligned}\langle \Sigma_Y^{\varphi_1,\varphi_2}\Sigma_Z^{\varphi_3,\varphi_4} f, g\rangle &= V_{\varphi_3} f(Z) V_{\varphi_1}\big(\Pi(Z)\varphi_4\big)(Y)\overline{V_{\varphi_2} g(Y)}\\ r &= \int_{\mathbb{R}^{2d}} \big\langle \Pi(Z)\varphi_4, \Pi(Y)\varphi_1\big\rangle\big(\Pi(Y)\varphi_2 \otimes \overline{\Pi(Z)\varphi_3}\big)(u,v)\bar{g}(u) f(v)\,du\,dv\end{aligned}$$

that is an integral operator with kernel

$$k(u,v) = \big\langle \Pi(Z)\varphi_4, \Pi(Y)\varphi_1\big\rangle\big(\Pi(Y)\varphi_2 \otimes \overline{\Pi(Z)\varphi_3}\big)(u,v).$$

The Weyl symbol is given by [12, Thm. 14.3.5]

$$\begin{aligned}p_{Y,Z}(u,v) &= \mathcal{F}_2\mathcal{T}_s k(u,v) = \big\langle \Pi(Z)\varphi_4, \Pi(Y)\varphi_1\big\rangle \mathcal{F}_2\mathcal{T}_s\big(\Pi(Y)\varphi_2 \otimes \overline{\Pi(Z)\varphi_3}\big)\\ &= V_{\varphi_1}\big(\Pi(Z)\varphi_4\big)(Y) W\big(\Pi(Y)\varphi_2, \Pi(Z)\varphi_3\big)(u,v),\end{aligned}$$

then by (12), (16) and (21) and making easy computations one gets (32). □

Lemma 4.2. *Let* $Y \in \mathbb{R}^{2d}$, $f, g \in \mathcal{S}(\mathbb{R}^d)$, $\alpha = (\alpha_1,\alpha_2)$, $\beta = (\beta_1,\beta_2) \in \mathbb{Z}_+^d \times \mathbb{Z}_+^d$, *then*

$$Y^\alpha W(f,g)(Y) = \frac{1}{2^{|\alpha|}(2\pi i)^{|\alpha_2|}} \sum_{\beta\le\alpha} \binom{\alpha}{\beta} (-1)^{|\beta_2|} W(t^{\alpha_1-\beta_1}\partial^{\alpha_2-\beta_2} f, t^{\beta_1}\partial^{\beta_2} g)(Y). \tag{34}$$

Proof. Let $Y = (x, \omega) \in \mathbb{R}^{2d}$, we shall deal with the *time* variable x and the *frequency* variable separately. Since

$$W(t_j f, g)(x,\omega) = x_j W(f,g)(x,\omega) + \frac{1}{2}\int_{\mathbb{R}^d} t_j e^{-2\pi i t\omega} f\left(x+\frac{t}{2}\right)\overline{g\left(x-\frac{t}{2}\right)}\, dt,$$
$$W(f, t_j g)(x,\omega) = x_j W(f,g)(x,\omega) - \frac{1}{2}\int_{\mathbb{R}^d} t_j e^{-2\pi i t\omega} f\left(x+\frac{t}{2}\right)\overline{g\left(x-\frac{t}{2}\right)}\, dt$$

by summing up the two previous relations,

$$x_j W(f,g)(Y) = \frac{1}{2}\big[W(t_j f, g)(Y) + W(f, t_j g)(Y)\big]$$

and by induction we get

$$x^{\alpha_1} W(f,g)(x,\omega) = \frac{1}{2^{|\alpha_1|}} \sum_{\beta_1 \le \alpha_1} \binom{\alpha_1}{\beta_1} W(t^{\alpha_1-\beta_1} f, t^{\beta_1} g)(x,\omega). \tag{35}$$

