| Title | BP_*(BP) and typical formal groups |
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| Author(s) | Landweber, Peter S. |
| Citation | Osaka Journal of Mathematics. 1975, 12(2), p. 357-363 |
| Version Type | VoR |
| URL | https://doi.org/10.18910/5492 |
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Landweber, P.S.
Osaka J. Math.
12 (1975), 357-363

# $BP_*(BP)$ AND TYPICAL FORMAL GROUPS

PETER S. LANDWEBER[1)]

(Received July 22, 1974)

**1. Introduction.** D. Quillen showed in [6] that the formal group law of complex cobordism is a universal formal group, hence for a commutative ring $R$ there is a natural bijection between ring homomorphisms $MU_* \to R$ and formal groups over $R$, where $MU_*$ is the coefficient ring of complex cobordism. Similarly, S. Araki [4] has shown that for a fixed prime $p$, the formal group law of Brown-Peterson cohomology is universal for typical group laws over commutative $Z_{(p)}$-algebras. Thus if $R$ is a commutative $Z_{(p)}$-algebra, there is a natural bijection between ring homomorphisms $BP_* \to R$ and typical formal groups over $R$, where $BP_*$ is the coefficient ring of Brown-Peterson cohomology.

In this note we shall show that $BP_*(BP)$ represents isomorphisms between typical formal groups over $Z_{(p)}$-algebras. This places $BP_*(BP)$ in a purely algebraic setting, as was done for $MU_*(MU)$in the Appendix to [5]. We show how the structure maps for $BP_*(BP)$ arise in this context, and use our point of view to derive the formulas of J.F. Adams [2, Theorem 16.1] for these structure maps.

All this works as well for $MU_*(MU)$,by omitting mention of *typical* formal groups; this gives a description of $MU_*(MU)$which is somewhat different from the one given in [5]. In the $BP$-case it is essential to use coordinates for curves over a typical formal group $\mu$ which depend on $\mu$. But in the $MU$- case, it is optional whether one uses "moving coordinates" (as we do here) or "absolute coordinates" as in [5].

The ideas in this note grew out of musings over D. Ravenel's paper [7] on multiplicative operations in $BP^*(BP)$.

*2.* **Recollections** (Araki [3, §1] and [4]) For the most part we follow Araki's notation. All rings and algebras are to be commutative. By an *isomor-*

*Abstract.* It is shown that $BP_*(BP)$ represents the functor which assigns to a commutative $Z_{(p)}$- algebra $R$ the set of isomorphisms between typical formal groups over $R$. The structure maps of the Hopf algebra $BP_*(BP)$ all arise naturally from this point of view, and one can easily derive the formulas of Adams [2, Theorem 16.1] for them.

1) Research supported in part by an N.S.F. grant.

*phism* $\phi$: $\mu\to\mu'$ between formal groups we mean a homomorphism satisfying $\phi(T)\equiv T \bmod \deg 2$ (Araki calls this a *strict* isomorphism).

For a formal group $\mu$, let $C_\mu$ denote the additive group of *curves* over $\mu$, i.e. power series $\gamma(T)$ with zero constant term and with addition $(\gamma_1+^{\mu}\gamma_2)(T) =\mu(\gamma_1(T),\gamma_2(T))$. The *identity* curve is $\gamma_0(T)=T$. A homomorphism $\varphi$: $\mu\to\mu'$ induces a homomorphism $\phi_\#$: $C_\mu\to C_{\mu'}$, by $\phi_\#(\gamma)=\phi\circ\gamma$. Similarly a ring homomorphism $/$: $R\to R'$ sends $\mu$ to a formal group $f_*\mu$ and induces a homomorphism $f_*$: $C_\mu\to C_{f_*\mu}$ on curves, by applying $f$ to the coefficients of power series over $R$.

Fix a prime $p$, and let $Z_{(p)}$ denote the integers localized at $p$. Let $R$ be a $Z_{(p)}$-algebra and $\mu$ a formal group over $R$. From [4, 2.5] we recall the Frobenius operators $f_n$ on curves; these satisfy $f_n\phi_\#=\phi_\# f_n$ A curve $\gamma$ over $\mu$ is called *typical* if $f_q\gamma=0$ for all primes $q\neq p$. The formal group $\mu$ is called *typical* if the identity curve $\gamma_0$ over $\mu$ is typical. Theorem 3.6 of [4] states that for a typical formal group $\mu$ over a $Z_{(p)}$-algebra $R$, a curve 7 over $\mu$ is typical if and only if it has a series expansion in $C_\mu$ of the form

$$\gamma(t) = \sum_{k=1}^{\infty}{}^{\mu} c_k T^{p^k}$$

with (uniquely determined) coefficients $c_k\in R$.

