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<td>Author(s)</td>
<td>Favini, Angelo; Lorenzi, Alfredo; Tanabe, Hiroki et al.</td>
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<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 2005, 42(2), p. 385-406</td>
</tr>
<tr>
<td>Version Type</td>
<td>VoR</td>
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<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/5512">https://doi.org/10.18910/5512</a></td>
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Osaka University
AN $L^p$-APPROACH TO SINGULAR LINEAR PARABOLIC EQUATIONS IN BOUNDED DOMAINS

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(Received August 19, 2002)

Abstract

Singular means here that the parabolic equation is not in normal form neither can it be reduced to such a form. For this class of problems, following the operator approach used in [1], we prove global in time existence and uniqueness theorems related to (spatial) $L^p$-spaces. Various improvements to [2], [3] are given.

1. Introduction

In this paper we will consider the following boundary value problem

\begin{align}
&D_t[m(x)u(x, t)] + A(x, D_x)u(x, t) = f(x, t), \quad \forall(x, t) \in \Omega \times [0, \tau], \\
&u(x, t) = 0, \quad \forall(x, t) \in \partial\Omega \times [0, \tau], \\
&m(x)u(x, t) \to m(x)u_0(x), \quad \text{for a.e. } x \in \Omega, \text{ as } t \to 0+,
\end{align}

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a boundary of class $C^2$, while $A(x, D_x)$ is the following second-order uniformly elliptic operator in divergence form

\begin{equation}
A(x, D) = -\sum_{i,j=1}^{n} D_{x_i} [a_{ij}(x)D_{x_j}] + a_0(x).
\end{equation}

Moreover, $0 \neq m \in L^\infty(\Omega)$ is a non-negative function which need not to be bounded away from 0. Consequently, our parabolic equation is, in general, singular.

Particular cases of (1.1) are discussed in the monograph [3], pp.74–80. See also [2]. Note that in [3], p.80, the restriction $p \in (2, +\infty)$ should be made.

Using the theoretical results in [3] and the fundamental approach in [4] we can develop an $L^p$-theory, $p \in (1, +\infty)$, also in the present degenerate case\(^a\). The keystone in order to apply the results in [1] and [3], Theorem 3.28, p.69, to (1.1)–(1.4)

\(^a\)We note that in this case the initial condition (1.3) should be more correctly meant as the following $L^p$-limit: $\|m(\cdot)u(\cdot, t) - m(\cdot)u_0(\cdot)\|_{L^p(\Omega)} \to 0$ as $t \to 0+$.
consists in showing an operator estimate of the form
\begin{equation}
\|L(\lambda M + L)^{-1}\|_{L^2(\Omega)} \leq C(1 + |\lambda|)^{1-\beta}, \quad \forall \lambda \in \Sigma_\alpha,
\end{equation}
where \( X = L^p(\Omega), \ 0 < \beta \leq \alpha \leq 1, \ \alpha + \beta > 1, \)
\begin{equation}
\Sigma_\alpha = \{ \lambda \in \mathbb{C} : \text{Re}\lambda \geq -c(1 + |\text{Im}\lambda|)^{\rho} \}, \quad (c > 0),
\end{equation}
and
\begin{align}
\mathcal{D}(L) &= \mathcal{D}(L_p) = W^{2, p}(\Omega) \cap W^{1, p}(\Omega), \quad Lu(x) = A(x, D)u(x), \quad u \in \mathcal{D}(L), \\
\mathcal{D}(M) &= L^p(\Omega), \quad Mu(x) = m(x)u(x).
\end{align}
We in fact show that (1.5) holds with \( \alpha = 1, \ \beta = 1/p, \ p \in (1, +\infty). \)
Moreover, when \( m \) is \( \rho \)-regular, i.e.
\begin{equation}
m \in C^1(\overline{\Omega}), \quad |\nabla m(x)| \leq C_1 m(x)^{\rho}, \quad \forall x \in \overline{\Omega}, \quad \text{for some } \rho \in (0, 1],
\end{equation}
\( C_1 \) being a positive constant, we can improve the index \( \beta \) in estimate (1.5) from \( \beta = 1/p \) to
\begin{equation}
\beta = \begin{cases}
\frac{2 - \rho}{2}, & \text{if } p \in (1, 2), \ \rho \in (2 - p, 1], \\
\frac{2p(2 - \rho)}{2p - 2}, & \text{if } p \in [2, +\infty), \ \rho \in (0, 1].
\end{cases}
\end{equation}
The result proved in this paper will be applied, in a subsequent paper, to identify the unknown kernel \( k \) in the integro-differential singular equation of parabolic type
\begin{equation}
D_t[m(x)u(x, t)] + A(x, D_x)u(x, t) = \int_0^t k(t - s)B(x, D_x)u(x, s)\, ds + f(x, t), \quad \forall (x, t) \in \Omega \times [0, \tau],
\end{equation}
\( B(x, D_x) \) being a linear second-order differential operator.
We stress that the present paper was originated by a requirement of additional smoothness of solution \( u \) of (1.11) needed to recover the unknown kernel \( k \). This occurrence is in accordance with the well-known fact that inverse problems usually force deeper, and sometimes, unexpected insights in direct problems.

2. Solving the spectral problem \((\lambda M + L)u = f\)

The basic aim of this section consists in showing that estimate (1.5) holds when the linear operators \( M \) and \( L \) are defined by (1.7) and (1.8), respectively. To this aim we assume that the coefficients \( \alpha_{i,j} \) and \( a_0 \) satisfy the properties
\begin{equation}
a_{i,j} \in C^1(\overline{\Omega}), \quad a_0 \in C(\overline{\Omega}), \quad a_{i,j} = a_{j,i}, \quad i, j = 1, \ldots, n,
\end{equation}
\[ (2.2) \quad c_0 |\xi|^2 \leq \sum_{i,j=1}^{n} a_{i,j}(x) \xi_i \xi_j \leq c_1 |\xi|^2, \quad \forall x \in \overline{\Omega}, \forall \xi \in \mathbb{R}^n, \quad a_0(x) \geq \gamma, \quad \forall x \in \Omega, \]

\[ c_0, c_1 \] and \( \gamma \) being three positive constants.  

A remarkable result by Okazawa [4, p.702] provides, for any \( u \in D(L) \),

\[ \Re((L - a_0)u, u |u|^{p-2}) \]

\[ \geq \begin{cases} 
  c_0 \int_{\Omega} |u|^{p-2} |\nabla u|^2 \, dx \geq 0, & \text{if } p \in [2, \infty), \\
  c_0(p-1) \int_{\Omega} (|u|^2 + \delta)^{(p-2)/2} |\nabla u|^2 \, dx \geq 0, & \text{if } p \in (1, 2),
\end{cases} \]

\[ |\Im(Lu, u |u|^{p-2})| \leq \frac{|p-2|}{2p-1} \Re((L - a_0)u, u |u|^{p-2}), \]

where the brackets denote

\[ (f, g) = \int_{\Omega} f(x) \overline{g(x)} \, dx, \quad f \in L^p(\Omega), \quad g \in L^p'(\Omega), \quad \frac{1}{p} + \frac{1}{p'} = 1, \]

\( u |u|^{p-2} \) is assumed to vanish whenever \( u \) does, and \( \delta > 0 \) is arbitrary.

**Remark 2.1.** It is important to observe that bound (2.4) holds even in the degenerate elliptic case (cf. [4, p. 702] and the following Lemma 3.3).

From (2.4) we immediately deduce the estimate

\[ |\Im(Lu, u |u|^{p-2})| + \frac{|p-2|}{2p-1} \int_{\Omega} a_0(x) |u(x)|^p \, dx \]

\[ \leq \frac{|p-2|}{2p-1} \Re(Lu, u |u|^{p-2}). \]

Consider now the spectral problem

\[ (2.6) \quad u \in D(L), \quad \lambda nu + Lu = f \in L^p(\Omega). \]

Taking the real and imaginary parts of the scalar product of both sides in (2.6) with \( u |u|^{p-2} \), we get

\[ \Re \lambda \int_{\Omega} m |u|^p \, dx + \Re(Lu, u |u|^{p-2}) = \Re \int_{\Omega} f \overline{u} |u|^{p-2} \, dx, \]

\[ \Im \lambda \int_{\Omega} m |u|^p \, dx + \Im(Lu, u |u|^{p-2}) = \Im \int_{\Omega} f \overline{u} |u|^{p-2} \, dx. \]
From (2.8) we deduce the inequalities

\[(2.9) \quad |\text{Im} \lambda| \int_{\Omega} m |u|^p \, dx \leq |\text{Im}(Lu, u|u|^{p-2})| + |\text{Im} \int_{\Omega} f \overline{u} |u|^{p-2} \, dx|.
\]

Multiply then both sides in (2.9) by a positive constant \(k\) and add the obtained inequality to equation (2.7). From (2.5) we get

\[(2.10) \quad \left(\text{Re} \lambda + k |\text{Im} \lambda|\right) \int_{\Omega} m |u|^p \, dx + \left(1 - k \frac{|p - 2|}{2\sqrt{p - 1}}\right) \text{Re}(Lu, u|u|^{p-2}) \leq \text{Re} \int_{\Omega} f \overline{u} |u|^{p-2} \, dx + k |\text{Im} \int_{\Omega} f \overline{u} |u|^{p-2} \, dx| \leq (1 + k) \|f\|_p \|u\|_p^{p-1}.
\]

