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$BP_*(BP)$ AND TYPICAL FORMAL GROUPS

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1. Introduction. D. Quillen showed in [6] that the formal group law of complex cobordism is a universal formal group, hence for a commutative ring R there is a natural bijection between ring homomorphisms $MU_* \rightarrow R$ and formal groups over R , where MU_* is the coefficient ring of complex cobordism. Similarly, S. Araki [4] has shown that for a fixed prime p , the formal group law of Brown-Peterson cohomology is universal for typical group laws over commutative $Z_{(p)}$ -algebras. Thus if R is a commutative $Z_{(p)}$ -algebra, there is a natural bijection between ring homomorphisms $BP_* \rightarrow R$ and typical formal groups over R , where BP_* is the coefficient ring of Brown-Peterson cohomology.

In this note we shall show that $BP_*(BP)$ represents isomorphisms between typical formal groups over $Z_{(p)}$ -algebras. This places $BP_*(BP)$ in a purely algebraic setting, as was done for $MU_*(MU)$ in the Appendix to [5]. We show how the structure maps for $BP_*(BP)$ arise in this context, and use our point of view to derive the formulas of J.F. Adams [2, Theorem 16.1] for these structure maps.

All this works as well for $MU_*(MU)$, by omitting mention of *typical* formal groups; this gives a description of $MU_*(MU)$ which is somewhat different from the one given in [5]. In the BP -case it is essential to use coordinates for curves over a typical formal group μ which depend on μ . But in the MU -case, it is optional whether one uses "moving coordinates" (as we do here) or "absolute coordinates" as in [5].

The ideas in this note grew out of musings over D. Ravenel's paper [7] on multiplicative operations in $BP_*(BP)$.

2. Recollections (Araki [3, §1] and [4]) For the most part we follow Araki's notation. All rings and algebras are to be commutative. By an *isomor-*

Abstract. It is shown that $BP_*(BP)$ represents the functor which assigns to a commutative $Z_{(p)}$ -algebra R the set of isomorphisms between typical formal groups over R . The structure maps of the Hopf algebra $BP_*(BP)$ all arise naturally from this point of view, and one can easily derive the formulas of Adams [2, Theorem 16.1] for them.

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phism $\phi: \mu \rightarrow \mu'$ between formal groups we mean a homomorphism satisfying $\phi(T) \equiv T \pmod{\text{deg } 2}$ (Araki calls this a *strict* isomorphism).

For a formal group μ , let C_μ denote the additive group of curves over μ , i.e. power series $\gamma(T)$ with zero constant term and with addition $(\gamma_1 + {}^\mu\gamma_2)(T) = {}^\mu(\gamma_1(T), \gamma_2(T))$. The *identity* curve is $\gamma_0(T) = T$. A homomorphism $\phi: \mu \rightarrow \mu'$ induces a homomorphism $\phi_\# : C_\mu \rightarrow C_{\mu'}$, by $\phi_\#(\gamma) = \phi \circ \gamma$. Similarly a ring homomorphism $f: R \rightarrow R'$ sends μ to a formal group $f_*\mu$ and induces a homomorphism $f_* : C_\mu \rightarrow C_{f_*\mu}$ on curves, by applying f to the coefficients of power series over R .

Fix a prime p , and let $Z_{(p)}$ denote the integers localized at p . Let R be a $Z_{(p)}$ -algebra and μ a formal group over R . From [4, 2.5] we recall the Frobenius operators f_n on curves; these satisfy $f_n \phi_\# = \phi_\# f_n$. A curve γ over μ is called *typical* if $f_q \gamma = 0$ for all primes $q \neq p$. The formal group μ is called *typical* if the identity curve γ_0 over μ is typical. Theorem 3.6 of [4] states that for a typical formal group μ over a $Z_{(p)}$ -algebra R , a curve γ over μ is typical if and only if it has a series expansion in C_μ of the form

$$\gamma(t) = \sum_{k=1}^{\infty} {}^\mu c_k T^{p^k}$$

with (uniquely determined) coefficients $c_k \in R$.

From Theorems 4.6 and 5.6 of [4], we see that the formal group μ_{BP} over BP_* of Brown-Peterson cohomology is typical and universal for typical formal groups over $Z_{(p)}$ -algebras.

