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THE RIEMANN-ROCH THEOREM FOR MANIFOLDS WITH CONICAL SINGULARITIES

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0. Introduction

The classical Riemann-Roch theorem reads as follows. Suppose M is a compact Riemann surface. Let p be a point in M and m an integer. For a holomorphic form u in a punctured neighbourhood of p, we write $\mathrm{ord}\,(\mathrm{u},\mathrm{p}) \geq -\mathrm{m}$ if the product $(z-p)^m u$ extends to a holomorphic form in a whole neighbourhood of p. Thus, u is allowed to have a pole of order $\leq m$ at p, if $m > 0$, and is required to have a zero of order $\geq -m$ at p, if $m \leq 0$. Given any point divisor $\delta = p_1^{m_1} \dots p_N^{m_N}$ on M, denote by $L(\delta, \bar{\partial})$ the space of all holomorphic functions on $M \setminus \mathrm{supp}\,\delta$ such that $\mathrm{ord}\,(\mathrm{u},\mathrm{p}_\nu) \geq -\mathrm{m}_\nu$, for each $\nu = 1, \dots, N$, and by $L(\delta^{-1}, \bar{\partial}')$ the space of all holomorphic forms of bidegree $(0,1)$ on $M \setminus \mathrm{supp}\,\delta$ such that $\mathrm{ord}\,(\mathrm{u},\mathrm{p}_\nu) \geq \mathrm{m}_\nu$, for each $\nu = 1, \dots, N$. Then (cf. Springer [20] and elsewhere),

$$\dim_{\mathbb{C}} L(\delta, \bar{\partial}) = (1-g) + \sum_\nu m_\nu + \dim_{\mathbb{C}} L(\delta^{-1}, \bar{\partial}'), \tag{0.1}$$

where g is the genus of the Riemann surface M equal to the number of "handles" of M. The quantity $\deg \delta = \sum_\nu m_\nu$ is known as the degree of the divisor δ.

Being a very particular case of the Atiyah-Singer index theorem, this result illustrates rather strikingly how the index theorem applies to proving the existence of solutions of elliptic equations. Indeed, (0.1) implies the *Riemann inequality* $\dim_{\mathbb{C}} L(\delta, \bar{\partial}) \geq (1-g) + \deg \delta$, whence it follows that the space $L(\delta, \bar{\partial})$ is not trivial provided $\deg \delta > g-1$. On the other hand, if $\deg \delta > 2(g-1)$, then the space $L(\delta^{-1}, \bar{\partial}')$ proves to contain only the zero form, and so the Riemann inequality in fact becomes the equality determining the dimension of $L(\delta, \bar{\partial})$.

The classical Riemann-Roch theorem has been generalized in different ways to higher-dimensional complex varieties. The best known generalizations are the Hirzebruch Riemann-Roch theorem and the Grothendieck Riemann-Roch theorem (cf. Fulton and Lang [5] and the references given there). In fact, the Hirzebruch Riemann-Roch theorem served as a starting point and a source of technical tools for the Atiyah-Singer theorem.

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In the paper [6], Gromov and Shubin suggested a generalization that was motivated by the classical analysis of solutions with point singularities to general elliptic equations. Namely, let $A \in \mathrm{Diff}^{a}(V, \tilde{V})$ be an elliptic differential operator of order a between sections of vector bundles V and $\tilde{V}$ over a smooth manifold M of dimension n. Pick a point p in M and an integer m. If $m \leq 0$, then, for a solution u of $Au = 0$ in a punctured neighbourhood of p, we write $\mathrm{ord}\,(u, p) \geq -m$ if u extends to a solution on the whole neighbourhood of p and $D^{\alpha}u(p) = 0$, $|\alpha| \leq -m - 1$. If $m > 0$, we proceed as follows. Let u be a solution of $Au = 0$ in $O \setminus \{p\}$, O being a neighbourhood of p. After shrinking O, we may assume that O lies within a local chart on M, both V and $\tilde{V}$ are trivial over O and A has a fundamental solution Φ in O. Were M, V, $\tilde{V}$ and A real analytic, the well-known result on the structure of hyperfunctions with a point support, if applied to Au, would allow us to conclude that $u(y) = \sum_{\alpha} D_y^{\alpha}\Phi(y, p)c_{\alpha}$ modulo solutions on the entire neighbourhood O, the series converging uniformly in y on compact subsets of $O \setminus \{p\}$ (cf. [24, 9.1.11]). In the C^{∞} case we draw the same conclusion for those u which are extendable to a distribution on the whole neighbourhood O. Now, we write $\mathrm{ord}\,(u, p) \geq -m$ if u extends to a distribution on O and

$$u(y) = \sum_{|\alpha| \leq m-1} D_y^{\alpha}\Phi(y, p)c_{\alpha}, \quad y \in O \setminus \{p\}, \tag{0.2}$$

modulo solutions to $Au = 0$ on O. Since the singularity of $\Phi(y, y')$ on the diagonal of $O \times O$ is actually the same as that of the standard fundamental solution for the $(a/2)$ th power of the Laplace operator in $\mathbb{R}^n$, we easily deduce that $\mathrm{ord}\,(u, p) \geq -m$ if and only if $u = u_r + u_s$ in $O \setminus \{p\}$, where $u_r \in C^{\infty}_{loc}(O, V)$ and $u_s(y) = o(|y - p|^{a-n-m})$ as $y \to p$. We write $\mathrm{ord}\,(u, p) = -m$ if $\mathrm{ord}\,(u, p) \geq -m$ but it is not true that $\mathrm{ord}\,(u, p) \geq -m+1$. Obviously, it is immaterial which local coordinates on O and local trivialisations of V and $\tilde{V}$ we choose to define $\mathrm{ord}\,(u, p)$. This definition is compatible with the standard definition of the order of a pole or a zero for a meromorphic form on a Riemann surface (this corresponds to the case $n = 2$, $A = \bar{\partial}$ and $a = 1$). By a *point divisor* on M is meant any element of the free Abelian group generated by points of this manifold. We write a point divisor in the multiplicative form $\delta = p_1^{m_1} \dots p_N^{m_N}$, with $m_\nu \in \mathbb{Z} \setminus \{0\}$. Set $\mathrm{supp}\,\delta = \{\mathrm{p}_1, \dots, \mathrm{p}_N\}$. It is customary to write $p^0 = \emptyset$ that corresponds to the unity of the group. The 'inverse' divisor is defined by $\delta^{-1} = p_1^{-m_1} \dots p_N^{-m_N}$, and so $\mathrm{supp}\,\delta^{-1} = \mathrm{supp}\,\delta$. The degree of a divisor δ is defined to be

$$\deg \delta = k \sum_{\nu=1}^{N} \mathrm{sgn}\, \mathrm{m}_\nu \left(\binom{|m_\nu|+n-1}{n} - \binom{|m_\nu|-a+n-1}{n} \right), \tag{0.3}$$

k being the rank of V, where $\binom{J}{j} = \frac{J!}{j!(J-j)!}$ if $j \leq J$ and 0 otherwise. Note that $\deg \delta$ depends also on the order a of A and on the fibre dimension k of V (or $\tilde{V}$, which is clear from the ellipticity of A). Having disposed of these preliminary steps, we introduce two spaces

$$L(\delta, A) = \{u \in C^\infty_{loc}(M \setminus \operatorname{supp}\delta, \mathrm{V})\colon \mathrm{Au} = 0,\ \mathrm{ord}\,(\mathrm{u}, \mathrm{p}_\nu) \geq -\mathrm{m}_\nu\},$$
$$L(\delta^{-1}, A') = \{g \in C^\infty_{loc}(M \setminus \operatorname{supp}\delta, \tilde{\mathrm{V}}')\colon \mathrm{A'g} = 0,\ \mathrm{ord}\,(\mathrm{g}, \mathrm{p}_\nu) \geq \mathrm{m}_\nu\},$$

where $A' \in \mathrm{Diff}^{\mathrm{a}}(\tilde{\mathrm{V}}', \mathrm{V}')$ is the transpose of A. These are spaces of "meromorphic" solutions to the equation $Au = 0$ and its transpose $A'g = 0$, respectively, depending on a given divisor; the solutions are allowed to have some poles (at points that enter into the corresponding divisor with positive degrees) and are required to have zeros (at points that enter into the corresponding divisor with negative degrees).

Theorem 0.1 (cf. [6]). *If M is a compact smooth closed manifold, then*

$$\dim_{\mathbb{C}} L(\delta, A) = \mathrm{ind}\,\mathrm{A} + \deg\delta + \dim_{\mathbb{C}} \mathrm{L}(\delta^{-1}, \mathrm{A}'). \tag{0.4}$$

Since the index of the Cauchy-Riemann operator on a compact Riemann surface is equal to $1 - g$, g being the genus, the classical Riemann-Roch theorem (cf. (0.1)) is a very particular case of Theorem 0.1.

Let us mention yet another particular case of Theorem 0.1. If A is a selfadjoint elliptic operator on a compact smooth closed manifold M, then $\mathrm{ind}\,\mathrm{A} = 0$, which yields $\dim_{\mathbb{C}} L(\delta, A) = \deg\delta + \dim_{\mathbb{C}} L(\delta^{-1}, A)$. This result for the scalar Laplacian on a Riemannian manifold goes back at least as far as Nadirashvili [12].

It is worth pointing out that the index of the operator A can be evaluated in each Sobolev space $H^s(M, V)$, $s \in \mathbb{R}$. The elliptic theory on a compact smooth closed manifold M shows that the mapping $A\colon H^s(M, V) \to H^{s-a}(M, \tilde{V})$ is Fredholm and its index is independent of s. This index can be calculated by the Atiyah-Singer formula (cf. [1]).

Gromov and Shubin [6] gave also a generalization of Theorem 0.1 to non-compact smooth manifolds with compact boundary. In this case one imposes appropriate boundary conditions and conditions at infinity in order to ensure that the given elliptic operator defines a Fredholm operator in suitable spaces.

In this paper we derive a generalization of the classical Riemann-Roch theorem that is motivated by the analysis of solutions of elliptic equations on manifolds with conical singularities (cf. Kondrat'ev [8], Melrose and Mendoza [9], Plamenevskii [14], Schulze [17, 18, 19]). On such a manifold M live differential operators which are usual over the smooth part of M and of so-called Fuchs type close to singular points. They act naturally in weighted Sobolev spaces of distributions on the smooth part of M, the weight functions being powers of the distance to the set of singular points. The concept of ellipticity relies on two symbolic levels, the first of the two is the usual principal symbol defined over the smooth part of M up to the singular points, and the second of these is the conormal symbol defined over the set of singular points. The conormal symbol at a singular point $v \in M$ is a family of usual differential operators acting in Sobolev spaces over a cross-section of M close to v. The parameter z substituting the Fuchs-type derivative along the geodesic at v varies over a horizontal line $\Im z = -\gamma_v$ in the complex plane, γ_v being the weight exponent at v. Thus, the ellipticity depends on the weighted Sobolev spaces to be domains of the operator in question. Elliptic operators are Fredholm and have parametrices within the so-called *cone algebra* of pseudodifferential operators on M (cf. *ibid*). In this setting we prove equality (0.4) both for divisors δ supported away from the set of singular points and for those meeting this set.

The idea of using the calculus of b-pseudodifferential operators on a manifold with boundary to deduce the classical Riemann-Roch theorem goes back to the book of Melrose [10, 6.3].

1. Manifolds with Singular Points

Let M be a (topological) manifold of dimension n with a singular point v, and let M have a C^∞ structure away from v. We are going to induce a singular C^∞ structure on M at the point v.

To this end, let us fix the type of the singular point v by specifying a model object in an Euclidean space. Namely, consider the model surface in $\mathbb{R}^{n+1}$ given by $C_0 = \{rS(f(r)x) \colon r \in [0,1), x \in X\}$, where S is a diffeomorphism of a star-shaped domain $\Omega \subset \mathbb{R}^n$ onto an open subset of the unit sphere S^n in $\mathbb{R}^{n+1}$, X is a compact closed submanifold of dimension $n-1$ in Ω, and f is a C^∞ function on $(0,1)$ with values in $(0,1]$, continuous up to $r=0$. This surface is smooth away from the origin $0 \in \mathbb{R}^{n+1}$ and the origin is a *conical point* of C_0, if $f \equiv const$, and a *cusp* of a higher order, if $f(0+) = 0$. Under the mapping $\pi(r,x) = rS(f(r)x)$, the smooth part $C_0 \setminus \{0\}$ of C_0 is identified with the cylinder $(0,1) \times X$ over X, while the origin is blown up to the base $\{0\} \times X$ of this cylinder. Moreover,

$$\det \frac{\partial \pi}{\partial (r,x)} = \det \frac{\partial\, rS(\omega)}{\partial (r,\omega)} \,|_{\omega = f(r)x}$$

vanishes only for $r = 0$ where $\operatorname{rank}_{\mathbb{R}} \det \frac{\partial \pi}{\partial (r,x)} = 1$.

Since C_0 is embedded into $\mathbb{R}^{n+1}$, there is a natural way to define a *singular* C^∞ structure on this surface. Namely, by a C^∞ function on C_0 we mean the restriction, to C_0, of some C^∞ function in a neighbourhood of C_0. Were C_0 smooth at 0, then a familiar result would yield that such functions have inner description in terms of local coordinates on C_0, which serves as an additional argument in favour of our definition. If u is a C^∞ function on C_0, then the pull-back $\pi^* u\,(r,x) = u(rS(f(r)x))$ is a C^∞ function on the cylinder $[0,1) \times X$, i.e., up to $r = 0$. The converse is not true as shows any component of $\pi^{-1}(y)$, $y \in C_0$.

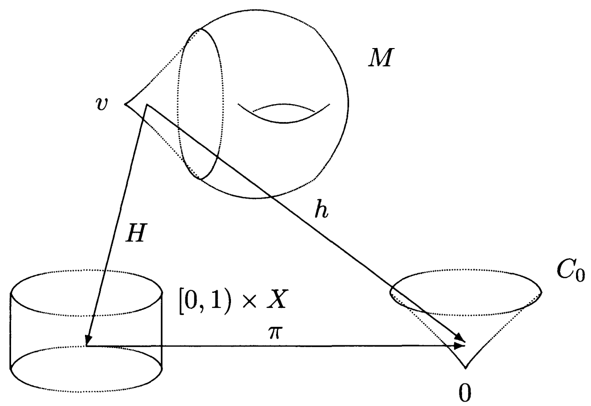

Fig. 1: A manifold with a cusp at v.

Now, a homeomorphism h of C_0 is said to be a *diffeomorphism* if $h(0) = 0$ and there is a diffeomorphism of a neighbourhood of C_0 in $\mathbb{R}^{n+1}$ whose restriction to C_0 coincides with h.

Returning to the original manifold M, we call M a manifold with a cusp at v if there is a neighbourhood O of v and a homeomorphism $h : O \to C_0$ such that $h(v) = 0$ and the restriction $h : O \setminus \{v\} \to C_0 \setminus \{0\}$ is a diffeomorphism. Any two such homeomorphisms h_1 and h_2 are said to be *equivalent* if the composition $h_2 h_1^{-1}$ is a diffeomorphism of C_0. Then, the C^∞ *cusp structure* on M close to v is defined by any class of equivalent homeomorphisms $O \to C_0$, as above. Our next goal is to give an alternative description of the "model object" which still makes sense for not necessarily embedded manifolds X. Set $H = \pi^{-1} \circ h$, the composition being regarded as a multivalent mapping $O \to [0,1) \times X$. This is a diffeomorphism of $O \setminus \{v\}$ onto $(0,1) \times X$ and the image of v by H is the base $\{0\} \times X$ of the cylinder. An easy consideration shows that H is actually a homeomorphism of O onto the topological cone

$$C_t(X) = \frac{[0,1) \times X}{\{0\} \times X}$$

over X. We call any two homeomorphisms H_1 and H_2 of O onto $C_t(X)$ with these properties *equivalent* if the restriction of $H_2 H_1^{-1}$ to $(0,1) \times X$ extends to a diffeomorphism of a neighbourhood of $[0,1) \times X$ in $\mathbb{R} \times X$. Classes of equivalent homeomorphisms $H : O \to C_t(X)$ give M various singular C^∞ structures close to the point v. Let us elucidate the relevance of the C^∞ cusp structures among them.

