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AN $L^p$-APPROACH TO SINGULAR LINEAR PARABOLIC EQUATIONS IN BOUNDED DOMAINS

ANGELO FAVINI, ALFREDO LORENZI, HIROKI TANABE and ATSUSHI YAGI

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Abstract

Singular means here that the parabolic equation is not in normal form neither can it be reduced to such a form. For this class of problems, following the operator approach used in [1], we prove global in time existence and uniqueness theorems related to (spatial) $L^p$-spaces. Various improvements to [2], [3] are given.

1. Introduction

In this paper we will consider the following boundary value problem

\begin{align}
\tag{1.1}
D_t[m(x)u(x,t)] + A(x, D_x)u(x,t) &= f(x, t), \quad \forall (x, t) \in \Omega \times [0, \tau], \\
\tag{1.2}
u(x, t) &= 0, \quad \forall (x, t) \in \partial \Omega \times [0, \tau], \\
\tag{1.3}
m(x)u(x, t) &\to m(x)u_0(x), \quad \text{for a.e. } x \in \Omega, \text{ as } t \to 0+,
\end{align}

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a boundary of class $C^2$, while $A(x, D_x)$ is the following second-order uniformly elliptic operator in divergence form

\begin{equation}
\tag{1.4}
A(x, D) = - \sum_{i,j=1}^n D_{x_i}[a_{ij}(x)D_{x_j}] + a_0(x).
\end{equation}

Moreover, $0 \neq m \in L^\infty(\Omega)$ is a non-negative function which need not to be bounded away from 0. Consequently, our parabolic equation is, in general, singular.

Particular cases of (1.1) are discussed in the monograph [3], pp.74–80. See also [2]. Note that in [3], p.80, the restriction $p \in (2, +\infty)$ should be made.

Using the theoretical results in [3] and the fundamental approach in [4] we can develop an $L^p$-theory, $p \in (1, +\infty)$, also in the present degenerate case*. The key-stone in order to apply the results in [1] and [3], Theorem 3.28, p.69, to (1.1)–(1.4)


\*We note that in this case the initial condition (1.3) should be more correctly meant as the following $L^p$-limit: $\|m(\cdot)\omega(\cdot, t) - m(\cdot)u_0(\cdot)\|_{L^p(\Omega)} \to 0$ as $t \to 0+$.
consists in showing an operator estimate of the form
\begin{equation}
||L(\lambda M + L)^{-1}||_{L^\infty(\Omega)} \leq C(1 + |\lambda|)^{1-\beta}, \quad \forall \lambda \in \Sigma_{\alpha},
\end{equation}
where \( X = L^p(\Omega), \) \( 0 < \beta \leq \alpha \leq 1, \alpha + \beta > 1, \)
\begin{equation}
\Sigma_{\alpha} = \{ \lambda \in \mathbb{C} : \text{Re} \lambda \geq -c(1 + |\text{Im} \lambda|^2) \}, \quad (c > 0),
\end{equation}
and
\begin{align}
(1.7) & \quad \mathcal{D}(L) = \mathcal{D}(L_p) = W^{2, p}(\Omega) \cap W^{1, p}_0(\Omega), \quad Lu(x) = A(x, D)u(x), \quad u \in \mathcal{D}(L), \\
(1.8) & \quad \mathcal{D}(M) = L^p(\Omega), \quad Mu(x) = m(x)u(x).
\end{align}
We in fact show that (1.5) holds with \( \alpha = 1, \beta = 1/p, \ p \in (1, +\infty). \)
Moreover, when \( m \) is \( \rho \)-regular, i.e.
\begin{equation}
m \in C^1(\overline{\Omega}), \quad |\nabla m(x)| \leq C_1 m(x)^\rho, \quad \forall x \in \overline{\Omega}, \quad \text{for some } \rho \in (0, 1],
\end{equation}
\( C_1 \) being a positive constant, we can improve the index \( \beta \) in estimate (1.5) from \( \beta = 1/p \) to
\begin{equation}
\beta = \begin{cases} 
(2 - \rho)^{-1}, & \text{if } p \in (1, 2), \ \rho \in (2 - p, 1], \\
2[p(2 - \rho)]^{-1}, & \text{if } p \in [2, +\infty), \ \rho \in (0, 1].
\end{cases}
\end{equation}
The result proved in this paper will be applied, in a subsequent paper, to identify the unknown kernel \( k \) in the integro-differential singular equation of parabolic type
\begin{equation}
D_t[m(x)u(x, t)] + A(x, D_x)u(x, t) = \int_0^t k(t - s)B(x, D_x)u(x, s) \, ds + f(x, t),
\end{equation}
\( \forall (x, t) \in \Omega \times [0, \tau], \)
\( B(x, D_x) \) being a linear second-order differential operator.
We stress that the present paper was originated by a requirement of additional smoothness of solution \( u \) of (1.11) needed to recover the unknown kernel \( k \). This occurrence is in accordance with the well-known fact that inverse problems usually force deeper, and sometimes, unexpected insights in direct problems.

2. Solving the spectral problem \((\lambda M + L)u = f\)

The basic aim of this section consists in showing that estimate (1.5) holds when the linear operators \( M \) and \( L \) are defined by (1.7) and (1.8), respectively. To this aim we assume that the coefficients \( a_{i,j} \) and \( a_0 \) satisfy the properties
\begin{equation}
a_{i,j} \in C^1(\overline{\Omega}), \quad a_0 \in C(\overline{\Omega}), \quad a_{i,j} = a_{j,i}, \quad i, j = 1, \ldots, n,
\end{equation}
(2.2) \[ c_0 |\xi|^2 \leq \sum_{i,j=1}^{n} a_{i,j}(x) \xi_i \xi_j \leq c_1 |\xi|^2, \quad \forall x \in \Omega, \forall \xi \in \mathbb{R}^n, \quad a_0(x) \geq \gamma, \quad \forall x \in \overline{\Omega}, \]

\( c_0, c_1 \) and \( \gamma \) being three positive constants.

A remarkable result by Okazawa [4, p.702] provides, for any \( u \in \mathcal{D}(L) \),

\[
\text{Re}( (L - a_0)u, u |u|^{p-2}) \geq \left\{ \begin{array}{ll}
 c_0 \int_{\Omega} |u|^{p-2} |\nabla u|^2 \, dx \geq 0, & \text{if } p \in [2, \infty), \\
 c_0 (p-1) \int_{\Omega} (|u|^2 + \delta)^{p-2} |\nabla u|^2 \, dx \geq 0, & \text{if } p \in (1, 2),
\end{array} \right.
\]

(2.3)

\[
|\text{Im}(Lu, u |u|^{p-2})| \leq \frac{|p-2|}{2 \sqrt{p-1}} \text{Re}( (L - a_0)u, u |u|^{p-2}),
\]

(2.4)

where the brackets denote

\[
(f, g) = \int f(x) \overline{g(x)} \, dx, \quad f \in L^p(\Omega), \quad g \in L^{p'}(\Omega), \quad \frac{1}{p} + \frac{1}{p'} = 1,
\]

\( u |u|^{p-2} \) is assumed to vanish whenever \( u \) does, and \( \delta > 0 \) is arbitrary.

Remark 2.1. It is important to observe that bound (2.4) holds even in the degenerate elliptic case (cf. [4, p. 702] and the following Lemma 3.3).

From (2.4) we immediately deduce the estimate

\[
|\text{Im}(Lu, u |u|^{p-2})| + \frac{|p-2|}{2 \sqrt{p-1}} \int_{\Omega} a_0(x) |u(x)|^p \, dx \\
\leq \frac{|p-2|}{2 \sqrt{p-1}} \text{Re}(Lu, u |u|^{p-2}).
\]

(2.5)

Consider now the spectral problem

(2.6) \[ u \in \mathcal{D}(L), \quad \lambda nu + Lu = f \in L^p(\Omega). \]

Taking the real and imaginary parts of the scalar product of both sides in (2.6) with \( u |u|^{p-2} \), we get

\[
\text{Re} \lambda \int_{\Omega} |u|^p \, dx + \text{Re}(Lu, u |u|^{p-2}) = \text{Re} \int_{\Omega} f \overline{u} |u|^{p-2} \, dx,
\]

(2.7)

\[
\text{Im} \lambda \int_{\Omega} |u|^p \, dx + \text{Im}(Lu, u |u|^{p-2}) = \text{Im} \int_{\Omega} f \overline{u} |u|^{p-2} \, dx.
\]

(2.8)
From (2.8) we deduce the inequalities

$$\text{Im} \lambda \int_\Omega m|u|^p \, dx \leq |\text{Im}(Lu, u|u|^{p-2})| + |\text{Im} \int_\Omega f|u|^{p-2} \, dx|.$$  

(2.9)

Multiply then both sides in (2.9) by a positive constant $k$ and add the obtained inequality to equation (2.7). From (2.5) we get

$$\text{Re} \lambda + k|\text{Im} \lambda| \int_\Omega m|u|^p \, dx + \left(1 - k \frac{|p-2|}{2\sqrt{p-1}} \right) \text{Re}(Lu, u|u|^{p-2})$$

$$\leq \text{Re} \int_\Omega f|u|^{p-2} \, dx + k \left|\text{Im} \int_\Omega f|u|^{p-2} \, dx\right| \leq (1 + k)\|f\|_p\|u\|_p^{p-1}.$$  

