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ON p-RADICAL GROUPS G AND THE NILPOTENCY INDICES OF J (kG) II

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1. Introduction

Let G be a finite p-solvable group with a Sylow p-subgroup of order p^a, k an algebraically closed field of characteristic $p>0$, kG the group algebra of G over k, $t(G)$ the nilpotency index of the radical $J(kG)$ of kG and $l(M)$ the Loewy length of (right) kG-modules M.

S. Koshitani [4] and D.A.R. Wallace [14] proved that $a(p-1)+1 \leq t(G) \leq p^a$. Y. Tsushima [12] proved that the second equality $t(G)=p^a$ holds if and only if a Sylow p-subgroup of G is cyclic. Here we shall study the structure of G with $t(G)=a(p-1)+1$. If G has p-length 1, then by K. Motose and Y. Ninomiya [9] $t(G)=a(p-1)+1$ if and only if a Sylow p-subgroup of G is elementary abelian. Therefore we shall be interested in the structure of G of p-length 2 with $t(G)=a(p-1)+1$. As such examples, we know the followings.

Let $F=GF(p^n)$ be a finite field of p^n elements for some integer n with $p \mid n$, λ a generater of the multiplicative group F^* of F, $v=\lambda^{p^{n/p}-1}$ and V be the additive group of F. Let $T(p^n)$ be the set of semilinear transformations on V of the form $v \mapsto \alpha v^{\sigma}$, $\alpha \in F^*$, σ a field automorphism of F (see [11, p.229]). Then we can consider semidirect product $V \rtimes T(p^n)$ of V by $T(p^n)$. Let $T_0=\{v \rightarrow \alpha v^{\sigma} \mid \alpha \in \langle v \rangle, \sigma \in \mathrm{Gal}(F/GF(p^{n/p}))\} \subseteq T(p^n)$. Then we define $A_{p,n,p}=V \rtimes T_0 \subseteq V \rtimes T(p^n)$ (see [3]). A Sylow p-subgroup of $A_{p,n,p}$ is of order p^{n+1}. In [7] K. Motose proved $t(A_{p,n,p})=(n+1)(p-1)+1$.

Now, following K. Motose and Y. Ninomiya [8] we call G p-radical if $J(kG) \subseteq (kP)kG$, where P is a Sylow p-subgroup of G. Then $A_{p,n,p}$ is p-radical (see [13]). So we consider the structure of p-radical group G with $t(G)=a(p-1)+1$. In [3] we proved that such groups G satisfy $G=0_{p,p',p,p'}(G)$. In this paper, we shall prove the following result.

Theorem. *For a p-radical group G the following conditions are equivalent.*
(1) $t(G)=a(p-1)+1$.
(2) $l(P_G(k))=a(p-1)+1$, *where* $P_G(k)$ *is the projective cover of the* 1-*dimentional trivial* kG-*module* k_G.
(3) *The following conditions hold.*

(i) $O^{p'}(G)=N\rtimes H$ *for some elementary abelian p-group N and a (p-radical) group H, where* $O^{p'}(G)$ *is the minimal normal subgroup of G of index prime to p.*

(ii) $H=M\rtimes P$, *where M is a p'-group and P is an elementary abelian p-group.*

(iii) $P\subseteq {}_MC_H(x)$ *for all* $x\in N$.

(4) *The following conditions hold.*

(i) $O^{p'}(G/O_{p'}(G))=G_0\times N_0$, *where* N_0 *is an elementary abelian p-group and* G_0 *is isomorphic to a normal subgroup of* $\Pi_i A_{p,n_i,p}$ *with p-power index for some* n_i*'s.*

(ii) $O_{p',p}(G_1)=O_{p'}(G)\times O_p(G_1)$, *where* G_1 *is an inverse image of* G_0 *in G.*

2. Preliminaries

In this section we shall prepare some lemmas and propositions which will be used to show the theorem.

Throughout this section, except in the last four lemmas we shall treat a group G of the form $G=N\rtimes H$, a semidirect product of a p-group N by a group H. Under the conjugation action, kN can be viewed as a kH-module.

Clearly $J(kN)^l$ is a kH-submodule of kN for all integer $l\geq 0$. We put

$$grkN:=\bigotimes_{l\geq 0} J(kN)^l/J(kN)^{l+1},$$

where $J(kN)^0$ means kN.

Lemma 1.1. *Assume* $N=N_1\times N_2$. *Then the map* $J(kN_1)\otimes J(kN_2)\mapsto J(kN)$ $(a\otimes b\mapsto ab)$ *induces a k-isomorphism*

$$\bigoplus_{l_1+l_2=l} J(kN_1)^{l_1}/J(kN_1)^{l_1+1}\otimes_k J(kN_2)^{l_2}/J(kN_2)^{l_2+1}\simeq J(kN)^l/J(kN)^{l+1}.$$

This isomorphism is a kH-isomorphism if each N_i *is H-invariant.*

Proof. For simplicity, put $J:=J(kN)$, $J_1:=J(kN_1)$ and $J_2:=J(kN_2)$. Notice that the canonical map $J_1\otimes_k J_2\rightarrow J$ $(a\otimes b\mapsto ab)$ induces an epimorphism

$$\bigoplus_{l_1+l_2=l} J_1^{l_1}/J_1^{l_1+1}\otimes_k J_2^{l_2}/J_2^{l_2+1}\rightarrow J^l/J^{l+1}.$$

Since $\sum_{l\geq 0}\sum_{l_1+l_2=l}\dim_k(J_1^{l_1}/J_1^{l_1+1}\otimes_k J_2^{l_2}/J_2^{l_2+1})=\dim_k kN=\sum_{l\geq 0}\dim_k J^l/J^{l+1}$, the above map must be an isomorphism. From the construction, the last statement is clear.