Suppose h_1, h_2 are two equivalent homeomorphisms $O \to C_0$, thus defining the same C^∞ cusp structure on M at v. Write $H_1 = \pi^{-1} \circ h_1$ and $H_2 = \pi^{-1} \circ h_2$

and consider the composition $H_2 \circ H_1^{-1} = \pi^{-1}(h_2 \circ h_1^{-1})\pi$. Since $h_2 \circ h_1^{-1}$ is a diffeomorphism of C_0, it follows that the restriction of $H_2 \circ H_1^{-1}$ to $(0,1) \times X$ extends to a diffeomorphism of a neighbourhood of $[0,1) \times X$, i.e., H_1, H_2 are equivalent homeomorphisms $O \to C_t(X)$. Thus, each C^∞ cusp structure on M determines in a natural way some singular C^∞ structure on M via the "model object" $C_t(X)$.

As the topological cone $C_t(X)$ has no canonical singular C^∞ structure, it is not to be expected that C^∞ cusp structures on M at v can be specified by singular C^∞ structures on M via $C_t(X)$. In other words, different C^∞ cusp structures on M can determine the same singular C^∞ structure on M via $C_t(X)$ because, for a diffeomorphism Δ of $[0,1) \times X$, the composition $\pi \circ \Delta \circ \pi^{-1}$ need not be a diffeomorphism of C_0. However, each singular C^∞ structure on M at v via $C_t(X)$ originates with some C^∞ cusp structure, as every homeomorphism $H \colon O \to C_t(X)$ factors through C_0, i.e., $H = \pi^{-1}h$ for some homeomorphism $h \colon O \to C_0$ (cf. Fig. 1).

We deduce that in order to specify a C^∞ cusp structure on M at v within a singular C^∞ structure defined by a homeomorphism of O onto $C_t(X)$, one needs an additional information on the original cusp structure. As such an information can serve either a Riemannian (cusp) metric on the cylinder $[0,1) \times X$ or a class of typical vector fields near the base $r = 0$ of the cylinder.

The concept of a manifold with cusps extends in a natural way to the case of several singular points.

2. Cusp Algebras

We begin by showing the class of Riemannian metrics on the cylinder $[0,1) \times X$ specifying C^∞ cusp structures on M close to a singular point v.

The diffeomorphism $\pi \colon (0,1) \times X \to C_0 \setminus \{0\}$ pulls back the Riemannian metric $dy_1^2 + \ldots + dy_{n+1}^2$ from the smooth part of C_0 to the cylinder $(0,1) \times X$, thus giving

$$
\begin{aligned}
&\pi^* \left(dy_1^2 + \ldots + dy_{n+1}^2 \right) \\
(2.1) \qquad &= dr^2 + \sum_{j=1}^{n+1} \left((rf') \left(\sum_{\iota=1}^{n} x_\iota \frac{\partial S_j}{\partial \omega_\iota}(fx) \right) dr + (rf) \sum_{\iota=1}^{n} \frac{\partial S_j}{\partial \omega_\iota}(fx)\, d\omega_\iota \right)^2
\end{aligned}
$$

followed by restricting the differentials $d\omega_1, \ldots, d\omega_n$ to tangential vectors to X. Of course, (2.1) degenerates at the base $\{0\} \times X$ of the cylinder.

EXAMPLE 2.1. Let $n = 2$ and let C_0 be the surface with a cusp at the origin given in the polar coordinates of $\mathbb{R}^3$ by

$$
\begin{cases}
y_1 = r \cos \varphi \sin f(r) \psi_0, \\
y_2 = r \sin \varphi \sin f(r) \psi_0, \\
y_3 = r \cos f(r) \psi_0,
\end{cases}
$$

where $r \in [0,1)$, $\varphi \in [0,2\pi)$ and ψ_0 is a fixed angle in the interval $(0,\pi)$. Then, a trivial verification shows that

$$\pi^* \left(dy_1^2 + dy_2^2 + dy_3^2\right) = (1 + (rf'(r)\psi_0)^2)dr^2 + (r \sin f(r)\psi_0)^2 d\varphi^2,$$

$d\varphi^2$ being the Riemannian metric along the unit circle $\mathbb{S}^1$. □

Vector fields along the base X of the cylinder $(0,1)\times X$ endowed with Riemannian metric (2.1) are of the form $(rf)^{-1}\sum_{\iota=1}^{n-1} a_\iota(r,x)\,\partial/\partial x_\iota$ in local coordinates of X. The coefficients a_ι are of class C^∞ up to $r=0$ if so is f. We are thus lead to typical vector fields on a manifold with a C^∞ cusp structure. These are

$$\begin{aligned} & a_0(r,t)\,\tfrac{\partial}{\partial r} + \tfrac{1}{rf}\textstyle\sum_{\iota=1}^{n-1} a_\iota(r,x)\,\tfrac{\partial}{\partial x_\iota} \\ & \quad = \tfrac{1}{rf}\left(a_0(r,x)\left(rf\tfrac{\partial}{\partial r}\right) + \textstyle\sum_{\iota=1}^{n-1} a_\iota(r,x)\,\tfrac{\partial}{\partial x_\iota}\right) \end{aligned} \tag{2.2}$$

close to the cusp.

If f satisfies the condition $\sup|r^j f^{(j)}(r)| < \infty$, for all j, then such vector fields behave properly under composition. Modulo the weight factor $(rf)^{-1}$, they are section of a vector bundle ${}^bTM$ over M called the *compressed tangent bundle*. When restricted to the smooth part of M, this latter is isomorphic to the usual tangent bundle over $M \setminus \{v\}$. On the other hand, the weight factor $(rf)^{-1}$ can be managed via suitable weighted Sobolev spaces on M.

The microlocalisation of this Lie algebra of vector fields leads to an algebra of pseudodifferential operators on M called a "cusp algebra." For more details we refer the reader to [21] and [15].

In particular, if v is a conical point of M, i.e., $f \equiv 1$, then the Riemannian metric close to v becomes

$$dr^2 + r^2 \sum_{\iota,\kappa=1}^{n} \left(\sum_{j=1}^{n+1} \frac{\partial S_j}{\partial \omega_\iota}\frac{\partial S_j}{\partial \omega_\kappa}\right) d\omega_\iota d\omega_\kappa,$$

resulting in the Fuchs-type derivative $\boldsymbol{D} = r\frac{1}{i}\frac{\partial}{\partial r}$ and in the cone algebra of Melrose and Mendoza [9] and Schulze [17, 18, 19].

3. The Riemann-Roch Theorem

Let us consider a compact closed manifold M with a finite set of conical points $\mathrm{sing\,M} = \{\mathrm{v}_1,\dots,\mathrm{v}_\mathrm{I}\}$. As described above, such a manifold has a C^∞ structure away from the set $\mathrm{sing\,M}$ and a C^∞ *cone structure* close to each point $v \in \mathrm{sing\,M}$. Alternatively, M can be thought of as a compact smooth manifold with cylindrical 'ends', i.e., close to a point $v \in \operatorname{sing} \mathrm{M}$, we identify M with a cylinder $C_v = [0,1) \times X_v$ over a compact smooth closed manifold X_v of dimension $n-1$, each C_v being endowed with a cone metric $dr^2 + r^2 g_{X_v}(r)$ where $g_{X_v}(r)$ is a family of Riemannian metrics on X_v, smooth in $r \in [0,1)$ up to $r = 0$.

The cone metric gives rise to the Lie algebra of vector fields on C_v spanned by $r\partial/\partial r$ and $\partial/\partial x_j$, where $x = (x_1, \ldots, x_{n-1})$ are local coordinates on X_v. It follows that the typical differential operators on C_v are of the so-called Fuchs type

$$A = \frac{1}{r^a} \sum_{j=0}^{a} a_j(r) \boldsymbol{D}^j, \tag{3.1}$$

where $\boldsymbol{D} = r \frac{1}{i} \frac{\partial}{\partial r}$ and $a_j \in C^\infty_{loc}([0,1), \mathrm{Diff}^{\mathrm{a-j}}(\mathrm{X_v}))$. The class of Fuchs-type operators is invariant under local diffeomorphisms of M preserving the C^∞ cone structure.

To each operator (3.1) we assign its principal Fourier symbol $\sigma^a_{\mathcal{F}}(A)$ away from $r = 0$ as well as its Mellin symbol at $r = 0$,

$$\sigma_{\mathcal{M}}(A)(v,z) = \sum_{j=0}^{a} a_j(0) z^j. \tag{3.2}$$

This latter is regarded as a family of differential operators over X_v acting in Sobolev spaces $H^s(X_v) \to H^{s-a}(X_v)$ and parametrised by the complex variable z varying along a horizontal line $\Gamma_\gamma = \{z \in \mathbb{C} \colon \Im z = \gamma\}$, $\gamma \in \mathbb{R}$.

Now, by a differential operator of order a on M, we mean any differential operator of order a on the smooth part $M \setminus \operatorname{sing} \mathrm{M}$ of M which is of Fuchs-type (3.1) close to singular points. In just the same way we define differential operators A between sections of smooth vector bundles V and $\tilde{V}$ over M. When "pulled back" to a cylindrical end C_v, both V and $\tilde{V}$ are trivial over the boundary $\{0\} \times X_v$, which allows us to regard the Mellin symbol at $r = 0$, $\sigma_{\mathcal{M}}(A)$, as a mapping $H^s(X_v) \otimes V_v \to H^{s-a}(X_v) \otimes \tilde{V}_v$. We continue to write $\mathrm{Diff}^{\mathrm{a}}(\mathrm{V}, \tilde{\mathrm{V}})$ for the space of all differential operators of order a between sections of V and $\tilde{V}$.

The natural domain of an operator $A \in \mathrm{Diff}^{\mathrm{a}}(\mathrm{V}, \tilde{\mathrm{V}})$ is a weighted Sobolev space $H^{s,\gamma}(M, V)$ of sections of V over M, where $s \in \mathbb{R}$ and $\gamma = (\gamma_1, \ldots, \gamma_I)$ is a tuple of real numbers. This space is modeled on the usual Sobolev space $H^s_{loc}(M \setminus \operatorname{sing} \mathrm{M}, \mathrm{V})$ away from the singular points and on a weighted Sobolev space $H^{s,\gamma_i}(C_{v_i}, V)$ close to the singular point v_i. The definition of $H^{s,\gamma_i}(C_{v_i}, V)$ invokes the Mellin transform in $r \in \mathbb{R}_+$ and the Fourier transform in $x \in \mathbb{R}^{n-1}$, along with the weight factor $r^{-\gamma_i}$ (cf. Schulze [18, 1.1.1]).

Each operator $A \in \mathrm{Diff}^{\mathrm{a}}(\mathrm{V}, \tilde{\mathrm{V}})$ is known to extend to a continuous mapping $H^{s,\gamma}(M, V) \to H^{s-a,\gamma-a}(M, \tilde{V})$, for all $s \in \mathbb{R}$ and $\gamma \in \mathbb{R}^I$, where we set $\gamma - a = (\gamma_1 - a, \ldots, \gamma_I - a)$.

The weight tuple γ enters into the concept of ellipticity on a manifold with conical singularity in the following way. An operator $A \in \mathrm{Diff}^{\mathrm{a}}(\mathrm{V}, \tilde{\mathrm{V}})$ is said to be *elliptic* with respect to a weight tuple $\gamma \in \mathbb{R}^I$, if A is elliptic in the usual sense away from the set $\operatorname{sing} \mathrm{M}$ and, for each $i = 1, \ldots, I$, the Mellin symbol of A at the singular point v_i is an isomorphism $H^s(X_{v_i}) \otimes V_{v_i} \overset{\cong}{\to} H^{s-a}(X_{v_i}) \otimes \tilde{V}_{v_i}$, for any one $s \in \mathbb{R}$ and all $z \in \Gamma_{-\gamma_i}$.

Note that if A is elliptic with respect to $\gamma \in \mathbb{R}^I$, then its transpose $A' \in \mathrm{Diff}^{\mathrm{a}}(\tilde{\mathrm{V}}', \mathrm{V}')$ under the pairing $H^{-s,-\gamma}(M, V') \times H^{s,\gamma}(M, V) \to \mathbb{C}$ is elliptic with respect to the weight tuple $a - \gamma$.

A basic result of the analysis on manifolds with conical singularities is that, given any $\gamma \in \mathbb{R}^I$, the mapping $A\colon H^{s,\gamma}(M, V) \to H^{s-a,\gamma-a}(M, \tilde{V})$ is Fredholm for all $s \in \mathbb{R}$ if and only if A is elliptic with respect to γ (cf. *ibid*, 1.2.2). Moreover, if A is elliptic, then the kernel and the cokernel of the mapping A are independent of s (but not of γ), and so the index of A can be evaluated in the space $H^{\infty,\gamma}(M, V)$.

We are now in a position to introduce our first version of the Riemann-Roch theorem for a manifold M with conical singularities. To this end, given any point divisor $\delta = p_1^{m_1} \ldots p_N^{m_N}$ with $\operatorname{supp} \delta \cap \operatorname{sing} \mathrm{M} = \emptyset$, we consider two spaces

$$\begin{aligned} L(\delta, A) &= \{u \in H^{\infty,\gamma}_{loc}(M \setminus \operatorname{supp} \delta, \mathrm{V})\colon \mathrm{Au} = 0,\ \operatorname{ord}(\mathrm{u}, \mathrm{p}_\nu) \geq -\mathrm{m}_\nu\}, \\ L(\delta^{-1}, A') &= \{g \in H^{\infty,a-\gamma}_{loc}(M \setminus \operatorname{supp} \delta, \tilde{\mathrm{V}}')\colon \mathrm{A'g} = 0,\ \operatorname{ord}(\mathrm{g}, \mathrm{p}_\nu) \geq \mathrm{m}_\nu\}, \end{aligned}$$

$\operatorname{ord}(\mathrm{u}, \mathrm{p})$ being defined as above.

Theorem 3.1. *Suppose A is a differential operator on M, elliptic with respect to a weight tuple $\gamma \in \mathbb{R}^I$, and δ is a point divisor on M supported away from the set of singular points. Then,*

$$\dim_{\mathbb{C}} L(\delta, A) = \operatorname{ind} \mathrm{A} + \deg \delta + \dim_{\mathbb{C}} \mathrm{L}(\delta^{-1}, \mathrm{A}'). \tag{3.3}$$

We emphasize that $\operatorname{ind} \mathrm{A}$ means the index of the operator A evaluated in any one Sobolev space $H^{s,\gamma}(M, V)$, $s \in \mathbb{R}$. The problem of finding an explicit index formula for Fredholm differential operators on a compact closed manifold with conical singularities has not been solved in a completely satisfactory way (however, see the work of Fedosov and the authors [3] and the references given there for partial results). Theorem 3.1 can be useful anyway, for explicit index formulas are known for particular operators.