Next, we consider the *frequency* variable

$$\begin{aligned}
\omega^{\alpha_2} W(f,g)(x,\omega) &= \frac{1}{(-2\pi i)^{|\alpha_2|}} \int_{\mathbb{R}^d} \partial^{\alpha_2} e^{-2\pi i t\omega} f\left(x+\frac{t}{2}\right)\overline{g\left(x-\frac{t}{2}\right)}\, dt \\
&= \frac{1}{(2\pi i)^{|\alpha_2|}} \int_{\mathbb{R}^d} e^{-2\pi i t\omega} \partial^{\alpha_2}\left(f\left(x+\frac{t}{2}\right)\overline{g\left(x-\frac{t}{2}\right)}\right) dt \\
&= \frac{1}{(4\pi i)^{|\alpha_2|}} \sum_{\beta_2\le\alpha_2} \binom{\alpha_2}{\beta_2} (-1)^{|\beta_2|} W(\partial^{\alpha_2-\beta_2} f, \partial^{\beta_2} g)(x,\omega)
\end{aligned}$$

By gluing together the previous relations we get (4.2). □

We now have all the instruments needed to prove our main result.

Proof of Theorem 1.1. By means of (6) we have

$$A_a^{\varphi_1,\varphi_2} A_b^{\varphi_3,\varphi_4} = \int_{\mathbb{R}^{2d}} a(Y) b(Z) \Sigma_Y^{\varphi_1,\varphi_2} \Sigma_Z^{\varphi_3,\varphi_4}\, dY\, dZ. \tag{36}$$

Applying Taylor's formula to the symbol b

$$b(Z) = b(Y) + \sum_{|\alpha|=1}^{N-1} \partial^\alpha b(Y) \frac{(Z-Y)^\alpha}{\alpha!} + b_N(Y,Z),$$

where

$$b_N(Y,Z) = N \sum_{|\alpha|=N} \int_0^1 (1-\theta)^{N-1} \partial^\alpha b\big(Y + \theta(Z-Y)\big)\, d\theta\, \frac{(Z-Y)^\alpha}{\alpha!}.$$

We substitute the previous expansion of the symbol b in (36) and we get

$$A_a^{\varphi_1,\varphi_2} A_b^{\varphi_3,\varphi_4} = \sum_{|\alpha|=0}^{N-1} \Omega_\alpha + R_N \tag{37}$$

where

$$\Omega_\alpha = \frac{1}{\alpha!} \int_{\mathbb{R}^{2d}} a(Y)\partial^\alpha b(Y)(Z-Y)^\alpha \Sigma_Y^{\varphi_1,\varphi_2} \Sigma_Z^{\varphi_3,\varphi_4}\, dY\, dZ, \tag{38}$$

and

$$R_N = \int_{\mathbb{R}^{2d}} a(Y) b_N(Y,Z) \Sigma_Y^{\varphi_1,\varphi_2} \Sigma_Z^{\varphi_3,\varphi_4}\, dY\, dZ.$$

First we find the Weyl symbol $r = \sigma(R_N)$ of R_N. By Lemma 4.1(ii), we have

$$r = \sigma(R_N) = \int_{\mathbb{R}^{2d}} a(Y) b_N(Y,Z) p_{Y,Z}\, dY\, dZ$$

we therefore have to show that $r \in S^{m_1+m_2-N}$, which means, for $\gamma \in \mathbb{Z}_+^d$, to estimate $\partial^\gamma r$. We make the change of variables $(X-Y, X-Z) \to (-Y,-Z)$ in the previous integral and we estimate $\partial^\gamma(ab_N) = \sum_{\delta\le\gamma} \binom{\gamma}{\delta} \partial^{\gamma-\delta} a \partial^\delta b_N$. Since $a \in S^{m_1}$ and by (30)

$$|\partial^{\gamma-\delta} a(X+Y)| \lesssim \Lambda(X+Y)^{m_1-|\gamma|+|\delta|} \lesssim \Lambda(X)^{m_1-|\gamma|+|\delta|} \langle Y\rangle^{|m_1-|\gamma|+|\delta||}.$$

Analogously, since $b \in S^{m_2}$ and by means of (30) and (29) we get, for $|\alpha| = N$,

$$\begin{aligned}\left|\partial^{\alpha+\delta} b\big(X+Y+\theta(Z-Y)\big)\right| &\lesssim \Lambda\big(X+\theta Z+(1-\theta)Y\big)^{m_2-N-|\delta|} \\ &\lesssim \Lambda(X)^{m_2-N-|\delta|} \langle Y\rangle^{|m_2-N-|\delta||} \big\langle (Z-Y)\big\rangle^{|m_2-N-|\delta||}.\end{aligned}$$