(2.10)

Choose now $k = k_1(p)$ so small as to satisfy

$$h_1(p) = 1 - k_1(p) \frac{|p-2|}{2\sqrt{p-1}} > 0, \quad \forall p \in (1, +\infty).$$  

(2.11)

Observe that

$$\text{Re}(Lu, u|u|^{p-2}) = \text{Re}((L - a_0)u, u|u|^{p-2})$$

$$+ \frac{1}{2} \text{Re}(a_0 u, u|u|^{p-2}) + \frac{1}{2} \text{Re}(a_0 u, u|u|^{p-2})$$

$$\geq \text{Re}((L - a_0)u, u|u|^{p-2}) + \frac{\kappa}{2\|m\|_\infty} \|u\|_p^p + \frac{\gamma}{2\|m\|_\infty} \int_\Omega m|u|^p \, dx,$$  

(2.12)

since $m(x) \leq \|m\|_\infty$ implies

$$\frac{m(x)}{\|m\|_\infty} \leq \frac{a_0(x)}{2}.$$  

In view of (2.11), (2.12) and (2.3), we obtain from (2.10) that

$$\left(\text{Re} \lambda + k_1(p)|\text{Im} \lambda| + \frac{\gamma h_1(p)}{2\|m\|_\infty}\right) \int_\Omega m|u|^p \, dx$$

$$+ \frac{\gamma h_1(p)}{2}\|u\|_p^p + h_1(p) \text{Re}((L - a_0)u, u|u|^{p-2}) \leq [k_1(p) + 1]\|f\|_p\|u\|_p^{p-1}.$$  

(2.13)

Introduce now the sector

$$\Sigma_1 = \left\{ \mu \in \mathbb{C} : \text{Re} \mu + \frac{k_1(p)}{2}|\text{Im} \mu| + \frac{\gamma h_1(p)}{4\|m\|_\infty} \geq 0 \right\}.$$  

Then, for $\lambda \in \Sigma_1$,

$$\text{Re}((L - a_0)u, u|u|^{p-2}) \leq \frac{k_1(p) + 1}{h_1(p)}\|f\|_p\|u\|_p^{p-1},$$  

(2.14)
We can now summarize the results proved in this section in Theorem 2.1.

\begin{equation}
||u||_p \leq \frac{2(k_1(p) + 1)}{\gamma h_1(p)} ||f||_p^p.
\end{equation}

Consequently,

\begin{equation}
\left( \text{Re} \lambda + k_1(p) | \text{Im} \lambda | + \frac{\gamma h_1(p)}{2 \|m\|_{\infty}} \right) \int_\Omega m |u|^p \, dx \leq C_1(p) ||f||_p^p,
\end{equation}

We now need a simple proposition. For the proof see Section 6.

**Proposition 2.1.** Let \( k > 0 \) and \( \varepsilon > 0 \) be two positive constants, and let \( \Sigma_{k, \varepsilon} \) be a sectorial domain given by

\[ \Sigma_{k, \varepsilon} = \left\{ \mu \in \mathbb{C} : \text{Re} \mu + \frac{k}{2} | \text{Im} \mu | + \frac{\varepsilon}{2} \geq 0 \right\}. \]

Then it holds that

\[ |\lambda| + 1 \leq \left( \frac{2}{k} + \frac{2}{\varepsilon} + 1 \right) (\text{Re} \lambda + k | \text{Im} \lambda | + \varepsilon), \quad \lambda \in \Sigma_{k, \varepsilon}. \]

Since \( \Sigma_1 = \Sigma_{k(p), h_1(p), 2 \|m\|_{\infty}, 1} \), this proposition then yields

\begin{equation}
(\|\lambda\| + 1) \int_\Omega m |u|^p \, dx \leq C_2(p) \|f\|_p \|u\|_{L^p}^{p-1} \leq C_3(p) \|f\|_p^p, \quad \lambda \in \Sigma_1.
\end{equation}

To show that \( (\lambda M + L)^{-1} \) is a bounded operator on \( L^p(\Omega) \) for \( \lambda \in \Sigma_1 \), it now suffices to verify that \( \mathcal{R}(\lambda M + L) = L^p(\Omega) \). But this is verified by the usual techniques without difficulty. In fact, for each \( \lambda \in \Sigma_1 \), we already know that \( \mathcal{R}(\Lambda + \lambda M + L) = L^p(\Omega) \) provided \( \Lambda > 0 \) is a sufficiently large number. Let \( 0 \leq \theta \leq 1 \) be a parameter, and consider the family of closed linear operators \( A(\theta) = \theta \Lambda + \lambda M + L, \) \( 0 \leq \theta \leq 1 \). Then the desired result is obtained by the following proposition the proof of which will be given in the final section.

**Proposition 2.2.** Let \( A(\theta), \) \( 0 \leq \theta \leq 1, \) be a family of closed linear operators acting on a Banach space \( X \) with constant domain \( \mathcal{D}(A(\theta)) \equiv \mathcal{D} \). Assume that the family satisfies the conditions

\begin{align}
\delta \|u\| & \leq \|A(\theta)u\|, \quad u \in \mathcal{D}, \\
\|A(\theta) - A(\theta')u\| & \leq N |\theta - \theta'| \|u\|, \quad u \in \mathcal{D}
\end{align}

with some constants \( \delta > 0 \) and \( N > 0 \) independent of \( \theta, \theta' \in [0, 1] \). Then, \( \mathcal{R}(A(1)) = X \) implies \( \mathcal{R}(A(\theta)) = X \) for every \( \theta \in [0, 1] \).

We can now summarize the results proved in this section in Theorem 2.1.
Theorem 2.1. Let $L$ and $M$ be the linear operators defined by (1.7) and (1.8), the coefficients $a_{i,j}$, $i,j = 1, \ldots, n$, $\alpha$ enjoying properties (2.1) and (2.2) and $m$ being a non-negative function in $L^\infty(\Omega)$. Then the spectral equation $\lambda M u + L u = f$, with $f \in L^p(\Omega)$, admits, for any $\lambda \in \Sigma_1 = \{ \mu \in \mathbb{C} : \text{Re} \mu + (k_1(p)/2) |\mu| |\gamma h_1(p)/(4|m|_\infty) \geq 0 \}$ and $p \in (1, +\infty)$, a unique solution $u \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ satisfying the estimates

$$
\|u\|_p \leq C_4(p)\|f\|_p, \quad \|Mu\|_p \leq C_5(p)|\lambda|^{-1/p}\|f\|_p, \quad \lambda \in \Sigma_1,
$$

$$
\|Lu\|_p \leq C_6(p)(1 + |\lambda|^{1/p^2})\|f\|_p, \quad \lambda \in \Sigma_1.
$$

3. The case when $m$ is $\beta$-regular and $p \in [2, +\infty)$

We will show that when the multiplier $m$ is more regular, i.e. it satisfies (1.9), our $\beta$ can be chosen larger than $1/p$. We recall that all the previous estimates (2.6)–(2.16) hold for any $p \in (1, +\infty)$.

First of all we need the following lemma concerning the computation of the gradient of the function $|u|^{p-2}$ when $p \in [2, +\infty)$. For this purpose we need some lemmata.

Lemma 3.1. Let $u \in W^{1,p}_0(\Omega)$ with $p \in [2, +\infty)$. Then the function $|u|^{p-2}$ belongs to $W^{1,p}_0(\Omega)$ and the following formulae hold

$$
D_x |u|^{p-2} = |u|^{p-2} D_x \overline{u} + (p - 2) g_p(u) \text{Re}(g_p(u) D_x u),
$$

(3.1)

where

$$
g_p(u)(x) = \begin{cases} 
\overline{u(x)}|u(x)|^{p-1}/2, & \text{if } u(x) \neq 0, \\
0, & \text{if } u(x) = 0.
\end{cases}
$$

(3.2)

Proof. Let $\phi$ be any function in $C_0^\infty(\Omega)$. Then the following equalities hold:

$$
\langle D_x \phi, \overline{u}|u|^{p-2} \rangle = \lim_{\varepsilon \to 0^+} \langle D_x \phi, \overline{u}(|u|^2 + \varepsilon)^{(p-2)/2} \rangle
$$

$$
= -\lim_{\varepsilon \to 0^+} \langle \phi, (|u|^2 + \varepsilon)^{(p-2)/2} D_x \overline{u} + \frac{p-2}{2} \overline{u} |u|^2 + \varepsilon (p-4)/2 (\overline{u} D_x u + u D_x \overline{u}) \rangle
$$

$$
= -\lim_{\varepsilon \to 0^+} \langle \phi, (|u|^2 + \varepsilon)^{(p-2)/2} D_x \overline{u} + (p - 2) \overline{u} |u|^2 + \varepsilon (p-4)/2 \text{Re}(\overline{u} D_x u) \rangle
$$

$$
= -\lim_{\varepsilon \to 0^+} \langle \phi, (|u|^2 + \varepsilon)^{(p-2)/2} D_x \overline{u} + (p - 2) \overline{u} |u|^2 + \varepsilon (p-4)/4 \text{Re}(\overline{u} |u|^2 + \varepsilon (p-4)/4 D_x u) \rangle
$$

$$
= -\langle \phi, |u|^{p-2} D_x \overline{u} + (p - 2) g_p(u) \text{Re}(g_p(u) D_x u) \rangle.
$$

(3.3)

We have used here the relation $\lim_{\varepsilon \to 0^+} \overline{u}(x)(|u(x)|^2 + \varepsilon)^{(p-4)/4} = g_p(u)(x)$, which takes advantage of the assumption $p \in [2, +\infty)$.
Remark 3.1. From definition (3.2) we easily deduce the identity

\[
|g_p(u)(x)| = |u(x)|^{(\rho - 2)/2}.
\]

We can now prove the following Lemma 3.2.

