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AN $L^p$-APPROACH TO SINGULAR LINEAR PARABOLIC EQUATIONS IN BOUNDED DOMAINS

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Abstract

Singular means here that the parabolic equation is not in normal form neither can it be reduced to such a form. For this class of problems, following the operator approach used in [1], we prove global in time existence and uniqueness theorems related to (spatial) $L^p$-spaces. Various improvements to [2], [3] are given.

1. Introduction

In this paper we will consider the following boundary value problem

\[ D_t \{m(x)u(x,t)\} + A(x,D_x)u(x,t) = f(x,t), \quad \forall (x,t) \in \Omega \times [0, \tau), \]

\[ u(x,t) = 0, \quad \forall (x,t) \in \partial \Omega \times [0, \tau), \]

\[ m(x)u(x,t) \to m(x)u_0(x), \quad \text{for a.e. } x \in \Omega, \text{ as } t \to 0+, \]

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a boundary of class $C^2$, while $A(x,D_x)$ is the following second-order uniformly elliptic operator in divergence form

\[ A(x,D) = - \sum_{i,j=1}^{n} D_{x_i} \{a_{ij}(x)D_x x_j\} + a_0(x). \]

Moreover, $0 \neq m \in L^\infty(\Omega)$ is a non-negative function which need not to be bounded away from 0. Consequently, our parabolic equation is, in general, singular.

Particular cases of (1.1) are discussed in the monograph [3], pp.74–80. See also [2]. Note that in [3], p.80, the restriction $p \in (2, +\infty)$ should be made.

Using the theoretical results in [3] and the fundamental approach in [4] we can develop an $L^p$-theory, $p \in (1, +\infty)$, also in the present degenerate case*. The keystone in order to apply the results in [1] and [3], Theorem 3.28, p.69, to (1.1)–(1.4)

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*We note that in this case the initial condition (1.3) should be more correctly meant as the following $L^p$-limit: $\|m(\cdot)u(\cdot,t) - m(\cdot)u_0(\cdot)\|_{L^p(\Omega)} \to 0$ as $t \to 0+$.
consists in showing an operator estimate of the form

\begin{equation}
\|L(\lambda M + L)^{-1}\|_{L^2(\Omega)} \leq C(1 + |\lambda|^{1-\beta}), \quad \forall \lambda \in \Sigma_\alpha,
\end{equation}

where \( X = L^p(\Omega), 0 < \beta \leq \alpha \leq 1, \alpha + \beta > 1, \)

\begin{equation}
\Sigma_\alpha = \{ \lambda \in \mathbb{C} : \text{Re} \lambda \geq -c(1 + |\text{Im} \lambda|^{\rho}) \}, \quad (c > 0),
\end{equation}

and

\begin{align}
\mathcal{D}(L) &= \mathcal{D}(L_\rho) = W^{2,\rho}(\Omega) \cap W_0^{1,\rho}(\Omega), & Lu(x) &= A(x, D)u(x), & u \in \mathcal{D}(L), \\
\mathcal{D}(M) &= L^p(\Omega), & Mu(x) &= m(x)u(x).
\end{align}

We in fact show that (1.5) holds with \( \alpha = 1, \beta = 1/p, p \in (1, +\infty). \)

Moreover, when \( m \) is \( \rho \)-regular, i.e.

\begin{equation}
m \in C^1(\Omega), \quad |\nabla m(x)| \leq C_1 m(x)^\rho, \quad \forall x \in \Omega,
\end{equation}

\( C_1 \) being a positive constant, we can improve the index \( \beta \) in estimate (1.5) from \( \beta = 1/p \) to

\begin{equation}
\beta = \begin{cases} 
2 - \rho \end{cases}, & \text{if } p \in (1, 2), \quad \rho \in (2 - p, 1], \\
\frac{2}{p(2 - \rho)} & \text{if } p \in [2, +\infty), \quad \rho \in (0, 1].
\end{cases}
\end{equation}

The result proved in this paper will be applied, in a subsequent paper, to identify the unknown kernel \( k \) in the integro-differential singular equation of parabolic type

\begin{equation}
D_t[m(x)u(x, t)] + A(x, D_x)u(x, t) = \int_0^t k(t - s)B(x, D_x)u(x, s) \, ds + f(x, t), \quad \forall (x, t) \in \Omega \times [0, \tau],
\end{equation}

\( B(x, D_x) \) being a linear second-order differential operator.