Recall that $W(\varphi_2,\varphi_3)$ and $W(\varphi_4,\mathcal{I}\varphi_1) \in \mathcal{S}(\mathbb{R}^{2d})$, therefore we have the boundedness of the integrals

$$\begin{aligned} I_{\gamma\delta} = \int_{\mathbb{R}^{2d}} &\langle Y\rangle^{|m_1+m_2-N-|\gamma||} \langle (Z-Y)\rangle^{m_2-|\delta|} e^{2\pi i [Z,Y]} \\ &\times W(\varphi_4,\mathcal{I}\varphi_1)\left(\frac{Y-Z}{2}\right) W(\varphi_2,\varphi_3)\left(-\frac{Y+Z}{2}\right) dY\, dZ < \infty, \end{aligned}$$

and we obtain, for all $\gamma \in \mathbb{Z}_+^d$, the estimate $|\partial^\gamma r(X)| \lesssim \sum_{\delta\le\gamma} \binom{\gamma}{\delta} I_{\gamma\delta} \Lambda(X)^{m_1+m_2-N-|\gamma|}$, that is $r \in S^{m_1+m_2-N}$.

Now, we compute the Weyl symbol $\sigma(\Omega_0)$ of the operator Ω_0. Again, by

Lemma 4.1(ii), the change of variables $(X-Y, X-Z) \to (-Y, -Z)$ and (23),

$$\begin{aligned}\sigma(\Omega_0)(X) &= \int_{\mathbb{R}^{2d}} a(X+Y)b(X+Y)\big(\langle\varphi_4,\varphi_3\rangle W(\mathcal{I}\varphi_2,\mathcal{I}\varphi_1)(Y)\big)\, dY \\ &= \langle\varphi_4,\varphi_3\rangle \int_{\mathbb{R}^{2d}} a(X+Y)b(X+Y)W(\varphi_2,\varphi_1)(-Y)\, dY \\ &= \langle\varphi_4,\varphi_3\rangle ab * W(\varphi_2,\varphi_1)(X),\end{aligned}$$

then, by (8), we get $\Omega_0 = \langle\varphi_4,\varphi_3\rangle A_{ab}^{\varphi_2,\varphi_1}$, which means that the first term of the composition formula (36) is a localization operator with symbol given by the product of the two symbols a and b.

Next, we compute the Weyl symbol $\sigma(\Omega_\alpha)$ of the operator Ω_α. By the same computations as the previous Ω_0, we get

$$\begin{aligned}\sigma(\Omega_\alpha)(X) &= 2^{-d}\frac{1}{\alpha!}\int_{\mathbb{R}^d} a(X+Y)\partial^\alpha b(X+Y)\bigg(\int_{\mathbb{R}^d}(Z-Y)^\alpha W(\varphi_4,\mathcal{I}\varphi_1)\left(\frac{Y-Z}{2}\right) \\ &\qquad \times W(\varphi_2,\varphi_3)\left(-\frac{Y+Z}{2}\right)e^{2\pi i[Z,Y]}\, dZ\bigg)\, dY \\ &= 2^{-d}\frac{(-2)^{|\alpha|}}{\alpha!}\int_{\mathbb{R}^d} a(X+Y)\partial^\alpha b(X+Y)\bigg(\int_{\mathbb{R}^d}\left(\frac{Y-Z}{2}\right)^\alpha W(\varphi_4,\mathcal{I}\varphi_1) \\ &\qquad \times \left(\frac{Y-Z}{2}\right) W(\varphi_2,\varphi_3)\left(-\frac{Y+Z}{2}\right)e^{2\pi i[Z,Y]}\, dZ\bigg)\, dY.\end{aligned}$$

For computing the previous expression we do not use the technique developed in [1, Thm. 2.5] which consists of expanding the power $(Z-Y)^\alpha$ and then working separately on the two variables Y and Z. The reason is that in that context they could exploit the fact of having the Gaussian functions as windows, that we do not have any more. We then use formula (34) which let us get rid of the powers in the following way: the cross-Wigner distribution is replaced by sums of cross-Wigner distributions of products of powers with derivatives of the previous windows. Namely, using (34) and, secondly, (23) we have