Choose now \(k = k_1(p)\) so small as to satisfy

\[(2.11) \quad h_1(p) := 1 - k_1(p) \frac{|p - 2|}{2\sqrt{p - 1}} > 0, \quad \forall p \in (1, +\infty).
\]

Observe that

\[
\text{Re}(Lu, u|u|^{p-2}) = \text{Re}((L - a_0)u, u|u|^{p-2}) + \frac{1}{2} \text{Re}(a_0 u, u|u|^{p-2}) + \frac{1}{2} \text{Re}(a_0 u, u|u|^{p-2})
\]

\[(2.12) \quad \geq \text{Re}((L - a_0)u, u|u|^{p-2}) + \frac{\gamma}{2} \|u\|_p^2 + \frac{\gamma}{2\|m\|_\infty} \int_{\Omega} m |u|^p \, dx,
\]

since \(m(x) \leq \|m\|_\infty\) implies

\[
\frac{m(x)}{\|m\|_\infty} \frac{a_0(x)}{2} \leq \frac{a_0(x)}{2}.
\]

In view of (2.11), (2.12) and (2.3), we obtain from (2.10) that

\[
\left(\text{Re} \lambda + k_1(p) |\text{Im} \lambda| + \frac{\gamma h_1(p)}{2\|m\|_\infty}\right) \int_{\Omega} m |u|^p \, dx
\]

\[(2.13) \quad + \frac{\gamma h_1(p)}{2} \|u\|_p^p + h_1(p) \text{Re}((L - a_0)u, u|u|^{p-2}) \leq [k_1(p) + 1] \|f\|_p \|u\|_p^{p-1}.
\]

Introduce now the sector

\[
\Sigma_1 = \left\{ \mu \in \mathbb{C} : \text{Re} \mu + \frac{k_1(p)}{2} |\text{Im} \mu| + \frac{\gamma h_1(p)}{4\|m\|_\infty} \geq 0 \right\}.
\]

Then, for \(\lambda \in \Sigma_1\),

\[(2.14) \quad \text{Re}((L - a_0)u, u|u|^{p-2}) \leq \frac{k_1(p) + 1}{h_1(p)} \|f\|_p \|u\|_p^{p-1},
\]
We now need a simple proposition. For the proof see Section 6.

Proposition 2.1. Let \( k > 0 \) and \( \varepsilon > 0 \) be two positive constants, and let \( \Sigma_{k,\varepsilon} \) be a sectorial domain given by

\[
\Sigma_{k,\varepsilon} = \left\{ \mu \in \mathbb{C} : \text{Re}\, \mu + \frac{k}{2} |\text{Im}\, \mu| + \frac{\varepsilon}{2} \leq 0 \right\}.
\]

Then it holds that

\[
|\lambda| + 1 \leq \left( \frac{2}{k} + \frac{2}{\varepsilon} + 1 \right) (\text{Re}\, \lambda + k |\text{Im}\, \lambda| + \varepsilon) , \quad \lambda \in \Sigma_{k,\varepsilon}.
\]

Since \( \Sigma_1 = \Sigma_{k(p),h_1(p)/2||m||_{\infty}} \), this proposition then yields

\[
(2.16) \quad (|\lambda| + 1) \int_{\Omega} |m| |u|^p \, dx \leq C_2(p) ||f||_p ||u||_p^{p-1} \leq C_3(p) ||f||_p^p , \quad \lambda \in \Sigma_1.
\]

To show that \( (\lambda M + L)^{-1} \) is a bounded operator on \( L^p(\Omega) \) for \( \lambda \in \Sigma_1 \), it now suffices to verify that \( \mathcal{R}(\lambda M + L) = L^p(\Omega) \). But this is verified by the usual techniques without difficulty. In fact, for each \( \lambda \in \Sigma_1 \), we already know that \( \mathcal{R}(\Lambda + \lambda M + L) = L^p(\Omega) \) provided \( \Lambda > 0 \) is a sufficiently large number. Let \( 0 \leq \theta \leq 1 \) be a parameter, and consider the family of closed linear operators \( A(\theta) = \theta \Lambda + \lambda M + L, \ 0 \leq \theta \leq 1 \). Then the desired result is obtained by the following proposition the proof of which will be given in the final section.

Proposition 2.2. Let \( A(\theta), \ 0 \leq \theta \leq 1 \), be a family of closed linear operators acting on a Banach space \( X \) with constant domain \( \mathcal{D}(A(\theta)) \equiv \mathcal{D} \). Assume that the family satisfies the conditions

\[
(2.17) \quad \delta ||u|| \leq ||A(\theta)u||, \quad u \in \mathcal{D},
\]

\[
(2.18) \quad ||[A(\theta) - A(\theta')]u|| \leq N||\theta - \theta'||||u||, \quad u \in \mathcal{D}
\]

with some constants \( \delta > 0 \) and \( N > 0 \) independent of \( \theta, \theta' \in [0,1] \). Then, \( \mathcal{R}(A(1)) = X \) implies \( \mathcal{R}(A(\theta)) = X \) for every \( \theta \in [0,1] \).

We can now summarize the results proved in this section in Theorem 2.1.
Theorem 2.1. Let $L$ and $M$ be the linear operators defined by (1.7) and (1.8), the coefficients $a_{i,j}$, $i, j = 1, \ldots, n$, $a_0$ enjoying properties (2.1) and (2.2) and $m$ being a non-negative function in $L^\infty(\Omega)$. Then the spectral equation $\lambda M u + L u = f$, with $f \in L^p(\Omega)$, admits, for any $\lambda \in \Sigma_1 = \{ \mu \in C : \Re \mu + (k_1(p)/2)[\Im \mu + \gamma h_1(p)/(4|m|_\infty)] \geq 0 \}$ and $p \in (1, +\infty)$, a unique solution $u \in W^2,p(\Omega) \cap W^1,p(\Omega)$ satisfying the estimates

$$
\|u\|_p \leq C_4(p)\|f\|_p, \quad \|M u\|_p \leq C_5(p)\|\lambda\|^{-1/p}\|f\|_p, \quad \lambda \in \Sigma_1,
$$

$$
\|L u\|_p \leq C_6(p)(1 + |\lambda|^{1/p})\|f\|_p, \quad \lambda \in \Sigma_1.
$$

3. The case when $m$ is $\rho$-regular and $p \in [2, +\infty)$

We will show that when the multiplier $m$ is more regular, i.e. it satisfies (1.9), our $\beta$ can be chosen larger than $1/p$. We recall that all the previous estimates (2.6)–(2.16) hold for any $p \in (1, +\infty)$.

First of all we need the following lemma concerning the computation of the gradient of the function $\overline{u}|u|^{p-2}$ when $p \in [2, +\infty)$. For this purpose we need some lemmata.

Lemma 3.1. Let $u \in W_0^{1,p}(\Omega)$ with $p \in [2, +\infty)$. Then the function $\overline{u}|u|^{p-2}$ belongs to $W_0^{1,p}(\Omega)$ and the following formulae hold

$$
D_j \overline{u}|u|^{p-2} = |u|^{p-2}D_j \overline{u} + (p-2)g_p(u) \Re(g_p(u)D_j u),
$$
a.e. in $\Omega$, $j = 1, \ldots, n$,

(3.1)

where

(3.2)

$$
g_p(u)(x) = \begin{cases} 
\overline{u}(x)|u(x)|^{p-4}/2, & \text{if } u(x) \neq 0, \\
0, & \text{if } u(x) = 0.
\end{cases}
$$

Proof. Let $\phi$ be any function in $C_0(\Omega)$. Then the following equalities hold:

$$
\langle D_j \phi, \overline{u}|u|^{p-2}\rangle = \lim_{\varepsilon \to 0^+} \langle D_j \phi, \overline{u}(|u|^2 + \varepsilon)^{(p-2)/2}\rangle
$$

$$
= -\lim_{\varepsilon \to 0^+} \langle \phi, (|u|^2 + \varepsilon)^{(p-2)/2}D_j \overline{u} + \frac{p-2}{2} \overline{u}(|u|^2 + \varepsilon)^{(p-4)/2}D_j u + uD_j \overline{u}\rangle
$$

$$
= -\lim_{\varepsilon \to 0^+} \langle \phi, (|u|^2 + \varepsilon)^{(p-2)/2}D_j \overline{u} + (p-2)|u|^2 + \varepsilon)^{(p-4)/2}D_j u\rangle
$$

$$
= -\lim_{\varepsilon \to 0^+} \langle \phi, (|u|^2 + \varepsilon)^{(p-2)/2}D_j \overline{u} + (p-2)|u|^2 + \varepsilon)^{(p-4)/4}D_j u\rangle
$$

$$
= -(\phi, |u|^{p-2}D_j \overline{u} + (p-2)g_p(u) \Re(g_p(u)D_j u)).
$$

(3.3)

We have used here the relation $\lim_{\varepsilon \to 0^+} \overline{u}(x)(|u(x)|^2 + \varepsilon)^{(p-4)/4} = g_p(u)(x)$, which takes advantage of the assumption $p \in [2, +\infty)$. \qed
REMARK 3.1. From definition (3.2) we easily deduce the identity

\[ |g_p(u)(x)| = |u(x)|^{(p-2)/2}. \]

We can now prove the following Lemma 3.2.