Lemma 1.2. *Let* $H=\langle s\rangle$ *be a cyclic group of order p and* $N=\langle x_1\rangle\times\cdots\times\langle x_p\rangle$ *be an elementary aberian p-group of rank p. Assume that the action of H on N is given by* $x_i^s=x_{i+1}$ $(1\leq i\leq p-1)$ *and* $x_p^s=x_1$. *Then,*

$$J(kN)^l / J(kN)^{l+1} \simeq \begin{cases} a\ projective\ kH\text{-}module & if \quad l \not\equiv 0 \pmod p \\ K_H \oplus (a\ projective\ kH\text{-}module) & if \quad l \equiv 0 \pmod p \end{cases}$$

for $0 \le l \le p(p-1)$.

In particular, $grkN \simeq pk_H \oplus$ *(a projective kH-module)*

Proof. For simplicity, put $J := J(kN)$. By the assumption on N and Lemma 1.1, we have a k-isomorphism $J^l / J^{l+1} \simeq \bigoplus_{l_1 + \cdots + l_p = l} k(x_1 - 1)^{l_1} \otimes \cdots \otimes (x_p - 1)^{l_p}$. We shall examine the action of H on J^l / J^{l+1}.

Let $\alpha_{(l_1, \cdots, l_p)}$ be element in J^l / J^{l+1} which corresponds to $(x_1 - 1)^{l_1} \otimes \cdots \otimes (x_p - 1)^{l_p}$. Then $\{\alpha_{(l_1, \cdots, l_p)}; l_1 + \cdots + l_p = l\}$ forms a k-basis of J^l / J^{l+1}. $(\alpha_{(l_1, \cdots, l_p)})^s = \alpha_{(l_p, l_1, \cdots, l_{p-1})}$, so s permutes the above k-basis and we see that $\alpha_{(l_1, \cdots, l_p)}$ is stabilized by s if and only if $l_1 = l_2 = \cdots = l_p (= l/p)$. Thus the assertion follows.

Lemma 1.3. *Let N be an elementary abelian p-group and H be an abelian p'-group with $C_H(N) = 1$. Then there exists an element $x \in N$ such that $C_H(x) = 1$.*

Proof. If H acts on N irreducibly, then H is cyclic and $C_H(x) = 1$ for all $1 \ne x \in N$.

Since H is a p'-group, we have a decomposition where $N = N_1 \times N_2$, where N_1 is H-irreducible and N_2 is H-invariant. Assume $N_2 \ne 1$ and we shall use induction on $|N|$.

Since $H / C_H(N_1)$ acts on N_1 irreducibly and faithfuly, there exists $x_1 \in N_1$ such that $C_H(x_1) = C_H(N_1)$. Since the action of $C_H(N_1)$ on N_2 is faithful, there exists $x_2 \in N_2$ such that $C_H(x_2) \cap C_H(N_1) = 1$ by induction. For $x = x_1 x_2 \in N$, $C_H(x) = 1$ as desired.

Proposition 1.4. *Let $H = M \rtimes \langle s \rangle$ be a Frobenius group with p'-group M and $|s| = p$. Assume N is an elementary abelian p-group. If grkN is a semisimple kH-module, then $s \in {}_M C_H(x)$ for all $x \in N$.*

Proof. Let n be the number of M-conjugacy classes on N. We shall show $|C_N(s)| = n$. First we shall prove that $grkN \simeq n(k_H) \oplus$ (a projective kH-module). Since kM is semisimple, we have $\mathrm{Inv}_M(grkN) \simeq \mathrm{Inv}_M(kN)$ and $\dim_k \mathrm{Inv}_M(kN) = n$. As $M \lhd H$, $\mathrm{Inv}_M(grkN)$ is a kH-submodule of $grkN$ and is semisimple. Thus, $\mathrm{Inv}_M(grkN) = \mathrm{Inv}_H(grkN)$. H is Frobenius and p-nilpotent, so its simple modules are trivial or projective ones. Hence we have $grkN \simeq \mathrm{Inv}_H(grkN) \oplus$ (a projective kH-module) $\simeq n(k_H) \oplus$ (a projective kH-module) as desired.

Secondly, we shall give another decomposition of $grkN$. Notice that $N = [N, M] \times C_N(M)$ and put $N_0 := C_N(M)$. By Lemma 1.1, $grkN_0$ is a semisimple kH-module and therefore has a trivial $\langle s \rangle$-action. So, $grkN_0 = |N_0|(k_{\langle s \rangle})$ as

$k\langle s\rangle$-modules. Furthermore, $t(N_0\langle s\rangle)=t(N_0)+t(\langle s\rangle)-1$ by Theorem 2.7 of [6]. Hence $N_0\langle s\rangle$ is elementary abelian, and so $\langle s\rangle$ centralizes N_0.