$$\begin{aligned}\sigma(\Omega_\alpha)(X) &= 2^{-d}\frac{(-1)^{|\alpha|}}{\alpha!}\int_{\mathbb{R}^d} a(X+Y)\partial^\alpha b(X+Y)\Bigg(\int_{\mathbb{R}^d}\frac{1}{(2\pi i)^{|\alpha_2|}}\sum_{\beta\le\alpha}\binom{\alpha}{\beta}(-1)^{|\beta_2|} \\ &\qquad \times W(t^{\alpha_1-\beta_1}\partial^{\alpha_2-\beta_2}\varphi_4, t^{\beta_1}\partial^{\beta_2}\mathcal{I}\varphi_1)\left(\frac{Y-Z}{2}\right)W(\varphi_2,\varphi_3)\left(-\frac{Y+Z}{2}\right) \\ &\qquad \times e^{2\pi i[Z,Y]}\, dZ\Bigg)\, dY \\ &= \frac{(-1)^{|\alpha|}}{(2\pi i)^{|\alpha_2|}\alpha!}\sum_{\beta\le\alpha}\binom{\alpha}{\beta}(-1)^{|\beta_1|}\langle t^{\alpha_1-\beta_1}\partial^{\alpha_2-\beta_2}\varphi_4,\varphi_3\rangle\int_{\mathbb{R}^d} a(X+Y)\end{aligned}$$

$$\times\, \partial^\alpha b(X+Y) W\big(\mathcal{I}\varphi_2, \mathcal{I}(t^{\beta_1}\partial^{\beta_2}\varphi_1)\big)(Y)\, dY$$

therefore

$$\sigma(\Omega_\alpha)(X) = \frac{(-1)^{|\alpha|}}{(2\pi i)^{|\alpha_2|}\alpha!} \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} (-1)^{|\beta_1|} \langle t^{\alpha_1-\beta_1}\partial^{\alpha_2-\beta_2}\varphi_4, \varphi_3\rangle \times \big[(a\partial^\alpha b) * W(\varphi_2, t^{\beta_1}\partial^{\beta_2}\varphi_1)\big](X).$$

Now, by means of (8), we have that

$$\Omega_\alpha = \frac{(-1)^{|\alpha|}}{(2\pi i)^{|\alpha_2|}\alpha!} \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} (-1)^{|\beta_1|} \langle t^{\alpha_1-\beta_1}\partial^{\alpha_2-\beta_2}\varphi_4, \varphi_3\rangle A_{a\partial^\alpha b}^{t^{\beta_1}\partial^{\beta_2}\varphi_1,\varphi_2}. \tag{39}$$

Finally, we plug Ω_0 and (39) in (37) and we obtain the assumption. $\square$

REMARK. It might be useful to write (10) in the equivalent form

$$A_a^{\varphi_1,\varphi_2} A_b^{\varphi_3,\varphi_4} = \sum_{|\alpha|=0}^{N-1} \frac{(-1)^{|\alpha|}}{(2\pi i)^{|\alpha_2|}\alpha!} \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} (-1)^{|\beta_1|} \times \left(\int_{\mathbb{R}^{2d}} W(t^{\alpha_1-\beta_1}\partial^{\alpha_2-\beta_2}\varphi_4, \varphi_3)(Y)\, dY\right) A_{a\partial^\alpha b}^{t^{\beta_1}\partial^{\beta_2}\varphi_1,\varphi_2} + L_r. \tag{40}$$

Let us point out that the remainder term L_r can also be handled and estimated in the way developed in [15, 1]. Moreover, also the symbol classes we consider can be replaced by the ones of [15, 1].

5. The Gaussian windows

It is possible to reset formula (10) or (40) in terms of Wick operators corresponding to the same pair of windows $\varphi_1, \varphi_2 = \varphi$, when φ is the Gaussian function. This is the classical Wick case [15, 1].

Theorem 5.1. *Let $\varphi_j = \varphi = 2^{d/4}e^{-\pi t^2}$ for $j = 1, \dots, 4$, $a \in S^{m_1}$, $b \in S^{m_2}$. Let N be a positive integer, $\alpha \in \mathbb{Z}_+^d \times \mathbb{Z}_+^d$, $Y, Z \in \mathbb{R}^{2d}$. If we set*

$$c = \sum_{|\alpha|=0}^{N-1} \frac{(-1)^{|\alpha|}2^d}{\alpha!(4\pi)^{|\alpha|}} \left[\left(\sum_{q=0}^{\infty} \frac{(4\pi\Delta_Z)^q}{q!} Z^\alpha \Big|_{Z=\partial_Y - iH_Y}\right) a\partial^\alpha b\right], \tag{41}$$

then relation (40) *becomes*

$$A_a^{\varphi,\varphi} A_b^{\varphi,\varphi} = A_c^{\varphi,\varphi} + L_r, \tag{42}$$

where L_r is a Weyl operator with Weyl symbol $r \in S^{m_1+m_2-N}$.