**Lemma 3.2.** Let \((b_{i,j})_{i,j=1,\ldots,n}\) be a matrix of functions in \(C^1(\Omega, \mathbb{R})\) such that

\[ b_{i,j} = b_{j,i}, \quad i, j = 1, \ldots, n, \]

\[ c_0 |\xi|^2 \mu(x) \leq \sum_{i,j=1}^n b_{i,j}(x) \xi_i \xi_j \leq c_1 |\xi|^2 \mu(x), \quad \forall x \in \Omega, \forall \xi \in \mathbb{R}^n, \]

where \(\mu \in C(\Omega)\) is a non-negative function and \(c_0, c_1\) are two positive constants. Then for any \(p \in [2, +\infty)\), the linear operator \(K = -\sum_{i,j=1}^n D_x [b_{i,j}(x)D_{x_j}]\) with \(\mathcal{D}(K) = \mathcal{D}(L)\) (cf. (1.7)) satisfies the relations

\[ c_0 \left( \int_{\Omega} |u|^{p-2} |Du|^2 \, dx + \int_{\Omega} \mu \sum_{j=1}^n |\text{Re}(g_p(u)D_{x_j}u)|^2 \, dx \right) \]

\[ \leq \text{Re}(Ku, \overline{u}) |u|^{p-2} \leq c_1 \left( \int_{\Omega} |u|^{p-2} |Du|^2 \, dx + \int_{\Omega} \mu \sum_{j=1}^n |\text{Re}(g_p(u)D_{x_j}u)|^2 \, dx \right), \]

\[ \text{Im}(Ku, \overline{u}) |u|^{p-2} = (p-2) \int_{\Omega} \sum_{i,j=1}^n b_{i,j} |\text{Re}(g_p(u)D_{x_j}u)||\text{Im}(g_p(u)D_{x_i}u)| \, dx. \]

Proof. From Lemma 3.1 and an integration by parts we easily deduce the identity

\[ (Ku, \overline{u}) |u|^{p-2} = \int_{\Omega} \sum_{i,j=1}^n b_{i,j} D_{x_j}u D_{x_i}(\overline{u}) |u|^{p-2} \, dx \]

\[ = \int_{\Omega} \sum_{i,j=1}^n |u|^{p-2} b_{i,j} D_{x_j}u D_{x_i} \overline{u} \, dx \]

\[ + (p-2) \int_{\Omega} \sum_{i,j=1}^n b_{i,j} g_p(u)D_{x_j}u \text{Re}(g_p(u)D_{x_i}u) \, dx. \]

Relations (3.7) and (3.8) follow immediately from (3.9) taking the real and the imaginary parts.

**Lemma 3.3.** Under the assumptions in the statement of Lemma 3.2 operator \(K\) satisfies inequalities (2.3) and (2.4) with \(K\) in the place of \(L = a\).
Proof. This lemma has essentially been proved in [4], although a slight modification is needed in its proof. For any \( \varepsilon > 0 \) define \( \delta_{i,j} = b_{i,j} + \varepsilon \delta_{i,j}, \ i, j = 1, \ldots, n, \) and set \( K_\varepsilon = K - \varepsilon \Delta. \) Since the matrix \((a_{i,j})_{i,j=1}^n\) is uniformly positive definite, from (2.3) and (2.4), with \( u \in \mathcal{D}(L_0) \), we obtain the inequalities

\[
0 \leq \Re(K_\varepsilon u, u|u|^{p-2}) = \Re(K u, u|u|^{p-2}) + \varepsilon \Re(-\Delta u, u|u|^{p-2}),
\]

\[
|\Im(K_\varepsilon u, u|u|^{p-2})| = |\Im(K u, u|u|^{p-2}) + \varepsilon \Im(-\Delta u, u|u|^{p-2})|
\]

(3.10) \[ \leq \frac{|p-2|}{2p-1} [\Re(K u, u|u|^{p-2}) + \varepsilon \Re(-\Delta u, u|u|^{p-2})]. \]

Taking the limit as \( \varepsilon \to 0^+ \) in (3.10) and (3.11), we easily deduce that \( K \) satisfies (2.3) and (2.4).

We shall use also the following identity

\[
(Lu, m^{p-1}u|u|^{p-2}) = (m^{p-1}Lu, u|u|^{p-2})
\]

(3.12) \[ = (K_0 u, u|u|^{p-2}) + (p - 1) \left( m^{p-2} \sum_{i,j=1}^n a_{i,j} D_{x_i}m D_{x_j}u, u|u|^{p-2} \right), \quad u \in \mathcal{D}(L), \]

where

\[
K_0 = -\sum_{i,j=1}^n D_{x_i} [m(x)^{p-1}a_{i,j}(x)D_{x_j}] + m(x)^{p-1}a_0(x).
\]

Let now \( u \) be a solution to equation (2.6). Taking the scalar product of both sides in (2.6) with \( m^{p-1}u|u|^{p-2} \) and using (3.12), we easily get the equalities

\[
(f, m^{p-1}u|u|^{p-2}) = (\lambda mu + Lu, m^{p-1}u|u|^{p-2})
\]

(3.13) \[ = \lambda ||Mu||^p_p + (K_0 u, u|u|^{p-2}) + (p - 1) \left( m^{p-2} \sum_{i,j=1}^n a_{i,j} D_{x_i}m D_{x_j}u, u|u|^{p-2} \right) . \]

Taking the real and imaginary parts in (3.13) and using (2.4) with \( L - a_0 \) replaced by \( K = K_0 - m^{p-1}a_0 \), we easily deduce the inequalities

\[
\Re \lambda ||Mu||^p_p + \gamma \int_{\Omega} m^{p-1} |u|^p \, dx + \Re((K_0 - m^{p-1}a_0)u, u|u|^{p-2})
\]

(3.14) \[ \leq |(f, m^{p-1}u|u|^{p-2})| + (p - 1) \left| \left( m^{p-2} \sum_{i,j=1}^n a_{i,j} D_{x_i}m D_{x_j}u, u|u|^{p-2} \right) \right| ,
\]

\[
|\Im \lambda ||Mu||^p_p | \leq |\Im((K_0 - m^{p-1}a_0)u, u|u|^{p-2})| + |(f, m^{p-1}u|u|^{p-2})| + (p - 1) \left| \left( m^{p-2} \sum_{i,j=1}^n a_{i,j} D_{x_i}m D_{x_j}u, u|u|^{p-2} \right) \right| \]
\[ \leq \frac{|p - 2|}{2\sqrt{p - 1}} \text{Re}((K_0 - m^{p-1}a_0)u, u|u|^{p-2}) + |(f, m^{p-1}u|u|^{p-2})| 
\]

\[(3.15) \quad + (p - 1) \left| \left( m^{p-2} \sum_{i,j=1}^n a_{i,j} D_{x_i}m D_{x_j}u, u|u|^{p-2} \right) \right|. \]

Multiply now by \(k_1(p)\) (cf. (2.11)) the first and last sides in (3.15) and add to the first and last sides in (3.14). We get the estimate

\[
[\text{Re} \lambda + k_1(p) \text{Im} \lambda] + \gamma \|m\|_\infty^{-1} \|Mu\|_p^p
\]

\[+ \left( 1 - k_1(p) \frac{|p - 2|}{2\sqrt{p - 1}} \right) \text{Re}((K_0 - m^{p-1}a_0)u, u|u|^{p-2})\]

\[\leq [1 + k_1(p)] \left\{ |(f, m^{p-1}u|u|^{p-2})| + (p - 1) \left| \left( m^{p-2} \sum_{i,j=1}^n a_{i,j} D_{x_i}m D_{x_j}u, u|u|^{p-2} \right) \right| \right\},\]

\[(3.16)\]

where we have made use of the elementary inequality

\[m(x)^p \leq \|m\|_\infty m(x)^{p-1}, \quad x \in \Omega.\]

We now estimate the last term in (3.16) with the aid of (1.9). Using twice Hölder’s inequality, we get

\[\left| \left( m^{p-2} \sum_{i,j=1}^n a_{i,j} D_{x_i}m D_{x_j}u, u|u|^{p-2} \right) \right| \leq \int_{\Omega} m^{p-2} |u|^{p-1} \left| \sum_{i,j=1}^n a_{i,j} D_{x_i}m D_{x_j}u \right| \, dx\]

\[\leq \int_{\Omega} m^{p-2} |u|^{p-1} \left| \sum_{i,j=1}^n a_{i,j} D_{x_i}m D_{x_j}u \right|^{1/2} \left| \sum_{i,j=1}^n a_{i,j} D_{x_i}m D_{x_j}u \right|^{1/2} \, dx\]

\[\leq C_7 \int_{\Omega} m^{p-2+p} |u|^{p-1} |\nabla u| \, dx = C_7 \int_{\Omega} m^{pp/2} m^{(p-2)(2-p)/2} |u|^{-1+p/2} |\nabla u| \, dx\]

\[\leq C_7 \left( \int_{\Omega} m^{pp/2} |u|^{pp+1-p} \, dx \right)^{1/2} \left( \int_{\Omega} m^{p-2+p} |u|^{p-2} |\nabla u|^2 \, dx \right)^{1/2}\]

\[\leq C_7 ||Mu||_p^{pp/2} ||u||_{p(1-\rho)/p}^{(1-\rho)p/2} ||m||_{\infty}^{p-2(2-p)/2} \left( \int_{\Omega} |u|^{p-2} |\nabla u|^2 \, dx \right)^{1/2}.\]

\[(3.17)\]