**Lemma 3.2.** Let \((b_{i,j})_{i,j=1,\ldots,n}^n\) be a matrix of functions in \(C^1(\Omega, \mathbb{R})\) such that

\[
b_{i,j} = b_{j,i}, \quad i, j = 1, \ldots, n,
\]

\[
c_0 |\xi|^2 \mu(x) \leq \sum_{i,j=1}^n b_{i,j}(x) \xi_i \xi_j \leq c_1 |\xi|^2 \mu(x), \quad \forall x \in \Omega, \forall \xi \in \mathbb{R}^n,
\]

where \(\mu \in C(\Omega)\) is a non-negative function and \(c_0, c_1\) are two positive constants. Then for any \(p \in [2, +\infty)\), the linear operator \(K = -\sum_{i,j=1}^n D_x [b_{i,j}(x) D_x] \) with \(\mathcal{D}(K) = \mathcal{D}(L)\) (cf. (1.7)) satisfies the relations

\[
c_0 \left( \int_{\Omega} \mu |u|^{p-2} |Du|^2 \, dx + \int_{\Omega} \mu \sum_{j=1}^n |\text{Re}(g_p(u) D_x u)|^2 \, dx \right)
\]

\[
\leq \text{Re}(Ku, \overline{u}|u|^{p-2}) \leq c_1 \left( \int_{\Omega} \mu |u|^{p-2} |Du|^2 \, dx + \int_{\Omega} \mu \sum_{j=1}^n |\text{Re}(g_p(u) D_x u)|^2 \, dx \right),
\]

\[
\text{Im}(Ku, \overline{u}|u|^{p-2}) = (p - 2) \sum_{i,j=1}^n b_{i,j} \text{Re}(g_p(u) D_x u) |\text{Re}(g_p(u) D_x u)| \, dx.
\]

Proof. From Lemma 3.1 and an integration by parts we easily deduce the identity

\[
(Ku, \overline{u}|u|^{p-2}) = \int_{\Omega} \sum_{i,j=1}^n b_{i,j} D_x u D_x (\overline{u}|u|^{p-2}) \, dx
\]

\[
= \int_{\Omega} \sum_{i,j=1}^n |u|^{p-2} b_{i,j} D_x u D_x \overline{u} \, dx
\]

\[
+ (p - 2) \int_{\Omega} \sum_{i,j=1}^n b_{i,j} g_p(u) D_x u \text{Re}(g_p(u) D_x u) \, dx.
\]

Relations (3.7) and (3.8) follow immediately from (3.9) taking the real and the imaginary parts.

**Lemma 3.3.** Under the assumptions in the statement of Lemma 3.2 operator \(K\) satisfies inequalities (2.3) and (2.4) with \(K\) in the place of \(L - a\).
Proof. This lemma has essentially been proved in [4], although a slight modification is needed in its proof. For any \( \varepsilon > 0 \) define \( a_{i,j} = b_{i,j} + \varepsilon \delta_{i,j}, \ i, j = 1, \ldots, n, \) and set \( K_\varepsilon = K - \varepsilon \Delta. \) Since the matrix \((a_{i,j})_{i,j=1}^n\) is uniformly positive definite, from (2.3) and (2.4), with \( u \in D(L_0), \) we obtain the inequalities

\[
0 \leq \text{Re}(K_\varepsilon u, u|u|^{p-2}) = \text{Re}(K u, u|u|^{p-2}) + \varepsilon \text{Re}(\Delta u, u|u|^{p-2}),
\]

\[
|\text{Im}(K_\varepsilon u, u|u|^{p-2})| = |\text{Im}(K u, u|u|^{p-2}) + \varepsilon \text{Im}(\Delta u, u|u|^{p-2})|
\leq \frac{|p-2|}{2\sqrt{p-1}}[\text{Re}(K u, u|u|^{p-2}) + \varepsilon \text{Re}(\Delta u, u|u|^{p-2})].
\]

Taking the limit as \( \varepsilon \to 0^+ \) in (3.10) and (3.11), we easily deduce that \( K \) satisfies (2.3) and (2.4).

We shall use also the following identity

\[
(L u, m^{p-1} u|u|^{p-2}) = (m^{p-1} L u, u|u|^{p-2})
\]

\[
= (K_0 u, u|u|^{p-2}) + (p-1) \left( m^{p-2} \sum_{i,j=1}^n a_{i,j} D_{x_i} m D_{x_j} u, u|u|^{p-2} \right), \ u \in D(L),
\]

where

\[
K_0 = - \sum_{i,j=1}^n D_{x_i} \left[ m(x)|p-1| a_{i,j}(x) D_{x_j} \right] + m(x)|p-1| \phi_0(x).
\]

Let now \( u \) be a solution to equation (2.6). Taking the scalar product of both sides in (2.6) with \( m^{p-1} u|u|^{p-2} \) and using (3.12), we easily get the equalities

\[
(f, m^{p-1} u|u|^{p-2}) = (\lambda m u + L u, m^{p-1} u|u|^{p-2})
\]

\[
= \lambda \|M u\|_p^p + (K_0 u, u|u|^{p-2}) + (p-1) \left( m^{p-2} \sum_{i,j=1}^n a_{i,j} D_{x_i} m D_{x_j} u, u|u|^{p-2} \right).
\]

Taking the real and imaginary parts in (3.13) and using (2.4) with \( L - \phi_0 \) replaced by \( K = K_0 - m^{p-1} \phi_0, \) we easily deduce the inequalities

\[
\text{Re} \lambda \|M u\|_p^p + \gamma \int \Omega m^{p-1}|u|^p \, dx + \text{Re}(K_0 - m^{p-1} \phi_0) u, u|u|^{p-2}
\]

\[
\leq \left| (f, m^{p-1} u|u|^{p-2}) \right| + (p-1) \left| \left( m^{p-2} \sum_{i,j=1}^n a_{i,j} D_{x_i} m D_{x_j} u, u|u|^{p-2} \right) \right|
\]

\[
|\text{Im} \lambda \|M u\|_p^p | \leq |\text{Im}(K_0 - m^{p-1} \phi_0) u, u|u|^{p-2}|
\]

\[
+ |(f, m^{p-1} u|u|^{p-2})| + (p-1) \left| \left( m^{p-2} \sum_{i,j=1}^n a_{i,j} D_{x_i} m D_{x_j} u, u|u|^{p-2} \right) \right|
\]
\[ \leq \frac{|p-2|}{2\sqrt{p-1}} \text{Re}(\langle K_0 - m^{p-1}a_0 \rangle u, u|u|^{p-2}) + |(f, m^{p-1}u|u|^{p-2})| \]

(3.15) \[ + (p-1) \left| \left( m^{p-2} \sum_{i,j=1}^{n} a_{i,j} D_{x_i} m D_{x_j} u, u|u|^{p-2} \right) \right|. \]

Multiply now by \( k_1(p) \) (cf. (2.11)) the first and last sides in (3.15) and add to the first and last sides in (3.14). We get the estimate

\[ [\text{Re}\lambda + k_1(p) |\text{Im}\lambda| + \gamma\|m\|^{-1}_{\infty} \|Mu\|_p^p] \]

\[ + \left( 1 - k_1(p) \frac{|p-2|}{2\sqrt{p-1}} \right) \text{Re}(\langle K_0 - m^{p-1}a_0 \rangle u, u|u|^{p-2}) \]

\[ \leq [1 + k_1(p)] \left\{ |(f, m^{p-1}u|u|^{p-2})| + (p-1) \left| \left( m^{p-2} \sum_{i,j=1}^{n} a_{i,j} D_{x_i} m D_{x_j} u, u|u|^{p-2} \right) \right| \right\}, \]

(3.16)

where we have made use of the elementary inequality

\[ m(x)^p \leq \|m\|_{\infty} m(x)^{p-1}, \quad x \in \Omega. \]