Assume that $[N,M]=1$. Then $N=N_0$, and so $grkN=|N|(k_{\langle s\rangle})$ as $k_{\langle s\rangle}$-modules. Hence $n=|N|=|C_N(s)|$.

Next assume that $[N,M]\neq 1$. Since H acts on $[N,M]$ by conjugation, we can regard $[N,M]$ as an H-module. Every H-constituent of $[N,M]$ doesn't contain M in its kernel, and so is projective by the assumption on H. Hence $[N,M]$ is a projective H-module. Hence we have an $\langle s\rangle$-invariant decomposition $[N,M]=N^{(1)}\times\cdots\times N^{(m)}$, where each $N^{(i)}$ has the same expression as that of "N in Lemma 1.2". So, by Lemma 1.1 and Lemma 1.2, $grk[N,M]\simeq p^m(k_{\langle s\rangle})\oplus$ (a projective $k\langle s\rangle$-module) as $k\langle s\rangle$-modules. Therefore, by Lemma 1.1, we have $grkN\simeq p^m|N_0|(k_{\langle s\rangle})\oplus$ (a projective $k\langle s\rangle$-module) as $k\langle s\rangle$-modules.

The above two decompositions of $grkN$ imply $n=p^m|N_0|$. Notice that $p^m|N_0|=|C_N(s)|$ since $|C_{N^{(i)}}(s)|=p$ for all i. Consequently we have $|C_N(s)|=n$.

Now, two distinct elements of $C_N(s)$ are not M-conjugate each other: otherwise they are $C_M(s)$-conjugate but this does not happen since $C_M(s)=1$. Therefore $C_N(s)$ is a set of representive of M-conjugacy classes of N as $|C_N(s)|=n$ and we get the result.

Proposition 1.5. *Let H, M and s be as in Proposition 1.4. If N is elementary abelian and $s\in {}_MC_H(x)$ for all $x\in N$, then $M/C_M(N)$ is abelian.*

Proof. We imitate the proof of Theorem 3.3 in [15] and shall use induction on $|H|$. We may assume that $C_M(N)=1$. And we may also assume that p is odd as H is a Frobenius group. Notice that M is nilpotent.

Step 1. *N and N_M are irreducible.*

Proof. Let N_0 be an irreducible M-submodule of N and $N_0=\langle x^M\rangle$. Then by our assumption $N_0=\langle x^H\rangle$ and is H-invariant. This implies that N is completely reducible as an H-module and the result follows by induction.

Step 2. *N_K is homogeneous for all $K\lhd H$.*

Proof. Suppose not and choose $K\lhd H$ maximal such that N_K is not homogeneous. Let $N_K=N_1\times\cdots\times N_l$, where N_i's are the homogeneous components of N_K. By Step 1 $K\subsetneq M$ as H is Frobenius. Let $L\lhd H$ with $1\neq L/K\subseteq Z(M/K)$. By the maximality of K, N_L is homogeneous and therefore L acts on N_i transitively. In particular, $H=N_H(N_1)L$. Thus $N_M(N_1)\lhd H$ as $K\subseteq N_M(N_1)\lhd N_H(N_1)$ and $L/K\subseteq Z(M/K)$. N is not a homogeneous $N_M(N_1)$-module since $N_M(N_1)$ can not act on $\{N_i\}$ transitively and $N_M(N_1)=K$ by the maximality of K. So $L=M$ and $N_M(N_i)=K$ for all i. We may assume that $s\in N_H(N_1)$. We claim that $N_H(N_1)\cap N_H(N_2)=K$. Suppose $N_H(N_1)=N_H(N_2)$. Then $s\in N_H(N_2)$. On the other hand, $N_2 = N^t$ for some t in $M \backslash K$ and therfore $s, s^{t^{-1}} \in N_H(N_1)$. So $[s,t] \in N_H(N_1) \cap M = K$ and this contradicts to the hypothesis that H is Frobenius. Now take $1 \neq x \in N_1$ and $1 \neq y \in N_2$. For $h \in C_H(xy)$, $h^2 \in C_H(x) \cap C_H(y)$ as h permutes $\{N_i\}$ and $x^h = x$, $y^h = y$ or $x^h = y$, $y^h = x$. Thus $h^2 \in N_H(N_1) \cap N_H(N_2) = K \subseteq M$. Because we are assuming that p is odd, $h \in M$ and $C_H(xy) \subseteq M$. This contradicts to our assumption and Step 2 follows.

Step 3. *If $A \lhd H$ is abelian, then $A \subseteq Z(M)$. Furthermore, $Z(M)$ is cyclic.*

Proof. By Step 2 A is cyclic. Thus $\operatorname{Aut} A$ is abelian and $M \subseteq C_H(A)$ as H is a Frobenius group with kernel M.

Step 4. (*Conclusion*) $M = Z(M)$.