On account of (2.3), (2.14) and (2.15), we easily observe the estimate

\[\int_{\Omega} |u|^{p-2} |\nabla u|^2 \, dx \leq C_8(p) ||f||_p.\]

\[(3.18)\]
From (2.15), (3.17) and (3.18) we finally deduce the estimates

\[ (3.19) \quad \left( m^{p-2} \sum_{i,j=1}^{n} a_{i,j} D_{x_i} m D_{x_j} u, u |u|^{p-2} \right) \leq C_{0}(p) \| f \|_{p}^{(p-\rho)/2} \| M u \|_{p}^{\rho/2}. \]

Moreover, we have

\[ (3.20) \quad |(f, m^{p-1} u |u|^{p-2})| \leq \| f \|_{p} \| M u \|_{p}^{p-1}. \]

Finally, from (3.16), (3.19), (3.20) and Lemma 3.2 with \( K = K_{0} - m^{p-1} a_{0} \) (which makes use of the assumption \( p \in [2, +\infty) \)) we deduce the inequality

\[ (3.21) \quad \left[ \text{Re} \, \lambda + k_{1}(p) \, \text{Im} \, \lambda \right] + \gamma \| m \|_{\infty}^{-1} \| M u \|_{p}^{p} + \left( 1 - k_{1}(p) \frac{|p-2|}{2 \sqrt{p-1}} \right) \text{Re} ((K_{0} - m^{p-1} a_{0}) u, u |u|^{p-2}) \]

\[ \leq C_{10}(p) \| f \|_{p} \| M u \|_{p}^{p-1} + \| f \|_{p}^{(p-\rho)/2} \| M u \|_{p}^{\rho/2}, \quad \lambda \in \Sigma_{1}. \]

We now introduce the sector

\[ \Sigma_{2} = \left\{ \lambda \in \mathbb{C} : \text{Re} \, \lambda + \frac{k_{1}(p)}{2} \left( \text{Im} \, \lambda \right) + \frac{\gamma}{2 \| m \|_{\infty}} \geq 0 \right\}. \]

Since \( h_{1}(p) \in (0, 1) \), (cf. (2.11)), we immediately deduce the inclusion \( \Sigma_{2} \subset \Sigma_{1} \) (see the definition of \( \Sigma_{2} \)).

Then, recalling that \( \text{Re} ((K_{0} - m^{p-1} a_{0}) u, u |u|^{p-2}) \) is non-negative (cf. Lemma 3.2) and applying Proposition 2.1, we obtain

\[ (|\lambda| + 1) \| M u \|_{p}^{p} \leq \gamma \int_{\Omega} m^{p-1} |u|^{p} \, dx + \text{Re} ((K_{0} - m^{p-1} a_{0}) u, u |u|^{p-2}) \]

\[ \leq C_{11}(p) \| f \|_{p} \| M u \|_{p}^{p-1} + \| f \|_{p}^{(p-\rho)/2} \| M u \|_{p}^{\rho/2}, \quad \lambda \in \Sigma_{2}. \]

Consequently, since \( \| u \|_{p} \leq C_{12}(p) \| f \|_{p} \) (cf. (2.15)), (3.15) and (3.22) imply

\[ (|\lambda| + 1) \| M u \|_{p}^{p(2-\rho)/2} \leq C_{13}(p) \| f \|_{p} \| M u \|_{p}^{p-1} \| f \|_{p}^{(p-\rho)/2} + \| f \|_{p}^{(p-\rho)/2}, \quad \lambda \in \Sigma_{2}. \]

By Proposition 2.2, it is verified that \( \lambda M + L \) is surjective on \( L^{p}(\Omega) \). Hence, estimate (1.5) holds with \( \alpha = 1 \) and \( \beta = 2[p(2-\rho)]^{-1} \).

We can summarize the results in this section in Theorem 3.1.

**Theorem 3.1.** Let \( L \) and \( M \) be the linear operators defined by (1.7) and (1.8), the coefficients \( a_{i,j} \) \( i, j = 1, \ldots, n \), \( a_{0} \) enjoying properties (2.1) and (2.2) and \( m \) being
a non-negative function satisfying (1.9). Then the spectral equation $\lambda Mu + Lu = f$, with $f \in L^p(\Omega)$, admits, for any $\lambda \in \Sigma = \{ \mu \in \mathbb{C} : \text{Re} \mu + (k_1(p)/2)\text{Im} \mu + (\gamma/2||m||_{\infty}) \geq 0 \}$ and $p \in [2, +\infty)$, a unique solution $u \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ satisfying the estimates

$$
\|u\|_p \leq C_\lambda(p)||f||_p, \quad ||Mu||_p \leq C_\lambda(p)||\lambda||^{-2/(p^2-\rho^2)}||f||_p, \quad \lambda \in \Sigma,
$$

$$
||Lu||_p \leq C_\lambda(p)(1 + ||\lambda||^{(p^2-\rho^2)/2(p^2-\rho^2)})||f||_p, \quad \lambda \in \Sigma.
$$

**Example 3.1.** Let $\Omega$ be a bounded domain and let $x_0$ be a fixed point in $\partial \Omega$. Define then $r = \max_{x \in \Omega} |x - x_0|$ and choose

$$
m(x) = \left( |x - x_0| r - |x - x_0| - r_1 \right)^q, \quad q \in (1, +\infty).
$$

An elementary computation shows that

$$
|\nabla m(x)| = q |(x - x_0) (r - |x - x_0|)|^{q-1} \leq q r m(x)^{(q-1)/q}, \quad x \in \Omega.
$$

Consequently, function $m$ satisfies condition (1.9).

We notice that for any open interval $\Omega \subset \mathbb{R}$ we have $r = \text{length}(\Omega)$.

**4. The case when $p \in (1, 2)$**

In this section we are going to considering the case $p \in (1, 2)$. From (2.4) we immediately deduce that the estimate

$$
|\text{Im}(Lu, u|u|^{p-2})| + \gamma ||u||_p^p \leq |\text{Im}(Lu, u|u|^{p-2})| + \int_\Omega a_0(x)|u(x)|^p \, dx
$$

(4.1)

holds true for any $p \in (1, +\infty)$.

Consider again the spectral problem

$$
u \in \mathcal{D}(L), \quad \lambda Mu + Lu = f \in L^p(\Omega).
$$

(4.2)

Multiplying both sides in (4.2) by $u|u|^{p-2}$ and integrating over $\Omega$, we get

$$
\lambda ||m^{1/p}u||_p^p + (Lu, u|u|^{p-2}) = (f, u|u|^{p-2}).
$$

(4.3)

Taking the real and imaginary parts, from (4.3) we deduce

$$
\text{Re} \lambda ||m^{1/p}u||_p^p + \text{Re}(L_0u, u|u|^{p-2}) + (a_0u, u|u|^{p-2}) = \text{Re}(f, u|u|^{p-2}),
$$

(4.4)

$$
\text{Im} \lambda ||m^{1/p}u||_p^p + \text{Im}(L_0u, u|u|^{p-2}) = \text{Im}(f, u|u|^{p-2}),
$$

(4.5)
where we have set
\[ L_0 = L - a_0. \]

Then from Okazawa [4, p.703] we get
\[ (L_0 u, u|u|^{p-2}) = \lim_{\delta \to 0+} I_p(u, \delta), \]
where \( \delta > 0 \) and
\[ I_p(u, \delta) = -\int_{\Omega} (|u(x)|^2 + \delta)^{(p-2)/2} \frac{\overline{u(x)}}{|u(x)|^2} \sum_{j,k=1}^{n} D_{x_k}[a(x)D_{x_j}u(x)] \, dx. \]

As mentioned at the beginning of Section 2, we have
\[ \text{Re}(L_0 u, u|u|^{p-2}) \geq c_0 \int_{\Omega} |u(x)|^{p-2} |\nabla u(x)|^2 \, dx, \quad \text{if } p \in [2, +\infty), \]
\[ \text{Re}(L_0 u, u|u|^{p-2}) \geq c_0(p-1) \int_{\Omega} (|u(x)|^2 + \delta)^{(p-2)/2} |\nabla u(x)|^2 \, dx, \quad \text{if } p \in (1, 2). \]

From (4.1) and (4.5) we deduce the inequalities
\[ |\text{Im} \lambda||m^{1/p}u||_p^p \leq |\text{Im}(Lu, u|u|^{p-2})| + \|f\|_p\|u\|_{p-1}^{p-1}, \]
\[ \leq \frac{|p-2|}{2\sqrt{p-1}} \text{Re}(Lu, u|u|^{p-2}) + \|f\|_p\|u\|_{p-1}^{p-1}. \]