The theorem is still true for elliptic differential operators on compact closed manifolds with cusps (cf. Schulze and Tarkhanov [22]). As is observed by Melrose [11], various problems for cusp and cone pseudodifferential operators are essentially the same. The index problem for pseudodifferential operators is, as yet, unsolved for all the fibred cusp algebras except the scattering algebra (cf. *ibid*).

If the set of singular points of M is empty, equality (3.3) gives (0.4), and so Theorem 3.1 contains Theorem 0.1 as a very particular case. However, the proof of Theorem 3.1 is similar in spirit to that of Gromov and Shubin [6].

We postpone the proof of Theorem 3.1 until Section 5 while showing that artificial conical points do not affect the index of the operator A if γ is properly chosen.

Lemma 3.2. *Let $v \in M \setminus \operatorname{sing M}$ and let M' be the manifold obtained from M by regarding v as new conical point. Suppose $\gamma' = (\gamma, \gamma_v)$, where $\gamma \in \mathbb{R}^I$ and $\gamma_v \in (a-n, 0)$. Then the index of $A \colon H^{s,\gamma'}(M', V) \to H^{s-a,\gamma'-a}(M', \tilde{V})$ is equal to the index of $A \colon H^{s,\gamma}(M, V) \to H^{s-a,\gamma-a}(M, \tilde{V})$.*

Note that in order to a γ_v fulfilling the condition of the lemma exist it is necessary and sufficient that $a < n$.

Proof. Pick a neighbourhood O of the point v such that O does not meet the set $\operatorname{sing M}$. If $-\gamma_v < n - a$, then each section $u \in H^{s,\gamma_v}_{loc}(O, V)$ satisfying $Au = 0$ in $O \setminus \{v\}$ extends to a solution of this equation on the whole neighbourhood O. This is a kind of the theorem on removable singularities for solutions of elliptic equations. For the proof, use expansion (0.2) together with the observation that the elements of $H^{s,-\infty}_{loc}(O, V)$ admit extensions to distributions on O. Conversely, if $0 < -\gamma_v$, then each solution to $Au = 0$ in O belongs to $H^{s,\gamma_v}_{loc}(O, V)$. We turn now to the transposed equation $A'g = 0$. Recall that the transpose A' is defined with respect to the pairing $H^{-s,-\gamma}(M, V') \times H^{s,\gamma}(M, V) \to \mathbb{C}$ induced by the inner product in $H^{0,0}(M, V)$, As $H^{0,0}(M, V)$ is different from $L^2(M, V)$, the transpose A' is no longer elliptic in the usual sense close to the point v. However, A' can be easily expressed by means of the transpose of A with respect to the pairing $H^{-s,-\gamma-n}(M, V') \times H^{s,\gamma}(M, V) \to \mathbb{C}$ induced by the inner product in $H^{0,-\frac{n}{2}}(M, V)$. When localised to O, this latter inner product corresponds to that in the space $L^2(O, V)$ up to the choice of a positive density, and hence the corresponding transpose of A is an elliptic operator in the neighbourhood of v. Summarising, we conclude that the conjugation of A' by the nth power of the distance to v, i.e. $r^{-n}A'r^n$, is an elliptic operator in the usual sense in O. Thus, if $0 < n - (a - \gamma_v) < n - a$, then we may apply the above argument to solutions of the transposed equation $A'g = 0$. The inequalities

$$\begin{array}{ccccc} 0 & < & -\gamma_v & < & n - a, \\ 0 & < & -(a - \gamma_v - n) & < & n - a \end{array}$$

are easily verified to coincide, thus resulting in $a - n < \gamma_v < 0$. Since we still have $\operatorname{ind A} = \dim\ker \mathrm{A} - \dim\ker \mathrm{A}'$, the lemma follows. □

4. A Duality Theorem

Let $\delta = p_1^{m_1} \dots p_N^{m_N}$ be a point divisor with a support away from the set of singular points of M.

We introduce the *positive* and *negative* parts of δ as divisors

$$\begin{aligned} \delta^+ &= p_1^{m_1^+} \dots p_N^{m_N^+}, \\ \delta^- &= p_1^{m_1^-} \dots p_N^{m_N^-}, \end{aligned}$$

where $m^+ = \max(m, 0)$, $m^- = \min(m, 0)$. Here all factors of the form p_ν^m with $m = 0$ have to be omitted.

It is clear that $\delta = \delta^+ \delta^-$ and

$$\begin{aligned} (\delta^{-1})^+ &= (\delta^-)^{-1}, \\ (\delta^{-1})^- &= (\delta^+)^{-1}. \end{aligned}$$

We next introduce new spaces which play an important role in the proof of Theorem 3.1 but on the other hand allow us to formulate a duality theorem which is important by itself.

Namely, $L'(\delta, A)$ is defined to consist of all sections $u \in H_{loc}^{\infty,\gamma}(M \setminus \operatorname{supp} \delta^+, \mathrm{V})$ such that u vanishes at p_ν up to order $-m_\nu - 1$, if $p_\nu \in \operatorname{supp} \delta^-$, and, for each $p_\nu \in \operatorname{supp} \delta^+$, there exist a neighbourhood O of p_ν and sections $u_r \in C_{loc}^\infty(O, V)$ and $u_s \in C_{loc}^\infty(O \setminus \{p_\nu\}, V)$ with the property that $u = u_r + u_s$ in $O \setminus \{p_\nu\}$ and $Au_s = 0$ in $O \setminus \{p_\nu\}$, $\mathrm{ord}\,(\mathrm{u_s}, \mathrm{p}_\nu) \geq -\mathrm{m}_\nu$.

Thus, we allow merely singularities that occur as singularities of solutions to $Au = 0$. The space $L'(\delta, A)$ consists of sections with the same zeros and singularities as allowed in the definition of $L(\delta, A)$. However, the definition of $L'(\delta, A)$ contains no global restrictions on u, so all possible local singularities and zeros can be present at each point $p_\mu \in \operatorname{supp} \delta$ independently from what happens at other points.

Now we introduce the reduced divisor

$$\tilde{\delta} = p_1^{\tilde{m}_1} \dots p_N^{\tilde{m}_N}, \tag{4.1}$$

where $\tilde{m}_\nu = \mathrm{sgn}\,\mathrm{m}_\nu\,(|\mathrm{m}_\nu| - \mathrm{a})^+$ and the factors $p_\nu^{\tilde{m}_\nu}$ with $\tilde{m}_\nu = 0$ have to be omitted.

Thus, compared with δ, the absolute value of every exponent decreases by a (or becomes 0 if it was initially less than a). Note that $(\delta^{-1})^\sim = (\tilde{\delta})^{-1}$ and

$$\begin{aligned} (\delta^+)^\sim &= (\tilde{\delta})^+, \\ (\delta^-)^\sim &= (\tilde{\delta})^-, \end{aligned}$$

hence the designations $\tilde{\delta}^{-1}$, $\tilde{\delta}^+$ and $\tilde{\delta}^-$ will cause no confusion.

We define the space $L''(\tilde{\delta}, A)$ to consist of all sections $f \in H^{\infty,\gamma-a}(M, \tilde{V})$ such that f vanishes at p_ν up to order $-\tilde{m}_\nu - 1$, if $p_\nu \in \operatorname{supp} \tilde{\delta}^-$. Note that $L''(\tilde{\delta}, A)$ actually depends on $\tilde{\delta}^-$ only, and so $L''(\tilde{\delta}, A) = L''(\tilde{\delta}^-, A)$.

Lemma 4.1. *The differential operator A extends in a natural way to a mapping $\tilde{A}: L'(\delta, A) \to L''(\tilde{\delta}, A)$.*

Proof. Indeed, pick $u \in L'(\delta, A)$. It follows from the definition of $L'(\delta, A)$ that Au, being defined on $M \setminus \operatorname{supp} \delta^+$, extends by continuity to a section $f \in H^{\infty,\gamma-a}(M, \tilde{V})$. Moreover, $f \in L''(\tilde{\delta}, A)$, as is easy to check. Hence, setting $\tilde{A}u = f$ yields the required extension of A. □

We now apply these arguments again, with the operator A with domain $H^{\infty,\gamma}(M, V)$ replaced by the transpose A' with domain $H^{\infty,a-\gamma}(M, \tilde{V}')$, to introduce the spaces $L'(\delta^{-1}, A')$ and $L''(\tilde{\delta}^{-1}, A')$.

In fact, $L'(\delta^{-1}, A')$ consists of all sections $g \in H^{\infty,a-\gamma}_{loc}(M \setminus \operatorname{supp} \delta^-, \tilde{V}')$ such that g vanishes at p_ν up to order $m_\nu - 1$, if $p_\nu \in \operatorname{supp} \delta^+$, and, for each $p_\nu \in \operatorname{supp} \delta^-$, there exist a neighbourhood O of p_ν and sections $g_r \in C^\infty_{loc}(O, \tilde{V}')$ and $g_s \in C^\infty_{loc}(O \setminus \{p_\nu\}, \tilde{V}')$ with the property that $g = g_r + g_s$ in $O \setminus \{p_\nu\}$ and $A'g_s = 0$ in $O \setminus \{p_\nu\}$, $\operatorname{ord}(g_s, p_\nu) \geq m_\nu$.

Furthermore, $L''(\tilde{\delta}^{-1}, A')$ is generated by sections $v \in H^{\infty,-\gamma}(M, V')$ such that v vanishes at p_ν up to order $\tilde{m}_\nu - 1$, if $p_\nu \in \operatorname{supp} \tilde{\delta}^+$.

Lemma 4.2. *The differential operator A' extends in a natural way to a mapping $\tilde{A}': L'(\delta^{-1}, A') \to L''(\tilde{\delta}^{-1}, A')$.*

Proof. This follows by the same method as in Lemma 4.1. □

Our next objective is to introduce an important duality in the spaces defined before. To this end, we recall the definition of the dual bundle on a manifold with conical singularities.

By a density on M we mean any density ω over the smooth part of M, which takes the form $\omega = r^{n-1} dr dx$ modulo factors smooth up to $r = 0$, close to each conical point v. Obviously, this definition is independent of the particular splitting of coordinates (r, x) near v. As is customary, we denote by Ω the bundle of complex densities on M. For every $\omega \in C^\infty(M, \Omega)$, the integral $\int_M \omega$ is well-defined. Note that this integral still makes sense for all integrable densities on M. If V is a vector bundle over M, then the bundle $V' = \operatorname{Hom}_{\mathbb{C}}(V, \Omega)$ is called the *dual bundle*. There is a natural pairing of bundles $V' \otimes V \to \Omega$ which gives the pairing in sections $\langle \cdot, \cdot \rangle: C^\infty(M, V') \times C^\infty(M, V) \to \mathbb{C}$ by means of $(v, u) \mapsto \int_{y \in M} \langle v, u \rangle_y$. Here, $\langle v, u \rangle_y \in \Omega_y$ is obtained by use of the pairing between V'_y and V_y. When restricted to $C^\infty_{comp}(M, V') \times C^\infty_{comp}(M, V)$, this pairing extends to $H^{-\gamma-n}(M, V') \times H^{0,\gamma}(M, V)$, to each real γ. Indeed, if $v \in H^{0,-\gamma-n}(M, V')$ and $u \in H^{0,\gamma}(M, V)$, then $r^{\gamma+n} v \in H^{0,0}(M, V')$ and $r^{-\gamma} u \in H^{0,0}(M, V)$, where r is thought of as the distance to conical points. Hence it follows that the density $\langle v, u \rangle_y = r^{-n} \langle r^{\gamma+n} v, r^{-\gamma} u \rangle_y$ is integrable over M. As usually, this subtends that the pairing $V' \otimes V \to \Omega$ relies on fixed volume form on M and Hermitian metric on V. Since our reference space is $H^{0,0}(M,V)$, we modify the pairing in question by including the weight factor r^{-n}, thus arriving at

$$\langle v,u\rangle = \int_{y\in M} r^{-n}\,\langle v,u\rangle_y$$

for $v\in H^{0,-\gamma}(M,V')$, $u\in H^{0,\gamma}(M,V)$.

In contrast to the case of a compact closed C^∞ manifold M, the transposed operator A' does not fulfil the property $\langle g,Au\rangle = \langle A'g,u\rangle$ for all smooth u and g, but only for those with $(\operatorname{supp} u \cap \operatorname{supp} g)\cap \operatorname{sing} M = \emptyset$. However, the following is what we really need.

Lemma 4.3. *For each $u\in H^{a,\gamma}(M,V)$ and $g\in H^{a,a-\gamma}(M,\tilde V')$,*

$$\langle g,Au\rangle = \langle A'g,u\rangle. \tag{4.2}$$

Proof. By a property of weighted Sobolev spaces, we can choose a sequence $(u_\nu)_{\nu=1,2,\dots}$ in $C^\infty_{comp}(M\setminus \operatorname{sing} M, V)$ approximating u in the norm of $H^{a,\gamma}(M,V)$. Then $Au_\nu\to Au$ in the norm of $H^{0,\gamma-a}(M,V)$, whence

$$\begin{aligned}\langle g,Au\rangle &= \lim_{\nu\to\infty}\langle g,Au_\nu\rangle\\ &= \lim_{\nu\to\infty}\langle A'g,u_\nu\rangle\\ &= \langle A'g,u\rangle,\end{aligned}$$

as required. □

We thus deduce that the natural domain of the transpose A' is the Sobolev space $H^{s,a-\gamma}(M,\tilde V')$.

Lemma 4.4. *For each point divisor δ supported away from* sing M, *the pairings*

$$\begin{aligned}H^{\infty,-\gamma}(M,V')\times H^{\infty,\gamma}(M,V) &\to \mathbb{C},\\ H^{\infty,a-\gamma}(M,\tilde V')\times H^{\infty,\gamma-a}(M,\tilde V) &\to \mathbb{C}\end{aligned}$$

extend to pairings

$$\begin{aligned}L''(\tilde\delta^{-1},A')\times L'(\delta,A) &\to \mathbb{C},\\ L'(\delta^{-1},A')\times L''(\tilde\delta,A) &\to \mathbb{C}.\end{aligned} \tag{4.3}$$

Proof. We claim that pairings (4.3) are in fact defined by integration over $M\setminus \operatorname{supp}\delta$. We only need to show that the integrals which appear here really converge.

Indeed, suppose $u \in L'(\delta, A)$ and $v \in L''(\tilde{\delta}^{-1}, A')$. Then, near a point p_ν with $m_\nu > 0$, we have $u(y) = O(|y - p_\nu|^{a-n-m_\nu+\epsilon})$ where ϵ is any number in the interval $(0, 1)$. The case $a - m_\nu \geq 0$ does not evoke any problem. In the opposite case we have $v(y) = O(|y - p_\nu|^{m_\nu - a})$. Hence $\langle v(y), u(y)\rangle = O(|y - p_\nu|^{-n+\epsilon})$ and the integral $\int_{M \setminus \operatorname{supp} \delta} \langle v(y), u(y)\rangle$ converges near all points p_ν with $m_\nu > 0$ which are the only possible singularities.

The same reasoning applies to the second pairing in (4.3) which completes the proof. □

Now let $\langle \cdot, \cdot \rangle : H' \times H \to \mathbb{C}$ be a bilinear pairing of two complex vector spaces H and H'. Given a vector subspace Σ of H, we define the *annihilator* or *orthogonal complement* $\Sigma^\perp$ of Σ with respect to the pairing $\langle \cdot, \cdot \rangle$ to consist of all $v \in H'$ such that $\langle v, u\rangle = 0$ for each $u \in \Sigma$. Thus, $\Sigma^\perp$ is a vector subspace in H'. And vice versa, if Σ' is a vector subspace of H', then $\Sigma'^\perp$ is defined as a vector subspace in H'.