Proof. Notice that the prime factors of $|M|$ and $|Z(M)|$ coincide as M is nilpotent. Suppose $M \neq Z(M)$ and let $A/Z(M)$ be a chief factor of H in $Z(M/Z(M))$. Then $A/Z(M)$ is an elementary abelian q-group for some prime q. $p \mid (q-1)$ since $\langle s \rangle$ acts on $Z(M)$ regularly and $q \mid |Z(M)|$. Hence $GF(q)$ is a splitting field for $\langle s \rangle$. Thus an $\langle s \rangle$-invariant minimal subgroup of $A/Z(M)$ is of order q and it is M-invariant as $A/Z(M) \subseteq Z(M/Z(M))$. Thus $|A/Z(M)| = q$ and A is abelian. By Step 3 this is a contradiction.

We close this section with the following four lemmas.

Lemma 1.6. *Let P be an abelian group and P act on a group M with $(|P|,|M|) = 1$. If $[M,x]$ is abelian for all $x \in P$, then $[M,P]$ is abelian.*

Proof. It suffices to show that $M_0 := [M,x][M,y]$ is abelian for all $x,y \in P$. Notice that $M_0 = [M, \langle x \rangle \langle y \rangle]$ as P is abelian. $[M,y]$ is an $\langle x \rangle$-invariant normal subgroup of M_0. Let $\bar{M}_0 = M/[M,y]$, then $\bar{M}_0 = \overline{[M,x]} = \overline{[M_0,x]}$. Since $\bar{M}_0$ is abelian, $C_{\bar{M}_0}(x) = 1$. This implies $C_{M_0}(x) \subseteq [M,y]$. Similarly, we can show $C_{M_0}(x) \subseteq [M,xy]$. Hence, $C_{M_0}(x) \subseteq [M,y] \cap [M,xy]$. Since $M_0 = [M,y][M,xy]$ and $[M,y],[M,xy]$ are abelian, $C_{M_0}(x) \subseteq Z(M_0)$. Therefore, $M_0 = C_{M_0}(x)[M_0,x]$ is abelian.

Lemma 1.7. *Let M be a p'-group, P a p-group and $H = M \rtimes P$ a semidirect product of M by P. If H is p-radical, then $J(kH) = \Sigma_{Q \subseteq P} J(kQ)\widehat{[M,Q]}kH$, where $\widehat{[M,Q]}$ is the sum of all elements of $[M,Q]$ in kH.*

Moreover, if P is abelian, then $J(kH)^n = \Sigma_{Q \subseteq P} J(kQ)^n \widehat{[M,Q]}kH$ for all $n \in N$.

Proof. If P is abelian, then for a subgroup Q of P. $\widehat{[M,Q]} \in Z(kH)$ and $[M,Q]Q \lhd H$. Thus for $Q,R \subseteq P$, $J(kQ)^{\lambda}\widehat{[M,Q]}J(kR)^{\mu}\widehat{[M,R]} \subseteq J(kQR)^{\lambda+\mu}\widehat{[M,QR]}$

for all $\lambda,\mu\in N$. So it suffices to show the first statement.

($\supseteq$) Let Q be any subgroup of P. Since $(1-x)[\widehat{M,Q}]=[\widehat{M,Q}](1-x)$ for all $x\in Q$ and Q is a p-group, $J(kQ)[\widehat{M,Q}]\subseteq J(k[M,Q]Q)$. Now $J(k[M,Q]Q)kH \subseteq J(kH)$ as $[M,Q]Q\lhd\lhd H$. Thus $J(kH)\supseteq\Sigma_{Q\subseteq P}J(kQ)[\widehat{M,Q}]kH$.

($\subseteq$) Let $\Sigma_{i=1}^{l}e_i=1$ be a decomposition of 1 into the orthogonal sum of primitive idempotents of kM, T_i denote the inertial group of e_ikM in H and Q_i be a Sylow p-subgroup of T_i in P. So Q_i is a defect group of the unique block of kT_i which covers e_ikM. Clifford's theorem says that $J(kH)=\Sigma_{i=1}^{l}(e_ikH)J(kH)$ $=\Sigma_{i=1}^{l}e_iJ(kT_i)kH$ as H is p-nilpotent. Now $T_i=M\rtimes Q_i$ is p-radical and p-nilpotent as H is p-radical and p-nilpotent. Thus, $[M,Q_i]\subseteq \mathrm{Ker}(e_ikM)$ by [13, Lemma 7] and $\bar{T}_i:=T_i/[M,Q_i]\simeq Q_i\times C_M(Q_i)$ by [13, Theorem 2]. Then $J(k\bar{T}_i)=J(k\bar{Q}_i)k\bar{T}_i$. Therefore, $e_iJ(kT_i)kH=e_i[\widehat{M,Q_i}]J(k\bar{T}_i)kH=e_i[\widehat{M,Q_i}]J(kQ_i)kH\subseteq[\widehat{M,Q_i}]J(kQ_i)kH$, so $J(kH)\subseteq\Sigma_{Q\subseteq P}J(kQ)[\widehat{M,Q}]kH$.

Let $\mathscr{F}_0$ be the family of all finite group G such that $l(P_G(k))=a(p-1)+1$, where p^a is the order of a Sylow p-subgroup of G.