Multiply then both sides in (4.11) by a positive constant \( \nu \) and add the obtained inequality to equation (4.4) to get (cf. (2.2))
\[ (\text{Re} \lambda + \nu|\text{Im} \lambda|)||m^{1/p}u||_p^p + \left(1 - \nu \frac{|p-2|}{2\sqrt{p-1}}\right) \text{Re}(L_0 u, u|u|^{p-2}) + \frac{\nu}{2}\|u\|_p^p + \frac{\nu}{2}\|u\|_p^p \]
\[ \leq (\text{Re} \lambda + \nu|\text{Im} \lambda|)||m^{1/p}u||_p^p + \left(1 - \nu \frac{|p-2|}{2\sqrt{p-1}}\right) \text{Re}(L_0 u, u|u|^{p-2}) + (a_0 u, u|u|^{p-2}) \]
\[ \leq \text{Re}(f, u|u|^{p-2}) + \nu\|f\|_p\|u\|_{p-1}^{p-1} \leq (1 + \nu)\|f\|_p\|u\|_{p-1}^{p-1}. \]
\[ (4.12) \]

Choose now \( \nu = \nu(p) \) so small as to satisfy
\[ \nu_1(p) := 1 - \nu(p) \frac{|p-2|}{2\sqrt{p-1}} > 0, \quad \forall p \in (1, +\infty). \]

On the other hand, since \( m \in L^\infty(\Omega), \|u\|_p \geq ||m||_{\infty}^{-1/p}||m^{1/p}u||_p. \) Then (4.12) and (4.13) imply
\[ \left( \frac{\nu}{2\|m\|_{\infty}} + \frac{\text{Re} \lambda + \nu|\text{Im} \lambda|}{2\|m\|_{\infty}} \right) ||m^{1/p}u||_p^p + \nu_1(p) \text{Re}(L_0 u, u|u|^{p-2}) + \frac{\nu}{2}\|u\|_p^p \]
(4.14) \[ \leq [1 + v(p)]\|f\|_p\|u\|^{p-1}_p. \]

In other words, there exist two positive constants \(C_{18}\) and \(C_{19}\) such that

\[
\left( \frac{\gamma}{2\|m\|_{\infty}} + \text{Re} \lambda + v|\text{Im} \lambda| \right) \|m^{1/p}u\|_p^p + C_{18} \text{Re}(L_0u, u|u|^{p-2}) + \frac{\gamma}{2}\|u\|_p^p
\]

(4.15) \[ \leq C_{19}\|f\|_p\|u\|^{p-1}_p, \quad \lambda \in \Sigma, \]

the sector \(\Sigma\) being defined by

\[ \Sigma = \{ \lambda \in \mathbb{C} : \text{Re} \lambda + v|\text{Im} \lambda| + \frac{\gamma}{2\|m\|_{\infty}} \geq \varepsilon_0 > 0 \}. \]

Notice that (4.7), (4.8), (4.10), (4.15) yield, in particular, the basic bounds

(4.16) \[ \|u\|_p \leq \frac{2}{\gamma}C_{19}\|f\|_p, \quad \text{Re}(L_0u, u|u|^{p-2}) \leq C_{20}\|f\|_p^p, \]

and

\[
(p-1)c_0 \lim_{\delta \to 0^+} \int_{\Omega} \left( |u(x)|^2 + \delta \right)^{(p-2)/2} |\nabla u(x)|^2 \, dx
\]

(4.17) \[ \leq \lim_{\delta \to 0^+} \text{Re} I_\rho(u, \delta) \leq C_{20}\|f\|_p\|u\|^{p-1}_p, \quad \lambda \in \Sigma. \]

From (4.3) we deduce the estimates

\[
|\lambda|\|m^{1/p}u\|_p^p \leq |(Lu, u|u|^{p-2})| + \|f\|_p\|u\|^{p-1}_p
\]

\[
\leq \left( 1 + \frac{|p-2|}{2\sqrt{p-1}} \right) \text{Re}(Lu, u|u|^{p-2}) + \|f\|_p\|u\|^{p-1}_p
\]

(4.18) \[ \leq C_{21}\|f\|_p\|u\|^{p-1}_p \leq C_{22}^p\|f\|_p^p. \]

Consequently, (4.18) immediately yields

(4.19) \[ |\lambda|^{1/p}\|Mu\|_p \leq C_{23}\|f\|_p. \]

This, in turn, implies that (1.5) holds with \(\alpha = 1\) and \(\beta = 1/p\) and provides a different proof to (1.5).

Now we focus our attention to the case when \(m \in C^1(\overline{\Omega})\) satisfies inequality (1.9) with

(4.20) \[ \rho \in (2-p, 1]. \]

Multiplying both sides in (4.2) by \(m(\chi)^{p-1}|\overline{u(\chi)}|u(\chi)|^{p-2}\) and integrating over \(\Omega\), we
easily get
\[
\lambda \|Mu\|_p^p - \lim_{\delta \to 0^+} \int_\Omega m(x)^{p-1} \overline{u(x)}(|u(x)|^2) + \delta^{(p-2)/2} \sum_{j,k=1}^n D_{x_j}[a_{j,k}(x)D_{x_k}u(x)] \, dx
\]
\[
+ \int_\Omega a_0(x)m(x)^{p-1} |u(x)|^p \, dx = \int_\Omega f(x)m(x)^{p-1} \overline{u(x)}|u(x)|^{p-2} \, dx.
\]
(4.21)

An integration by parts in the integral appearing in the limit, which takes into account (4.20) and (4.21), easily yields
\[
- \int_\Omega m(x)^{p-1} \overline{u(x)}(|u(x)|^2) + \delta^{(p-2)/2} \sum_{j,k=1}^n D_{x_j}[a_{j,k}(x)D_{x_k}u(x)] \, dx
\]
\[
= \int_\Omega (|u(x)|^2 + \delta)^{(p-2)/2} m(x)^{p-1} \sum_{j,k=1}^n a_{j,k}(x)D_{x_j}D_{x_k}u(x) \, dx
\]
\[
+ (p-1) \int_\Omega \overline{u(x)}(|u(x)|^2 + \delta)^{(p-2)/2} \sum_{j,k=1}^n m(x)^{p-2} D_{x_j}m(x)a_{j,k}(x)D_{x_k}u(x) \, dx
\]
\[
+ (p-2) \int_\Omega m(x)^{p-1}(|u(x)|^2 + \delta)^{(p-1)/2} \sum_{j,k=1}^n a_{j,k}(x) \text{Re} \left( \frac{u(x)}{D_{x_j}u(x)} \overline{u(x)}D_{x_k}u(x) \right) \, dx
\]
\[
=: I_1(\delta) + (p-1)I_2(\delta) - (2-p)I_3(\delta).
\]
(4.22)

We have made use here of the following Proposition 4.1 whose proof is postponed to Section 6.

**Proposition 4.1.** Let \( m \) satisfy property (1.9). Then for any \( \beta \in (1 - \rho, 1) \), the function \( m(\cdot)^\beta \) belongs to \( C^1(\overline{\Omega}) \) and \( \nabla[m(\cdot)^\beta](x) = m_1(x) \) for any \( x \in \overline{\Omega} \), where
\[
m_1(x) = \begin{cases} 0, & x \in Z(m), \\ \beta m(x)^{\beta - 1}\nabla m(x), & x \notin Z(m), \end{cases}
\]
and \( Z(m) \) denotes the zero-set of \( m \). Moreover,
\[
|\nabla[m(\cdot)^\beta](x)| \leq Cm(x)^{\beta - 1 + \rho}, \quad x \in \overline{\Omega}.
\]

Since the matrix \((a_{j,k}(x))_{j,k=1,...,n}\) is real-valued and positive definite, from (4.22) we immediately deduce that
\[
I_1(\delta) \text{ and } \text{Re} \, I_3(\delta) \text{ are positive for any } \delta \in \mathbb{R}_+.
\]
(4.24)
Then we observe that $I_2(\delta)$ has a limit as $\delta \to 0+$ and

\[(4.25) \quad \lim_{\delta \to 0+} I_2(\delta) = \int_{\Omega} \overline{u(x)} \overline{u(x)} \left| m(x) \right|^{p-2} \sum_{j,k=1}^{n} m(x) a_{jk}(x) D_{ij} m(x) D_{\alpha} u(x) \, dx.\]

Note that the integral in the right-hand side is well-defined on the whole of $W^{1,p}(\Omega)$ since $\overline{u(x)} \overline{u(x)} \in L^p(\Omega)$, $m(x) a_{jk}(x) D_{ij} m(x) \in L^\infty(\Omega)$ and $D_{\alpha} u \in L^p(\Omega)$.

Further, (4.25) implies that there exists also $\lim_{\delta \to 0+} \left[ I_1(\delta) - (2 - p) I_3(\delta) \right]$. From (4.24) we deduce that there exist the limits

\[
\lim_{\delta \to 0+} \text{Im} I_1(\delta) \quad \text{and} \quad \lim_{\delta \to 0+} \left[ I_1(\delta) - (2 - p) \text{Re} I_3(\delta) \right].
\]

We can now prove the following Lemma 4.1.