We now estimate the last term in (3.16) with the aid of (1.9). Using twice Hölder’s inequality, we get

\[ \left| \left( m^{p-2} \sum_{i,j=1}^{n} a_{i,j} D_{x_i} m D_{x_j} u, u|u|^{p-2} \right) \right| \leq \int_{\Omega} m^{p-2} |u|^{p-1} \left| \sum_{i,j=1}^{n} a_{i,j} D_{x_i} m D_{x_j} u \right| dx \]

\[ \leq \int_{\Omega} m^{p-2} |u|^{p-1} \left| \sum_{i,j=1}^{n} a_{i,j} D_{x_i} m D_{x_j} u \right|^{1/2} \left| \sum_{i,j=1}^{n} a_{i,j} D_{x_i} u D_{x_j} u \right|^{1/2} dx \]

\[ \leq C_7 \int_{\Omega} m^{p-2+\rho} |u|^{p-1} \|\nabla u\|_p dx = C_7 \int_{\Omega} m^{p(p-2)(2-\rho)/2} m^{(p-2)(2-\rho)/2} |u|^{-1+p/2} |\nabla u| dx \]

\[ \leq C_7 \left( \int_{\Omega} m^{p(p-2)(2-\rho)/2} |u|^{p-1+\rho} dx \right)^{1/2} \left( \int_{\Omega} m^{p(p-2)(2-\rho)/2} |\nabla u|^2 dx \right)^{1/2} \]

\[ \leq C_7 \|Mu\|_p^{p/2} \|u\|_p^{p(1-\rho)/2} m^{(p-2)(2-\rho)/2} \left( \int_{\Omega} |u|^{p-2} |\nabla u|^2 dx \right)^{1/2}. \]

(3.17)

On account of (2.3), (2.14) and (2.15), we easily observe the estimate

\[ \int_{\Omega} |u|^{p-2} |\nabla u|^2 dx \leq C_8(p) \|f\|_p^p. \]

(3.18)
From (2.15), (3.17) and (3.18) we finally deduce the estimates

\[(3.19) \quad \left| \left( m^{p-2} \sum_{i,j=1}^{n} a_{i,j} D_{x_i} m D_{x_j} u, u |u|^{p-2} \right) \right| \leq C_{0}(p) \| f \|_p^{\rho(2-p)/2} \| Mu \|_p^{\rho(2-p)/2}. \]

Moreover, we have

\[(3.20) \quad |(f, m^{p-1} u |u|^{p-2})| \leq \| f \|_p \| Mu \|_p^{p-1}. \]

Finally, from (3.16), (3.19), (3.20) and Lemma 3.2 with \( K = K_0 - m^{p-1} a_0 \) (which makes use of the assumption \( p \in [2, +\infty) \)) we deduce the inequality

\[(3.21) \quad [\text{Re} \, \lambda + k_1(p) \text{Im} \lambda] + \gamma \| m \|_\infty^{-1} \| Mu \|_p^{p-1} + \| f \|_p^{p(2-p)/2} \| Mu \|_p^{\rho(2-p)/2}, \quad \lambda \in \Sigma_1. \]

We now introduce the sector

\[ \Sigma_2 = \left\{ \lambda \in \mathbb{C} : \text{Re} \, \lambda + \frac{k_1(p)}{2} \text{Im} \lambda + \frac{\gamma}{2\| m \|_\infty} \geq 0 \right\}. \]

Since \( h_1(p) \in (0, 1) \), (cf. (2.11)), we immediately deduce the inclusion \( \Sigma_2 \subset \Sigma_1 \) (see the definition of \( \Sigma_2 \)).

Then, recalling that \( \text{Re}((K_0 - m^{p-1} a_0) u, u |u|^{p-2}) \) is non-negative (cf. Lemma 3.2) and applying Proposition 2.1, we obtain

\[(3.22) \quad (|\lambda| + 1) \| Mu \|_p^{p-1} \leq \gamma \int_\Omega m^{p-1} |u|^{p} \, dx + \text{Re}((K_0 - m^{p-1} a_0) u, u |u|^{p-2}) \leq C_{11}(p) \| f \|_p \| Mu \|_p^{p-1} + \| f \|_p^{p(2-p)/2} \| Mu \|_p^{\rho(2-p)/2}, \quad \lambda \in \Sigma_2. \]

Consequently, since \( \| u \|_p \leq C_{12}(p) \| f \|_p \) (cf. (2.15)), (3.15) and (3.22) imply

\[(3.23) \quad (|\lambda| + 1) \| Mu \|_p^{p(2-p)/2} \leq C_{13}(p) \| f \|_p \| Mu \|_p^{p-1} + \| f \|_p^{p(2-p)/2}, \quad \lambda \in \Sigma_2. \]

By Proposition 2.2, it is verified that \( \lambda M + L \) is surjective on \( L^p(\Omega) \). Hence, estimate (1.5) holds with \( \alpha = 1 \) and \( \beta = 2(1/p - 1). \)

We can summarize the results in this section in Theorem 3.1.

**Theorem 3.1.** Let \( L \) and \( M \) be the linear operators defined by (1.7) and (1.8), the coefficients \( a_{i,j} \), \( i, j = 1, \ldots, n \), \( a_0 \) enjoying properties (2.1) and (2.2) and \( m \) being
a non-negative function satisfying (1.9). Then the spectral equation \( \lambda Mu + Lu = f \), with \( f \in L^p(\Omega) \), admits, for any \( \lambda \in \Sigma_2 = \{ \mu \in \mathbb{C} : \Re \mu + (k_1(p)/2) \| \mu \| + (\gamma/2)\|m\|_\infty \geq 0 \} \) and \( p \in [2, +\infty) \), a unique solution \( u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \) satisfying the estimates

\[
\|u\|_p \leq C_{14}(p)\|f\|_p, \quad \|Mu\|_p \leq C_{15}(p)\|\lambda\|^{-2/(p^2-\rho)}\|f\|_p, \quad \lambda \in \Sigma_2, \\
\|Lu\|_p \leq C_{16}(p)(1 + \|\lambda\|^{(p(2-\rho)-2)/(p^2-\rho)})\|f\|_p, \quad \lambda \in \Sigma_2.
\]

**Example 3.1.** Let \( \Omega \) be a bounded domain and let \( x_0 \) be a fixed point in \( \partial \Omega \). Define then \( r = \max_{x \in \Omega} |x - x_0| \) and choose

\[
m(x) = \max_{x \in \Omega} |x - x_0| - r_1, \quad q \in (1, +\infty).
\]

An elementary computation shows that

\[
|\nabla m(x)| = q|x - x_0| |r - |x - x_0||^{q-1}|2|x - x_0| - r| \leq qrm(x)^{q-1}/q, \quad x \in \Omega.
\]

Consequently, function \( m \) satisfies condition (1.9).

We notice that for any open interval \( \Omega \subset \mathbb{R} \) we have \( r = \text{length}(\Omega) \).

**4. The case when \( p \in (1, 2) \)**

In this section we are going to considering the case \( p \in (1, 2) \). From (2.4) we immediately deduce that the estimate

\[
|\text{Im}(Lu, u|u|^{p-2})| + \gamma\|u\|_p^p \leq |\text{Im}(Lu, u|u|^{p-2})| + \int_\Omega a_0(x)|u(x)|^p \, dx \\
\leq \frac{|p-2|}{2\sqrt{p-1}} \text{Re}(Lu, u|u|^{p-2}), \quad u \in \mathcal{D}(L)
\]

holds true for any \( p \in (1, +\infty) \).

Consider again the spectral problem

\[
\lambda Mu + Lu = f \in L^p(\Omega).
\]

Multiplying both sides in (4.2) by \( u|u|^{p-2} \) and integrating over \( \Omega \), we get

\[
\lambda \|m^{1/p}u\|_p^p + (Lu, u|u|^{p-2}) = (f, u|u|^{p-2}).
\]

Taking the real and imaginary parts, from (4.3) we deduce

\[
\text{Re} \lambda \|m^{1/p}u\|_p^p + \text{Re}(Lu, u|u|^{p-2}) + (a_0 u, u|u|^{p-2}) = \text{Re}(f, u|u|^{p-2}),
\]

\[
\text{Im} \lambda \|m^{1/p}u\|_p^p + \text{Im}(Lu, u|u|^{p-2}) = \text{Im}(f, u|u|^{p-2}),
\]

\[
|\text{Im}(Lu, u|u|^{p-2})| + \gamma\|u\|_p^p \leq |\text{Im}(Lu, u|u|^{p-2})| + \int_\Omega a_0(x)|u(x)|^p \, dx \\
\leq \frac{|p-2|}{2\sqrt{p-1}} \text{Re}(Lu, u|u|^{p-2}), \quad u \in \mathcal{D}(L)
\]