Lemma 1.8. *Let G be a p-solvable group and $N\lhd G$. If $G\in\mathscr{F}_0$, then G/N, $N\in\mathscr{F}_0$.*

Proof. Let p^a, p^b be the orders of Sylow p-subgroups of G and N, respectively. By [15, Corollary 3.6] and [6, Lemma 1.1], $b(p-1)+1+(a-b)(p-1)+1-1 \le l(P_N(k))+l(P_{G/N}(k))-1 \le l(P_G(k)) = a(p-1)+1$. Hence $l(P_{G/N}(k))=(a-b)(p-1)+1$ and $l(P_N(k))=b(p-1)+1$, and so G/N, $N\in\mathscr{F}_0$.

Lemma 1.9. *Let G be a p-group with $G\in\mathscr{F}_0$. Then G is elementary abelian.*

Proof. Since $P_G(k)=kG$, G is elementary abelian by [9, Theorem 1].

3. Proof of theorem

In this section we shall prove the theorem stated in the introduction. By [15, Corollary 3.6], the condition (1) implies the condition (2) in the theorem. Now we shall prove the condition (2) implies (3).

In the proof of [3, Theorem 3], if we reset $\mathscr{F}_0$ instead of $\mathscr{F}$ and reset Lemma 1.8 (respectively, Lemma 1.8, 1.9) instead of Lemma 2.6, 2.7(respectively, Theorem 3.1 of [14]), then we have the following result.

If G is a p-radical group with $G/O_{p'}(G)\in\mathscr{F}_0$, then $G=O_{p,p',p,p'}(G)$. Therefore, if $l(P_G(k))=a(p-1)+1$ and G is a p-radical group with $O^{p'}(G)=G$, then $G=O_{p,p',p}(G)$. Let M be a Hall p'-subgroup of G and let $H=N_G(M)$. By the Frattini argument, $G=O_p(G)H$. By [13, Theorem 2], $[O_p(G),M]\cap C_{O_p(G)}(M)=1$. Since $C_{O_p(G)}(M)=O_p(G)\cap H$, $G=[O_p(G),M]\rtimes H$. Let P be a Sylow p-subgroup

of H and let $N=[O_p(G),M]$. By Lemma 1.8, 1.9, G has the following form.
(*) $G=N\rtimes H$ and $H=M\rtimes P$, where N and P are elementary abelian p-groups, M is a p'-group and H is p-radical with $[M,P]=M$.

Theorem A. *In the above notations if $l(P_G(k))=a(p-1)+1$, then $P\subseteq {}_M C_H(x)$ for all $x\in N$.*

Proof. $grkN$ is semisimple as a kH-module by [6, Lemma 1.4]. For $1\neq s\in P$, $[M,s]\langle s\rangle$ is a normal subgroup of H and Frobenius by [13, Theorerm 2]. As $[M,s]\langle s\rangle \lhd H$, $grkN$ is also semisimple as a $k([M,s]\langle s\rangle)$-module. Thus $s\in {}_M C_H(x)$ for all $x\in N$ by Proposition 1.4. Hence $s\in C_H(x)M$ and $P\subseteq C_H(x)M$, so $P\subseteq {}_M C_H(x)$.

Next we consider the condition (3) and (4) in the theorem.

Theorem B. *The following conditions are equivalent.*
(1) *G satisfies (*) and $P\subseteq {}_M C_H(x)$ for all $x\in N$.*
(2) *The following conditions hold.*
(i) *G is p-radical with $O^{p'}(G)=G$.*
(ii) *$G/O_{p'}(G)$ is a direct product of an elementary abelian p-group and G_0, which is isomorphic to a normal subgroup of $\tilde{G}$ containing $O^p(\tilde{G})$, where $\tilde{G}\simeq A_{p,n_1,p}\times\cdots\times A_{p,n_m,p}$ for some $n_1,\cdots,n_m$ and $O_{p',p}(G_1)=O_{p'}(G)\times O_p(G_1)$, where G_1 is the inverse image of G_0 in G.*

Proof. First we prove that the condition (1) implies (2). Assume that the condition (1) is sastisfied. Then G is p-radical and $O^{p'}(G)=G$. $C_P(M)$ is a direct factor of G and therefore we may assume $C_P(M)=1$. Then $O_{p'}(G)=C_M(N)$, $O_p(G)=N$. We shall prove that the condition (2)(ii) is satisfied for G in the following steps.

Step 1. *We may assume $C_M(N)=1$.*

Proof. If $O_{p'}(G)=C_M(N)\neq 1$, then by using induction on $|G|$ for $\bar{G}:=G/O_{p'}(G)$, $\bar{G}=G_0\times N_0$, where G_0 is isomorphic to a normal subgroup of $\Pi_i A_{p,n_i,p}$ containing $O^p(\Pi_i A_{p,n_i,p})$ for some n_i's and N_0 is an elementary abelian p-group. Let G_1 be the inverse image of G_0 in G. So, $O_{p',p}(G_1)=O_{p'}(G)P_1$ for some p-group P_1. In the definition of $A_{p,n,p}$, $[V,v]=V$. Hence $[\bar{P}_1,\bar{M}]=\bar{P}_1$, so $P_1\subseteq O_{p'}(G)[P_1,M]$. On the other hand, $[P_1,M]\subseteq MN$, so $P_1\subseteq MN$ and $P_1\subseteq N$. Therefore, $O_{p',p}(G_1)$ $=O_{p'}(G)\times O_p(G_1)$, so we may assume $C_M(N)=1$.