Proof. We shall divide it into four steps.

STEP 1. Let $\gamma, \delta \in \mathbb{Z}_+^d$, $Y = (x, \omega) \in \mathbb{R}^{2d}$, we need to compute $W(\varphi, t^\gamma \partial^\delta \varphi)(x, \omega)$. Let $\psi_1, \psi_2 \in \mathcal{S}$, then by straightforward computations one gets

$$W(\psi_1, t^\gamma \psi_2)(x, \omega) = \left(x + i\frac{H_x}{4\pi}\right)^\gamma W(\psi_1, \psi_2)(x, \omega). \tag{43}$$

By Lemma 2.2(iii) and the previous (43) we obtain

$$\begin{aligned} W(\psi_1, \partial^\delta \psi_2)(x, \omega) &= (-2\pi i)^{|\delta|} W(\widehat{\psi}_1, \xi^\delta \widehat{\psi}_2)(\omega, -x) \\ &= (-2\pi i)^{|\delta|} \left(\omega + i\frac{H_\omega}{4\pi}\right)^\delta W(\psi_1, \psi_2)(x, \omega). \end{aligned} \tag{44}$$

Now, by (43) and (44) we get

$$W(\psi_1, t^\gamma \partial^\delta \psi_2)(x, \omega) = (-2\pi i)^{|\delta|} \left(x + i\frac{H_x}{4\pi}\right)^\gamma \left(\omega + i\frac{H_\omega}{4\pi}\right)^\delta W(\psi_1, \psi_2)(x, \omega). \tag{45}$$

STEP 2. For every $\beta = (\beta_1, \beta_2) \in \mathbb{Z}_+^d \times \mathbb{Z}_+^d$, we define

$$\left(Y + i\frac{H_Y}{4\pi}\right)^\beta := \left(x + i\frac{H_x}{4\pi}\right)^{\beta_1} \left(\omega + i\frac{H_\omega}{4\pi}\right)^{\beta_2}.$$

In terms of the Gaussian φ, we have $W(\varphi, \varphi)(x, \omega) = 2^d e^{-2\pi(x^2+\omega^2)}$ and

$$\begin{aligned} 2^d \int_{\mathbb{R}^{2d}} \left(Y + i\frac{H_Y}{4\pi}\right)^{\alpha-\beta} e^{-2\pi Y^2}\, dY &= 2^d \int_{\mathbb{R}^{2d}} Y^{\alpha-\beta} e^{-2\pi Y^2}\, dY \\ &= \begin{cases} \dfrac{(\alpha-\beta)!}{(8\pi)^{|\alpha-\beta|/2}((\alpha-\beta)/2)!}, & \text{if } \alpha - \beta \in 2(\mathbb{Z}_+^d \times \mathbb{Z}_+^d); \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

STEP 3. We use the previous relation and (45) to compute (40). We have

$$\begin{aligned} (40) = &\sum_{|\alpha|=0}^{N-1} \frac{(-1)^{|\alpha|}}{(2\pi i)^{|\alpha_2|}\alpha!} \sum_{\beta \le \alpha} \binom{\alpha}{\beta} (-1)^{|\beta_1|} \left(\int_{\mathbb{R}^{2d}} (2\pi i)^{|\alpha_2-\beta_2|} \left(Y - i\frac{H_Y}{4\pi}\right)^{\alpha-\beta} \right. \\ &\left. \times\, W(\varphi, \varphi)(Y)\, dY \right) L_{(a\partial^\alpha b)*\left((-2\pi i)^{|\beta_2|}(Y+i(H_Y/4\pi))^\beta W(\varphi,\varphi)\right)} + L_r \\ = &\sum_{|\alpha|=0}^{N-1} \frac{1}{\alpha!} 2^d \sum_{\substack{\beta \le \alpha \\ \alpha-\beta \in 2(\mathbb{Z}_+^d \times \mathbb{Z}_+^d)}} \binom{\alpha}{\beta} \frac{(\alpha-\beta)!}{(8\pi)^{|\alpha-\beta|/2}((\alpha-\beta)/2)!} \end{aligned}$$

$$\times L_{[(Y+i(H_Y/4\pi))^\beta(e^{-2\pi Y^2})]*(a\partial^\alpha b)} + L_r.$$