2. Isomorphisms between typical formal groups

Let R be a $Z_{(p)}$ -algebra and consider triples (μ, ϕ, μ') where μ and μ' are typical formal groups over R and $\phi: \mu' \rightarrow \mu$ is an isomorphism. We write $TI(R)$ for the set of these triples, and $TF(R)$ for the set of typical formal groups over R . We know that

$$TF(R) \cong \text{Hom}(BP_*, R)$$

on the category of $Z_{(p)}$ -algebras, and plan to show that

$$TI(R) \cong \text{Hom}(BP_*(BP), R)$$

on this category.

Lemma 1. *Let R be a $Z_{(p)}$ -algebra and $\phi: \mu' \rightarrow \mu$ an isomorphism of formal groups over R . Then μ' is typical if and only if ϕ is a typical curve over μ .*

Proof. ϕ induces an isomorphism $\phi_\# : C_{\mu'} \rightarrow C_\mu$ commuting with the Frobenius operators, and $\phi_\#(\gamma_0) = \phi$; the result is now immediate. QED

Notice that $\phi: \mu' \rightarrow \mu$ an isomorphism implies that

$$\mu'(X, Y) = \phi^{-1}(\mu(\phi(X), \phi(Y)))$$

or $\mu' = \mu^\phi$ in the notation of [4, 2.11]. Thus μ' is determined by μ and ϕ , and we may view $TI(R)$ as the pairs (μ, ϕ) where μ is a typical formal group over R and ϕ is a typical curve over μ with $\phi(T) = T \bmod \text{deg } 2$.

From [2, Theorem 16.1], we know that $BP_*(BP)$ is a polynomial algebra

$$BP_*[t_1, t_2, \dots] = BP_* \otimes Z_{(p)}[t_1, t_2, \dots].$$

We agree to put $t_0 = 1$.

Theorem 1. *There is a natural bijection $TI(R) \cong \text{Hom}(BP_*(BP)\mathbb{R})$ on the category of $Z_{(p)}$ -algebras.*

Proof. Let $(\mu, \phi, \mu^\phi) \in TI(R)$, so μ is a typical formal group over R and ϕ is a typical curve over μ of the form

$$\phi(T) = \sum_{k=0}^{\infty} c_k T^{p^k}$$

with $c_k \in R$ and $c_0 = 1$. To μ we can associate a homomorphism

$$/: BP_* \rightarrow R$$

with $f_*(\mu_{BP}) = \mu$. And then to μ we associate a homomorphism

$$g: Z_{(p)}[t_1, t_2, \dots] \rightarrow R$$

with $g(t_k) = c_k$ for all k . Together we obtain a homomorphism

$$f \otimes g: BP_*(BP) = BP_* \otimes Z_{(p)}[t_i] \rightarrow R$$

from which we can recover f and g and so also μ and ϕ . Since any homomorphism $BP_* \otimes Z_{(p)}[t_i] \rightarrow R$ has the form $f \otimes g$, the result is proved. QED

3. The structure maps. We shall now account for the structure maps of the Hopf algebra $BP_*(BP)$ [1, Lecture 3] and the formulas given for them by Adams in [2, Theorem 16.1]. We begin by defining natural maps:

- $\eta_L: TI(R) \rightarrow TF(R), (\mu_1, \phi, \mu_2) \mapsto \mu_1$
- $\eta_R: TI(R) \rightarrow TF(R), (\mu_1, \phi, \mu_2) \mapsto \mu_2$
- $\varepsilon: TF(R) \rightarrow TI(R), \mu \mapsto (\mu, \gamma_0, \mu)$
- $C: TI(R) \rightarrow TI(R), (\mu_1, \phi, \mu_2) \mapsto (\mu_2, \phi^{-1}, \mu_1)$
- $\psi: TI^2(R) \rightarrow TI(R)$, where $TI^2(R)$ is defined by the pull-back diagram

$$\begin{array}{ccc} TI^2(R) & \xrightarrow{\pi_2} & TI(R) \\ \downarrow \pi_1 & & \downarrow \eta_L \\ TI(R) & \xrightarrow{\eta_R} & TF(R) \end{array}$$

and the map is

$$((\mu_1, \phi, \mu_2), (\mu_2, \phi', \mu_3)) \mapsto (\mu_1, \phi\phi', \mu_3)$$

On general grounds, these give rise to ring homomorphisms which we give the same names:

$$\begin{aligned} \eta_L, \eta_R &: BP_* \rightarrow BP_*(BP) \\ \varepsilon &: BP_*(BP) \rightarrow BP_* \\ c &: BP_*(BP) \rightarrow BP_*(BP) \\ \psi &: BP_*(BP) \rightarrow BP_*(BP) \otimes_{BP_*} BP_*(BP) \end{aligned}$$

where the tensor product is formed by viewing the left copy of $BP_*(BP)$ as a BP_* -module via η_R , and the right copy as a BP_* -module via η_L . One sees immediately that

$$\varepsilon\eta_L = 1,$$

$$c\eta_L = \eta_R$$

and $\psi\eta_L$ is η_L followed by the inclusion of the left copy of $BP_*(BP)$ into the tensor product.