holds true for any \( p \in (1, +\infty) \).
where we have set
\[ L_0 = L - a_0. \]
Then from Okazawa [4, p.703] we get
\[ (L_0u, u|u|^{p-2}) = \lim_{\delta \to 0^+} I_\delta(u, \delta), \]
where \( \delta > 0 \) and
\[ I_\delta(u, \delta) = -\int_\Omega (|u(x)|^2 + \delta)^{(p-2)/2} \frac{u(x)}{|u(x)|} \sum_{j,k=1}^N D_{jk} [a_{jk}(x) D_{jk} u(x)] \, dx. \]
As mentioned at the beginning of Section 2, we have
\[ \text{Re}(L_0 u, u|u|^{p-2}) \geq c_0 \int_\Omega |u(x)|^2 |\nabla u(x)|^2 \, dx, \quad \text{if } p \in [2, +\infty), \]
\[ \text{Re}(L_0 u, u|u|^{p-2}) \geq c_0 (p - 1) \int_\Omega (|u(x)|^2 + \delta)^{(p-2)/2} |\nabla u(x)|^2 \, dx, \quad \text{if } p \in (1, 2). \]
From (4.1) and (4.5) we deduce the inequalities
\[ |\text{Im} \lambda||m^{1/p}u||^p_p \leq |\text{Im}(Lu, u|u|^{p-2})| + \|f\|_p \|u||^{p-1}_p, \]
\[ \leq \frac{|p - 2|}{2\sqrt{p - 1}} \text{Re}(Lu, u|u|^{p-2}) + \|f\|_p \|u||^{p-1}_p. \]
Multiply then both sides in (4.11) by a positive constant \( \nu \) and add the obtained inequality to equation (4.4) to get (cf. (2.2))
\[ (\text{Re} \lambda + \nu |\text{Im} \lambda||m^{1/p}u||^p_p + \left( 1 - \nu \frac{|p - 2|}{2\sqrt{p - 1}} \right) \text{Re}(L_0 u, u|u|^{p-2}) + \frac{\nu}{2} ||u||^p_p + \frac{\nu}{2} ||u||^p_p \]
\[ \leq (\text{Re} \lambda + \nu |\text{Im} \lambda||m^{1/p}u||^p_p + \left( 1 - \nu \frac{|p - 2|}{2\sqrt{p - 1}} \right) \text{Re}(L_0 u, u|u|^{p-2}) + (a_0u, u|u|^{p-2}) \]
\[ \leq \text{Re}(f, u|u|^{p-2}) + \|f\|_p \|u||^{p-1}_p \leq (1 + \nu)||f\|_p \|u||^{p-1}_p. \]
\[ (4.12) \]
Choose now \( \nu = \nu(p) \) so small as to satisfy
\[ \nu_1(p) := 1 - \nu(p) \frac{|p - 2|}{2\sqrt{p - 1}} > 0, \quad \forall p \in (1, +\infty). \]
On the other hand, since \( m \in L^\infty(\Omega), ||u||_p \geq ||m||_\infty^{1/p} ||m^{1/p}u||_p. \) Then (4.12) and (4.13) imply
\[ \left( \frac{\nu}{2 ||m||_\infty} + \text{Re} \lambda + \nu |\text{Im} \lambda| \right) ||m^{1/p}u||^p_p + \nu_1(p) \text{Re}(L_0 u, u|u|^{p-2}) + \frac{\nu}{2} ||u||^p_p \]
(4.14) \[ \leq [1 + v(p)]
\|f\|_p \|u\|_p^{p-1}. \]

In other words, there exist two positive constants \( C_{18} \) and \( C_{19} \) such that

\[
\left( \frac{Y}{2\|m\|_{\infty}} + \text{Re} \lambda + v|\text{Im} \lambda| \right) \|m^{1/p}u\|_p^p + C_{18} \text{Re}(L_0 u, u|u|^{p-2}) + \frac{Y}{2} \|u\|_p^p \\
\leq C_{19}\|f\|_p \|u\|_p^{p-1}, \quad \lambda \in \Sigma,
\]

the sector \( \Sigma \) being defined by

\[
\Sigma = \{ \lambda \in \mathbb{C} : \text{Re} \lambda + v|\text{Im} \lambda| + \frac{Y}{2\|m\|_{\infty}} \geq \varepsilon_0 > 0 \}.
\]

Notice that (4.7), (4.8), (4.10), (4.15) yield, in particular, the basic bounds

(4.16) \[ \|u\|_p \leq \frac{2}{Y} C_{19}\|f\|_p, \quad \text{Re}(L_0 u, u|u|^{p-2}) \leq C_{20}\|f\|_p, \]

and

\[
(p - 1)C_{0} \lim_{\delta \to 0^+} \int_{\Omega} (|u(x)|^2 + \delta)^{(p-2)/2} \|
abla u(x)\|^2 \, dx \\
\leq \lim_{\delta \to 0^+} \text{Re} I_{p}(u, \delta) \leq C_{20}\|f\|_p \|u\|_p^{p-1}, \quad \lambda \in \Sigma.
\]

From (4.3) we deduce the estimates

(4.18) \[ \|\lambda\| m^{1/p}u\|_p^p \leq [(L u, u|u|^{p-2}) + \|f\|_p \|u\|_p^{p-1} \\
\leq \left( 1 + \frac{|p - 2|}{2\sqrt{p - 1}} \right) \text{Re}(L_0 u, u|u|^{p-2}) + \|f\|_p \|u\|_p^{p-1} \\
\leq C_{21}\|f\|_p \|u\|_p^{p-1} \leq C_{22}\|f\|_p^p.
\]

Consequently, (4.18) immediately yields

(4.19) \[ \|\lambda\|^{1/p} M u\|_p \leq C_{23}\|f\|_p. \]

This, in turn, implies that (1.5) holds with \( \alpha = 1 \) and \( \beta = 1/p \), and provides a different proof to (1.5).

Now we focus our attention to the case when \( m \in C^1(\overline{\Omega}) \) satisfies inequality (1.9) with

(4.20) \[ \rho \in (2 - p, 1]. \]

Multiplying both sides in (4.2) by \( m(x)^{p-1}|u(x)|u(x)|u(x)|^{p-2} \) and integrating over \( \Omega \), we
easily get

\[ \lambda \|Mu\|_p^p - \lim_{\delta \to 0^+} \int_\Omega m(x)^{p-1}u(x)|u(x)|^p + \delta^{(p-2)/2} \sum_{j,k=1}^n D_{ij}^j[a_{j,k}(x)D_{nk}u(x)] \, dx \]

(4.21)  

+ \int_\Omega \alpha_0(x)m(x)^{p-1}|u(x)|^p \, dx = \int_\Omega f(x)m(x)^{p-1}u(x)|u(x)|^{p-2} \, dx.

An integration by parts in the integral appearing in the limit, which takes into account (4.20) and (4.21), easily yields

\[ - \int_\Omega m(x)^{p-1}u(x)|u(x)|^2 + \delta^{(p-2)/2} \sum_{j,k=1}^n D_{ij}^j[a_{j,k}(x)D_{nk}u(x)] \, dx \]

\[ = \int_\Omega (|u(x)|^2 + \delta)^{(p-2)/2}m(x)^{p-1} \sum_{j,k=1}^n a_{j,k}(x)D_{ij}u(x)D_{nk}u(x) \, dx \]

(4.22)

\[ + (p-1)\int \Omega \frac{u(x)|u(x)|^2 + \delta^{(p-2)/2} \sum_{j,k=1}^n m(x)^{p-2}D_{ij}m(x)a_{j,k}(x)D_{nk}u(x) \, dx \]

\[ + (p-2)\int \Omega m(x)^{p-1}(|u(x)|^2 + \delta)^{(p-2)/2} \sum_{j,k=1}^n a_{j,k}(x)Re\left(\frac{u(x)D_{ij}u(x)}{u(x)}D_{nk}u(x)\right) \, dx \]

(4.23)

\[ =: I_1(\delta) + (p-1)I_2(\delta) - (2-p)I_3(\delta). \]

We have made use here of the following Proposition 4.1 whose proof is postponed to Section 6.

**Proposition 4.1.** Let \( m \) satisfy property (1.9). Then for any \( \beta \in (1 - \rho, 1) \), the function \( m(\cdot)^\beta \) belongs to \( C^1(\overline{\Omega}) \) and \( \nabla[m(\cdot)^\beta](x) = m_1(x) \) for any \( x \in \overline{\Omega} \), where

\[ m_1(x) = \begin{cases} 0, & x \in Z(m), \\ \beta m(x)^{\beta-1}\nabla m(x), & x \notin Z(m), \end{cases} \]

and \( Z(m) \) denotes the zero-set of \( m \). Moreover,

\[ |\nabla[m(\cdot)^\beta](x)| \leq Cm(x)^{\beta-1+\rho}, \quad x \in \overline{\Omega}. \]

Since the matrix \((a_{j,k}(x))_{j,k=1}^n\) is real-valued and positive definite, from (4.22) we immediately deduce that

\[ I_1(\delta) \text{ and } Re I_3(\delta) \text{ are positive for any } \delta \in \mathbb{R}_+. \]
Then we observe that $I_2(\delta)$ has a limit as $\delta \to 0+$ and

$$\lim_{\delta \to 0+} I_2(\delta) = \int_{\Omega} \frac{u(x)|u(x)|^{p-2}}{\Omega} \sum_{j,k=1}^{n} m(x)^{p-2} D_{x_j} m(x) a_{j,k}(x) D_{x_k} u(x) \, dx.$$ 

Note that the integral in the right-hand side is well-defined on the whole of $W^{1,p}(\Omega)$ since $\Omega|u|^{p-2} \in L^p(\Omega)$, $m^{p-2} D_{x_j} m \in L^\infty(\Omega)$ and $D_{x_j} u \in L^p(\Omega)$.

Further, (4.25) implies that there exists also $\lim_{\delta \to 0+} [I_1(\delta) - (2 - p) I_3(\delta)]$.