In the following theorem the annihilator is with respect to the second pairing in (4.3).

Theorem 4.5. 1) *For each* $u \in L'(\delta, A)$ *and* $g \in L'(\delta^{-1}, A')$, *it follows that* $\langle g, \tilde{A}u\rangle = \langle \tilde{A}'g, u\rangle$.

2) $\operatorname{im} \tilde{A} = \left(\ker \tilde{A}'\right)^\perp$, *i.e.,* $f \in \operatorname{im} \tilde{A}$ *if and only if* $f \in L''(\tilde{\delta}, A)$ *and* $\langle g, f\rangle = 0$ *for all* $g \in \ker \tilde{A}'$.

3) $\dim \operatorname{coker} \tilde{A} = \dim \ker \tilde{A}'$.

The relevance of Theorem 4.5 to Theorem 3.1 is clear from the fact that $\ker \tilde{A} = L(\delta, A)$ and $\ker \tilde{A}' = L(\delta^{-1}, A')$. Both the theorems will be proved in parallel in the next section.

Note that part 2) gives solvability conditions for the equation $Au = f$ in the class $L'(\delta, A)$ that consists of sections with prescribed orders of zeros and poles.

5. Proofs

Theorems 3.1 and 4.5 will be proved simultaneously because these proofs intertwine (cf. Gromov and Shubin [6]).

In the sequel $H^{-\infty,\gamma}(M, V)$ stands for the union of the spaces $H^{s,\gamma}(M, V)$ over all $s \in \mathbb{R}$. Obviously,

$$\begin{array}{rcl} H^{-\infty,\gamma}(M, V) & \hookrightarrow & \mathcal{D}'(M \setminus \operatorname{sing} M, V), \\ \mathcal{E}'(M \setminus \operatorname{sing} M, V) & \hookrightarrow & H^{-\infty,\gamma}(M, V), \end{array}$$

for each $\gamma \in \mathbb{R}$.

Denote by $\mathcal{E}'_\delta(M,V)$ the subspace of $\mathcal{E}'(M \setminus \operatorname{sing} M, V)$ consisting of sections u such that $\operatorname{supp} u \subset \operatorname{supp} \delta^+$ and, near $\operatorname{supp} \delta^+$, u can be written as

$$u(y) = \sum_{p_\nu \in \operatorname{supp} \delta^+} \sum_{|\alpha| \le m_\nu - 1} c_{\nu\alpha} D^\alpha \delta(y - p_\nu), \tag{5.1}$$

where $\delta(y)$ is the Dirac measure and $c_{\nu\alpha} \in V_{p_\nu}$. Clearly, $\mathcal{E}'_\delta(M,V) = \mathcal{E}'_{\delta^+}(M,V)$.

Similar spaces will be used for the bundle $\tilde{V}$ and other divisors occurring in the proof.

For every $u \in L'(\delta, A)$ we can find a "regularisation" $\tilde{u} \in H^{-\infty,\gamma}(M,V)$ such that $\tilde{u} = u$ on $M \setminus \operatorname{supp} \delta^+$ and $A\tilde{u} = f_r + f_s$ with $f_r \in H^{\infty,\gamma-a}(M,\tilde{V})$ and $f_s \in \mathcal{E}'_\delta(M,\tilde{V})$. Denote by $\tilde{L}'(\delta, A)$ the space of all such regularisations. Due to the elliptic regularity result and the structure of fundamental solutions (cf. Introduction), the space $\tilde{L}'(\delta, A)$ can be equivalently described as the set of all $\tilde{u} \in H^{-\infty,\gamma}(M,V)$ such that $\tilde{u}$ is of class C^∞ in a neighbourhood of $\operatorname{supp} \delta^-$, $\tilde{u}$ vanishes at each point $p_\nu \in \operatorname{supp} \delta^-$ up to order $-m_\nu - 1$, and $A\tilde{u} = f_r + f_s$ with $f_r \in H^{\infty,\gamma-a}(M,\tilde{V})$ and $f_s \in \mathcal{E}'_\delta(M,\tilde{V})$.

Lemma 5.1. *The sequence*

$$0 \longrightarrow \mathcal{E}'_{\bar{\delta}}(M,V) \xrightarrow{i} \tilde{L}'(\delta, A) \xrightarrow{r} L'(\delta, A) \longrightarrow 0 \tag{5.2}$$

is exact. Here i and r are the natural inclusion and restriction mappings.

Proof. The surjectivity of r means the existence of a regularisation as mentioned before, the injectivity of i is evident. So we must only prove the exactness in the middle term which actually means that if $u \in \mathcal{E}'(M \setminus \operatorname{sing} M, V)$ is supported on $\operatorname{supp} \delta^+$ and $Au \in \mathcal{E}'_\delta(M,\tilde{V})$, then $u \in \mathcal{E}'_{\bar{\delta}}(M,V)$. This is a local assertion, and so it suffices to consider the case $\delta = p^m$ with $m < 0$. But then the statement easily follows from the ellipticity of A. □

Lemma 5.2. *We have*

$$\dim \mathcal{E}'_\delta(M,V) = q \sum_{p_\nu \in \operatorname{supp} \delta^+} \binom{m_\nu + n - 1}{n}. \tag{5.3}$$

Proof. Since $\mathcal{E}'_\delta(M,V) = \bigoplus_{m_\nu > 0} \mathcal{E}'_{p_\nu^{m_\nu}}(M,V)$, it is sufficient to prove that, for every $m_\nu > 0$,

$$\dim \mathcal{E}'_{p_\nu^{m_\nu}}(M,V) = q \binom{m_\nu + n - 1}{n},$$

which reduces to a well-known combinatorial exercise. □

Lemma 5.3. *As defined in Lemma 4.1, the operator $\tilde{A}$ is Fredholm and its index satisfies*

$$\operatorname{ind} \tilde{\mathrm{A}} = \operatorname{ind} \mathrm{A} + \deg \delta. \tag{5.4}$$

Proof. Consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E}'_{\tilde{\delta}}(M,V) & \xrightarrow{i} & \tilde{L}'(\delta,A) & \xrightarrow{r} & L'(\delta,A) & \longrightarrow & 0 \\
 & & \Big\downarrow A^{+} & & \Big\downarrow \hat{A} & & \Big\downarrow \tilde{A} & & \\
0 & \longrightarrow & \mathcal{E}'_{\delta}(M,\tilde{V}) & \xrightarrow{i} & L''(\tilde{\delta},A) \oplus \mathcal{E}'_{\delta}(M,\tilde{V}) & \xrightarrow{\pi} & L''(\tilde{\delta},A) & \longrightarrow & 0
\end{array}$$

where the first row is sequence (5.2), the mappings i and π in the second row are natural inclusion and projection, respectively, and A^{+}, $\hat{A}$ are the restrictions of A to the corresponding spaces of distributions. Since both rows in the diagram are exact, we can assert, by the well-known algebraic property of the Euler characteristic, that

$$\operatorname{ind} \tilde{\mathrm{A}} = \operatorname{ind} \hat{\mathrm{A}} + \operatorname{ind} \mathrm{A}^{+}.$$

On the other hand, A^{+} operates in finite-dimensional spaces, and so its index is equal to the difference of the dimensions of the spaces. Thus,

$$\begin{aligned}
\operatorname{ind} \mathrm{A}^{+} &= \dim \mathcal{E}'_{\tilde{\delta}}(M,V) - \dim \mathcal{E}'_{\delta}(M,\tilde{V}) \\
&= -q \sum_{p_\nu \in \operatorname{supp} \delta^{+}} \left(\binom{m_\nu + n - 1}{n} - \binom{m_\nu - a + n - 1}{n} \right) \\
&= -\deg \delta^{+},
\end{aligned}$$

the second equality being due to (5.3). Hence

$$\operatorname{ind} \tilde{\mathrm{A}} = \operatorname{ind} \hat{\mathrm{A}} + \deg \delta^{+}. \tag{5.5}$$

Now consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{\infty,\gamma}(M,V;\delta) & \xrightarrow{i} & \tilde{L}'(\delta,A) & \xrightarrow{\pi \circ \hat{A}} & \mathcal{E}'_{\delta}(M,\tilde{V}) & \longrightarrow & 0 \\
 & & \Big\downarrow A^{-} & & \Big\downarrow \hat{A} & & \Big\downarrow \mathrm{Id} & & \\
0 & \longrightarrow & L''(\tilde{\delta},A) & \xrightarrow{i} & L''(\tilde{\delta},A) \oplus \mathcal{E}'_{\delta}(M,\tilde{V}) & \xrightarrow{\pi} & \mathcal{E}'_{\delta}(M,\tilde{V}) & \longrightarrow & 0
\end{array}$$

where $H^{\infty,\gamma}(M,V;\delta)$ is defined to consist of all sections $u \in H^{\infty,\gamma}(M,V)$ such that u vanishes at p_ν up to order $-m_\nu - 1$ if $p_\nu \in \operatorname{supp} \delta^{-}$. The operator A^{-} is the restriction of A. Once again, the rows are exact whence

$$\operatorname{ind} \hat{\mathrm{A}} = \operatorname{ind} \mathrm{A}^{-}. \tag{5.6}$$

Finally, consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{\infty,\gamma}(M,V;\delta) & \xrightarrow{i} & H^{\infty,\gamma}(M,V) & \xrightarrow{q} & J_\delta(V) & \longrightarrow & 0 \\ & & \downarrow A^- & & \downarrow A & & \downarrow J(A) & & \\ 0 & \longrightarrow & L''(\tilde\delta,A) & \xrightarrow{i} & H^{\infty,\gamma-a}(M,\tilde V) & \xrightarrow{q} & J_{\tilde\delta}(\tilde V) & \longrightarrow & 0 \end{array}$$

where

$$\begin{aligned} J_\delta(V) &= \frac{H^{\infty,\gamma}(M,V)}{H^{\infty,\gamma}(M,V;\delta)}, \\ J_{\tilde\delta}(\tilde V) &= \frac{H^{\infty,\gamma-a}(M,\tilde V)}{H^{\infty,\gamma-a}(M,\tilde V;\tilde\delta)}, \end{aligned}$$

i and q are the natural inclusion and quotient mappings, and $J(A)$ is the natural quotient mapping. Then we deduce

$$\operatorname{ind} \mathrm{A}^- = \operatorname{ind} \mathrm{A} - \operatorname{ind} \mathrm{J}(\mathrm{A}).$$

Since $J_\delta(V) = \bigoplus_{m_\nu<0} J_{p_\nu^{m_\nu}}(V)$ and, for each $m_\nu < 0$,

$$\dim J_{p_\nu^{m_\nu}}(V) = q \begin{pmatrix} -m_\nu + n - 1 \\ n \end{pmatrix}$$

(cf. (5.3)), we obtain

$$\begin{aligned} \operatorname{ind} \mathrm{J}(\mathrm{A}) &= \dim J_\delta(V) - \dim J_{\tilde\delta}(\tilde V) \\ &= q \sum_{p_\nu \in \operatorname{supp} \delta^-} \left(\begin{pmatrix} -m_\nu + n - 1 \\ n \end{pmatrix} - \begin{pmatrix} -m_\nu - a + n - 1 \\ n \end{pmatrix} \right) \\ &= -\deg \delta^-. \end{aligned}$$

Hence $\operatorname{ind} \mathrm{A}^- = \operatorname{ind} \mathrm{A} + \deg \delta^-$, and so applying (5.5) and (5.6) yields

$$\begin{aligned} \operatorname{ind} \tilde{\mathrm{A}} &= \operatorname{ind} \mathrm{A} + \deg \delta^- + \deg \delta^+ \\ &= \operatorname{ind} \mathrm{A} + \deg \delta, \end{aligned}$$

which completes the proof. □

Equality (5.4) means that $\dim \ker \tilde A = \operatorname{ind} \mathrm{A} + \deg \delta + \dim \operatorname{coker} \tilde{\mathrm{A}}$, and so Theorem 3.1 will be proved once we prove part 3) in Theorem 4.5. We begin with the proof of part 1) in Theorem 4.5.

Lemma 5.4. *For each* $u \in L'(\delta, A)$ *and* $g \in L'(\delta^{-1}, A')$, *we have*

$$\langle g, \tilde{A}u \rangle = \langle \tilde{A}' g, u \rangle. \tag{5.7}$$

Proof. Let us first assume that equality (5.7) holds for all $u \in L'(\delta, A)$ and $g \in L'(\delta^{-1}, A')$ such that $\operatorname{supp} u \cap \operatorname{supp} g$ does not meet the set $\operatorname{sing} M$. Pick a function $\chi \in C^\infty_{comp}(M \setminus \operatorname{sing} M)$ with the property that $\chi \equiv 1$ in a neighbourhood of $\operatorname{supp} \delta$. Then, for each $u \in L'(\delta, A)$ and $g \in L'(\delta^{-1}, A')$, we obtain

$$\begin{aligned}
\langle g, \tilde{A}u \rangle &= \langle g, \tilde{A}(\chi u) \rangle + \langle g, \tilde{A}((1-\chi)u) \rangle \\
&= \langle \tilde{A}' g, \chi u \rangle + \langle g, A((1-\chi)u) \rangle \\
&= \langle \tilde{A}' g, \chi u \rangle + \langle \tilde{A}' g, (1-\chi)u \rangle \\
&= \langle \tilde{A}' g, u \rangle
\end{aligned}$$

the third equality being a consequence of Lemma 4.3. We are thus reduced to proving (5.7) for $u \in L'(\delta, A)$ and $g \in L'(\delta^{-1}, A')$ supported on the smooth part of M.

This latter case is actually treated in Lemma 3.4 of Gromov and Shubin [6]. For the convenience of the reader we repeat the relevant material from [6].

Let us take a function $\omega \in C^\infty_{comp}(\mathbb{R}^n)$ such that $\omega(y) = 1$ if $|y| < \frac{1}{2}$, and $\omega(y) = 0$ if $|y| > 1$. For each $\varepsilon > 0$, set $\omega_\varepsilon(y) = \omega\left(\frac{y}{\varepsilon}\right)$, so that ω_ε is a C^∞ function with a support in the ball $|y| \le \varepsilon$, satisfying $\omega_\varepsilon(y) = 1$ if $|y| < \frac{\varepsilon}{2}$, and $|D^\alpha \omega_\varepsilon(y)| \le c_\alpha \varepsilon^{-|\alpha|}$ for all $y \in \mathbb{R}^n$.

For each point $p_\nu \in \operatorname{supp} \delta$, we fix local coordinates in a neighbourhood O_ν of p_ν. Using these local coordinates we define

$$\chi_\varepsilon(y) = 1 - \sum_{\nu=1}^{N} \omega_\varepsilon(y - p_\nu),$$

for small $\varepsilon > 0$. It follows that $\chi_\varepsilon = 0$ in a neighbourhood of $\operatorname{supp} \delta$, $\chi_\varepsilon = 1$ outside a small neighbourhood of $\operatorname{supp} \delta$, and $|D^\alpha \chi_\varepsilon(y)| \le c_\alpha \varepsilon^{-|\alpha|}$, the derivative being taken in chosen local coordinates. Now using the definition of the transposed operator and the convergence of the integrals defining both sides in (5.7) we get

$$\begin{aligned}
\langle g, \tilde{A}u \rangle &= \lim_{\varepsilon \to 0} \langle \chi_\varepsilon g, Au \rangle \\
&= \lim_{\varepsilon \to 0} \langle A'(\chi_\varepsilon g), u \rangle \\
&= \lim_{\varepsilon \to 0} \langle \chi_\varepsilon A' g, u \rangle + \lim_{\varepsilon \to 0} \langle [A', \chi_\varepsilon] g, u \rangle \\
&= \langle \tilde{A}' g, u \rangle + \lim_{\varepsilon \to 0} \langle [A', \chi_\varepsilon] g, u \rangle,
\end{aligned}$$

where $[A', \chi_\varepsilon] = A' \chi_\varepsilon - \chi_\varepsilon A'$ is the commutator of A' and χ_ε.