STEP 4. Now, by induction and exploiting the same arguments as those used to obtain Relations (2.17) and (2.19) in [1], we get the following equality (details of the proof are omitted for brevity)

$$\frac{1}{\alpha!}2^d \sum_{\substack{\beta\leq\alpha \\ \alpha-\beta\in 2(\mathbb{Z}_+^d\times\mathbb{Z}_+^d)}} \binom{\alpha}{\beta} \frac{(\alpha-\beta)!}{(8\pi)^{|\alpha-\beta|/2}((\alpha-\beta)/2)!} \left(Y+i\frac{H_Y}{4\pi}\right)^\beta (e^{-2\pi Y^2})$$
$$= \frac{(-1)^{|\alpha|}2^d}{\alpha!(4\pi)^{|\alpha|}} \left(\sum_{q=0}^\infty \frac{(4\pi\,\Delta_Z)^q}{q!}(Z^\alpha)\,\bigg|_{Z=\partial_Y-iH_Y}\right) e^{-2\pi Y^2}.$$

The last differential operator can hence be moved from the left to the right-hand side of the convolution fold, and this passage leads to (41). ☐

REMARKS. We address to [1] for a further elaboration of (41), see [1, Thm. 2.5], giving an expansion whose terms are well-ordered as standard.

It is clear from the previous proof that to pass from the general form of Theorem 1.1 to the explicit one, we need only the following two facts, evident for the Wigner distribution of the Gaussian:

1. One can exactly calculate $\int_{\mathbb{R}^{2d}} Y^{\alpha-\beta}W(\varphi_4,\varphi_3)(Y)\,dY := C(\alpha-\beta)$, which always vanishes for $\alpha-\beta\in 2(\mathbb{Z}_+^d\times\mathbb{Z}_+^d)$ if φ_3 and φ_4 are even functions.
2. The formula $Y_jW(\varphi_2,\varphi_1)(Y) = c_j\partial_jW(\varphi_2,\varphi_1)(Y)$ holds (or not) for some $c_j\in\mathbb{C}$.

6. Miscellaneous remarks

One would like to reset formula (10) or (40) in terms of Wick operators corresponding to the same pair of windows φ_1, φ_2 also in the general case. This is possible, in principle, by means of Corollary 6.4 below, allowing to pass from a pair of windows to another. Let us first list some properties of the Wick operators in our context.

Theorem 6.1. *Let $A_a^{\varphi_1,\varphi_2}$ be an operator with symbol $a(x,\omega)\in S^m$ and windows φ_1, $\varphi_2\in\mathcal{S}(\mathbb{R}^d)$ satisfying (see (22))*

$$\int_{\mathbb{R}^{2d}} W(\varphi_2,\varphi_1)(x,\omega)\,dx\,d\omega = \langle\varphi_2,\varphi_1\rangle \neq 0. \tag{46}$$

Then its Weyl symbol $\sigma(x,\omega) = a*W(\varphi_2,\varphi_1)$ belongs to S^m with asymptotic expansion

$$\sigma(x,\omega) \sim \sum_{\alpha,\beta} c_{\alpha\beta}\partial_\omega^\alpha\partial_x^\beta a(x,\omega), \tag{47}$$

where

$$\begin{aligned} c_{00} &= \langle \varphi_2, \varphi_1 \rangle \\ c_{\alpha\beta} &= \frac{1}{\alpha!\beta!} \int x^\beta \omega^\alpha \mathcal{I} W(\varphi_2, \varphi_1)(x, \omega)\, dx\, d\omega = \frac{1}{\alpha!\beta!(2\pi i)^{|\alpha+\beta|}} \partial_\omega^\alpha \partial_x^\beta \mathcal{F}[W(\varphi_2, \varphi_1)](0). \end{aligned}$$

Proof. It is a straightforward modification of [3, Theorem 6.1]. One must substitute the Gaussian function with the Wigner distribution $W(\varphi_2, \varphi_1)$ and exploit the property: $W(\varphi_2, \varphi_1) \in \mathcal{S}(\mathbb{R}^{2d})$ for $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$. □

If we drop condition (46), then Theorem (6.1) can be rephrased as follows.