From (4.24) we deduce that there exist the limits

$$\lim_{\delta \to 0+} \text{Im} I_3(\delta) \quad \text{and} \quad \lim_{\delta \to 0+} [I_1(\delta) - (2 - p) \text{Re} I_3(\delta)].$$

We can now prove the following Lemma 4.1.

**Lemma 4.1.** The following estimates hold for any $\delta \in \mathbb{R}_+$, $p \in (1,2)$ and $\eta \in (0,2(p-1)(2-p)^{-1})$:

$$I_1(\delta) - (2-p) \text{Re} I_3(\delta) - \eta(2-p) |\text{Im} I_3(\delta)| \geq 0,$$

$$I_1(\delta) + (p-1) \text{Re} I_2(\delta) - (2-p) \text{Re} I_3(\delta)$$

$$\geq -(p-1)(1 + \eta^2)^{1/2} |I_2(\delta)|,$$

$$\lim_{\delta \to 0+} [I_1(\delta) + (p-1) \text{Re} I_2(\delta) - (2-p) \text{Re} I_3(\delta)]$$

$$- \eta \lim_{\delta \to 0+} |(p-1) \text{Im} I_3(\delta) - (2-p) \text{Im} I_3(\delta)|$$

$$\geq -C_{24} ||f||_{p}^{2} ||M u||_{p}^{2} ||u||_{p}^{2-p} - p/2,$$

$C_{24}$ being a suitable positive constant.

**Proof.** Since the matrix $(a_{j,k}(x))_{j,k=1,...,n}$ is real-valued and positive definite, we immediately deduce the equality

$$\sum_{j,k=1}^{n} a_{j,k}(x) \zeta_j \bar{\zeta}_k = \sum_{j,k=1}^{n} a_{j,k}(x) [\text{Re}(\zeta_j) \text{Re}(\bar{\zeta}_k) + \text{Im}(\zeta_j) \text{Im}(\bar{\zeta}_k)], \quad \forall \zeta \in \mathbb{C}^n.$$

Consider now the formulae

$$I_1(\delta) = \int_{\Omega} (|u(x)|^2 + \delta)^{(p-4)/2} m(x)^{p-1} \sum_{j,k=1}^{n} a_{j,k}(x) \overline{u(x)} D_{x_j} \overline{u(x)} D_{x_k} u(x) \, dx$$

$$+ \delta \int_{\Omega} (|u(x)|^2 + \delta)^{(p-4)/2} m(x)^{p-1} \sum_{j,k=1}^{n} a_{j,k}(x) D_{x_j} u(x) \overline{D_{x_k} u(x)} \, dx$$

$$= \int_{\Omega} (|u(x)|^2 + \delta)^{(p-4)/2} m(x)^{p-1} \left\{ \sum_{j,k=1}^{n} a_{j,k}(x) \text{Re} \left[ \overline{\overline{u(x)} D_{x_j} \overline{u(x)}} \right] \text{Re} \left[ u(x) D_{x_k} u(x) \right] \right\} \, dx$$
\[ + \sum_{j,k=1}^{n} a_{j,k}^{(x)} \text{Im} \left[ \overline{u(x)} D_{x_j} u(x) \right] \text{Im} \left[ \overline{u(x)} D_{x_k} u(x) \right] \text{d}x \]

(4.29) \[ + \delta \int_{\Omega} \left( |u(x)|^2 + \delta (p-4)/2 m(x)^{p-1} \right) \sum_{j,k=1}^{n} a_{j,k}^{(x)} D_{x_j} u(x) \overline{D_{x_k} u(x)} \text{d}x, \quad \forall \delta \in \mathbb{R}_+ \]

\[ I_1(\delta) = (2 - p) \text{Re} I_3(\delta) = \int_{\Omega} \left( |u(x)|^2 + \delta (p-4)/2 m(x)^{p-1} \right) \]

\[ \times \left\{ (p - 1) \sum_{j,k=1}^{n} a_{j,k}^{(x)} \text{Re} \left[ \overline{u(x)} D_{x_j} u(x) \right] \text{Re} \left[ \overline{u(x)} D_{x_k} u(x) \right] \right. \]

\[ + \sum_{j,k=1}^{n} a_{j,k}^{(x)} \text{Im} \left[ \overline{u(x)} D_{x_j} u(x) \right] \text{Im} \left[ \overline{u(x)} D_{x_k} u(x) \right] \text{d}x \]

(4.30) \[ + \delta \int_{\Omega} \left( |u(x)|^2 + \delta (p-4)/2 m(x)^{p-1} \right) \sum_{j,k=1}^{n} a_{j,k}^{(x)} D_{x_j} u(x) \overline{D_{x_k} u(x)} \text{d}x, \quad \forall \delta \in \mathbb{R}_+ \]

\[ \text{Im} I_3(\delta) = \left| \int_{\Omega} \left( |u(x)|^2 + \delta (p-4)/2 m(x)^{p-1} \right) \sum_{j,k=1}^{n} a_{j,k}^{(x)} \text{Re} \left[ \overline{u(x)} D_{x_j} u(x) \right] \text{Im} \left[ \overline{u(x)} D_{x_k} u(x) \right] \text{d}x \right| \]

\[ \leq \frac{1}{2} \int_{\Omega} \left( |u(x)|^2 + \delta (p-4)/2 m(x)^{p-1} \right) \sum_{j,k=1}^{n} a_{j,k}^{(x)} \left\{ \text{Re} \left[ \overline{u(x)} D_{x_j} u(x) \right] \text{Re} \left[ \overline{u(x)} D_{x_k} u(x) \right] \right\} \text{d}x \]

(4.31) \[ + \text{Im} \left[ \overline{u(x)} D_{x_j} u(x) \right] \text{Im} \left[ \overline{u(x)} D_{x_k} u(x) \right] \text{d}x, \quad \forall \delta \in \mathbb{R}_+ \]

We have here used the Cauchy–Schwarz inequality and the geometric-arithmetic mean, i.e.

\[ \left| \sum_{j,k=1}^{n} a_{j,k}^{(x)} \xi_j \eta_k \right| \leq \left( \sum_{j,k=1}^{n} a_{j,k}^{(x)} \xi_j \xi_k \right)^{1/2} \left( \sum_{j,k=1}^{n} a_{j,k}^{(x)} \eta_j \eta_k \right)^{1/2} \]

\[ \leq \frac{1}{2} \left( \sum_{j,k=1}^{n} a_{j,k}^{(x)} \xi_j \xi_k + \sum_{j,k=1}^{n} a_{j,k}^{(x)} \eta_j \eta_k \right) = \frac{1}{2} \sum_{j,k=1}^{n} a_{j,k}^{(x)} [\xi_j \xi_k + \eta_j \eta_k], \quad \forall \xi, \eta \in \mathbb{R}^n. \]

From (4.24) and (4.31) we deduce the following inequality, where we take advantage of the membership \( \eta \in (0, 2(p - 1)(2 - p)^{-1}) \):

\[ I_1(\delta) = (2 - p) \text{Re} I_3(\delta) - \eta(2 - p)|\text{Im} I_3(\delta)| = \int_{\Omega} \left( |u(x)|^2 + \delta (p-4)/2 m(x)^{p-1} \right) \]

\[ \times \left\{ \left[ p - 1 - \frac{1}{2} \eta(2 - p) \right] \sum_{j,k=1}^{n} a_{j,k}^{(x)} \text{Re} \left[ \overline{u(x)} D_{x_j} u(x) \right] \text{Re} \left[ \overline{u(x)} D_{x_k} u(x) \right] \right\} \]
\[ + \left[ 1 - \frac{1}{2} \eta (2 - p) \right] \sum_{j,k=1}^{n} a_{j,k}(x) \text{Im} \left[ \overline{u(x)} D_{x_j} u(x) \right] \text{Im} \left[ \overline{u(x)} D_{x_k} u(x) \right] \text{d}x \]

\[ (4.32) + \delta \int_{\Omega} |u(x)|^2 + \delta (1+p)/2 m(x)^p \sum_{j,k=1}^{n} a_{j,k}(x) D_{x_j} u(x) D_{x_k} u(x) \text{d}x \geq 0, \quad \forall \delta \in \mathbb{R}^+. \]

We have thus proved (4.26).