**Lemma 4.1.** The following estimates hold for any $\delta \in \mathbb{R}_+, \, p \in (1, 2)$ and $\eta \in (0, 2(p-1)/(2-p)^{-1})$:

\[(4.26) \quad I_1(\delta) - (2 - p) \text{Re} I_3(\delta) - \eta(2 - p) | \text{Im} I_3(\delta) | \geq 0,
\]

\[I_1(\delta) + (p-1) \text{Re} I_2(\delta) - (2 - p) \text{Re} I_3(\delta) \]

\[(4.27) \quad - \eta (p-1) \text{Im} I_2(\delta) - (2 - p) \text{Im} I_3(\delta) \geq -(p-1)(1 + \eta^2)^{1/2} | I_2(\delta) |,
\]

\[\lim_{\delta \to 0+} \left[ I_1(\delta) + (p-1) \text{Re} I_2(\delta) - (2 - p) \text{Re} I_3(\delta) \right]
\]

\[= - \eta \lim_{\delta \to 0+} | (p-1) \text{Im} I_2(\delta) - (2 - p) \text{Im} I_3(\delta) | \]

\[(4.28) \quad \geq - C_{24} \| f \|_p^2 \| M u \|_p^{2-p} \| u \|_p^{2-p}, \]

$C_{24}$ being a suitable positive constant.

**Proof.** Since the matrix $(a_{jk}(x))_{j,k=1,\ldots,n}$ is real-valued and positive definite, we immediately deduce the equality

\[\sum_{j,k=1}^{n} a_{jk}(x) \overline{\zeta_j} \overline{\zeta_k} = \sum_{j,k=1}^{n} a_{jk}(x) \left[ \text{Re}(\zeta_j) \text{Re}(\zeta_k) + \text{Im}(\zeta_j) \text{Im}(\zeta_k) \right], \quad \forall \zeta \in \mathbb{C}^n.\]

Consider now the formulae

\[I_1(\delta) = \int_{\Omega} (|u(x)|^2 + \delta)^{(p-2)/2} m(x)^{p-1} \sum_{j,k=1}^{n} a_{jk}(x) \overline{u(x)} \overline{u(x)} D_{ij} u(x) D_{\alpha} u(x) \, dx \]

\[+ \delta \int_{\Omega} (|u(x)|^2 + \delta)^{(p-4)/2} m(x)^{p-1} \sum_{j,k=1}^{n} a_{jk}(x) D_{ij} u(x) D_{\alpha} u(x) \, dx \]

\[= \int_{\Omega} (|u(x)|^2 + \delta)^{(p-2)/2} m(x)^{p-1} \left\{ \sum_{j,k=1}^{n} a_{jk}(x) \text{Re} \left[ \overline{u(x)} D_{ij} u(x) \right] \text{Re} \left[ \overline{u(x)} D_{\alpha} u(x) \right] \right\} \]
\[ + \sum_{j,k=1}^{n} a_{j,k}(x) \text{Im} \left[ \overline{u(x)} D_{x_j} u(x) \right] \text{Im} \left[ \overline{u(x)} D_{x_k} u(x) \right] \, \text{dx} \]

\[ (4.29) \quad + \delta \int_{\Omega} \left( |u(x)|^2 + \delta (p-1)/2 m(x) \right)^{p-1} \sum_{j,k=1}^{n} a_{j,k}(x) D_{x_j} u(x) D_{x_k} \overline{u(x)} \, \text{dx}, \quad \forall \delta \in \mathbb{R}_+ \]

\[ I_1(\delta) = (2 - p) \text{Re} I_3(\delta) = \int_{\Omega} \left( |u(x)|^2 + \delta (p-1)/2 m(x) \right)^{p-1} \]
\[ \times \left\{ (p - 1) \sum_{j,k=1}^{n} a_{j,k}(x) \text{Re} \left[ \overline{u(x)} D_{x_j} u(x) \right] \text{Re} \left[ \overline{u(x)} D_{x_k} u(x) \right] \right. \]
\[ + \sum_{j,k=1}^{n} a_{j,k}(x) \text{Im} \left[ \overline{u(x)} D_{x_j} u(x) \right] \text{Im} \left[ \overline{u(x)} D_{x_k} u(x) \right] \, \text{dx} \]

\[ (4.30) \quad + \delta \int_{\Omega} \left( |u(x)|^2 + \delta (p-1)/2 m(x) \right)^{p-1} \sum_{j,k=1}^{n} a_{j,k}(x) D_{x_j} u(x) D_{x_k} \overline{u(x)} \, \text{dx}, \quad \forall \delta \in \mathbb{R}_+ \]

\[ \text{Im} I_3(\delta) = \left| \int_{\Omega} \left( |u(x)|^2 + \delta (p-1)/2 m(x) \right)^{p-1} \sum_{j,k=1}^{n} a_{j,k}(x) \text{Re} \left[ \overline{u(x)} D_{x_j} u(x) \right] \text{Im} \left[ \overline{u(x)} D_{x_k} u(x) \right] \, \text{dx} \right| \]

\[ \leq \frac{1}{2} \int_{\Omega} \left( |u(x)|^2 + \delta (p-1)/2 m(x) \right)^{p-1} \sum_{j,k=1}^{n} a_{j,k}(x) \left\{ \text{Re} \left[ \overline{u(x)} D_{x_j} u(x) \right] \text{Re} \left[ \overline{u(x)} D_{x_k} u(x) \right] \right\} \]

\[ (4.31) \quad + \text{Im} \left[ \overline{u(x)} D_{x_j} u(x) \right] \text{Im} \left[ \overline{u(x)} D_{x_k} u(x) \right] \, \text{dx}, \quad \forall \delta \in \mathbb{R}_+ \]

We have here used the Cauchy–Schwarz inequality and the geometric-arithmetic mean, i.e.

\[ \left| \sum_{j,k=1}^{n} a_{j,k}(x) \xi_j \eta_k \right| \leq \left( \sum_{j,k=1}^{n} a_{j,k}(x) \xi_j \xi_k \right)^{1/2} \left( \sum_{j,k=1}^{n} a_{j,k}(x) \eta_j \eta_k \right)^{1/2} \]

\[ \leq \frac{1}{2} \left( \sum_{j,k=1}^{n} a_{j,k}(x) \xi_j \xi_k + \sum_{j,k=1}^{n} a_{j,k}(x) \eta_j \eta_k \right) = \frac{1}{2} \sum_{j,k=1}^{n} a_{j,k}(x) \left[ \xi_j \xi_k + \eta_j \eta_k \right], \quad \forall \xi, \eta \in \mathbb{R}^n. \]

From (4.24) and (4.31) we deduce the following inequality, where we take advantage of the membership \( \eta \in (0, 2(p-1)/(2 - p)) \):

\[ I_1(\delta) = (2 - p) \text{Re} I_3(\delta) - \eta(2 - p) |\text{Im} I_3(\delta)| = \int_{\Omega} \left( |u(x)|^2 + \delta (p-1)/2 m(x) \right)^{p-1} \]
\[ \times \left\{ \left[ p - 1 - \frac{1}{2} \eta(2 - p) \right] \sum_{j,k=1}^{n} a_{j,k}(x) \text{Re} \left[ \overline{u(x)} D_{x_j} u(x) \right] \text{Re} \left[ \overline{u(x)} D_{x_k} u(x) \right] \right\} \]
\[ + \left[ 1 - \frac{1}{2} \eta(2-p) \right] \sum_{j,k=1}^{n} a_{j,k}(x) \text{Im} \left[ \bar{u(x)}D_{x_j}u(x) \right] \text{Im} \left[ u(x)D_{x_k}u(x) \right] \right] \ dx \]

\[ (4.32) + \delta \int_{\Omega} |u(x)|^2 + \delta^{(p-1)/2} m(x)^{p-1} \sum_{j,k=1}^{n} a_{j,k}(x) D_{x_j}u(x) D_{x_k}u(x) \ dx \geq 0, \quad \forall \delta \in \mathbb{R}_+ . \]

We have thus proved (4.26).