Theorem 6.2. *If $k \in \mathbb{N}$ is the smallest index such that*

(48) $$\partial_\omega^\alpha \partial_x^\beta \mathcal{F}[W(\varphi_2, \varphi_1)](0) \neq 0, \quad \text{for some } (\alpha, \beta) \colon |\alpha + \beta| = k,$$

then, if $a(x, \omega) \in S^m$, we have that the Weyl symbol σ belongs to S^{m-k} (the order decreases!) and $A_a^{\varphi_1,\varphi_2}$ has the asymptotic expansion given by (47) with constants $c_{\alpha\beta} = 0$, for $|\alpha + \beta| < k$.

Let us give an example of windows φ_1, φ_2 such that condition (46) is not fulfilled. Let $\varphi_1(t) = 2^{d/4} t_j e^{-\pi t^2}$ and $\varphi_2(t) = 2^{d/4} e^{-\pi t^2}$, then by a straightforward computation we get $W(\varphi_2, \varphi_1)(x, \omega) = 2^d (x_j + i\omega_j) e^{-2\pi(x^2+\omega^2)}$ and we have $\int_{\mathbb{R}^{2d}} W(\varphi_2, \varphi_1)(x, \omega)\, dx\, d\omega = 0$, whereas it is easy to check that (48) is valid for $k = 1$.

Not every Weyl operator can be seen as a localization operator with symbol in S^m (see [3]). However, from the symbolic calculus we have a converse of Theorem 6.1 modulo regularizing operators, that is operators with kernels belonging to $\mathcal{S}(\mathbb{R}^{2d})$ [3, Thm. 6.2].

Theorem 6.3. *For every Weyl operator L_σ with Weyl symbol $\sigma \in S^m$, and for every pair of windows $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$ satisfying (46), there exists $a(x, \omega) \in S^m$ such that we have*

$$L_\sigma = A_a^{\varphi_1,\varphi_2} + L_{\sigma'}, \quad \text{with } \sigma' \in \mathcal{S}(\mathbb{R}^{2d}),$$

and

(49) $$a \sim \sum_{\gamma,\delta} \tilde{c}_{\gamma\delta} \partial_x^\gamma \partial_\omega^\delta \sigma$$

with constants $\tilde{c}_{\gamma,\delta} \in \mathbb{C}$. In particular,

(50) $$\tilde{c}_{00} = \frac{1}{\langle \varphi_2, \varphi_1 \rangle}.$$

The proof is again an easy modification of [3, Thm. 6.2].

Formally we have

$$\sum_{\gamma,\delta} \tilde{c}_{\gamma\delta}\partial_x^\gamma \partial_\omega^\delta = \tilde{c}_{00} \sum_{j=0}^{\infty} (-1)^j \left(\sum_{(\gamma,\delta)\neq(0,0)} \frac{c_{\gamma\delta}}{c_{00}} \partial_x^\gamma \partial_\omega^\delta \right)^j .$$

After developing the right-hand side, this gives the expression of $\tilde{c}_{\gamma\delta}$ in terms of the constants $c_{\gamma\delta}$.

Corollary 6.4. *Let $\varphi_1, \varphi_2, \varphi_3, \varphi_4 \in \mathcal{S}(\mathbb{R}^d)$ and let Condition* (46) *be satisfied for each pair of windows φ_1, φ_2 and φ_3, φ_4. If we consider a symbol $a \in S^m$, then there exists a symbol $a' \in S^m$ such that*

$$A_a^{\varphi_1,\varphi_2} = A_{a'}^{\varphi_3,\varphi_4} + L_\sigma, \quad \textit{with } \sigma \in \mathcal{S}(\mathbb{R}^{2d}),$$

and

$$a' \sim \sum_{\alpha,\beta,\gamma,\delta} \tilde{c}_{\gamma\delta} c_{\alpha\beta} \partial_x^\gamma \partial_\omega^\delta \partial_x^\alpha \partial_\omega^\beta a, \tag{51}$$

where $c_{\alpha\beta}$ are given by (47) *and $\tilde{c}_{\gamma\delta}$ are as in* (49) *corresponding to the windows φ_3, φ_4.*