It remains to prove that the last limit vanishes. To do this, we observe that $[A', \chi_\varepsilon]$ is a differential operator of order $a-1$ with coefficients supported in a small neighbourhood of $\operatorname{supp}\delta$. In fact,

$$[A', \chi_\varepsilon] = \sum_{|\alpha|\le o-1} A_{\alpha,\varepsilon}(y) D^\alpha$$

close to p_ν, with

$$\operatorname{supp} \mathrm{A}_{\alpha,\varepsilon} \subset \{y\colon \tfrac{\varepsilon}{2} \le |y-p_\nu| \le \varepsilon\}, \tag{5.8}$$

$$|A_{\alpha,\varepsilon}(y)| \le c\,\varepsilon^{-a+|\alpha|}. \tag{5.9}$$

We now proceed by considering two cases: $m_\nu < 0$ and $m_\nu > 0$.

Let $m_\nu < 0$. Then in O_ν we have

$$\begin{aligned} u(y) &= O(|y-p_\nu|^{-m_\nu}),\\ D^\alpha g(y) &= o(|y-p_\nu|^{a-n+m_\nu-|\alpha|}),\end{aligned}$$

and so, on $\operatorname{supp} \mathrm{A}_{\alpha,\varepsilon}$,

$$\begin{aligned} u(y) &= O(\varepsilon^{-m_\nu}),\\ D^\alpha g(y) &= o(\varepsilon^{a-n+m_\nu-|\alpha|}),\end{aligned}$$

which is due to (5.8). Hence (5.9) gives $\langle A_{\alpha,\varepsilon} D^\alpha g(y), u(y)\rangle = o(\varepsilon^{-n})$ and, since the volume of $\operatorname{supp} \mathrm{A}_{\alpha,\varepsilon}$ is $O(\varepsilon^n)$,

$$\int_{O_\nu} \langle [A', \chi_\varepsilon] g(y), u(y)\rangle = o(1) \quad \text{as} \quad \varepsilon \to 0, \tag{5.10}$$

as required.

Let $m_\nu > 0$. On the support of $A_{\alpha,\varepsilon}$, we similarly have

$$\begin{aligned} u(y) &= o(\varepsilon^{a-n-m_\nu}),\\ D^\alpha g(y) &= O(\varepsilon^{m_\nu-|\alpha|})\end{aligned}$$

whence $\langle A_{\alpha,\varepsilon} D^\alpha g(y), u(y)\rangle = o(\varepsilon^{-n})$. This clearly forces (5.10), and the proof is complete. □

Let $\langle\cdot,\cdot\rangle\colon H' \times H \to \mathbb{C}$ be a bilinear pairing of two complex spaces H and H'. We say that this pairing is *non-degenerate* if both $H^\perp$ and $H'^\perp$ are trivial, i.e., consist of zero elements only.

Lemma 5.5. *Pairings (4.3) are non-degenerate.*

Proof. The statement is evident because all spaces in (4.3) contain smooth sections of the corresponding bundles supported away from $\operatorname{sing} \mathrm{M} \cup \operatorname{supp} \delta$ and, on the other hand, the elements of these spaces are uniquely determined by their (smooth) restrictions to $M \setminus (\operatorname{sing} \mathrm{M} \cup \operatorname{supp} \delta)$. □

Now we need the following abstract lemma from [6] which we reproduce with the proof for the sake of completeness.

Lemma 5.6. *Suppose $\langle \cdot, \cdot \rangle : H' \times H \to \mathbb{C}$ is a non-degenerate bilinear pairing of complex spaces H and H'. Then, for each vector subspace Σ of H, we have $\Sigma \subset (\Sigma^{\perp})^{\perp}$ and*

$$
(5.11) \qquad \begin{array}{rcll} \dim \Sigma & = & \operatorname{codim} \Sigma^{\perp}, & \text{if } \dim \Sigma < \infty; \\ \operatorname{codim} \Sigma & \geq & \dim \Sigma^{\perp}, & \text{if } \dim \Sigma^{\perp} < \infty. \end{array}
$$

Proof. The inclusion $\Sigma \subset (\Sigma^{\perp})^{\perp}$ is obvious. From this we conclude that $\operatorname{codim} \Sigma \geq \operatorname{codim} (\Sigma^{\perp})^{\perp}$. Thus, the first formula of (5.11) implies the second one and we have only to prove the first formula.

To do this, we first observe that, since $H'^{\perp} = \{0\}$, for each finite linearly independent system $(h_i)_{i \in I}$ in H there is a system $(h'_i)_{i \in I}$ in H' parametrised with the same family of indices, such that $H' = \left(\mathcal{L}\,(h_i)_{i \in I}\right)^{\perp} \oplus \mathcal{L}\,(h'_i)_{i \in I}$. Here, $\mathcal{L}\,(h_i)_{i \in I}$ means the linear span of $(h_i)_{i \in I}$. It follows that $\dim \Sigma \geq \operatorname{codim} \Sigma^{\perp}$, and so it remains to prove the reverse inequality.

Consider the natural mapping $\Sigma \to \mathrm{Hom}_{\mathbb{C}}(\mathrm{H}'/\Sigma^{\perp}, \mathbb{C})$ given by $h \mapsto \mathcal{F}_h$, where

$$
\mathcal{F}_h \left(h' + \Sigma^{\perp}\right) = \langle h', h \rangle, \quad \text{for} \quad h' \in H'.
$$

From $H'^{\perp} = \{0\}$ we deduce that the mapping $h \mapsto \mathcal{F}_h$ is injective. Combining this with the fact that $\operatorname{codim} \Sigma^{\perp} < \infty$, we derive

$$
\begin{array}{rcl} \operatorname{codim} \Sigma^{\perp} & = & \dim H'/\Sigma^{\perp} \\ & = & \dim \mathrm{Hom}_{\mathbb{C}}(\mathrm{H}'/\Sigma^{\perp}, \mathbb{C}) \\ & \geq & \dim \Sigma, \end{array}
$$

as required. □

Lemma 5.7. *Under the second pairing of (4.3), we have*

$$
\left(\operatorname{im} \tilde{\mathrm{A}}\right)^{\perp} = \ker \tilde{A}'.
$$

Proof. By definition, $\ker \tilde{A}'$ consists of all sections $g \in L'(\delta^{-1}, A')$ such that $A'g = 0$ in $M \setminus \operatorname{supp} \delta$. On the other hand, $\operatorname{im} \tilde{\mathrm{A}}$ contains all sections of the form Au, with $u \in H^{\infty,\gamma}(M, V)$ supported away from $\operatorname{supp} \delta$. Combining this with Lemma 5.4, we arrive at the desired conclusion. □

Proofs of Theorems 3.1 and 4.5. Lemmas 5.6 and 5.7 imply

$$\operatorname{im} \tilde{\mathrm{A}} \subset \left(\ker \tilde{A}'\right)^{\perp}, \tag{5.12}$$

$$\operatorname{codim} \operatorname{im} \tilde{\mathrm{A}} \geq \dim \ker \tilde{A}', \tag{5.13}$$

and so we are left with the task of showing that both the inclusion and the inequality are actually equalities.

By (5.11), $\dim \ker \tilde{A}' = \operatorname{codim} \left(\ker \tilde{\mathrm{A}}'\right)^{\perp}$, hence equality in (5.13) implies equality in (5.12). Since $\operatorname{codim} \operatorname{im} \tilde{\mathrm{A}} = \dim \operatorname{coker} \tilde{\mathrm{A}}$, we only need to show that

$$\dim \operatorname{coker} \tilde{\mathrm{A}} = \dim \ker \tilde{\mathrm{A}}'. \tag{5.14}$$

For this purpose, we invoke the remark after Lemma 5.3 and (5.13) to see that

$$\begin{aligned} \dim \ker \tilde{A} &= \operatorname{ind} \mathrm{A} + \deg \delta + \dim \operatorname{coker} \tilde{\mathrm{A}} \\ &\geq \operatorname{ind} \mathrm{A} + \deg \delta + \dim \ker \tilde{\mathrm{A}}'. \end{aligned}$$

We now apply this argument again, with A replaced by A' and δ replaced by δ^{-1}, to obtain

$$\begin{aligned} \dim \ker \tilde{A}' &\geq \operatorname{ind} \mathrm{A}' + \deg \delta^{-1} + \dim \ker \tilde{\mathrm{A}} \\ &= -\operatorname{ind} \mathrm{A} - \deg \delta + \dim \ker \tilde{\mathrm{A}}. \end{aligned}$$

Combining these opposite inequalities yields

$$\operatorname{ind} \mathrm{A} + \deg \delta + \dim \operatorname{coker} \tilde{\mathrm{A}} = \operatorname{ind} \mathrm{A} + \deg \delta + \dim \ker \tilde{\mathrm{A}}',$$

which is equivalent to (5.14). This completes the proofs of Theorems 3.1 and 4.5. □

6. Contributions of Singular Points

The case where the support of a divisor δ is allowed to meet the set of singular points of M presents a much more delicate problem. The reason is that a solution

u to $Au = 0$ in a punctured neighbourhood of a point $p \in \operatorname{sing} \mathrm{M}$ need not have an expansion like (0.2). Hence the question arises of finding a proper substitute of solutions with $\operatorname{ord}(\mathrm{u}, \mathrm{p}) \geq -\mathrm{m}$ as well as of specifying the degree of a divisor δ with $\operatorname{supp} \delta \cap \operatorname{sing} \mathrm{M} \neq \emptyset$.

By private communication M. Gromov informed us that the contributions of conical points $p \in \operatorname{supp} \delta$ can be evaluated by expanding solutions as series in Bessel functions. But our approach is based on quite different ideas from the analysis on a manifold with conical points.

We begin with an equivalent description of the order (of "zero") of a solution at a point $p \in M \setminus \operatorname{sing} \mathrm{M}$ in terms of the weighted Sobolev spaces. In order to get asymptotic results, it is necessary to impose some restrictions on the order of A. Namely, we assume that $a < n$.

Let u be a solution of $Au = 0$ in $O \setminus \{p\}$, where O is a coordinate neighbourhood of p on the smooth part of M. If O is sufficiently small, then A has a fundamental solution $\Phi \in \Psi^{-\mathrm{a}}(\tilde{\mathrm{V}}|_{\mathrm{O}}, \mathrm{V}|_{\mathrm{O}})$ in O. Since $a < n$, the kernel of Φ bears the estimates

$$D_y^\alpha D_{y'}^\beta \Phi(y, y') = O(|y - y'|^{a-n-|\alpha|-|\beta|}), \quad \text{for all} \quad \alpha, \beta \in \mathbb{Z}_+^n, \tag{6.1}$$

uniformly on compact subsets of $O \times O$. Combining this with (0.2) yields the following assertion.

Lemma 6.1. *Let $m \in \mathbb{Z}$. In order that* $\operatorname{ord}(\mathrm{u}, \mathrm{p}) \geq -\mathrm{m}$ *it is necessary and sufficient that*

$$\begin{array}{rcll} u & \in & H_{loc}^{\infty, -m-0}(O, V), & \textit{for} \quad m \leq 0; \\ u & \in & H_{loc}^{\infty, a-n+1-m-0}(O, V), & \textit{for} \quad m > 0. \end{array}$$

Note that by $H_{loc}^{s,\gamma}(O, V)$ we mean the weighted Sobolev spaces as above, constructed as if p be an artificial conical point of M. The lemma is still true if we replace the exponent $s = \infty$ by $s = 0$, for u is a solution of $Au = 0$ away from p.

Proof. Indeed, from (0.2) and (6.1) it follows that $\operatorname{ord}(\mathrm{u}, \mathrm{p}) \geq -\mathrm{m}$ if and only if

$$\begin{array}{rcll} |y - p|^{-\frac{n}{2}+m+0} u & \in & L_{loc}^2(O, V), & \text{for} \quad m \leq 0; \\ |y - p|^{-a+\frac{n}{2}-1+m+0} u & \in & L_{loc}^2(O, V), & \text{for} \quad m > 0, \end{array}$$

the space $L_{loc}^2(O, V)$ being defined with respect to the volume form dy on O. Since in the polar coordinates with centre at p we have $dy = r^{n-1} dr dx$, where $r = |y - p|$ and dx is the area form on the unit sphere S^{n-1}, the lemma follows. □

As described in Lemma 6.1, the notion of the order can be extended also to the singular points of M. The obvious asymmetry in m in the above two conditions is explained by the fact that, for $m > 0$, the "weakest" singularity of u at p is due to the term $\Phi(y,p)c_0$, i.e., $O(|y-p|^{a-n})$. While the definition of $\operatorname{ord}(u,p) \geq -m$ for $m \leq 0$ is irrelevant to the concrete differential operator A and agrees with the heuristic concept of the multiplicity of a zero, the definition of $\operatorname{ord}(u,p) \geq -m$ for $m > 0$ invokes A and differs from the heuristic concept unless $a = n-1$. On the other hand, a solution $u \in H^{s,\gamma}_{loc}(O,V)$ to $Au = 0$ in a punctured neighbourhood $O \setminus \{v\}$ of a conical point v is known to bear asymptotics of the form

$$(6.2) \qquad u(r,x) = \omega(r) \sum_{\mu=1}^{M} \sum_{j=0}^{j_\mu} r^{iz_\mu} (\log r)^j c_{\mu j}(x) \mod H^{\infty,\gamma+l}_{loc}(O,V)$$

close to v, where $\omega \in C^\infty_{comp}(O)$ is a cut-off function for the point v, $z_\mu \in \mathbb{C}$ are non-bijectivity points of the conormal symbol $\sigma_{\mathcal{M}}(A)(v,z)$ lying in the strip $-\gamma - l < \Im z < -\gamma$, with $l > 0$, and $c_{\mu j}$ are functions of finite-dimensional subspaces Σ_μ of $C^\infty(X_v) \otimes V_v$ on the base X_v. Hence the "orders" of such solutions can fill in the interval $(a-n, 0)$, too. For this reason we choose in favour of the definition of $\operatorname{ord}(u,p) \geq -m$ for $m \leq 0$, thus removing the asymmetry in $m \in \mathbb{Z}$.

DEFINITION 6.2. Let $p \in M$ and $m \in \mathbb{R}$. For a solution u to $Au = 0$ in $O \setminus \{p\}$, we write $\operatorname{ord}(u,p) \geq m$ if $u \in H^{\infty, m-\frac{1}{2}}_{loc}(O,V)$.

The correction $\frac{1}{2}$ in the exponent is chosen by purely aesthetic reasons. What we do in the case $p \notin \operatorname{sing} M$ is actually that we regard p as an artificial conical point of M by blowing up M at p.