Then we note that (4.27) is a consequence of (4.26):

\[ I_1(\delta) + (p-1) \Re I_2(\delta) - (2-p) \Re I_3(\delta) - \eta(p-1) \Im I_2(\delta) - (2-p) \Im I_3(\delta) \]
\[ \geq I_1(\delta) - (2-p) \Re I_3(\delta) - \eta(2-p) \Im I_3(\delta) + (p-1)(\Re I_2(\delta) - \eta \Im I_2(\delta) ) \]
\[ \geq -(p-1)(1 + \eta^2)^{1/2} |I_2(\delta)|, \quad \forall \delta \in \mathbb{R}_+ . \]

(4.33)

To conclude the proof of the lemma we take into account the relations

\[ \lim_{\delta \to 0+} \left[ I_1(\delta) + (p-1) \Re I_2(\delta) - (2-p) \Re I_3(\delta) \right] \]
\[ - \eta \lim_{\delta \to 0+} [(p-1) \Im I_2(\delta) - (2-p) \Im I_3(\delta)] \]
\[ \geq \lim_{\delta \to 0+} \left\{ \Re I_1(\delta) + (p-1) \Re I_2(\delta) - (2-p) \Re I_3(\delta) \right\} \]
\[ \quad - \eta [(p-1) \Im I_2(\delta) - (2-p) \Im I_3(\delta)] \]
\[ \geq -(p-1)(1 + \eta^2)^{1/2} \lim_{\delta \to 0+} |I_2(\delta)|, \quad \forall \delta \in \mathbb{R}_+ . \]

(4.34)

Next, consider the following chain of inequalities, which holds for any \( \delta \in \mathbb{R}_+ \):

\[ \lim_{\delta \to 0+} |I_2(\delta)| \leq \limsup_{\delta \to 0+} \int_{\Omega} |u(x)|^2 + \delta^{(p-1)/2} m(x)^{p-2} |D_{x_j}m(x)| |a_{j,k}(x)D_{x_k}u(x)| \ dx \]
\[ \leq \limsup_{\delta \to 0+} \int_{\Omega} |u(x)|^2 + \delta^{p/4} \sum_{j,k=1}^{n} m(x)^{p-2} |D_{x_j}m(x)| |a_{j,k}(x)||D_{x_k}u(x)| \ dx \]
\[ \times |u(x)|^2 + \delta^{(p-2)/4} \sum_{j,k=1}^{n} m(x)^{p-2} |D_{x_j}m(x)| |a_{j,k}(x)||D_{x_k}u(x)| \ dx \]
\[ \leq C_1 \limsup_{\delta \to 0+} \left[ \int_{\Omega} m(x)^{2(p-2)+p}|u(x)|^2 + \delta^{p/2} \ dx \right]^{1/2} \]
\[ \times \limsup_{\delta \to 0+} \left[ \int_{\Omega} |a_{j,k}(x)D_{x_k}u(x)|^2 |u(x)|^2 + \delta^{(p-2)/2} \ dx \right]^{1/2} \]
\[ (\text{cf. (4.16), (4.17))} \]
\[ C_{25} \lim_{\delta \to 0^+} \left\{ \left[ \int_{\Omega} m(x)^{2(p-2+\rho)} |u(x)|^2 + \delta \right]^{p/2} \, dx \right\}^{1/2} \times \left[ \int_{\Omega} |u(x)|^2 + \delta |\nabla u(x)|^2 \, dx \right]^{1/2} \]
\[ \leq C_{26} \left[ \int_{\Omega} m(x)^{2(p-2+\rho)} |u(x)|^p \, dx \right]^{1/2} \| f \|_p^{p/2} \]
\[ \text{(4.35) } \leq C_{27} \| f \|_p^{p/2} \| Mu \|_p^{2-p+\rho} \| u \|_p^{2-p-\rho/2}. \]

To derive the last inequality we have applied Hölder’s inequality with index \( q = p[2(p-2+\rho)]^{-1} \) to the integral
\[ \int_{\Omega} [m(x)|u(x)|^{2(p-2+\rho)}|u(x)|^{-p+4-2\rho} \, dx. \]

From (4.34) and (4.35) we immediately conclude (4.28).

Taking now the real part and the modulus of the imaginary part in (4.21) and using (4.22), we easily derive the relations

\[ \text{Re} \lambda \| Mu \|_p^p + \lim_{\delta \to 0^+} \left[ I_1(\delta) + (p-1) \text{Re} I_2(\delta) - (2-p) \text{Re} I_3(\delta) \right] \]
\[ + \int_{\Omega} a_0(x)m(x)^{p-1}|u(x)|^p \, dx = \text{Re} \int_{\Omega} m(x)^{p-1}f(x)\overline{u(x)}|u(x)|^{p-2} \, dx, \]
\[ |\text{Im} \lambda \| Mu \|_p^p \| \leq \lim_{\delta \to 0^+} \| (p-1) \text{Im} I_2(\delta) - (2-p) \text{Im} I_3(\delta) \| \]
\[ + \left| \text{Im} \int_{\Omega} m(x)^{p-1}f(x)\overline{u(x)}|u(x)|^{p-2} \, dx \right|, \quad \forall \lambda \in \mathbb{C}. \]

Add now member by member (4.36) and (4.37) multiplied by \( \eta \in (0, 2\sqrt{p-1}(2-p)\gamma)] \) and use (4.28) and (2.2). We easily deduce the following estimate for any \( \lambda \in \Sigma = \{ \mu \in \mathbb{C} : \text{Re} \mu + \eta |\text{Im} \mu| \geq 0 \} \):

\[ \left[ \text{Re} \lambda + \eta |\text{Im} \lambda| + \frac{\gamma}{|\text{Im} \lambda|} \right] \| Mu \|_p^p \]
\[ \leq - \lim_{\delta \to 0^+} \left[ I_1(\delta) + (p-1) \text{Re} I_2(\delta) - (2-p) \text{Re} I_3(\delta) \right] \]
\[ - \eta \lim_{\delta \to 0^+} \left| (p-1) \text{Im} I_2(\delta) - (2-p) \text{Im} I_3(\delta) \right| \]
\[ + \text{Re} \int_{\Omega} f(x)m(x)^{p-1}\overline{u(x)}|u(x)|^{p-2} \, dx + \eta \left| \text{Im} \int_{\Omega} f(x)m(x)^{p-1}\overline{u(x)}|u(x)|^{p-2} \, dx \right| \]
\[ \leq - \lim_{\delta \to 0^+} \left[ I_1(\delta) + (p-1) \text{Re} I_2(\delta) - (2-p) \text{Re} I_3(\delta) \right] \]
\[ - \eta \left| (p-1) \text{Im} I_2(\delta) - (2-p) \text{Im} I_3(\delta) \right| \]
\[ + \text{Re} \int_\Omega f(x)m(x)\{\rho^{-1}u(x)\}u(x)|u(x)|^{p-2} \, dx + \eta \left| \text{Im} \int_\Omega f(x)m(x)\{\rho^{-1}u(x)\}u(x)|u(x)|^{p-2} \, dx \right| \]
\[ \leq C_{28}\|f\|_{p}^{2}\|Mu\|_{p}^{\frac{2\rho}{\rho-p}}|u|_{p}^{\frac{2\rho-p}{\rho}} + (1 + \eta^{2})^{1/2}\|f\|_{p}\|Mu\|_{p}^{1-\rho}. \]

(4.38)

Take \( \lambda \) in the sector

\[ \Sigma_{3} = \left\{ \mu \in \mathbb{C} : \text{Re} \mu + \frac{\eta}{2}|\text{Im} \mu| + \frac{\gamma}{2|m|_{\infty}} \geq 0 \right\}. \]

Then, since \( \|u\|_{p} \leq C_{19}\|f\|_{p} \) (cf. (2.11), (2.12) and our definition of \( \eta \)) and \( 2 - \rho - p/2 > 0 \) (cf. (4.20)), by Proposition 2.1 we immediately derive the inequality

\[ (|\lambda| + 1)\|Mu\|_{p}^{\frac{2\rho}{\rho-p}} \leq C_{34}[\|f\|_{p}^{2+\rho} + \|f\|_{p}\|Mu\|_{p}^{1-\rho}], \quad \text{if} \quad \lambda \in \Sigma_{3}. \]

Finally, \( \|Mu\|_{p} \leq \|m\|_{\infty}\|u\|_{p} \leq C_{19}\|m\|_{\infty}\|f\|_{p} \) implies

\[ (|\lambda| + 1)\|Mu\|_{p}^{\frac{2\rho}{\rho-p}} \leq C_{30}\|f\|_{p}^{2-\rho}, \quad \text{if} \quad \lambda \in \Sigma_{3}. \]

We can now collect the result in this section in the following Theorem 4.1.

**Theorem 4.1.** Let \( L \) and \( M \) be the linear operators defined by (1.7) and (1.8), the coefficients \( a_{i,j} \) \( i, j = 1, \ldots, n \), \( a_{0} \) enjoying properties (2.1) and (2.2) and \( m \) being a non-negative function satisfying (1.9). Then the spectral equation \( \lambda Mu + Lu = f \), with \( f \in L^{p}(\Omega) \), admits, for any \( \lambda \in \Sigma_{3} \) and \( p \in (1,2), \rho \in [2-p,1] \), a unique solution \( u \in W^{2,p}(\Omega) \cap W^{1,p}_{0}(\Omega) \) satisfying the estimates

\[ \|u\|_{p} \leq C_{30}\|f\|_{p}, \quad \|Mu\|_{p} \leq C_{31}(p)|\lambda|^{-2\rho}m^{\rho}\|f\|_{p}, \quad \lambda \in \Sigma_{3}, \]

\[ \|Lu\|_{p} \leq C_{32}(1 + |\lambda|^{(2-p)^{-1}}m^{\rho})\|f\|_{p}, \quad \lambda \in \Sigma_{3}. \]

**Example 4.1.** Let \( n = 1, m(x) = x^{q}(1-x)^{q}, q \in (1, +\infty), \Omega = (0, 1) \). Then

\[ m'(x) = q(1 - 2x)m(x)^{q-1}/q, \quad x \in (0, 1). \]

Hence (4.25) holds true for any \( q \in (1, +\infty) \). If we have to deal with \( L^{p}(0,1) \) with \( p \in (1,2), \) to satisfy (4.20) we are forced to assume \( q > (p - 1)^{-1}. \)

**5. Solving problem (1.1)–(1.3)**

Taking the spectral Theorems 2.1, 3.1, 4.1 into account, from Theorem 3.26 in [3] we can easily derive our existence and uniqueness result. For this purpose we need to introduce the following interpolation space

\[ L^{p}_{\theta,\infty} = \left\{ g \in L^{p}(\Omega) : \sup_{t \geq 1} t^{\theta}\|L(tM + L)^{-1}\|_{L^{p}(\Omega)} < +\infty \right\}. \]
In particular, any \( g = mh \) belongs to \( L^p_{\partial, \infty} \), whenever \( m \in L^\infty(\Omega) \) and \( h \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \). Notice that \( L^p_{\partial, \infty} \subset (X; D(LM^{-1}))_{\theta, \infty} \).