Then we note that (4.27) is a consequence of (4.26):

\[ I_1(\delta) + (p - 1) \text{Re} I_2(\delta) - (2 - p) \text{Re} I_3(\delta) - \eta (p - 1) \text{Im} I_2(\delta) - (2 - p) \text{Im} I_3(\delta) \]

\[ \geq I_1(\delta) - (2 - p) \text{Re} I_3(\delta) - \eta (2 - p) \text{Im} I_3(\delta) + (p - 1)[\text{Re} I_2(\delta) - \eta \text{Im} I_2(\delta)] \]

\[ \geq -(p - 1)(1 + \eta^2)^{1/2} |I_2(\delta)|, \quad \forall \delta \in \mathbb{R}^+. \]

(4.33)

To conclude the proof of the lemma we take into account the relations

\[ \lim_{\delta \to 0^+} \left[ I_1(\delta) + (p - 1) \text{Re} I_2(\delta) - (2 - p) \text{Re} I_3(\delta) \right] \]

\[ - \eta \lim_{\delta \to 0^+} \left[ (p - 1) \text{Im} I_2(\delta) - (2 - p) \text{Im} I_3(\delta) \right] \]

\[ \geq \lim_{\delta \to 0^+} \left\{ \text{Re} I_1(\delta) + (p - 1) \text{Re} I_2(\delta) - (2 - p) \text{Re} I_3(\delta) \right. \]

\[ - \left. \eta (p - 1) \text{Im} I_2(\delta) - (2 - p) \text{Im} I_3(\delta) \right\} \]

\[ \geq -(p - 1)(1 + \eta^2)^{1/2} \lim_{\delta \to 0^+} |I_2(\delta)|, \quad \forall \delta \in \mathbb{R}^+. \]

(4.34)

Next, consider the following chain of inequalities, which holds for any \( \delta \in \mathbb{R}_+: \)

\[ \lim_{\delta \to 0^+} |I_2(\delta)| \]

\[ \leq \limsup_{\delta \to 0^+} \int_{\Omega} (|u(x)|^2 + \delta)^{(p-1)/2} \sum_{j,k=1}^{n} m(x)^{p-2} |D_{x_j} m(x)||a_{j,k}(x)||D_{x_k} u(x)| \text{d}x \]

\[ \leq \limsup_{\delta \to 0^+} \int_{\Omega} (|u(x)|^2 + \delta)^{p/4} \]

\[ \times (|u(x)|^2 + \delta)^{(p-2)/4} \sum_{j,k=1}^{n} m(x)^{p-2} |D_{x_j} m(x)||a_{j,k}(x)||D_{x_k} u(x)| \text{d}x \]

\[ \leq C_1 \limsup_{\delta \to 0^+} \left[ \int_{\Omega} m(x)^{2(p-2)+p}(|u(x)|^2 + \delta)^{p/2} \text{d}x \right]^{1/2} \]

\[ \times \limsup_{\delta \to 0^+} \left[ \int_{\Omega} \sum_{j,k=1}^{n} |a_{j,k}(x)||D_{x_k} u(x)|^2 (|u(x)|^2 + \delta)^{(p-2)/2} \text{d}x \right]^{1/2} \]

(cf. (4.16), (4.17))
\[
\leq C_{25} \lim_{\delta \to 0^+} \left\{ \left[ \int_\Omega m(x) \left( \|u(x)\|^2 + \delta \right)^{p/2} \, dx \right] \right\}^{1/2} \\
\times \left[ \int_\Omega (|u(x)|^2 + \delta)^{(p-2)/2} \|\nabla u(x)\|^2 \, dx \right]^{1/2} \right\}
\leq C_{26} \left\{ \int_\Omega m(x)^{(p-2+\rho)} |u(x)|^p \, dx \right\}^{1/2} \|f\|_p^{p/2}
\]
\[\leq C_{27} \|f\|_p^{p/2} \|Mu\|_p^{p-2+\rho} \|u\|_p^{2-p-\rho/2}.\]

(4.35)

To derive the last inequality we have applied Hölder’s inequality with index \( q = p[2(p - 2 + \rho)]^{-1} \) to the integral
\[
\int_\Omega [m(x)|u(x)|]^{2(p-2+\rho)} |u(x)|^{-p+4-2\rho} \, dx.
\]

From (4.34) and (4.35) we immediately conclude (4.28).

Taking now the real part and the modulus of the imaginary part in (4.21) and using (4.22), we easily derive the relations

\[
\Re \lambda \|Mu\|_p^p + \lim_{\delta \to 0^+} \left[ I_1(\delta) + (p - 1) \Re I_2(\delta) - (2 - p) \Re I_3(\delta) \right] \\
+ \int_\Omega a_0(x)m(x)\lambda^{p-1} |u(x)|^p \, dx = \Re \int_\Omega m(x)\lambda^{p-1} f(x)\overline{u(x)}|u(x)|^p-2 \, dx,
\]

\[\|\Im \lambda\|_p^p \leq \lim_{\delta \to 0^+} \|(p - 1) \Im I_2(\delta) - (2 - p) \Im I_3(\delta)\|\]

(4.36)

\[
\Im \left[ \int_\Omega m(x)\lambda^{p-1} f(x)\overline{u(x)}|u(x)|^p-2 \, dx \right], \quad \forall \lambda \in \mathbb{C}.
\]

(4.37)

Add now member by member (4.36) and (4.37) multiplied by \( \eta \in (0, 2\sqrt{p-1}(2-p)^{-1}) \) and use (4.28) and (2.2). We easily deduce the following estimate for any \( \lambda \in \Sigma =: \{ \mu \in \mathbb{C} : \Re \mu + \eta |\Im \mu| \geq 0 \} \):

\[
\left[ \Re \lambda + \eta |\Im \lambda| + \frac{\eta}{\|m\|_{\infty}} \right] \|Mu\|_p^p \\
\leq - \left[ \lim_{\delta \to 0^+} \left[ I_1(\delta) + (p - 1) \Re I_2(\delta) - (2 - p) \Re I_3(\delta) \right] \\
- \eta \lim_{\delta \to 0^+} \|(p - 1) \Im I_2(\delta) - (2 - p) \Im I_3(\delta)\| \right] \\
+ \Re \int_\Omega f(x)m(x)\lambda^{p-1}|u(x)|^p-2 \, dx + \eta \left| \Im \left[ \int_\Omega f(x)m(x)\lambda^{p-1}\overline{u(x)}|u(x)|^p-2 \, dx \right] \right|
\]

\[
\leq - \lim_{\delta \to 0^+} \left[ I_1(\delta) + (p - 1) \Re I_2(\delta) - (2 - p) \Re I_3(\delta) \right] \\
- \eta \left[ (p - 1) \Im I_2(\delta) - (2 - p) \Im I_3(\delta) \right].
\]
\begin{align*}
+ \Re \int \Omega \frac{f(x) m(x) \partial x}{\partial x} |u(x)|^{2-2} \, dx + \eta \left| \int \Omega \frac{f(x) m(x) \partial x}{\partial x} |u(x)|^{2-2} \, dx \right| \\
\leq C_{28} \|f\|^{1/2}_p \|M u\|^{\frac{1}{2}}_p \|u\|^{\frac{1}{2} p - \frac{1}{2}}_p + (1 + \eta^2)^{1/2} \|f\|_p \|M u\|^{1-\rho}_p.
\end{align*}
(4.38)

Take \( \lambda \) in the sector
\begin{equation}
\Sigma_3 = \left\{ \mu \in \mathbb{C} : \Re \mu + \eta \mu + \frac{\gamma}{2 \|m\|_\infty} \geq 0 \right\}.
\end{equation}
(4.39)

Then, since \( \|u\|_p \leq C_{19} \|f\|_p \) (cf. (2.11), (2.12) and our definition of \( \eta \)) and \( 2 - \rho - p/2 > 0 \) (cf. (4.20)), by Proposition 2.1 we immediately derive the inequality
\begin{equation}
(\lambda + 1) \|M u\|^{\frac{2}{\rho}}_p \leq C_{24} (\|f\|^{2-\rho}_p + \|f\|_p \|M u\|^{1-\rho}_p), \quad \text{if} \quad \lambda \in \Sigma_3.
\end{equation}
(4.40)

Finally, \( \|M u\|_p \leq \|m\|_\infty \|u\|_p \leq C_{19} \|m\|_\infty \|f\|_p \) implies
\begin{equation}
(\lambda + 1) \|M u\|^{\frac{2}{\rho}}_p \leq C_{30} \|f\|^{\frac{2}{\rho}}_p, \quad \text{if} \quad \lambda \in \Sigma_3.
\end{equation}
(4.41)

We can now collect the result in this section in the following Theorem 4.1.

**Theorem 4.1.** Let \( L \) and \( M \) be the linear operators defined by (1.7) and (1.8), the coefficients \( a_{i,j} \), \( i, j = 1, \ldots, n \), \( a_0 \) enjoying properties (2.1) and (2.2) and \( m \) being a non-negative function satisfying (1.9). Then the spectral equation \( \lambda M u + L u = f \), with \( f \in L^p(\Omega) \), admits, for any \( \lambda \in \Sigma_3 \) and \( p \in (1, 2), \rho \in [2 - p, 1] \), a unique solution \( u \in W^{2, p}(\Omega) \cap W^{1, p}_0(\Omega) \) satisfying the estimates
\begin{equation}
\|u\|_p \leq C_{30} \|f\|_p, \quad \|M u\|_p \leq C_{31}(p) |\lambda|^{-2-p} \|f\|_p, \quad \lambda \in \Sigma_3,
\end{equation}
(4.42)
\begin{equation}
\|L u\|_p \leq C_{32}(1 + |\lambda|^{-p(2-\rho)-1}) \|f\|_p, \quad \lambda \in \Sigma_3.
\end{equation}

**Example 4.1.** Let \( n = 1, m(x) = x^q (1 - x)^q, \quad q \in (1, +\infty), \quad \Omega = (0, 1). \)

Then
\begin{equation}
m'(x) = q(1 - 2x)m(x)^{q-1}/q, \quad x \in (0, 1).
\end{equation}

Hence (4.25) holds true for any \( q \in (1, +\infty) \). If we have to deal with \( L^p(0, 1) \) with \( p \in (1, 2) \), to satisfy (4.20) we are forced to assume \( q > (p - 1)^{-1} \).