From Theorems 4.6 and 5.6 of [4], we see that the formal group $\mu_{BP}$ over $BP_*$ of Brown-Peterson cohomology is typical and universal for typical formal groups over $Z_{(p)}$-algebras.

## 2. Isomorphisms between typical formal groups

Let $R$ be a $Z_{(p)}$-algebra and consider triples $(\mu, \varphi, \mu')$ where $\mu$ and $\mu'$ are typical formal groups over $R$ and $\varphi$: $\mu'\to\mu$ is an isomorphism. We write $TI(R)$ for the set of these triples, and $TF(R)$ for the set of typical formal groups over $R$. We know that

$$TF(R)\cong \mathrm{Hom}\,(BP_*, R)$$

on the category of $Z_{(p)}$-algebras, and plan to show that

$$TI(R) \cong \mathrm{Hom}(BP_*(BP), R)$$

on this category.

**Lemma 1.** *Let $R$ be a $Z_{(p)}$-algebra and $\varphi$: $\mu'\to\mu$ cm isomorphism ojformal groups over $R$. Then $\mu!$ is typical if and only if $\varphi$ is a typical curve over $\mu$.*

Proof. $\varphi$ induces an isomorphism $\phi_\#$: $C_{\mu'}\to C_\mu$ commuting with the Frobenius operators, and $\phi_\#(\gamma_0)=\phi$; the result is now immediate. QED

Notice that $\phi$: $\mu\to\mu$ an isomorphism implies that

$$\mu'(X, Y) = \phi^{-1}(\mu(\phi(X), \phi(Y))$$

or $\mu'=\mu^{\phi}$ in the notation of [4, 2.11]. Thus $\mu'$ is determined by $\mu$ and φ, and we may view $TI(R)$as the pairs $(\mu, \phi)$ where $\mu$ is a typical formal group over $R$ and φ is a typical curve over $\mu$ with $\phi(T)= T$ mod deg 2.

From [2, Theorem 16.1], we know that $BP_*(BP)$ is a polynomial algebra

$$BP_*[t_1, t_{2,\dots}] = BP_*\otimes Z_{(p)}[t_1, t_{2,\dots}]\,.$$

We agree to put $t_0=1$.

**Theorem 1.** *There is a natural bijection* $TI(R)\cong\mathrm{Hom}(BP_*(BP),R)$ *on the category of $Z_{(p)}$-algebras.*

Proof. Let $(\mu, \varphi, \mu^{\phi})\in TI(R)$,so $\mu$ is a typical formal group over $R$ and φ is a typical curve over $\mu$ of the form

$$\phi(T) = \sum_{k=0}^{\infty} {}^{\mu}c_k T^{p^k}$$

with $c_k\in R$ and $c_0=1$. To $\mu$ we can associate a homomorphism

$$f: BP_* \to R$$

with$f_*(\mu_{BP})=\mu$. And then to $\mu$ we associate a homomorphism

$$g\colon Z_{(p)}[t_1, t_2, \cdots] \to R$$

with $g(t_k)=c_k$ for all $k$. Together we obtain a homomorphism

$$f\otimes g\colon\ BP_*(BP) = BP_*\otimes Z_{(p)}[t_i] \to R$$

from which we can recover $f$ and $g$ and so also μ and $\varphi$. Since any homomorphism $BP_*\otimes Z_{(p)}[t_i]\to R$ has the form $f\otimes g$, the result is proved. QED

**3. The structure maps.** We shall now account for the structure maps of the Hopf algebra $BP_*(BP)$ [1, Lecture 3] and the formulas given for them by Adams in [2, Theorem 16.1]. We begin by defining natural maps:

$\eta_L$: $TI(R)\to TF(R)$, $(\mu_1, \varphi, \mu_2)\mapsto\mu_1$
$\eta_R$: $TI(R) \to TF(R)$,$(\mu_1, \varphi, \mu_2)\mapsto\mu_2$
$\varepsilon$ : $TF(R)\to TI(R)$, $\mu\mapsto(\mu, \gamma_0, \mu)$
$c$ : $TI(R)\to TI(R)$, $(\mu_1, \phi, \mu_2)\mapsto(\mu_2, \phi^{-1}, \mu_1)$
$\psi$ : $TI^2(R)\to TI(R)$, where $TI^2(R)$is defined by the pull-back diagram

$$\begin{tikzcd} TI^2(R) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & TI(R) \arrow[d, "\eta_L"] \\ TI(R) \arrow[r, "\eta_R"] & TF(R) \end{tikzcd}$$

and the map is

$$((\mu_1,\phi,\ \mu_2),\ (\mu_2,\ \phi',\ \mu_3)) \mapsto (\mu_1,\phi\phi',\ \mu_3)$$

On general grounds, these give rise to ring homomorphisms which we give the same names:

$\eta_L, \eta_R$: $BP_* \to BP_*(BP)$
$\varepsilon$ : $BP_*(BP) \to BP_*$
$c$ : $BP_*(BP) \to BP_*(BP)$
$\psi$: $BP_*(BP) \to BP_*(BP) \otimes_{BP_*} BP_*(BP)$

where the tensor product is formed by viewing the left copy of $BP_*(BP)$ as a $BP_*$-module via $\eta_R$, and the right copy as a $BP_*$-module via $\eta_L$. One sees immediately that

$$\varepsilon\eta_L = 1\ ,$$
$$c\eta_L = \eta_R$$

and $\psi\eta_L$ is $\eta_L$ followed by the inclusion of the left copy of $BP_*(BP)$ into the tensor product.