The point divisors δ we have to deal with under this definition of $\operatorname{ord}(u,p)$ are still elements of a free abelian group generated by points of the manifold M. These are of the form $\delta = p_1^{m_1} \dots p_N^{m_N}$, now with $m_1, \dots, m_N$ real numbers. Under Definition 6.2, the inverse divisor occurring in (0.4) and (3.3) should be $\delta^{-1} = p_1^{-m_1+n-1-a} \dots p_N^{-m_N+n-1-a}$, which prompts us a group operation in the set of all point divisors. Namely, for $\delta' = p_1^{m'_1} \dots p_N^{m'_N}$ and $\delta'' = p_1^{m''_1} \dots p_N^{m''_N}$, we set

$$\delta'\delta'' = p_1^{m'_1 + m''_1 - (n-1-a)} \dots p_N^{m'_N + m''_N - (n-1-a)},$$

depending on the dimension of the underlying manifold M and the order of the differential operator A. This agrees with the usual operation in case A is of order $a = n-1$, as is the case for the Cauchy-Riemann operator on a Riemann surface.

A divisor $\delta = p_1^{m_1} \dots p_N^{m_N}$ is said to be *non-characteristic* for A if, for each ν, either $p_\nu \notin \operatorname{sing} M$ or $p_\nu \in \operatorname{sing} M$ and $\sigma_{\mathcal{M}}(A)(p_\nu, z)$ is invertible on the line $\Gamma_{m_\nu + \frac{1}{2}}$.

We are now in a position to extend Theorem 3.1 to the case of point divisors meeting the set of singular points. For a point divisor $\delta = p_1^{m_1} \dots p_N^{m_N}$, we consider two spaces

$$L(\delta, A) = \{u \in H^{\infty,\gamma}_{loc}(M \setminus \operatorname{supp}\delta, V) \colon Au = 0,\ \operatorname{ord}(u, p_\nu) \geq -m_\nu\},$$
$$L(\delta^{-1}, A') = \{g \in H^{\infty,a-\gamma}_{loc}(M \setminus \operatorname{supp}\delta, \tilde{V}') \colon A'g = 0,\ \operatorname{ord}(g, p_\nu) \geq m_\nu - n + 1 + a\}.$$

Theorem 6.3. *Let A be a differential operator on M elliptic with respect to a weight tuple $\gamma \in \mathbb{R}^I$. Assume that δ is a point divisor on M non-characteristic for A. Then,*

$$\dim_{\mathbb{C}} L(\delta, A) = \operatorname{ind} \mathrm{A} + \deg \delta + \dim_{\mathbb{C}} \mathrm{L}(\delta^{-1}, \mathrm{A}'). \tag{6.3}$$

Just as in (0.3) the degree of δ occurring in (6.3) is made up of contributions of the points p_ν, i.e., $\deg \delta = \sum_{\nu=1}^{N} \deg p_\nu^{m_\nu}$.

To describe the contributions of the points p_ν lying on the smooth part of M, denote by $[m]$ the integral part of $m \in \mathbb{R}$, i.e., the largest of the integers not exceeding m. Then,

$$\deg p_\nu^{m_\nu} = \begin{cases} -k\left(\binom{[-m_\nu-\frac{1}{2}]+n}{n} - \binom{[-m_\nu-\frac{1}{2}]-a+n}{n}\right), & \text{if } [-m_\nu-\frac{1}{2}] \geq -1; \\ 0, & \text{if } a-n \leq [-m_\nu-\frac{1}{2}] < -1; \\ k\left(\binom{-[-m_\nu-\frac{1}{2}]+a-1}{n} - \binom{-[-m_\nu-\frac{1}{2}]-1}{n}\right), & \text{if } [-m_\nu-\frac{1}{2}] < a-n \end{cases}$$

(cf. (0.3)).

To evaluate the contributions of points $p_\nu \in \operatorname{sing} \mathrm{M}$ we need more information on the conormal symbol $\sigma_{\mathcal{M}}(A)$. Pick a conical point v of M. The *spectrum* of $\sigma_{\mathcal{M}}(A)$ at v is said to consist of all points $z \in \mathbb{C}$ such that $\sigma_{\mathcal{M}}(A)(v,z)$ fails to be an isomorphism $H^s(X_v) \otimes V_v \to H^{s-a}(X_v) \otimes \tilde{V}_v$ for some $s \in \mathbb{R}$. Recall that X_v stands for a cross-section of X close to v, being a compact smooth closed manifold. We denote by $\operatorname{spec} \sigma_{\mathcal{M}}(\mathrm{A})(\mathrm{v}, \cdot)$ the spectrum of $\sigma_{\mathcal{M}}(A)$ at v. From the invertibility of the (compressed) principal symbol of A over the set of singular points of M it follows that $\sigma_{\mathcal{M}}(A)(v,z)$ is a holomorphic family of elliptic differential operators over X_v parametrised by $z \in \mathbb{C}$. Moreover, the restriction of this family to each horizontal line is an elliptic operator on X_v with the parameter z. Hence we deduce that the spectrum of $\sigma_{\mathcal{M}}(A)$ at v is a discrete set in the complex plane, whose intersection with each horizontal strip of finite width is finite. Away from the spectrum the inverse $\sigma_{\mathcal{M}}(A)(v,z)^{-1}$ is well known to be a holomorphic family of pseudodifferential operators in $\Psi^{-\mathrm{a}}(\mathrm{X_v}) \otimes \operatorname{Hom}_{\mathbb{C}}(\tilde{\mathrm{V}}_\mathrm{v}, \mathrm{V_v})$. A further observation is that $\sigma_{\mathcal{M}}(A)(v,z)^{-1}$ is actually a meromorphic family over the complex plane, with poles of a finite rank at the points of $\operatorname{spec} \sigma_{\mathcal{M}}(\mathrm{A})(\mathrm{v}, \cdot)$. This means that, for each $z_\mu \in \operatorname{spec} \sigma_{\mathcal{M}}(\mathrm{A})(\mathrm{v}, \cdot)$, we can write

$$\sigma_{\mathcal{M}}(A)(v,z)^{-1} = \sum_{j=1}^{order\, z_\mu} S_{\mu j}(v)\,(z - z_\mu)^{-j} + R_\mu(z), \tag{6.4}$$

where $S_{\mu j}(v)$ are smoothing operators of finite rank over X_v and $R_\mu(z)$ is holomorphic in a neighbourhood of $z = z_\mu$. It follows immediately that the singular range of

$\sigma_{\mathcal{M}}(A)(v,z)^{-1}$ at z_μ, i.e., the space

$$\left\{ u = \sum_{j=1}^{order\, z_\mu} u_j (z - z_\mu)^{-j} : \begin{array}{c} u_j \in C^\infty(X_v) \otimes V_v, \\ \sigma_{\mathcal{M}}(A)(v,z)u \text{ is holomorphic near } z_\mu \end{array} \right\},$$

is finite-dimensional. The dimension of this space is known as the *rank* of the pole z_μ and is denoted by $\operatorname{rank} z_\mu$ (following the notation in Melrose [10, 5.2]).

Now, for $p_\nu = v_i$ a conical point of M, we have

$$\deg p_\nu^{m_\nu} = \operatorname{sgn}\left(m_\nu + \frac{1}{2} + \gamma_i\right) \sum_{\substack{z \in \operatorname{spec} \sigma_{\mathcal{M}}(A)(p_\nu, \cdot) \\ \Im z \in \left(-\gamma_i, m_\nu + \frac{1}{2}\right)}} \operatorname{rank} z, \tag{6.5}$$

the sum in the right-hand side being 0 if $m_\nu + \frac{1}{2} + \gamma_i = 0$. Note that the lines $\Gamma_{-\gamma_i}$ and $\Gamma_{m_\nu + \frac{1}{2}}$ are, by assumption, free of the points of the spectrum of $\sigma_{\mathcal{M}}(A)(p_\nu, \cdot)$.

A particular case of Theorem 6.3 is the *Relative Index Theorem* of Melrose and Mendoza [9] (cf. also [17, 2.2.3], [10, 6.2]), which corresponds to the case $\operatorname{supp} \delta = \operatorname{sing} M$.

Proof of Theorem 6.3. We are going to deduce Theorem 6.3 from Theorem 3.1 and the Relative Index Theorem cited above.

To this end, we denote by A_β, for $\beta \in \mathbb{R}^I$, the operator A regarded as mapping $H^{s,\beta}(M,V) \to H^{s-a,\beta-a}(M,\tilde{V})$, the exponent s being immaterial in the sequel. If A is elliptic with respect to a weight tuple w, then $\operatorname{ind} A_\beta = \dim \ker A_\beta - \dim \ker A'_\beta$ where both $\ker A_\beta$ and $\ker A'_\beta$ are independent of s. Here, we abbreviate $(A_\beta)'$ to A'_β.

Consider a weight tuple $\gamma' = (\gamma'_1, \dots, \gamma'_I)$ in $\mathbb{R}^I$ defined as follows. Pick $i = 1, \dots, I$. If $v_i \notin \operatorname{supp} \delta$, then $\gamma'_i = \gamma_i$. If $v_i = p_\nu$ for some $\nu = 1, \dots, N$, then $\gamma'_i = -m_\nu - \frac{1}{2}$. Since δ is non-characteristic for A, we conclude that the differential operator A is elliptic with respect to γ'.

We next reduce the divisor $\delta = p_1^{m_1} \dots p_N^{m_N}$ to its part δ' supported away from the set of singular points. Namely, pick $\nu = 1, \dots, N$. If $p_\nu \notin \operatorname{sing} M$, then we allow p_ν to occur in the new divisor δ' with just the same weight m_ν. If $p_\nu \in \operatorname{sing} M$, then we assign the weight 0 to p_ν, thus omitting p_ν as part of δ'. The divisor δ' so obtained does not meet the set $\operatorname{sing} M$.

Now, a trivial verification shows that the space $L(\delta, A)$ referring to the operator $A = A_\gamma$ coincides with the space $L(\delta', A_{\gamma'})$, the operator A here being $A_{\gamma'}$. Thus,

$$\begin{aligned} L(\delta, A) &= L(\delta', A_{\gamma'}), \\ L(\delta^{-1}, A) &= L(\delta'^{-1}, A'_{\gamma'}). \end{aligned}$$

The operator $A_{\gamma'}$ and the divisor δ' fulfill the condition of Theorem 3.1, hence

(3.3) yields

$$\dim_{\mathbb{C}} L(\delta, A) = \operatorname{ind} A_{\gamma'} + \deg \delta' + \dim_{\mathbb{C}} L(\delta^{-1}, A'). \tag{6.6}$$

On the other hand, we can assert, by the Relative Index Theorem (cf. *ibid*), that

$$\operatorname{ind} A_{\gamma'} = \operatorname{ind} A + \sum_{i=1}^{I} \operatorname{sgn}\,(\gamma_i - \gamma_i') \sum_{\substack{z \in \operatorname{spec} \sigma_{\mathcal{M}}(A)(v_i,\cdot) \\ \Im z \in (-\gamma_i, -\gamma_i')}} \operatorname{rank} z. \tag{6.7}$$

Combining (6.6), (6.7) and Lemma 6.1, we arrive at equality (6.3), as required. □

From Theorem 0.1 and Lemma 3.2, one may conjecture that formula (6.5) is still true for the points p_ν lying on the smooth part of M, now with γ_i *any* number in the interval $\left(a - \frac{n}{2}, \frac{n}{2}\right)$. We postpone this discussion until Section 8.

7. Rigged Divisors

In this section we extend Theorem 6.3 to point divisors δ carrying information on asymptotics of solutions at the points occurring in δ. Since asymptotic expansions like (0.2) near points p lying on the smooth part of M are very special cases of those at singular points and since each point $p \in M \setminus \operatorname{sing} M$ can be thought of as an artificial conical point, we will restrict our attention to the divisors δ supported on the set $\operatorname{sing} M$.

As described in Section 6., a solution $u \in H^{s,\gamma}(M, V)$ of the equation $Au = 0$ bears asymptotics of the form (6.2) close to a conical point $v \in M$. The sum in (6.2) is over all points z_μ of the spectrum of the conormal symbol $\sigma_{\mathcal{M}}(A)(v, \cdot)$, which lie in the strip $-\gamma - l < \Im z < -\gamma$, while $j_\mu + 1$ is the order of the pole z_μ of the inverse symbol (cf. (6.4)). Thus, the number $(j_\mu + 1) \dim_{\mathbb{C}} \Sigma_\mu$ is in fact equal to the rank of the pole z_μ.

A divisor $\delta = p_1^{m_1} \dots p_N^{m_N}$ specifies only the strips in the complex plane, in which asymptotics at p_ν are allowed. As for the spaces Σ_μ at each point p_ν, they depend on the particular splitting of coordinates close to p_ν. Indeed, the representation of a solution in the form (6.2) depends on the choice of coordinates. It follows that in order to specify the spaces Σ_ν at the points occurring in the divisor we have to fix cylindrical structures near these points. When specifying the spaces Σ_ν, we arrive at what Gromov and Shubin [7] called the *rigged divisors*.

Let us recall the concept of an asymptotic type which is relevant to our theory (cf. Schulze [17, 1.2.1]). Pick a conical point $v \in M$. A *weight datum* at v is a pair $w = (\gamma, [-l, 0))$ consisting of a number $\gamma \in \mathbb{R}$ and a finite interval $[-l, 0)$, $l > 0$, on the real axis. By an *asymptotic type* associated with the weight datum w is meant any collection $\mathrm{as} = (\mathrm{z}_\mu, \mathrm{j}_\mu, \Sigma_\mu)_{\mu=1,\dots,\mathrm{M}}$, where z_μ are complex numbers in the strip $-\gamma - l \le \Im z < -\gamma$, j_μ are non-negative integers, and Σ_μ are finite-dimensional subspaces of $C^\infty(X_v) \otimes V_v$. For simplicity we ignore the dependence of 'as' on the vector bundle V, e.g. in notation.

DEFINITION 7.1. The rank of an asymptotic type $\mathrm{as} = (\mathrm{z}_\mu, \mathrm{j}_\mu, \Sigma_\mu)_{\mu=1,\dots,\mathrm{M}}$ is defined to be

$$\operatorname{rank} \mathrm{as} = \sum_{\mu=1}^{\mathrm{M}} (\mathrm{j}_\mu + 1) \dim_{\mathbb{C}} \Sigma_\mu .$$

If u is a section of V in a punctured neighbourhood of the point v, then we write $\mathrm{as}(\mathrm{u}, \mathrm{v}) \in \mathrm{as}$ if $u = \sum_{\mu=1}^{M} \sum_{j=0}^{j_\mu} r^{iz_\mu} (\log r)^j c_{\mu j}(x)$ modulo $H^{s,\gamma+l}_{loc}(O, V)$, for some $c_{\mu j} \in \Sigma_\nu$ and some neighbourhood O of v. To deal with such sections, we invoke a concept of weighted Sobolev spaces with asymptotics on a manifold with conical points.

Given an asymptotic type $\mathrm{as} = (\mathrm{z}_\mu, \mathrm{j}_\mu, \Sigma_\mu)_{\mu=1,\dots,\mathrm{M}}$, we denote by $\mathcal{A}_{\mathrm{as}}$ the finite-dimensional space spanned by the functions

$$\left(\omega(t)\, r^{iz_\mu} \,(\log r)^j \, c_{\mu j}(x)\right)_{\substack{\mu=1,\dots,M \\ j=0,1,\dots,j_\mu}},$$

with $c_{\mu j} \in \Sigma_\mu$ and ω a cut-off function for the point v. We can certainly assume that ω is supported in a sufficiently small neighbourhood O of v, and so $\mathcal{A}_{\mathrm{as}}$ can be identified within the space $H^{\infty,\gamma}_{comp}(O, V)$. Obviously, the dimension of $\mathcal{A}_{\mathrm{as}}$ is equal to the rank of the asymptotic type 'as'.