We stress that the present paper was originated by a requirement of additional smoothness of solution \( u \) of (1.11) needed to recover the unknown kernel \( k \). This occurrence is in accordance with the well-known fact that inverse problems usually force deeper, and sometimes, unexpected insights in direct problems.

### 2. Solving the spectral problem \((\lambda M + L)u = f\)

The basic aim of this section consists in showing that estimate (1.5) holds when the linear operators \( M \) and \( L \) are defined by (1.7) and (1.8), respectively. To this aim we assume that the coefficients \( a_{i,j} \) and \( a_0 \) satisfy the properties

\begin{equation}
a_{i,j} \in C^1(\Omega), \quad a_0 \in C(\Omega), \quad a_{i,j} = a_{j,i}, \quad i, j = 1, \ldots, n,
\end{equation}
(2.2) \( c_0|\xi|^2 \leq \sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \leq c_1|\xi|^2, \quad \forall x \in \overline{\Omega}, \forall \xi \in \mathbb{R}^n, \quad a_0(x) \geq \gamma, \quad \forall x \in \overline{\Omega}, \)

\( c_0, \ c_1 \) and \( \gamma \) being three positive constants.

A remarkable result by Okazawa [4, p.702] provides, for any \( u \in \mathcal{D}(L), \)

\[
\text{Re}((L - a_0)u, u|u|^{p-2}) \\
\geq \begin{cases} 
  c_0 \int_{\Omega} |u|^{p-2} |\nabla u|^2 \, dx \geq 0, & \text{if } p \in [2, \infty), \\
  c_0(p-1) \int_{\Omega} (|u|^2 + \delta |u|^{p-2}/2) |\nabla u|^2 \, dx \geq 0, & \text{if } p \in (1, 2), 
\end{cases}
\]

(2.3)

\[
|\text{Im}(Lu, u|u|^{p-2})| \leq \frac{|p-2|}{2\sqrt{p-1}} \text{Re}((L - a_0)u, u|u|^{p-2}),
\]

(2.4)

where the brackets denote

\[
(f, g) = \int f(x) \overline{g(x)} \, dx, \quad f \in L^p(\Omega), \quad g \in L^{p'}(\Omega), \quad \frac{1}{p} + \frac{1}{p'} = 1,
\]

\( u|u|^{p-2} \) is assumed to vanish whenever \( u \) does, and \( \delta > 0 \) is arbitrary.

**Remark 2.1.** It is important to observe that bound (2.4) holds even in the degenerate elliptic case (cf. [4, p. 702] and the following Lemma 3.3).

From (2.4) we immediately deduce the estimate

\[
|\text{Im}(Lu, u|u|^{p-2})| + \frac{|p-2|}{2\sqrt{p-1}} \int_{\Omega} a_0(x)|u(x)|^p \, dx \\
\leq \frac{|p-2|}{2\sqrt{p-1}} \text{Re}(Lu, u|u|^{p-2}).
\]

(2.5)

Consider now the spectral problem

\[
u \in \mathcal{D}(L), \quad \lambda \nu + Lu = f \in L^p(\Omega).
\]

(2.6)