**Theorem** 2. *These homomorphisms are the structure maps for the* **Hopf** *algebra* $BP_*$ $(BP)$—$\eta_L$*is the* **leftunit**, $\eta_R$ *is the right* **unit**, $\varepsilon$ *is the* **counit**, *c is the* **conjugation** *and* $\psi$ *is the coproduct.*

Let $\log_{BP}(T)=\sum_{k=0}^{\infty} m_{p^k-1}T^{p^k}(m_0=1)$be the logarithm for $BP$ over $BP_*\otimes Q$ [2, §16], so $\log_{BP}$: $\mu_{BP} \to G_a$ is an isomorphism to the additive **group**:

$$\log_{BP}(\mu_{BP}(X,\ Y)) = \log_{BP}(X)+\log_{BP}(Y)\ .$$

**Theorem** 3. *We have*

i) $\eta_R(m_{p^k-1})=\sum_{i+j=k} m_{p^i-1}(t_j)^{p^i}$

ii) $\eta_L$ *is the obvious inclusion of* $BP_*$ *into* $BP_*(BP)=BP_*[t_1, t_2,...]$

iii) $\varepsilon(t_i)=0$ for $i>0$

iv) *c satisfies* $\sum_{h+i+j=k} m_{p^h-1}(t_i)^{p^h}(ct_j)^{p^{h+i}}= m_{p^k-1}$

v) $\psi$ *atisfies* $\sum_{i+j=k} m_{p^i-1}(\psi t_j)^{p^i}= \sum_{h+i+j=k} m_{p^h-1}(t_i)^{p^h}\otimes(t_j)^{p^{h+i}}$.

Theorem 2 follows from Theorem 3 and the identical formulas of Adams [2, Theorem 16.1], in view of the identities preceeding the statement of Theorem 2 which determine the restrictions of *6, c* and $\psi$ to $BP_*$.

Proof of Theorem 3: ii) $\eta_L$ is a homomorphism $BP_*\to BP_*(BP)$ so that if $f\otimes g$: $BP_*(BP)=BP_*\otimes Z_{(p)}[t_i]\to R$represents $(\mu_1, \phi,\ \mu_2)$ then $(f\otimes g)\eta_L BP_*\to R$ represents $\mu_1$. *I.e.* this means that

$$(f\otimes g)\eta_L = f,$$

and is clearly satisfied by the obvious inclusion of $BP_*$ in $BP_*[t_1 t_2,\ldots]$.

i) $\eta_R$ is a homomorphism $BP_*\to BP_*(BP)$ so that if $f\otimes g$: $BP_*(BP)= BP_*\otimes Z_{(p)}[t_i]\to R$ represents $(\mu_1, \phi, \mu_2)$, then $(f\otimes g)\eta_R: BP_*\to R$ represents $\mu_2=\mu_1{}^\phi$.

Take $R=BP_*(BP)$, $\mu_1=\mu_{BP}$ (extended from $BP^*$ to $BP_*[t_i]$) and

$$\phi(T) = \sum_i {}^{\mu_{BP}} t_i T^{p^i}.$$

Then $f\otimes g$ is the identity, so $\eta_R$ represents the formal group $\mu_{BP}{}^\phi$ over $BP_*(BP)$: $\eta_{R*}(\mu_{BP})=\mu_{BP}{}^\phi$ Now over $BP_*\otimes Q$ we have an isomorphism $\log_{BP}$: $\mu_{BP}\to G_a$, hence also an isomorphism $\eta_{R^*}(\log_{BP})$: $\eta_{R^*}(\mu_{BP})\to G_a$. Noting that $\eta_{R^*}(\mu_{BP})=\mu_{BP}{}^\phi=(G_a{}^{\log_{BP}})^\phi$ we conclude that

$$\eta_{R^*}(\log_{BP}) = \log_{BP}\circ\phi.$$

Hence

$$\begin{aligned}\sum_k \eta_R(m_{p^k-1})T^{p^k} &= \log_{BP}(\textstyle\sum_j^{\mu_{BP}} t_j T^{p^j})\\ &= \sum_j \log_{BP}(t_j T^{p^j})\\ &= \sum_{i,j} m_{p^i-1}(t_j)^{p^i} T^{p^{i+j}}\end{aligned}$$

which proves i).