Step 2. *For any $Q\subseteq P$, $N\rtimes([M,Q]\rtimes Q)$ is normal in G and satisfies the condition* (1).

Proof. As H is p-radical and p-nilpotent, $[M,Q]\rtimes Q$ is so and $[[M,Q],Q]=[M,Q]$. For all $x\in N$, $P\subseteq {}_{M}C_H(x)$. So $Q\subseteq {}_{[M,Q]}C_H(x)$ as $M=C_M(Q)[M,Q]$.

Step 3. *M is abelian.*

Proof. Let $1\neq s\in P$ and consider the subgroup $N\rtimes([M,s]\rtimes\langle s\rangle)$. This satisfies the condition (1) by Step 2 and $[M,s]\rtimes\langle s\rangle$ is Frobenius by [13, Theorem 2]. Thus $[M,s]$ is abelian by Proposition 1.5 and the result follows from Lemma 1.6.

Step 4. *The following conditions hold.*
(1) *An M-invariant subgroup of N is H-invariant.*
(2) *Suppose $N=N_1\times N_2$, where N_i are M-invariant. Then $M=C_M(N_1)\times C_M(N_2)$.*

Proof. Let $N_0\subseteq N$ be M-invariant. To show that N_0 is H-invariant we may assume that N_0 is M-irreducible as M is a p'-group. Then $N_0=\langle x^M\rangle$ for some $x\in N_0$. By our assumption $P\subseteq {}_{M}C_H(x)$ and therefore $N_0=\langle x^H\rangle$ is H-invariant. Thus (1) follows. Suppose $N=N_1\times N_2$ for M-invariant subgroups N_i. By Lemma 1.3 $C_M(N_i)=C_M(x_i)$ for some $x_i\in N_i$. Because N_i are M-invariant we may take x_i with $P\subseteq C_H(x_i)$. Let $a\in M$ and apply our assumption on $x_1^a x_2$. There exists an element $b\in M$ such that $P^b\subseteq C_H(x_1^a x_2)=C_H(x_1)^a\cap C_H(x_2)$, here we used the fact that N_i are H-invariant. Thus $P^b=P^c$ for some element $c\in C_M(x_2)=C_M(N_2)$. Therefore $bc^{-1}\in N_M(P)=C_M(P)$. By [13, Theorem 2] $C_M(P)=1$ and it follows that $b=c\in C_M(N_2)$. We also have that $P^b=P^{ac'}$ for some $c'\in C_M(x_1)^a=C_M(N_1)^a=C_M(N_1)$ and $ba^{-1}\in C_M(N_1)$ by the similar argument in the above. Thus $a=(ab^{-1})b\in C_M(N_1)C_M(N_2)$ and the result follows.

Step 5. *We may assume $C_N(M)=1$.*

Proof. For all $x\in C_N(M)$, $P\subseteq C_H(x)$. Thus $C_N(M)\subseteq Z(G)$ and $G=C_N(M)\times([N,M]\rtimes H)$.

Let $N=N_1\times\cdots\times N_m$, where N_i's are M-irreducible and put $M_i:=C_M(\Pi_{j\neq i}N_j)$.

Step 6. *$M=M_1\times\cdots\times M_m$ and $N\rtimes M=(N_1\rtimes M_1)\times\cdots\times(N_m\rtimes M_m)$ and M_i's are P-invariant.*

Proof. Since $M=C_M(N_2)\times M_2$ by Step 4, $C_M(N_1)=M_2\times C_M(N_1\times N_2)$. Hence $M=M_1\times M_2\times C_M(N_1\times N_2)$. Similarly we have $C_M(N_1\times N_2)=M_3\times C_N(N_1\times N_2\times N_3)$, and hence $M=M_1\times M_2\times M_3\times C_M(N_1\times N_2\times N_3)$. By the similar argument, $M=M_1\times\cdots\times M_m$ since $C_M(N)=1$.

Step 7. *$N_i\rtimes M_iP/C_P(N_i)\simeq A_{p,n_i,p}$ for some n_i.*

Proof. N_i is M_i-irreducible and therefore M_i is cyclic as M_i is abelian. Also $[M_i, P]=M_i$ and $P \subseteq {}_{M_i}C_{M_iP}(x)$ for all $x \in N_i$. Thus [11, Proposition 19.8] implies $N_i \rtimes M_iP / C_P(N_i) \simeq A_{p,n_i,p}$ for some n_i. (see [2, Theorem])

Step 8. (*Conclusion.*)

Proof. $N_i \rtimes H / C_H(N_i) \simeq N_i \rtimes M_iP / C_P(N_i)$, $N \rtimes P$ is isomorphic to a subgroup of $\Pi_i N_i \rtimes H / C_H(N_i)$ containing $\Pi_i N_i \rtimes M_i C_H(N_i) / C_H(N_i) = \Pi_i N_i \rtimes M_i$ by Step 7. Now the result follows from Step 7.

The group $A_{p,n,p}$ satisfies (*) and the condition (1) in Theorem B. Thus it follows easily that the condition (2) implies (1) and Theorem B is proved.

By Theorem B the conditions (3) and (4) in our main theorem are equivalent. Theorem C in the following says that the condition (3)(and (4)) implies (1). Let $\mathscr{F}$ be the family of all finite group G such that $t(G)=a(p-1)+1$, where p^a is the order of a Sylow p-subgorup of G.