**5. Solving problem (1.1)–(1.3)**

Taking the spectral Theorems 2.1, 3.1, 4.1 into account, from Theorem 3.26 in [3] we can easily derive our existence and uniqueness result. For this purpose we need to introduce the following interpolation space
\begin{equation}
L^p_{\theta, \infty} = \left\{ g \in L^p(\Omega) : \sup_{t \geq 1} t^\theta \|L(tM + L)^{-1}\|_{L^p(\Omega)} < +\infty \right\}.
\end{equation}
(5.1)
In particular, any \( g = mh \) belongs to \( L^p_{\theta,\infty} \), whenever \( m \in L^\infty(\Omega) \) and \( h \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \). Notice that \( L^p_{\theta,\infty} \subset (X; D(LM^{-1}))_{\theta,\infty} \).

**Theorem 5.1.** Let \( p \in (1, +\infty) \), let \( m \in L^\infty(\Omega) \) be a non-negative function and let the coefficients \( a_{i,j}, i,j = 1, \ldots, n \), \( a_0 \) enjoy properties (2.1) and (2.2). Then for any

\[
u_0 \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \quad f \in C^0([0, T]; L^p(\Omega)), \quad \theta \in (1 - \beta, 1),
\]

with \( \beta = 1/p \) and

\[
-A(x, D_x)u_0 + f(0, \cdot) = g_0, \quad g_0 \in L^p_{\theta,\infty},
\]

problem (1.1)–(1.3) admits a unique solution

\[
u = C^{\theta+\beta}(0, T); L^p(\Omega)), \quad u = C^{\theta+\beta-1}(0, T); W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)).
\]

Moreover, if \( m \) is a non-negative function satisfying (1.9) and \( \beta \) is defined by (1.10), the same result holds under assumptions (5.1) and (5.2) on \((u_0, f)\).

\section{Proofs of the propositions}

\textbf{Proof of Proposition 2.1.} Let \( \lambda \in \Sigma_{k,\varepsilon} \) and \( \Re \lambda \geq 0 \). Then it is clear that

\[
|\Re \lambda| \leq \Re \lambda + k|\Im \lambda| + \varepsilon.
\]

On the other hand, if \( \lambda \in \Sigma_{k,\varepsilon} \) and \( \Re \lambda < 0 \), then

\[
|\Re \lambda| = -\Re \lambda \leq (k/2)|\Im \lambda| + (\varepsilon/2) \leq \Re \lambda + k|\Im \lambda| + \varepsilon.
\]

Therefore, \( |\Re \lambda| \leq \Re \lambda + k|\Im \lambda| + \varepsilon \) for any \( \lambda \in \Sigma_{k,\varepsilon} \). In the meantime it is obvious that

\[
|\Im \lambda| + 1 \leq 2[(1/k)+(1/\varepsilon)](k/2)|\Im \lambda| + \varepsilon \leq 2(1/k+(1/\varepsilon))(\Re \lambda + k|\Im \lambda| + \varepsilon)
\]

for any \( \lambda \in \Sigma_{k,\varepsilon} \). Hence we conclude that

\[
|\lambda| + 1 \leq |\Re \lambda| + |\Im \lambda| + 1 \leq (2/k+(2/\varepsilon)+1)(\Re \lambda + k|\Im \lambda| + \varepsilon), \quad \lambda \in \Sigma_{k,\varepsilon}.
\]

\hfill \Box

\textbf{Proof of Proposition 2.2.} We consider the set \( J = \{\theta \in [0, 1]; \mathcal{R}(A(\theta)) = X\} \) and shall prove that this set is an open and closed subset of the interval \([0, 1]\) under (2.17) and (2.18). In fact, let \( \theta \in J \); then, it follows from (2.17) that \( A(\theta)^{-1} \in \mathcal{L}(X) \) with \( \|A(\theta)^{-1}\| \leq \delta^{-1} \). Moreover, for any \( \theta' \in [0, 1] \), we have

\[
A(\theta') = [1 + \{A(\theta') - A(\theta)\}A(\theta)^{-1}]A(\theta).
\]

Since \( \|\{A(\theta') - A(\theta)\}A(\theta)^{-1}\| \leq N\delta^{-1}|\theta' - \theta| \), the operator \( 1 + \{A(\theta') - A(\theta)\}A(\theta)^{-1} \) is a linear isomorphism of \( X \) provided \( |\theta' - \theta| < N^{-1}\delta \). This then shows that \( \theta' \in J \) for any \( \theta' \) such that \( |\theta' - \theta| < N^{-1}\delta \); hence, \( J \) is an open set. Consider now a sequence \( \theta_n \in J \) and assume that \( \theta_n \to \theta \) as \( n \to +\infty \). Let \( f \in X \) be any vector; then, there exists a sequence \( u_n \in D \) such that \( A(\theta_n)u_n = f \). From (2.17) it follows that \( \|u_n\|_X \leq \delta^{-1}\|f\| \). Furthermore we observe that

\[
\| A(\theta)u_n - f \| \leq \| A(\theta)u_n - A(\theta_n)u_n\| + \| A(\theta_n)u_n - f \|
\]

\[ \leq N\delta^{-1}|\theta - \theta_n|\|f\|.
\]
therefore, \( A(\bar{\theta})u_n \to f \) as \( n \to +\infty \). In the meantime, \( \delta\|u_m-u_n\| \leq \|A(\bar{\theta})(u_m-u_n)\| \leq \|A(\bar{\theta})u_m-f\| + \|f-A(\bar{\theta})u_n\| \to 0 \) as \( m, n \to +\infty \). So, \( u_n \) has a limit \( u \in X \) as \( n \to +\infty \). Since \( A(\bar{\theta}) \) is a closed operator, \( u \in \mathcal{D} \) and \( A(\bar{\theta})u = f \); hence, \( \bar{\theta} \in J \). That is, \( J \) is a closed set. As \( 1 \in J \neq \emptyset \), we conclude that \( J = [0,1] \). \( \square \)

Proof of Proposition 4.1. According to (1.9), we have the inclusion \( Z(m) \subset Z(\nabla m) \). Moreover, formula (4.23) is trivial if \( x \notin Z(m) \). This therefore shows that we have to deal with the case \( x \in Z(m) \) only.

First we will consider the one-dimensional case \( (n=1) \). For this purpose assume \( x_0 \in Z(m) \). Our starting point is the following formula:

\[
\lim_{x \to x_0} \frac{m(x) - m(x_0)}{x - x_0} = \lim_{\varepsilon \to 0^+} \left| \frac{m(x) + \varepsilon - m(x)}{x - x_0} \right| = \lim_{x \to x_0} \left| \frac{\beta}{x - x_0} \int_{x_0}^x [m(t) + \varepsilon]^{\beta-1} m'(t) \, dt \right|.
\]

We next notice that \( \lim_{\varepsilon \to 0^+} [m(t) + \varepsilon]^{\beta-1} m'(t) = m_1(t) \) for any \( t \in \Omega \) and that

\[
[m(t) + \varepsilon]^{\beta-1} m'(t) \leq C[m(t) + \varepsilon]^{\beta-1} m(t)^\rho = C \left[ \frac{m(t)}{m(t) + \varepsilon} \right]^{1-\beta} m(t)^{\beta-1+\rho} \leq C m(t)^{\beta-1+\rho}, \quad \forall t \in \Omega.
\]

By virtue of the dominated convergence theorem and by the bound \( |m_1(t)| \leq C m(t)^{\beta-1+\rho} \) for any \( t \in \Omega \), we deduce the following relations:

\[
\lim_{x \to x_0^+} \frac{m(x) - m(x_0)}{x - x_0} = \lim_{x \to x_0^+} \left| \frac{1}{x - x_0} \int_{x_0}^x m_1(t) \, dt \right| \leq \lim_{x \to x_0^+} \frac{C}{x - x_0} \int_{x_0}^x m(t)^{\beta-1+\rho} \, dt = 0.
\]

Note here that \( m(\cdot)^{\beta-1+\rho} \) is continuous in \( \Omega \) and \( x_0 \in Z(m) \). An analogous argument holds for \( \lim_{x \to x_0^-} [(m(x)^\beta - m(x_0)^\beta) / (x - x_0)] \) also.

We have thus shown that there exists \( D_x[m(\cdot)^\beta](x_0) \) and coincides with \( 0 = m_1(x_0) \). Therefore the formula \( D_x[m(\cdot)^\beta](x) = m_1(x) \) holds for any \( x \in \Omega \). Since \( \beta \in (1-\rho,1) \), bound (1.9) and (4.23) immediately imply that \( m_1 \in C(\Omega) \). Consequently, \( m(\cdot)^\beta \in C(\Omega) \).

Finally, the multi-dimensional case is an immediate consequence of the case \( n = 1 \). \( \square \)
References


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