Theorem 2. *These homomorphisms are the structure maps for the Hopf algebra $BP_*(BP)$ — η_L is the left unit, η_R is the right unit, ε is the counit, c is the conjugation and ψ is the coproduct.*

Let $\log_{BP}(T) = \sum_{k=0}^{\infty} m_{p^k-1} T^{p^k}$ ($m_0=1$) be the logarithm for BP over $BP_* \otimes Q$ [2, §16], so $\log_{BP}: \mu_{BP} \rightarrow G_a$ is an isomorphism to the additive group:

$$\log_{BP}(\mu_{BP}(X, Y)) = \log_{BP}(X) + \log_{BP}(Y).$$

Theorem 3. *We have*

- i) $\eta_R(m_{p^k-1}) = \sum_{i+j=k} m_{p^i-1} (t_j)^{p^i}$
- ii) η_L is the obvious inclusion of BP_* into $BP_*(BP) = BP_*[t_1, t_2, \dots]$
- iii) $\varepsilon(t_i) = 0$ for $i > 0$
- iv) c satisfies $\sum_{h+i+j=k} m_{p^h-1} (t_i)^{p^h} (ct_j)^{p^{h+i}} = m_{p^k-1}$
- v) ψ satisfies $\sum_{i+j=k} m_{p^i-1} (\psi t_j)^{p^i} = \sum_{h+i+j=k} m_{p^h-1} (t_i)^{p^h} \otimes (t_j)^{p^{h+i}}$.

Theorem 2 follows from Theorem 3 and the identical formulas of Adams [2, Theorem 16.1], in view of the identities preceding the statement of Theorem 2 which determine the restrictions of ε , c and ψ to BP_* .

Proof of Theorem 3: ii) η_L is a homomorphism $BP_* \rightarrow BP_*(BP)$ so that $f \otimes g: BP_*(BP) = BP_* \otimes Z_{(p)}[t_i] \rightarrow R$ represents (μ_1, ϕ, μ_2) then $(f \otimes g)\eta_L: BP_* \rightarrow R$ represents μ_1 . I.e. this means that

$$(f \otimes g)\eta_L = f,$$

and is clearly satisfied by the obvious inclusion of BP_* in $BP_*[t_1, t_2, \dots]$.

i) η_R is a homomorphism $BP_* \rightarrow BP_*(BP)$ so that if $f \otimes g: BP_*(BP) = BP_* \otimes_{Z_{(p)}} [t_i] \rightarrow R$ represents (μ_1, ϕ, μ_2) , then $(f \otimes g)\eta_R: BP_* \rightarrow R$ represents $\mu_2 = \mu_1^\phi$.

Take $R = BP_*(BP)$, $\mu_1 = \mu_{BP}$ (extended from BP^* to $BP_*[t_i]$) and

$$\phi(T) = \sum_i \mu_{BP} t_i T^{p^i}.$$

Then $f \otimes g$ is the identity, so η_R represents the formal group μ_{BP}^ϕ over $BP_*(BP)$: $\eta_{R^*}(\mu_{BP}) = \mu_{BP}^\phi$. Now over $BP_* \otimes Q$ we have an isomorphism $\log_{BP}: \mu_{BP} \rightarrow G_a$, hence also an isomorphism $\eta_{R^*}(\log_{BP}): \eta_{R^*}(\mu_{BP}) \rightarrow G_a$. Noting that $\eta_{R^*}(\mu_{BP}) = \mu_{BP}^\phi = (G_a^{\log_{BP}})^\phi$ we conclude that

$$\eta_{R^*}(\log_{BP}) = \log_{BP} \circ \phi.$$

Hence

$$\begin{aligned} \sum_k \eta_R(m_{p^k-1}) T^{p^k} &= \log_{BP}(\sum_j \mu_{BP} t_j T^{p^j}) \\ &= \sum_j \log_{BP}(t_j T^{p^j}) \\ &= \sum_{i,j} m_{p^i-1}(t_j)^{p^i} T^{p^{i+j}} \end{aligned}$$

which proves i).