Theorem C. *If G satisfies the condition* (3) *in the theorem, then $G \in \mathscr{F}$.*

Proof. Put $|N| := p^b$, $|p| := p^c$ and $a=b+c$. It suffices to show $t(G) \leq a(p-1)+1$ by [4] and [14]. Notice that $N, H \in \mathscr{F}$.

Step 1. *We may assume $C_M(N)=1$.*

Proof. Let M_0 denote $C_M(N)$, B be a block of kG and b a block of kM_0 covered by B. Then b has the unique irreducible character, say φ and let Q be a Sylow p-subgroup of $I_H(\varphi)$. Notice that Q is a defect group of the block of $I_H(\varphi)$, which covers b. It suffices to prove that $t(B) \leq a(p-1)+1$ as $t(G) = \max_{B \in Bl_p(G)} t(B)$. Now, $[M,Q] \cap M_0 = [M_0, Q]([M,Q] \cap C_{M_0}(Q)) = [M_0, Q] \subseteq \operatorname{Ker} \varphi$ by [13, Lemma 7 and Theorem 2]. In particular, $[M_0, Q] \lhd H$, and so $[M_0, Q] \lhd G$. Therefore, we may assume $[M_0, Q]=1$ since φ is regarded as an irreducible character of $k(M_0/[M_0,Q])$ and B can be considered as a block of $k(G/[M_0,Q])$. In this case $[[M,Q], M_0]=1$, so $I_H(\varphi) \rhd ([M,Q]M_0 \rtimes Q)$ as $I_H(\varphi) \supseteq [M,Q]Q$. Therefore, $N \rtimes ([M,Q]M_0 \rtimes Q)$ is normal in $I_G(\varphi) = N \rtimes I_H(\varphi)$ and of p'-index. Now let $\tilde{b}$ denote the Fong correspondent of B w.r.t. $(G, I_G(\varphi))$, i.e., $\tilde{b}$ is the unique block of $k(I_G(\varphi))$ such that $\tilde{b}^G = B$. Then it suffices to show $I_G(\varphi) \in \mathscr{F}$ as $t(B) = t(\tilde{b})$ (see [5]). Now $I_G(\varphi) \in \mathscr{F} \Leftrightarrow N \rtimes [M,Q]M_0Q \in \mathscr{F} \Leftrightarrow N \rtimes [M,Q]Q) \in \mathscr{F}$, since $N \rtimes [M,Q]M_0Q = (N \rtimes [M,Q]Q) \times M_0$. $N \rtimes [M,Q]Q$ satisfies the assumption in Theorem C (see Step 2 in the proof of Theorem B) and $C_{[M,Q]}(N)=1$. So we may assume $M_0 = C_M(N) = 1$.

Step 2. *We may assume* $C_N(M)=1$.

Proof. Since $N=[N,M]\times C_N(M)$ the assumption in Theorem C implies that $G=([N,M]\rtimes H)\times C_N(M)$ (see Step 5 in the proof of Theorem B). Then $t(G)=t([N,M]\rtimes H)+t(C_N(M))-1$. Therefore, we may assume $C_N(M)=1$.

Step 3. $J(kH)^\lambda J(kN)^\mu J(kH)^\nu \subseteq \Sigma_{\mu_1+\mu_2=\mu} J(kN)^{\mu_1} J(kH)^{\lambda+\mu} J(kN)^{\mu_2}$.

Proof. We may use the same notations as in Step 6 in the proof of Theorem B. So, by Theorem B for all $Q\subseteq P$ and all i, $[M_i,Q]=\begin{cases} M_i & \text{if } Q\nsubseteq C_P(N_i) \\ 1 & \text{if } Q\subseteq C_P(N_i)\end{cases}$.

Therefore, $[M,Q]=\Pi_{i=1}^m [M_i,Q]=\prod_{Q\nsubseteq C_P(N_i)} M_i$.

For subgroups Q, R of P, set $\pi_Q=\{i\mid Q\nsubseteq C_P(N_i)\}$ and $\pi_R=\{i\mid R\nsubseteq C_P(N_i)\}$. Let $N=N_Q\times N_0\times N_R$ be a decomposition of N, where $N_Q=\prod_{i\notin\pi_Q} N_i$, $N_0=\prod_{i\in\pi_Q\cap\pi_R} N_i$ and $N_R=\prod_{i\in\pi_Q-(\pi_R\cap\pi_Q)} N_i$.