Taking the real and imaginary parts of the scalar product of both sides in (2.6) with \( u|u|^{p-2}, \) we get

\[
\text{Re} \lambda \int_{\Omega} m|u|^p \, dx + \text{Re}(Lu, u|u|^{p-2}) = \text{Re} \int_{\Omega} f \overline{|u|^{p-2}} \, dx, \tag{2.7}
\]

\[
\text{Im} \lambda \int_{\Omega} m|u|^p \, dx + \text{Im}(Lu, u|u|^{p-2}) = \text{Im} \int_{\Omega} f \overline{|u|^{p-2}} \, dx. \tag{2.8}
\]
From (2.8) we deduce the inequalities

\[ |\text{Im} \lambda| \int_\Omega m|u|^p \, dx \leq |\text{Im}(Lu, u|u|^{p-2})| + \left| \text{Im} \int_\Omega f|u|^{p-2} \, dx \right|. \tag{2.9} \]

Multiply then both sides in (2.9) by a positive constant \( k \) and add the obtained inequality to equation (2.7). From (2.5) we get

\[ \left( \text{Re} \lambda + k|\text{Im} \lambda| \right) \int_\Omega m|u|^p \, dx + \left( 1 - k \frac{|p-2|}{2\sqrt{p-1}} \right) \text{Re}(Lu, u|u|^{p-2}) \leq \text{Re} \int_\Omega f|u|^{p-2} \, dx \leq k \int_\Omega |f| |u|^{p-2} \, dx \leq (1 + k)\|f\|_p \|u\|_p^{p-1}. \tag{2.10} \]

Choose now \( k = k_1(p) \) so small as to satisfy

\[ h_1(p) = 1 - k_1(p) \frac{|p-2|}{2\sqrt{p-1}} > 0, \quad \forall p \in (1, +\infty). \tag{2.11} \]

Observe that

\[ \text{Re}(Lu, u|u|^{p-2}) = \text{Re}((L - \alpha_0)u, u|u|^{p-2}) + \frac{1}{2} \text{Re}(\alpha_0 u, u|u|^{p-2}) + \frac{1}{2} \text{Re}(\alpha_0 u, u|u|^{p-2}) \]

\[ \geq \text{Re}((L - \alpha_0)u, u|u|^{p-2}) \geq h_1(p) \|u\|_p^p + \frac{\gamma}{2\|m\|_\infty} \int_\Omega m|u|^p \, dx, \tag{2.12} \]

since \( m(x) \leq \|m\|_\infty \) implies

\[ \frac{m(x)}{\|m\|_\infty} \frac{\alpha_0(x)}{2} \leq \frac{\alpha_0(x)}{2}. \]

In view of (2.11), (2.12) and (2.3), we obtain from (2.10) that

\[ \left( \text{Re} \lambda + k_1(p)|\text{Im} \lambda| + \frac{\gamma h_1(p)}{2\|m\|_\infty} \right) \int_\Omega m|u|^p \, dx \]

\[ + \frac{\gamma h_1(p)}{2} \|u\|_p^p + h_1(p) \text{Re}((L - \alpha_0)u, u|u|^{p-2}) \leq [k_1(p) + 1]\|f\|_p \|u\|_p^{p-1}. \tag{2.13} \]

Introduce now the sector

\[ \Sigma_1 = \left\{ \mu \in \mathbb{C} : \text{Re} \mu + \frac{k_1(p)}{2} |\text{Im} \mu| + \frac{\gamma h_1(p)}{4\|m\|_\infty} \geq 0 \right\}. \]

Then, for \( \lambda \in \Sigma_1 \),

\[ \text{Re}((L - \alpha_0)u, u|u|^{p-2}) \leq \frac{k_1(p) + 1}{h_1(p)} \|f\|_p \|u\|_p^{p-1}. \tag{2.14} \]
(2.15) \[ ||u||_p \leq \frac{2(k_1(p) + 1)}{\gamma h_1(p)} ||f||_p. \]

Consequently,
\[
\left( \Re \lambda + k_1(p) |\Im \lambda| + \frac{\gamma h_1(p)}{2 ||m||_\infty} \right) \int_\Omega |u|^p \, dx \leq C_1(p) ||f||_p^p.
\]

We now need a simple proposition. For the proof see Section 6.