iii) $\varepsilon$ is a homomorphism $BP_*[t_i]\to BP_*$ such that if $f$: $BP_*\to R$ represents $\mu$ then $f\circ\varepsilon$: $BP_*[t_i]\to R$ represents $(\mu, \gamma_0, \mu)$, where $\gamma_0(T)=T$. Hence $f\circ\varepsilon(t_i)=0$ for $i>0$, from which it is immediate that $\varepsilon(t_i)=0$ for $i>0$.

iv) $c$ is a homomorphism $BP_*[t_i]\to BP_*[t_i]$ so that if $f\otimes g$: $BP_*[t_i]\to R$ represents $(\mu_1, \phi, \mu_2)$ then $(f\otimes g)\circ c$ represents $(\mu_2, \phi^{-1}, \mu_1)$.

Take $f\otimes g$ to be the identity, so / represents $\mu_{BP}$ with scalars extended to $BP_*[t_i]$ and $g$ represents

$$\phi(T) = \sum_i {}^{\mu_{BP}} t_i T^{p^i}.$$

Then $c=f'\otimes g$ where $f'$ represents $\mu_{BP}{}^\phi$ (giving $c\eta_L=\eta_R$ as noted above) and $g'$: $Z_{(p)}[t_i]\to BP_*[t_i]$ must be determined on the $t_i$'s. Now $g'$ represents $\phi^{-1}$: $\mu_{BP}\to\mu_{BP}{}^\phi$; thus

$$\phi^{-1}(T) = \sum_i {}^{\mu_{BP}{}^\phi} c(t_i)T^{p^i}$$

as a curve over $\mu_{BP}{}^\phi$. Applying $\phi_\#$: $C_{\mu_{BP}{}^\phi}\to C_{\mu_{BP}}$ and noting that

$$\phi_\#(\phi^{-1}) = \phi\circ\phi^{-1} = \gamma_0,$$

we compute:

$$T = \sum_j{}^{\mu_{BP}} \phi_*(c(t_j))T^{p^j}$$
$$= \sum_{i,j}{}^{\mu_{BP}} t_i(c(t_j))^{p^i} T^{p^{i+j}}.$$

Finally, we apply $\log_{BP}$ and obtain the desired formula.

v) $\psi$ is a homomorphism $BP_*(BP) \to BP_*(BP) \otimes_{BP_*} BP_*(BP)$ such that if

$$f \otimes g \text{ represents } (\mu_1, \phi, \mu_2)$$

and

$$f' \otimes g' \text{ represents } (\mu_2, \phi', \mu_3)$$

then

$$[(f \otimes g) \otimes (f' \otimes g')] \circ \psi = f'' \otimes g''$$

represents $(\mu_1, \phi\phi', \mu_3)$. Note that

$$\mu_3 = \mu_2{}^{\phi'} = (\mu_1{}^{\phi})^{\phi'} = \mu_1{}^{\phi\phi'}.$$

We seek a universal example. Take $R = BP_*(BP) \otimes_{BP_*} BP_*(BP)$, which is a polynomial algebra over $BP_*$ on generators $t_i \otimes 1$ and $1 \otimes t_j$ for $i$ and $j > 0$. Take

$$\mu_1 = \mu^{BP} \text{ (extended to } R)$$
$$\phi = \sum_i{}^{\mu_{BP}} (t_i \otimes 1) T^{p^i}$$
$$\mu_2 = (\mu^{BP})^{\phi}$$
$$\phi' = \sum_j{}^{\mu_2} (1 \otimes t_j) T^{p^j}$$
$$\mu_3 = \mu_2{}^{\phi'} = \mu_{BP}{}^{\phi\phi'}.$$

One verifies easily that $f \otimes g$ is the inclusion of the left copy of $BP_*(BP)$ in $R$, while $f' \otimes g'$ is the inclusion of the right copy. Thus $(f \otimes g) \otimes (f' \otimes g')$ is the identity, and so in this situation $\psi = f'' \otimes g''$. We want to find $\psi(t_j)$. The series $\sum_j{}^{\mu_{BP}} \psi(t_j) T^{p^j}$ must agree with the composition

$$\left(\sum_j{}^{\mu_{BP}} t_i \otimes 1\, T^{p^i}\right) \circ \left(\sum_{i,j}{}^{(\mu_{BP})^{\phi}} 1 \otimes t_j\, T^{p^j}\right),$$

hence

$$\sum_j{}^{\mu_P} \psi(t_j) T^{p^i} = \sum_{i,j}{}^{\mu_{BP}} t_i \otimes (t_j)^{p^i} T^{p^{i+j}}.$$

The formula v) now follows by applying $\log_{BP}$. This completes the verifications. QED

RUTGERS UNIVERSITY

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