Now, we assign a weight datum $w_i = (\gamma_i, [-l_i, 0))$ to every conical point v_i, $i = 1, \dots, I$. Let $\mathrm{as_i}$ be an asymptotic type associated with w_i and let $\mathrm{as} = (\mathrm{as_1}, \dots, \mathrm{as_I})$. For every $i = 1, \dots, I$, we fix a cut-off function ω_i close to v_i, such that $\operatorname{supp} \omega_{\mathrm{i}'} \cap \operatorname{supp} \omega_{\mathrm{i}''} = \emptyset$ unless $i' = i''$. Then, the sum $\mathcal{A}_{\mathrm{as}} = \mathcal{A}_{\mathrm{as_1}} \oplus \dots \oplus \mathcal{A}_{\mathrm{as_I}}$ is direct because the spaces involved are supported by disjoint sets, the space $\mathcal{A}_{\mathrm{as_i}}$ relying on ω_i. We set

$$H^{s,\gamma}_{\mathrm{as}}(M, V) = H^{s,\gamma+l}(M, V) \oplus \mathcal{A}_{\mathrm{as}}, \tag{7.1}$$

where $l = (l_1, \dots, l_I)$. We endow $H^{s,\gamma}_{\mathrm{as}}(M, V)$ with the topology of the direct sum of two normed spaces.

This definition of a space with asymptotics is slightly different from that in the cone theory (cf. Schulze [18, 1.1.2]). However, (7.1) seems to suit better the purposes of the present paper.

An elliptic differential operator $A \in \mathrm{Diff}^{\mathrm{a}}(\mathrm{V}, \tilde{\mathrm{V}})$ is known to behave properly in the spaces with asymptotics. Namely, suppose A is elliptic with respect to a weight tuple $\gamma \in \mathbb{R}^I$ and let $\mathrm{as} = (\mathrm{as_1}, \dots, \mathrm{as_N})$ be a tuple of asymptotic types, $\mathrm{as_i}$ being associated with a weight datum $w_i = (\gamma_i, [-l_i, 0))$. Then, there exists a unique tuple $\tilde{\mathrm{as}} =$

$(\tilde{\mathrm{as}}_1, \ldots, \tilde{\mathrm{as}}_N)$ of asymptotic types associated with weight data $\tilde{w}_i = (\gamma_i - a, [-l_i, 0))$, now for the bundle $\tilde{V}$, such that the operator $A\colon H^{s,\gamma}_{\mathrm{as}}(M,V) \to H^{s-a,\gamma-a}_{\tilde{\mathrm{as}}}(M,\tilde{V})$ is Fredholm and, moreover, $u \in H^{s,\gamma}(M,V)$ and $\mathrm{as}(\mathrm{Au}, \mathrm{v_i}) \in \tilde{\mathrm{as}}_{\mathrm{i}}$ imply $\mathrm{as}(\mathrm{u}, \mathrm{v_i}) \in \mathrm{as_i}$. For more details we refer the reader to Schulze [18, 1.2.2].

The point divisors we consider are of the form $\delta = p_1^{\mathrm{as}_1} \ldots p_N^{\mathrm{as}_N}$, where $p_\nu \in \operatorname{sing} \mathrm{M}$ and as_ν is an asymptotic type at p_ν associated with a weight datum $w_\nu = (\gamma_i - l_\nu, [-l_\nu, 0))$, the number i being defined by $p_\nu = v_i$. They are no longer elements of any natural group generated by points of M.

A divisor δ is said to be *non-characteristic* for A if $\sigma_{\mathcal{M}}(A)(p_\nu, z)$ is invertible on the line $\Gamma_{l_\nu - \gamma_i}$, for each $\nu = 1, \ldots, N$. If such is the case, we define the degree of δ to be

$$\deg \delta = \sum_{\nu=1}^{N} (\operatorname{rank} \mathrm{as}_\nu - \operatorname{rank} \tilde{\mathrm{as}}_\nu)\,,$$

$\tilde{\mathrm{as}}_\nu$ being chosen for as_ν as described above.

As mentioned, asymptotic expansion (6.2) is a good substitute for (0.2) in the case where p is a singular point of M. It is worth pointing out that the definition of $\deg \delta$ in this section agrees with (0.3).

We now proceed as we did before. For a point divisor $\delta = p_1^{\mathrm{as}_1} \ldots p_N^{\mathrm{as}_N}$, we introduce two spaces

$$\begin{aligned} L(\delta, A) &= \{u \in H^{\infty,\gamma}_{loc}(M \backslash \operatorname{supp} \delta, \mathrm{V})\colon \mathrm{Au} = 0,\ \mathrm{as}(\mathrm{u}, \mathrm{p}_\nu) \in \mathrm{as}_\nu\}, \\ L(\delta^{-1}, A') &= \{g \in H^{\infty, a-\gamma}_{loc}(M \backslash \operatorname{supp} \delta, \tilde{\mathrm{V}}')\colon \mathrm{A'g} = 0, \operatorname{ord}(\mathrm{g}, \mathrm{p}_\nu) \geq \mathrm{l}_\nu - \gamma_{\mathrm{i}} + \tfrac{1}{2} + \mathrm{a}\}, \end{aligned}$$

'ord' being as in Definition 6.2.

Theorem 7.2. *Suppose A is a differential operator on M elliptic with respect to a weight tuple $\gamma \in \mathbb{R}^I$. Then, for each point divisor δ supported on* $\operatorname{sing}\mathrm{M}$ *and non-characteristic for A, we have*

$$\dim_{\mathbb{C}} L(\delta, A) = \operatorname{ind} \mathrm{A} + \deg \delta + \dim_{\mathbb{C}} \mathrm{L}(\delta^{-1}, \mathrm{A}'). \tag{7.2}$$

The proof of Theorem 7.2 is completely independent of Theorem 3.1; in fact it is even simpler in this generality.

Proof. Let the weight tuple $\gamma' = (\gamma_1', \ldots, \gamma_I')$ be defined as follows. Pick $i = 1, \ldots, I$. If $v_i \in \operatorname{supp} \delta$, i.e., $v_i = p_\nu$ for some $\nu = 1, \ldots, N$, then $\gamma_i' = \gamma_i - l_\nu$. If $v_i \notin \operatorname{supp} \delta$, then $\gamma_i' = \gamma_i$.

In a similar way we define the tuple $\mathrm{as}' = (\mathrm{as}_1', \ldots, \mathrm{as}_{\mathrm{N}}')$ of asymptotic types. Namely, if $v_i = p_\nu$ for some $\nu = 1, \ldots, N$, then $\mathrm{as}_{\mathrm{i}}' = \mathrm{as}_\nu$. Otherwise we put $\mathrm{as}_{\mathrm{i}}' = 0$, the corresponding weight interval being empty.

Set

$$\begin{aligned} H_1 &= H^{s,\gamma'}_{\mathrm{as}'}(M,V), \\ H_2 &= H^{s-a,\gamma'-a}_{\tilde{\mathrm{as}}'}(M,\tilde{V}) \end{aligned}$$

and denote by T the operator $H_1 \to H_2$ induced by A.

Since δ is non-characteristic for A, the operator A is elliptic with respect to the weight tuple γ'. Hence it follows that T is a Fredholm operator, and so $\mathrm{ind\,T} = \mathrm{dim\,ker\,T} - \mathrm{dim\,coker\,T}$ is finite.

It is clear from the definition of the space $L(\delta, A)$ that $L(\delta, A) = \ker T$. Moreover, a simple verification shows that $L(\delta^{-1}, A')$ coincides with the kernel of the operator $A' \colon H^{-s+a,-\gamma'+a}(M, \tilde{V}') \to H^{-s,-\gamma'}(M, V')$.

We next claim that $\mathrm{dim\,coker\,T} = \mathrm{dim\,L}(\delta^{-1}, \mathrm{A}')$. To prove this, it is sufficient to show, by a familiar argument from functional analysis, that a section $f \in H_2$ belongs to the range of T if and only if f is "orthogonal" to $L(\delta^{-1}, A')$ under the pairing

$$H^{-s+a,-\gamma'+a}(M, \tilde{V}') \times H^{s-a,\gamma'-a}(M, \tilde{V}) \to \mathbb{C}.$$

Indeed, let $f = Au$ for some $u \in H_1$. Choose a sequence $(u_\nu)_{\nu=1,2,\dots}$ in $C^\infty_{comp}(M \setminus \mathrm{sing\,M}, \mathrm{V})$, such that $u_\nu \to u$ in the norm of $H^{s,\gamma'}(M, V)$. Then,

$$\begin{aligned} \langle g, f\rangle &= \langle g, Au\rangle \\ &= \lim_{\nu\to\infty} \langle g, Au_\nu\rangle \\ &= \lim_{\nu\to\infty} \langle A'g, u_\nu\rangle \\ &= 0 \end{aligned}$$

for all $g \in H^{-s+a,-\gamma'+a}(M, \tilde{V}')$ satisfying $A'g = 0$. Hence f is "orthogonal" to $L(\delta^{-1}, A')$.

On the other hand, suppose $f \in H_2$ is "orthogonal" to $L(\delta^{-1}, A')$. By the above, $L(\delta^{-1}, A')$ coincides with the annihilator of the range of the operator $A \colon H^{s,\gamma'}(M, V) \to H^{s-a,\gamma'-a}(M, \tilde{V})$. Hence we can assert that there is a section $u \in H^{s,\gamma'}(M, V)$ such that $Au = f$. However, $\mathrm{as(f, v_i)} \in \tilde{\mathrm{as}}'_{\mathrm{i}}$ implies $\mathrm{as(u, v_i)} \in \mathrm{as}'_{\mathrm{i}}$, showing $u \in H_1$, as required.

We have thus proved that $\dim_{\mathbb{C}} L(\delta, A) - \dim_{\mathbb{C}}(\delta^{-1}, A') = \mathrm{ind\,T}$. What is left is to show that $\mathrm{ind\,T} = \mathrm{ind\,A} + \deg \delta$.

Since

$$\begin{aligned} \deg \delta &= \sum_{i=1}^{I} \left(\mathrm{rank\,as}'_{\mathrm{i}} - \mathrm{rank}\,\tilde{\mathrm{as}}'_{\mathrm{i}}\right) \\ &= \sum_{i=1}^{I} \dim \mathcal{A}_{\mathrm{as}'_{\mathrm{i}}} - \sum_{i=1}^{I} \dim \mathcal{A}_{\tilde{\mathrm{as}}'_i}, \end{aligned}$$

we shall have established the desired equality if we prove the following:

$$\operatorname{ind} \mathrm{T} = \operatorname{ind} \mathrm{A} + (\dim \mathcal{A}_{\mathrm{as}'} - \dim \mathcal{A}_{\tilde{\mathrm{as}}'}) . \tag{7.3}$$

To this end, write

$$\begin{array}{rcl} H_1 & = & H^{s,\gamma'}(M,V) \oplus \mathcal{A}_{\mathrm{as}'}, \\ H_2 & = & H^{s-a,\gamma'-a}(M,\tilde{V}) \oplus \mathcal{A}_{\tilde{\mathrm{as}}'} \end{array}$$

and let

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

be the corresponding splitting of the operator T. Obviously, $T_{11} = A$. Moreover, $T_{21} = 0$ since the restriction of T to $H^{s,\gamma'}(M,V)$ operates to $H^{s-a,\gamma'-a}(M,\tilde{V})$.

Were T_{12} zero, this allow us to conclude immediately that

$$\operatorname{ind} \mathrm{T} = \operatorname{ind} \mathrm{T}_{11} + \operatorname{ind} \mathrm{T}_{22}, \tag{7.4}$$

which is just (7.3) because the index of $T_{22} : \mathcal{A}_{\mathrm{as}'} \to \mathcal{A}_{\tilde{\mathrm{as}}'}$ is equal to the difference $\dim \mathcal{A}_{\mathrm{as}'} - \dim \mathcal{A}_{\tilde{\mathrm{as}}'}$.

To derive equality (7.4) in the general case, we make use of the fact that $T_{11} = A$ is an elliptic, and consequently Fredholm, operator. Fix a parametrix T_{11}^{-1} for T_{11}, i.e., the inverse modulo compact operators. Then

$$\begin{pmatrix} 1 & 0 \\ -T_{21}T_{11}^{-1} & 1 \end{pmatrix} T \begin{pmatrix} 1 & -T_{11}^{-1}T_{12} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} T_{11} & 0 \\ 0 & T_{22} - T_{21}T_{11}^{-1}T_{12} \end{pmatrix}$$

holds modulo compact operators. As the first and the third factors on the left are isomorphisms, (7.4) follows (cf. Proposition 1.2.32 in Schulze [19]). This completes the proof. □

8. Spectrum of the Conormal Symbol at a Regular Point

When comparing (6.5) and (0.3), one may ask whether the spectrum of the conormal symbol of A at an artificial conical point is of a particular structure. The program be to identify the artificial conical points, via the spectrum of the conormal symbol, within a larger class of conical points which still bear the same property of the spectrum. Rather than discuss this in full generality, let us look at the spectrum of the conormal symbol of geometric differential operators.

Suppose that M is a two-dimensional manifold with conical singularities and $A \in \mathrm{Diff}^1(\mathrm{V},\tilde{\mathrm{V}})$ is an elliptic differential operator of geometric nature on M. Let p be a point on the smooth part of M. We will restrict our attention to a coordinate neighbourhood O of p over which both V and $\tilde{V}$ are trivial. Hence, we can identify A with a matrix of scalar partial differential operators in O, namely $A = \sum_{|\alpha|\le 1} A_\alpha(y) D^\alpha$, where A_α are matrices of smooth functions in O. We blow up M at p by introducing polar coordinates $y = \pi(r,\varphi)$ with centre at p, i.e., $\pi(r,\varphi) = p + r\,(\cos\varphi, \sin\varphi)$, where $r \in [0,\varepsilon)$, $\varphi \in [0, 2\pi)$. The pull-back of the differential operator A under this change of coordinates is

$$\pi^\sharp A = \sum_{|\alpha|\le 1} \pi^* A_\alpha \left(\cos\varphi D_r - \frac{1}{r}\sin\varphi D_\varphi\right)^{\alpha_1} \left(\sin\varphi D_r + \frac{1}{r}\cos\varphi D_\varphi\right)^{\alpha_2}$$

whence

$$\begin{aligned}
\sigma_{\mathcal{M}}(A)(p,z) &= \sum_{|\alpha|=1} A_\alpha(p)\,(\cos\varphi z - \sin\varphi D_\varphi)^{\alpha_1} (\sin\varphi z + \cos\varphi D_\varphi)^{\alpha_2} \\
&= \sigma^1_{\mathcal{F}}(A)\,(p,\,(-\sin\varphi,\cos\varphi))\,D_\varphi + \sigma^1_{\mathcal{F}}(A)\,(p,\,(\cos\varphi,\sin\varphi))\,z,
\end{aligned}$$

for $z \in \mathbb{C}$.