**Proposition 2.1.** Let \( k > 0 \) and \( \varepsilon > 0 \) be two positive constants, and let \( \Sigma_{k,\varepsilon} \) be a sectorial domain given by
\[
\Sigma_{k,\varepsilon} = \left\{ \mu \in \mathbb{C} : \Re \mu + \frac{k}{2} |\Im \mu| + \frac{\varepsilon}{2} \geq 0 \right\}.
\]

Then it holds that
\[
|\lambda| + 1 \leq \left( \frac{2}{k} + \frac{2}{\varepsilon} + 1 \right) (\Re \lambda + k|\Im \lambda| + \varepsilon), \quad \lambda \in \Sigma_{k,\varepsilon}.
\]

Since \( \Sigma_1 = \Sigma_{k_1(p)h_1(p)2||m||_\infty,1} \), this proposition then yields
(2.16) \[
(\lambda^i + 1) \int_\Omega |u|^p \, dx \leq C_2(p) ||f||_p ||u||_p^{p-1} \leq C_3(p) ||f||_p^p, \quad \lambda \in \Sigma_1.
\]

To show that \( (\lambda M + L)^{-1} \) is a bounded operator on \( L^p(\Omega) \) for \( \lambda \in \Sigma_1 \), it now suffices to verify that \( \mathcal{R}(\lambda M + L) = L^p(\Omega) \). But this is verified by the usual techniques without difficulty. In fact, for each \( \lambda \in \Sigma_1 \), we already know that \( \mathcal{R}(\Lambda + \lambda M + L) = L^p(\Omega) \) provided \( \Lambda > 0 \) is a sufficiently large number. Let \( 0 \leq \theta \leq 1 \) be a parameter, and consider the family of closed linear operators \( A(\theta) = \theta \Lambda + \lambda M + L, \quad 0 \leq \theta \leq 1 \). Then the desired result is obtained by the following proposition the proof of which will be given in the final section.

**Proposition 2.2.** Let \( A(\theta), \quad 0 \leq \theta \leq 1, \) be a family of closed linear operators acting on a Banach space \( X \) with constant domain \( \mathcal{D}(A(\theta)) \equiv \mathcal{D} \). Assume that the family satisfies the conditions
(2.17) \[
\delta ||u|| \leq ||A(\theta)u||, \quad u \in \mathcal{D},
\]
(2.18) \[
||[A(\theta) - A(\theta')]u|| \leq N|\theta - \theta'| ||u||, \quad u \in \mathcal{D}
\]
with some constants \( \delta > 0 \) and \( N > 0 \) independent of \( \theta, \theta' \in [0,1] \). Then, \( \mathcal{R}(A(1)) = X \) implies \( \mathcal{R}(A(\theta)) = X \) for every \( \theta \in [0,1] \).

We can now summarize the results proved in this section in Theorem 2.1.
Theorem 2.1. Let $L$ and $M$ be the linear operators defined by (1.7) and (1.8), the coefficients $a_{i,j}$, $i, j = 1, \ldots, n$, $a_0$ enjoying properties (2.1) and (2.2) and $m$ being a non-negative function in $L^\infty(\Omega)$. Then the spectral equation $\lambda Mu + Lu = f$, with $f \in L^p(\Omega)$, admits, for any $\lambda \in \Sigma_1 = \{ \mu \in C : \Re \mu + (k_1(p)/2) |\Im \mu + 2|/\lambda 1 \geq 0 \}$ and $p \in (1, +\infty)$, a unique solution $u \in W^2,p(\Omega) \cap W_0^1,p(\Omega)$ satisfying the estimates

$$
\|u\|_p \leq C_4(p)\|f\|_p, \quad \|Mu\|_p \leq C_5(p)\|\lambda^{-1/p}\|f\|_p, \quad \lambda \in \Sigma_1,
$$

$$
\|Lu\|_p \leq C_6(p)(1 + |\lambda|^{1/p^2})\|f\|_p, \quad \lambda \in \Sigma_1.
$$

3. The case when $m$ is $\rho$-regular and $p \in [2, +\infty)$

We will show that when the multiplier $m$ is more regular, i.e. it satisfies (1.9), our $\beta$ can be chosen larger than $1/p$. We recall that all the previous estimates (2.6)–(2.16) hold for any $p \in (1, +\infty)$.