iii) ε is a homomorphism $BP_*[t_i] \rightarrow BP_*$ such that if $f: BP_* \rightarrow R$ represents μ then $f \circ \varepsilon: BP_*[t_i] \rightarrow R$ represents (μ, γ_0, μ) , where $\gamma_0(T) = T$. Hence $f \circ \varepsilon(t_i) = 0$ for $i > 0$, from which it is immediate that $\varepsilon(t_i) = 0$ for $i > 0$.

iv) c is a homomorphism $BP_*[t_i] \rightarrow BP_*[t_i]$ so that if $f \otimes g: BP_*[t_i] \rightarrow R$ represents (μ_1, ϕ, μ_2) then $(f \otimes g) \circ c$ represents $(\mu_2, \phi^{-1}, \mu_1)$.

Take $f \otimes g$ to be the identity, so $/$ represents μ_{BP} with scalars extended to $BP_*[t_i]$ and g represents

$$\phi(T) = \sum_i \mu_{BP} t_i T^{p^i}.$$

Then $c = f' \otimes g$ where f' represents μ_{BP}^ϕ (giving $c\eta_L = \eta_R$ as noted above) and $g': Z_{(p)}[t_i] \rightarrow BP_*[t_i]$ must be determined on the t_i 's. Now g' represents $\phi^{-1}: \mu_{BP} \rightarrow \mu_{BP}^\phi$; thus

$$\phi^{-1}(T) = \sum_i \mu_{BP}^\phi c(t_i) T^{p^i}$$

as a curve over μ_{BP}^ϕ . Applying $\phi_*: C_{\mu_{BP}^\phi} \rightarrow C_{\mu_{BP}}$ and noting that

$$\phi_*(\phi^{-1}) = \phi \circ \phi^{-1} = \gamma_0,$$

we compute:

$$\begin{aligned}
 T &= \sum_j^{\mu_{BP}} \phi_*(c(t_j)) T^{p^j} \\
 &= \sum_{i,j}^{\mu_{BP}} t_i(c(t_j))^{p^i} T^{p^{i+j}}.
 \end{aligned}$$

Finally, we apply \log_{BP} and obtain the desired formula.

v) ψ is a homomorphism $BP_*(BP) \rightarrow BP_*(BP) \otimes_{BP_*} BP_*(BP)$ such that if

$$f \otimes g \text{ represents } (\mu_1, \phi, \mu_2)$$

and

$$f' \otimes g' \text{ represents } (\mu_2, \phi', \mu_3)$$

then

$$[(f \otimes g) \otimes (f' \otimes g')] \circ \psi = f'' \otimes g''$$

represents $(\mu_1, \phi\phi', \mu_3)$. Note that

$$\mu_3 = \mu_2^{\phi'} = (\mu_1 \phi)^{\phi'} = \mu_1^{\phi\phi'}.$$

We seek a universal example. Take $R = BP_*(BP) \otimes_{BP_*} BP_*(BP)$, which is a polynomial algebra over BP_* on generators $t_i \otimes 1$ and $1 \otimes t_j$ for i and $j > 0$. Take

$$\begin{aligned}
 \mu_1 &= \mu^{BP} \text{ (extended to } R) \\
 \phi &= \sum_i^{\mu_{BP}} (t_i \otimes 1) T^{p^i} \\
 \mu_2 &= (\mu^{BP})^\phi \\
 \phi' &= \sum_j^{\mu_2} (1 \otimes t_j) T^{p^j} \\
 \mu_3 &= \mu_2^{\phi'} = \mu_{BP}^{\phi\phi'}.
 \end{aligned}$$

One verifies easily that $f \otimes g$ is the inclusion of the left copy of $BP_*(BP)$ in R , while $f' \otimes g'$ is the inclusion of the right copy. Thus $(f \otimes g) \otimes (f' \otimes g')$ is the identity, and so in this situation $\psi = f'' \otimes g''$. We want to find $\psi(t_j)$. The series $\sum_j^{\mu_{BP}} \psi(t_j) T^{p^j}$ must agree with the composition

$$\left(\sum_j^{\mu_{BP}} t_i \otimes 1 T^{p^i} \right) \circ \left(\sum_{i,j}^{\mu_{BP}} \phi 1 \otimes t_j T^{p^j} \right),$$

hence

$$\sum_j^{\mu_P} \psi(t_j) T^{p^j} = \sum_{i,j}^{\mu_{BP}} t_i \otimes (t_j)^{p^i} T^{p^{i+j}}.$$

The formula v) now follows by applying \log_{BP} . This completes the verifications. QED

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