The spectrum of the conormal symbol at the point p is easily seen to consist of all $z \in \mathbb{C}$ such that the operator $\sigma_{\mathcal{M}}(A)(p,z)\colon C^\infty(\mathbb{S}^1)^k \to C^\infty(\mathbb{S}^1)^k$ is not invertible, k being the rank of V. Here, we identify $C^\infty(\mathbb{S}^1)$ with the space of all C^∞ functions on the real axis, periodic with period 2π. Since the transpose is induced by the differential operator

$$\begin{aligned}
&(\sigma_{\mathcal{M}}(A)(p,z))'\,u \\
&\quad = -D_\varphi\left(\sigma^1_{\mathcal{F}}(A)\,(p,\,(-\sin\varphi,\cos\varphi))\,u\right) + \sigma^1_{\mathcal{F}}(A)\,(p,\,(\cos\varphi,\sin\varphi))\,zu \\
&\quad = -\sigma^1_{\mathcal{F}}(A)\,(p,\,(-\sin\varphi,\cos\varphi))\,D_\varphi + \sigma^1_{\mathcal{F}}(A)\,(p,\,(\cos\varphi,\sin\varphi))\,(z-i)u \\
&\quad = -\sigma_{\mathcal{M}}(A)(p, i-z)\,u,
\end{aligned}$$

we are reduced to looking for complex values z such that the problem

(8.1)

$$\left\{\begin{array}{rcll}
\sigma^1_{\mathcal{F}}(A)\,(p,\,(-\sin\varphi,\cos\varphi))\,D_\varphi u &=& -z\,\sigma^1_{\mathcal{F}}(A)\,(p,\,(\cos\varphi,\sin\varphi))\,u, & \varphi\in\mathbb{R}, \\
u(\varphi+2\pi) &=& u(\varphi), & \varphi\in\mathbb{R}
\end{array}\right.$$

has a non-trivial solution in $C^\infty(\mathbb{S}^1)^k$.

To do this, we make use of the ellipticity of A at the point p, which enables us to conclude that the matrix $\sigma^1_{\mathcal{F}}(A)\,(p,\eta)$ is invertible on the unit circle in the cotangent plane $T^*_p M$. Set

$$M(\varphi) = \left(\sigma^1_{\mathcal{F}}(A)\,(p,\,(-\sin\varphi,\cos\varphi))\right)^{-1} \sigma^1_{\mathcal{F}}(A)\,(p,\,(\cos\varphi,\sin\varphi))\,;$$

it follows that $M(\varphi)$ is a $(k\times k)$-matrix of C^∞ functions on $\mathbb{R}$, periodic with period π. If each two values of $M(\varphi)$ on the interval $(0,\pi)$ commute, then the general solution of the differential equation in (8.1) is known to be

$$u(\varphi)=e^{-iz\int_0^\varphi M(\theta)d\theta}u_0, \qquad u_0\in\mathbb{C}^k.$$

We next substitute this solution u into the periodicity condition of (8.1). As

$$\begin{aligned} u(\varphi+2\pi) &= e^{-2iz\int_0^\pi M(\theta)d\theta}e^{-iz\int_0^\varphi M(\theta)d\theta}u_0 \\ &= e^{-2iz\int_0^\pi M(\theta)d\theta}u(\varphi), \end{aligned}$$

this gives

$$\left(e^{-2iz\int_0^\pi M(\theta)d\theta}-I_k\right)u(\varphi)=0$$

for all $\varphi\in\mathbb{R}$, I_k being the identity $(k\times k)$-matrix. In particular, we deduce that 1 is an eigenvalue of the matrix

$$e^{-2iz\int_0^\pi M(\theta)d\theta}, \tag{8.2}$$

for we require non-trivial solutions to (8.1).

Conversely, if 1 is an eigenvalue of matrix (8.2) and each two values of $M(\varphi)$ on the interval $(0,\pi)$ commute, then problem (8.1) has non-trivial solutions.

We now proceed with the study of matrix (8.2). The following properties of the matrix $M(\varphi)$ are straightforward:

$$\begin{aligned} M\left(\varphi+\tfrac{\pi}{2}\right)M(\varphi) &= -I_k, \\ M'(\varphi) &= (M(\varphi))^2+I_k. \end{aligned}$$

It follows from any one of these properties that $M(\varphi)$ is constant, i.e., independent of φ, if and only if it satisfies $(M(\varphi))^2=-I_k$. Such is the case for classical differential operators associated to a Riemannian metric.

Indeed, notice that the principal symbol of a geometric differential operator A satisfies

$$\left(\sigma_{\mathcal{F}}^1(A)(p,\eta)\right)^*\sigma_{\mathcal{F}}^1(A)(p,\eta)=|\eta|^2I_k, \quad \eta\in T_p^*M,$$

which is equivalent to the system of equalities

$$(A_\alpha(p))^*A_\beta(p)+(A_\beta(p))^*A_\alpha(p)=2\delta_{\alpha\beta}I_k, \quad |\alpha|=|\beta|=1,$$

$\delta_{\alpha\beta}$ being the Kronecker delta. In particular, the inverse of $\sigma_{\mathcal{F}}^1(A)(p,\eta)$ coincides with the adjoint $\left(\sigma_{\mathcal{F}}^1(A)(p,\eta)\right)^*$, for $\eta\in\mathbb{S}^1$. Now, an easy computation shows that

$$M(\varphi)=(A_{0,1}(p))^*A_{1,0}(p), \quad \varphi\in\mathbb{R},$$

as required.

Returning to (8.2), we expand the exponential function of a matrix as power series. Since M is independent of φ and satisfies $M^2 = -I_k$, we obtain

$$\begin{aligned} e^{-2zi\int_0^\pi M(\theta)d\theta} &= e^{-2\pi izM} \\ &= \sum_{\iota=0}^{\infty} \frac{1}{\iota!}(-2\pi izM)^\iota \\ &= \sum_{\iota=0}^{\infty}(-1)^\iota \frac{1}{(2\iota)!}(-2\pi iz)^{2\iota}\ I_k + \sum_{\iota=0}^{\infty}(-1)^\iota \frac{1}{(2\iota+1)!}(-2\pi iz)^{2\iota+1}\ M \\ &= \cos(-2\pi iz)\ I_k + \sin(-2\pi iz)\ M, \end{aligned}$$

for $\varphi \in \mathbb{R}$. Consequently, in order that 1 be an eigenvalue of matrix (8.2), it is necessary and sufficient that $z = 0, \pm i, \ldots$.

We have thus proved that $\operatorname{spec}\sigma_{\mathcal{M}}(\mathrm{A})(\mathrm{p}, \cdot) = \mathrm{i}\mathbb{Z}$ which is symmetric relative to each line $\Im z = i$ or $\Im z = i + \frac{1}{2}$, where $i \in \mathbb{Z}$.

In the general case the spectrum of $\sigma_{\mathcal{M}}(A)(p, \cdot)$ is among the roots of the characteristic equality

$$\det\left(e^{-2iz\int_0^\pi M(\theta)d\theta} - I_k\right) = 0. \tag{8.3}$$

Our last example demonstrates rather strikingly that even in the general case it is to be expected that the spectrum coincides with the set of all integers on the imaginary axis.

EXAMPLE 8.1. Suppose A is a scalar elliptic differential operator of order 1, i.e., $k = 1$. Then, $\sigma^1_{\mathcal{F}}(A)(p, \eta) = \eta_1 + c\eta_2$ up to a non-zero complex factor, where $c = a + ib$ is a complex number with $b \neq 0$. It follows that

$$M(\varphi) = \frac{\cos\varphi + c\sin\varphi}{-\sin\varphi + c\cos\varphi}$$

which is no longer constant in φ. We write this as $M(\varphi) = \Re M(\varphi) + i\Im M(\varphi)$, with

$$\begin{aligned} \Re M(\varphi) &= \frac{(a^2+b^2-1)\sin 2\varphi + 2a\cos 2\varphi}{(a^2+b^2-1)\cos 2\varphi - 2a\sin 2\varphi + (a^2+b^2+1)}; \\ \Im M(\varphi) &= \frac{-2b}{(a^2+b^2-1)\cos 2\varphi - 2a\sin 2\varphi + (a^2+b^2+1)}. \end{aligned}$$

Since

$$\Re M(\varphi) = -\frac{1}{2}\frac{\partial}{\partial\varphi}\log\left((a^2+b^2-1)\cos 2\varphi - 2a\sin 2\varphi + (a^2+b^2+1)\right)$$

and the expression under the 'log' sign is positive for all $\varphi \in [0, \pi]$, we conclude that

$$\int_0^\pi \Re M(\theta) d\theta = 0.$$

On the other hand,

$$\begin{aligned} \int_0^\pi \Im M(\theta) d\theta &= \int_{-\pi}^{\pi} \frac{-b\, d\phi}{(a^2+b^2-1)\cos\phi - 2a\sin 2\phi + (a^2+b^2+1)} \\ &= \int_{-\infty}^{\infty} \frac{-b\, dt}{(t-a)^2+b^2} \\ &= -\operatorname{sgn} \mathrm{b}\ \pi, \end{aligned}$$

as is easy to check. Thus, the integral of $M(\varphi)$ is an integer multiple of πi while the matrix $M(\varphi)$ itself is of rather general nature. Therefore, equation (8.3) becomes

$$e^{-2\pi \operatorname{sgn} \mathrm{b}\, \mathrm{z}} - 1 = 0,$$

showing $\operatorname{spec} \sigma_{\mathcal{M}}(\mathrm{A})(\mathrm{p}, \cdot) = \mathrm{i}\mathbb{Z}$. □

9. Applications

We will touch only a direct consequence of Theorems 3.1, 6.3 and 7.2 along the classical line.

Corollary 9.1 (Riemann inequality).

$$\dim_{\mathbb{C}} L(\delta, A) \geq \operatorname{ind} \mathrm{A} + \deg \delta. \tag{9.1}$$

In particular, if $\operatorname{ind} \mathrm{A} + \deg \delta > 0$, then the space $L(\delta, A)$ is non-trivial. So this space will be always non-trivial if we fix the orders of "zeros" and allow "poles" of sufficiently high order to make the degree $\deg \delta$ sufficiently large. For example, in the setting of Section 6, we can fix any set of points $p_1, \ldots, p_{N-1}$ and any weights $m_1, \ldots, m_{N-1}$, but take m_N sufficiently large to arrive at $\dim_{\mathbb{C}} L(\delta, A) > 0$.

In case A bears a *unique continuation property*, even the equality in (9.1) can be claimed if one has a sufficiently large number of "poles."

We finish the paper by showing a particular case of the Riemann-Roch Theorem for geometric differential operators on a two-dimensional manifold with conical points. Close to a conical point v, such an operator takes the form

$$A = \frac{1}{r} \left(a_1(r, \varphi)\, r D_r + a_{0,1}(r, \varphi)\, D_\varphi + a_{0,0}(r, \varphi) \right)$$

in polar coordinates $(r, \varphi) \in [0,1) \times \mathbb{S}^1$ with centre v, the coefficients being smooth up to $r = 0$ (cf. (3.1)). Moreover, the higher-order coefficients fulfil the 'Dirac quantisation' relations $a_1^{-1} a_{0,1} = -a_{0,1}^{-1} a_1$ at the conical point (i.e., for $r = 0$). For simplicity, we impose an additional condition that the lower-order term $a_{0,0}$ vanishes at v. Then just in the same way as in Section 8 we arrive at the equality $\operatorname{spec} \sigma_{\mathcal{M}}(\mathrm{A})(\mathrm{v}, \cdot) = \mathrm{i}\mathbb{Z}$, for each conical point $v \in M$. Since the rank of any pole z in the spectrum of $\sigma_{\mathcal{M}}(A)(v, \cdot)$ is equal to k, k being the rank of the bundle V, formula (6.5) becomes

$$\begin{aligned} \deg p_\nu^{m_\nu} &= \operatorname{sgn}\left(\mathrm{m}_\nu + \frac{1}{2} + \gamma_\mathrm{i}\right) \sum_{\substack{\mathrm{z} \in \mathrm{i}\mathbb{Z} \\ \Im \mathrm{z} \in \left(-\gamma_\mathrm{i}, \mathrm{m}_\nu + \frac{1}{2}\right)}} \mathrm{k} \\ &= k \operatorname{sgn}\left(\mathrm{m}_\nu + \frac{1}{2} + \gamma_\mathrm{i}\right) \mathrm{N} \end{aligned}$$

if $p_\nu = v_i$, where N is the number of integers in the interval $\left(-\gamma_i, m_\nu + \frac{1}{2}\right)$. We next observe that both $-\gamma_i$ and $m_\nu + \frac{1}{2}$ are not integer by assumption, since otherwise either of $\Gamma_{-\gamma_i}$ and $\Gamma_{m_\nu + \frac{1}{2}}$ meets the spectrum of $\sigma_{\mathcal{M}}(A)(v_i, \cdot)$. Therefore, N coincides with the absolute value of $[m_\nu + \frac{1}{2}] - [-\gamma_i]$. In what follows we assume that $\gamma_i \in (-1, 0)$ whence

$$\begin{aligned} \deg p_\nu^{m_\nu} &= k\left([m_\nu + \frac{1}{2}] - [-\gamma_i]\right) \\ &= k\,[m_\nu + \frac{1}{2}]. \end{aligned}$$

This coincides with the contribution of p_ν given after Theorem 6.3 as if p_ν were a point on the smooth part of M. Thus, in the case of geometric operators the contributions of the points $p_\nu \in \operatorname{supp} \delta$ lying on the smooth part and on the singular part of M are evaluated by the same formula. Hence it follows that, for any divisor $\delta = p_1^{m_1} \dots p_N^{m_N}$ on M, the degree of δ is equal to

$$\deg \delta = k \sum_{\nu=1}^{N} [m_\nu + \frac{1}{2}]. \tag{9.2}$$

Moreover, the index of the operator $A\colon H^{s,\gamma}(M, V) \to H^{s-1,\gamma-1}(M, \tilde{V})$ is independent of $\gamma \in \mathbb{R}^I$, provided that $\gamma_i \in (-1, 0)$ (cf. Lemma 3.2). It fact, it is given by

$$\operatorname{ind} \mathrm{A} = \int_{\mathrm{S}^*\mathrm{M}} \mathrm{AS}\,(\mathrm{A}),$$

where S^*M is the cosphere bundle of M and $\mathrm{AS}\,(\mathrm{A})$ is the Atiyah-Singer form of A suitably interpreted (see the work of Fedosov and the authors [4] for more details).

Thus, we arrive at the following consequence of Theorem 6.3. Set

$$L(\delta, A) = \{u \in H^{\infty,\gamma}_{loc}(M \setminus \operatorname{supp}\delta, \mathrm{V}) \colon \mathrm{Au} = 0,\ \operatorname{ord}(\mathrm{u}, \mathrm{p}_\nu) \geq -\mathrm{m}_\nu\},$$
$$L(\delta^{-1}, A') = \{g \in H^{\infty,1-\gamma}_{loc}(M \setminus \operatorname{supp}\delta, \tilde{\mathrm{V}}') \colon \mathrm{A'g} = 0,\ \operatorname{ord}(\mathrm{g}, \mathrm{p}_\nu) \geq \mathrm{m}_\nu\}.$$

Corollary 9.2. *Let A be a geometric operator as above and let $\gamma \in \mathbb{R}^I$ be such that $\gamma_i \in (-1, 0)$. Then, for each point divisor $\delta = p_1^{m_1} \dots p_N^{m_N}$ with $m_\nu \notin \mathbb{Z} + \frac{1}{2}$, we have*

$$\dim_{\mathbb{C}} L(\delta, A) = \int_{S^*M} \mathrm{AS}(\mathrm{A}) + \deg\delta + \dim_{\mathbb{C}} \mathrm{L}(\delta^{-1}, \mathrm{A}'),$$

$\deg\delta$ *being given by (9.2).*

It is worth mentioning that this corollary is the most immediate generalisation of the classical Riemann-Roch Theorem to the case of surfaces with singularities.

For a deeper discussion of applications of the Riemann-Roch Theorem we refer the reader to Gromov and Shubin [6].

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