First of all we need the following lemma concerning the computation of the gradient of the function $\overline{u} |u|^{p-2}$ when $p \in [2, +\infty)$. For this purpose we need some lemmata.

Lemma 3.1. Let $u \in W_0^1,p(\Omega)$ with $p \in [2, +\infty)$. Then the function $\overline{u} |u|^{p-2}$ belongs to $W_0^1,p(\Omega)$ and the following formulae hold

$$
D_x \overline{u} |u|^{p-2} = |u|^{p-2} D_x \overline{u} + (p-2)g_p(u) \Re (g_p(u) D_x u),
$$

a.e. in $\Omega$, $j = 1, \ldots, n$,

(3.1)

where

$$
g_p(u)(x) = \begin{cases} 
\overline{u(x)} |u(x)|^{p-1}/2, & \text{if } u(x) \neq 0, \\
0, & \text{if } u(x) = 0.
\end{cases}
$$

(3.2)

Proof. Let $\phi$ be any function in $C_0^\infty(\Omega)$. Then the following equalities hold:

$$
\langle D_x \phi, \overline{u} |u|^{p-2} \rangle = \lim_{\varepsilon \to 0^+} \langle D_x \phi, \overline{u} |u|^2 + \varepsilon |u|^{p-2}/(p-2) \rangle
$$

$$
= \lim_{\varepsilon \to 0^+} \langle \phi, (|u|^2 + \varepsilon |u|^{p-2})^{(p-2)/2} D_x \overline{u} + \frac{p-2}{2} |u|^2 + \varepsilon |u|^{p-4}/(p-4) \Re \overline{u} D_x u + u D_x \overline{u} \rangle
$$

$$
= \lim_{\varepsilon \to 0^+} \langle \phi, (|u|^2 + \varepsilon |u|^{p-2})^{(p-2)/2} D_x \overline{u} + (p-2) |u|^2 + \varepsilon |u|^{p-4}/(p-4) \Re (\overline{u} |u|^2 + \varepsilon |u|^{p-4}/4 D_x u) \rangle
$$

$$
= \langle \phi, |u|^{p-2} D_x \overline{u} + (p-2)g_p(u) \Re (g_p(u) D_x u) \rangle.
$$

(3.3)

We have used here the relation $\lim_{\varepsilon \to 0^+} \overline{u}(x)(|u(x)|^2 + \varepsilon |u(x)|^{p-4}/4)^{p-4} = g_p(u)(x)$, which takes advantage of the assumption $p \in [2, +\infty)$. \[ \square \]
Remark 3.1. From definition (3.2) we easily deduce the identity

\[ |g_p(u)(x)| = |u(x)|^{(p-2)/2}. \]

We can now prove the following Lemma 3.2.

**Lemma 3.2.** Let \((b_{i,j})_{i,j=1,n}\) be a matrix of functions in \(C^1(\overline{\Omega}, \mathbb{R})\) such that

\[ (3.5) \quad b_{i,j} = b_{j,i} \quad i, j = 1, \ldots, n, \]

\[ (3.6) \quad c_0 |\xi|^2 \mu(x) \leq \sum_{i,j=1}^{n} b_{i,j}(x) \xi_i \xi_j \leq c_1 |\xi|^2 \mu(x), \quad \forall x \in \overline{\Omega}, \forall \xi \in \mathbb{R}^n, \]

where \(\mu \in C(\overline{\Omega})\) is a non-negative function and \(c_0, c_1\) are two positive constants. Then for any \(p \in [2, +\infty)\), the linear operator \(K = -\sum_{i,j=1}^{n} D_{x_j}[b_{i,j}(x)D_{x_i}]\) with \(\mathcal{D}(K) = \mathcal{D}(L)\) (cf. (1.7)) satisfies the relations