**Theorem 5.1.** Let \( p \in (1, +\infty) \), let \( m \in L^\infty(\Omega) \) be a non-negative function and let the coefficients \( a_{i,j} \), \( i, j = 1, \ldots, n \), \( a_0 \) enjoy properties (2.1) and (2.2). Then for any

\[
(5.2) \quad u_0 \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega), \quad f \in C^0([0, T]; L^p(\Omega)), \quad \theta \in (1 - \beta, 1),
\]

with \( \beta = 1/p \) and

\[
(5.3) \quad -A(x, D_x)u_0 + f(0, \cdot) = g_0, \quad g_0 \in L^p_{\theta, \infty},
\]

problem (1.1)–(1.3) admits a unique solution

\[
(5.4) \quad mu \in C^{\theta + \beta}([0, T]; L^p(\Omega)), \quad u \in C^{\theta + \beta - 1}([0, T]; W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)).
\]

Moreover, if \( m \) is a non-negative function satisfying (1.9) and \( \beta \) is defined by (1.10), the same result holds under assumptions (5.1) and (5.2) on \((u_0, f)\).

### 6. Proofs of the propositions

**Proof of Proposition 2.1.** Let \( \lambda \in \Sigma_{k, \varepsilon} \) and \( \Re \lambda \geq 0 \). Then it is clear that \[|\Re \lambda| \leq \Re \lambda + k|\Im \lambda| + \varepsilon.\] On the other hand, if \( \lambda \in \Sigma_{k, \varepsilon} \) and \( \Re \lambda < 0 \), then \[|\Re \lambda| = -\Re \lambda \leq (k/2)|\Im \lambda| + (\varepsilon/2) \leq \Re \lambda + k|\Im \lambda| + \varepsilon.\] Therefore, \[|\Re \lambda| \leq \Re \lambda + k|\Im \lambda| + \varepsilon \] for any \( \lambda \in \Sigma_{k, \varepsilon} \). In the meantime it is obvious that \[|\Im \lambda| + 1 \leq 2[(1/k) + (1/\varepsilon)][(k/2)|\Im \lambda| + (\varepsilon/2)] \leq 2(1/k + 1/\varepsilon)(\Re \lambda + k|\Im \lambda| + \varepsilon)\] for any \( \lambda \in \Sigma_{k, \varepsilon} \). Hence we conclude that \[|\lambda| + 1 \leq |\Re \lambda| + |\Im \lambda| + 1 \leq [2/k + (2/\varepsilon) + 1](\Re \lambda + k|\Im \lambda| + \varepsilon), \lambda \in \Sigma_{k, \varepsilon}.\]

**Proof of Proposition 2.2.** We consider the set \( J = \{ \theta \in [0, 1]; \mathcal{R}(A(\theta)) = X \} \) and shall prove that this set is an open and closed subset of the interval \([0, 1]\) under (2.17) and (2.18). In fact, let \( \theta \in J \); then, it follows from (2.17) that \( A(\theta)^{-1} \in \mathcal{L}(X) \) with \( \|A(\theta)^{-1}\| \leq \delta^{-1} \). Moreover, for any \( \theta' \in [0, 1] \), we have

\[A(\theta') = [1 + \{A(\theta') - A(\theta)\}]A(\theta)^{-1}A(\theta).\]

Since \( \|\{A(\theta') - A(\theta)\}A(\theta)^{-1}\| \leq N\delta^{-1}|\theta' - \theta| \), the operator \( 1 + \{A(\theta') - A(\theta)\}A(\theta)^{-1} \) is a linear isomorphism of \( X \) provided \( |\theta' - \theta| < N^{-1}\delta \). This then shows that \( \theta' \in J \) for any \( \theta' \) such that \( |\theta' - \theta| < N^{-1}\delta \); hence, \( J \) is an open set. Consider now a sequence \( \theta_n \in J \) and assume that \( \theta_n \to \theta \) as \( n \to +\infty \). Let \( f \in X \) be any vector; then, there exists a sequence \( u_n \in D \) such that \( A(\theta_n)u_n = f \). From (2.17) it follows that \( \|u_n\| \leq \delta^{-1}\|f\| \). Furthermore we observe that \( \|A(\theta_n)u_n - f\| \leq \|A(\theta_n) - A(\theta)\|u_n\| \leq N\delta^{-1}|\theta_n - \theta|\|f\| \).
therefore, \( A(\bar{\theta})u_n \to f \) as \( n \to +\infty \). In the meantime, \( \delta \|u_m - u_n\| \leq \|A(\bar{\theta})(u_m - u_n)\| \leq \|A(\bar{\theta})u_m - f\| + \|f - A(\bar{\theta})u_n\| \to 0 \) as \( m, n \to +\infty \). So, \( u_n \) has a limit \( u \in X \) as \( n \to +\infty \). Since \( A(\bar{\theta}) \) is a closed operator, \( u \in D \) and \( A(\bar{\theta})u = f \); hence, \( \bar{\theta} \in J \). That is, \( J \) is a closed set. As \( 1 \in J \neq \emptyset \), we conclude that \( J = [0, 1] \).

Proof of Proposition 4.1. According to (1.9), we have the inclusion \( Z(m) \subset Z(\nabla m) \). Moreover, formula (4.23) is trivial if \( x \notin Z(m) \). This therefore shows that we have to deal with the case \( x \in Z(m) \) only.

First we will consider the one-dimensional case \( (n = 1) \). For this purpose assume \( x_0 \in Z(m) \). Our starting point is the following formula:

\[
\lim_{x \to x_0} \frac{m(x)^\beta - m(x_0)^\beta}{x - x_0} = \lim_{x \to x_0} \frac{[m(x) + \varepsilon]^\beta - \varepsilon^\beta}{x - x_0}
\]

\[
= \lim_{x \to x_0} \left| \lim_{\varepsilon \to 0^+} \frac{\beta}{x - x_0} \int_{x_0}^{x} [m(t) + \varepsilon]^{\beta - 1} m'(t) \, dt \right|.
\]

We next notice that \( \lim_{\varepsilon \to 0^+} [m(t) + \varepsilon]^{\beta - 1} m'(t) = m_1(t) \) for any \( t \in \Omega \) and that

\[
[|m(t) + \varepsilon|^{\beta - 1} m'(t)] \leq \beta [m(t) + \varepsilon]^{\beta - 1} m(t) \rho
\]

\[
= C \left[ \frac{m(t)}{m(t) + \varepsilon} \right]^{1-\beta} m(t)^{\beta - 1 + \rho} \leq C m(t)^{\beta - 1 + \rho}, \quad \forall t \in \Omega.
\]

By virtue of the dominated convergence theorem and by the bound \( |m_1(t)| \leq C m(t)^{\beta - 1 + \rho} \) for any \( t \in \overline{\Omega} \), we deduce the following relations:

\[
\lim_{x \to x_0^+} \frac{m(x)^\beta - m(x_0)^\beta}{x - x_0} = \lim_{x \to x_0^+} \left[ \frac{1}{x - x_0} \int_{x_0}^{x} m_1(t) \, dt \right]
\]

\[
\leq \lim_{x \to x_0^+} \left[ \frac{1}{x - x_0} \int_{x_0}^{x} |m_1(t)| \, dt \right] \leq \lim_{x \to x_0^+} \frac{C}{x - x_0} \int_{x_0}^{x} m(t)^{\beta - 1 + \rho} \, dt = 0.
\]

Note here that \( m(\cdot)^{\beta - 1 + \rho} \) is continuous in \( \Omega \) and \( x_0 \in Z(m) \). An analogous argument holds for \( \lim_{x \to x_0^-} \left[ (m(x)^\beta - m(x_0)^\beta)/(x - x_0) \right] \) also.

We have thus shown that there exists \( D_\varepsilon [m(\cdot)^\beta](x_0) \) and coincides with \( 0 = m_1(x_0) \). Therefore the formula \( D_\varepsilon [m(\cdot)^\beta](x) = m_1(x) \) holds for any \( x \in \Omega \) since \( \beta \in (1 - \rho, 1) \), bound (1.9) and (4.23) immediately imply that \( m_1 \in C(\Omega) \). Consequently, \( m(\cdot)^\beta \in C(\Omega) \).

Finally, the multi-dimensional case is an immediate consequence of the case \( n = 1 \).
References


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