Then $[M,Q]Q$ (resp. $[M,R]R$) acts on N_Q (resp. N_R) trivially. Now put $M_0:=[M,Q]\cap[M,R]$, then $P\subseteq {}_{M_0}C_H(x)$ for all $x\in N_0$. For all $\alpha\in J(kQ)^\lambda$ and $\beta\in J(kR)^\nu$,

$$\begin{aligned}
&\alpha\widehat{[M,Q]}J(kN)^\mu\beta\widehat{[M,R]}\\
&=\alpha\widehat{[M,Q]}(\Sigma_{\mu_1+\mu_2+\mu_3=\mu}J(kN_Q)^{\mu_1}J(kN_0)^{\mu_2}J(kN_R)^{\mu_3})\beta\widehat{[M,R]}\\
&=\Sigma_{\mu_1+\mu_2+\mu_3=\mu}\alpha\widehat{[M,Q]}J(kN_Q)^{\mu_1}J(kN_0)^{\mu_2}J(kN_R)^{\mu_3}\beta\widehat{[M,R]}\\
&=\Sigma_{\mu_1+\mu_2+\mu_3=\mu}J(kN_Q)^{\mu_1}\alpha\widehat{[M,Q]}J(kN_0)^{\mu_2}\beta\widehat{[M,R]}J(kN_R)^{\mu_3}.
\end{aligned}$$

Let $\gamma\in J(kN_0)^{\mu_2}$. Since $M_0=\prod_{i\in\pi_Q\cap\pi_R} M_i$, $M=M_0\times L$, where $L=\prod_{i\notin\pi_Q\cap\pi_R} M_i$. Then L centralizes M_0 and N_0, and so $(\sum_{a\in M_0}\gamma^a)^y=\sum_{a\in M_0}\gamma^a$ for all $y\in L$. Let $\gamma=\sum_{x\in N_0}c_x x$, where $c_x\in k$. Then $\sum_{a\in M_0}\gamma^a=\sum_{a\in M_0}\sum_{x\in N_0}c_x x^a=\sum_{x\in N_0}c_x\sum_{a\in M_0}x^a$. Since $P\subseteq {}_{M_0}C_H(x)$ for all $x\in N_0$, $(\sum_{a\in M_0}x^a)^y=\sum_{a\in M_0}x^a$, and so $(\sum_{a\in M_0}\gamma^a)^y=\sum_{a\in M_0}\gamma^a$ for all $y\in P$. Thus $\sum_{a\in M_0}\gamma^a\in Z(kG)$. Therefore $\hat{M}_0\gamma\hat{M}_0=M_0\sum_{a\in M_0}\gamma^a\in M_0(J(kN_0)^{\mu_2}\cap Z(kG))$ for all $\gamma\in J(kN_0)^{\mu_2}$.

Then,

$$\begin{aligned}
&\alpha\widehat{[M,Q]}J(kN_0)^{\mu_2}\beta\widehat{[M,R]}\\
&=\alpha\widehat{[M,Q]}\hat{M}_0J(kN_0)^{\mu_2}\hat{M}_0\beta\widehat{[M,R]}
\end{aligned}$$

$$\subseteq \alpha[\widehat{M,Q}]\beta[\widehat{M,R}]\hat{M}_0 J(kN_0)^{\mu_2}$$
$$=\alpha[\widehat{M,Q}]\beta[\widehat{M,R}]J(kN_0)^{\mu_2}=\alpha\beta[\widehat{M,Q}][\widehat{M,R}]J(kN_0)^{\mu_2}$$
$$\subseteq J(k(QR))^{\lambda+\mu}[\widehat{M,QR}]J(kN_0)^{\mu_2}$$
$$\subseteq J(kH)^{\lambda+\mu}J(kN_0)^{\mu_2} \text{ by Lemma 1.7.}$$

Thus,

$$\alpha[\widehat{M,Q}]J(kN)^{\mu}\beta[\widehat{M,R}]$$
$$=\Sigma_{\mu_1+\mu_2+\mu_3=\mu}J(kN_Q)^{\mu_1}\alpha[\widehat{M,Q}]J(kN_0)^{\mu_2}\beta[\widehat{M,R}]J(kN_R)^{\mu_3}$$
$$\subseteq \Sigma_{\mu_1+\mu_2+\mu_3=\mu}J(kN_Q)^{\mu_1}J(kH)^{\lambda+\mu}J(kN_0)^{\mu_2}J(kN_R)^{\mu_3}$$
$$\subseteq \Sigma_{\mu_1+(\mu_2+\mu_3)=\mu}J(kN)^{\mu_1}J(kH)^{\lambda+\mu}J(kN)^{\mu_2+\mu_3}.$$

Therefore, by Lemma 1.7,

$$J(kH)^{\lambda}J(kN)^{\mu}J(kH)^{\nu} \subseteq \Sigma_{\mu_1+\mu_2=\mu}J(kN)^{\mu_1}J(kH)^{\lambda+\mu}J(kN)^{\mu_2}.$$

Step 4. (*Conclusion.*) $t(G)\leq a(p-1)+1$.

Proof. Now $J(kG)=J(kH)+J(kN)kH$. So, we can easily show that $J(kG)^n =\Sigma_{\lambda+\mu+\nu=n}J(kN)^{\lambda}J(kH)^{\mu}J(kN)^{\nu}$, using Step 3 and induction on n. Let $\lambda+\mu+\nu =a(p-1)+1$. If $\mu\geq c(p-1)+1$, then $J(kN)^{\lambda}J(kH)^{\mu}J(kN)^{\nu}=0$ as $H\in\mathscr{F}$. If $\mu \leq c(p-1)$, then $\lambda+\mu \geq a(p-1)+1-c(p-1)=b(p-1)+1$. So, $J(kN)^{\lambda}J(kH)^{\mu}J(kN)^{\nu} \subseteq J(kN)^{\lambda+\mu}kG=0$ as $N\lhd G$ and $N\in\mathscr{F}$.

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