Proof. By Theorem 6.1 the localization operator can be written in term of Weyl operator as follows

$$A_a^{\varphi_1,\varphi_2} = L_\sigma + L_r, \quad \text{with } r \in \mathcal{S}(\mathbb{R}^{2d})$$

where $\sigma \in S^m$ with asymptotic expansion given by (47) and (48). Now, we apply Theorem 6.3 for the Weyl operator L_σ with respect to the windows φ_3, φ_4 and we get

$$L_\sigma = A_{a'}^{\varphi_3,\varphi_4} + L_{r'}, \text{with } r' \in \mathcal{S}(\mathbb{R}^{2d}),$$

where a' is recovered from the expansion of σ given by (49) and (50). □

Applying Corollary 6.4 to each term of the sum in the right hand side of (10), we may then rewrite the expression in terms of the same pair of windows φ_1, φ_2. In practice, for generic φ_1, φ_2, formula (10) seems preferable for applications, see below. Finally, we construct parametrices for elliptic Wick operators. Namely, assume that $p \in S^m$ is elliptic, i.e. for large $Z \in \mathbb{R}^{2d}$ we have

$$\Lambda(Z)^m \lesssim p(Z). \tag{52}$$

Note that $p^{-1}(Z)$ is then defined as element of S^{-m}, by cutting off for small Z. Fix $\varphi_j \in \mathcal{S}(\mathbb{R}^d)$, $j = 1, \dots, 4$, and let condition (46) be satisfied for each pair of windows φ_1, φ_2 and φ_3, φ_4. We furthermore add the normalization condition $\langle \varphi_3, \varphi_4 \rangle \cdot \langle \varphi_2, \varphi_1 \rangle = 1$. Consider then $A = A_p^{\varphi_3,\varphi_4}$; given $N \geq 1$, we want to construct a parametrix B, i.e.

$$BA = I + L_r, \tag{53}$$

where L_r is a Weyl operator with Weyl symbol $r \in S^{-N}$. We shall look for B given by a finite sum of operators of the form A_c^{Ψ,φ_2}, where Ψ consist of different linear combinations of windows of the type $t^\gamma \partial^\delta \varphi_1$ and the symbols c belong to S^{-m}. To this end, we begin to consider $A_{p^{-1}}^{\varphi_1,\varphi_2}$, so that applying Theorem 1.1,

$$A_{p^{-1}}^{\varphi_1,\varphi_2} A_p^{\varphi_3,\varphi_4} = I + \sum_{|\alpha|=1}^{N-1} \frac{(-1)^\alpha}{\alpha!} A_{p^{-1}\partial^\alpha p}^{\Phi_\alpha,\varphi_2} + L_s,$$

with $s \in S^{-N}$, and Φ_α as in (9). We may then set

$$B := \sum_{j=0}^{N-1} (-1)^j \left(\sum_{|\alpha|=1}^{N-1} \frac{(-1)^\alpha}{\alpha!} A_{p^{-1}\partial^\alpha p}^{\Phi_\alpha,\varphi_2} \right)^j A_{p^{-1}}^{\varphi_1,\varphi_2}.$$

Applying repeatedly Theorem 1.1 to the right-hand side we recognize that B has the required form; the remainder L_r in (53) is easily written in the Weyl form with $r \in S^{-N}$ by using Theorem 6.1 and the classical Weyl calculus.

Such an expression of the parametrix can be easily applied to the study of the regularity of the solutions in the frame of modulation spaces [9, 12], by using the boundedness properties of the operators $A_a^{\varphi_1,\varphi_2}$ contained in [6].

Let us finally point out that, concerning positivity properties of the classical Wick operators, they do not hold true in our context, if we consider two different windows φ_1, φ_2. In fact, if $\varphi_1 = \varphi_2 = \varphi$ and $a \geq 0$, we have

$$\langle A_a^{\varphi,\varphi} u, u \rangle = \langle a, \overline{V_\varphi u} V_\varphi u \rangle = \langle a, |V_\varphi u|^2 \rangle \geq 0,$$

and the positivity property is maintained. Now, take e.g. $\varphi_2 = i\varphi_1$, then

$$\langle A_a^{\varphi_1,\varphi_2} u, u \rangle = -i \langle a, |V_{\varphi_1} u|^2 \rangle.$$

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