\[ (3.7) \quad \text{Re}(Ku, \overline{u}|u|^{p-2}) \leq c_1 \left( \int_{\Omega} \mu |u|^{p-2} |Du|^2 \, dx + \int_{\Omega} \mu \sum_{j=1}^{n} \left| \text{Re}(g_p(u)D_{x_j}u) \right|^2 \, dx \right), \]

\[ (3.8) \quad \text{Im}(Ku, \overline{u}|u|^{p-2}) = (p-2) \int_{\Omega} \sum_{i,j=1}^{n} b_{i,j}[\text{Re}(g_p(u)D_{x_j}u)][\text{Im}(g_p(u)D_{x_i}u)] \, dx. \]

Proof. From Lemma 3.1 and an integration by parts we easily deduce the identity

\[ (Ku, \overline{u}|u|^{p-2}) = \int_{\Omega} \sum_{i,j=1}^{n} b_{i,j} D_{x_j} u D_{x_j} (\overline{u}|u|^{p-2}) \, dx \]

\[ = \int_{\Omega} \sum_{i,j=1}^{n} |u|^{p-2} b_{i,j} D_{x_j} u D_{x_i} \overline{u} \, dx \]

\[ + (p-2) \int_{\Omega} \sum_{i,j=1}^{n} b_{i,j} g_p(u) D_{x_j} u \text{Re}(g_p(u)D_{x_i}u) \, dx. \]

Relations (3.7) and (3.8) follow immediately from (3.9) taking the real and the imaginary parts.

\[ \square \]

**Lemma 3.3.** Under the assumptions in the statement of Lemma 3.2 operator \(K\) satisfies inequalities (2.3) and (2.4) with \(K\) in the place of \(L - a\).
Proof. This lemma has essentially been proved in [4], although a slight modification is needed in its proof. For any $\varepsilon > 0$ define $a_{i,j} = b_{i,j} + \varepsilon \delta_{i,j}$, $i, j = 1, \ldots, n$, and set $K_\varepsilon = K - \varepsilon \Delta$. Since the matrix $(a_{i,j})_{i,j=1}^{n}$ is uniformly positive definite, from (2.3) and (2.4), with $u \in \mathcal{D}(L_0)$, we obtain the inequalities

\begin{align}
0 \leq \text{Re}(K_\varepsilon u, u[u]^{p-2}) &= \text{Re}(Ku, u[u]^{p-2}) + \varepsilon \text{Re}(-\Delta u, u[u]^{p-2}), \\
|\text{Im}(K_\varepsilon u, u[u]^{p-2})| &= |\text{Im}(Ku, u[u]^{p-2}) + \varepsilon \text{Im}(-\Delta u, u[u]^{p-2})| \\
&\leq \frac{|p-2|}{2\sqrt{p-1}}[\text{Re}(Ku, u[u]^{p-2}) + \varepsilon \text{Re}(-\Delta u, u[u]^{p-2})].
\end{align}

(3.10) \quad (3.11)

Taking the limit as $\varepsilon \to 0+$ in (3.10) and (3.11), we easily deduce that $K$ satisfies (2.3) and (2.4).

We shall use also the following identity

\begin{align}
(Lu, m^{p-1}u[u]^{p-2}) &= (m^{p-1}Lu, u[u]^{p-2}) \\
&= (K_0 u, u[u]^{p-2}) + (p-1) \left( m^{p-2} \sum_{i,j=1}^{n} a_{i,j} D_{x_i} m D_{x_j} u, u[u]^{p-2} \right), \quad u \in \mathcal{D}(L),
\end{align}

(3.12)

where

\begin{align}
K_0 = - \sum_{i,j=1}^{n} D_{x_i} [m(x)^{p-1} a_{i,j}(x) D_{x_j}] + m(x)^{p-1} a_0(x).
\end{align}

Let now $u$ be a solution to equation (2.6). Taking the scalar product of both sides in (2.6) with $m^{p-1}u[u]^{p-2}$ and using (3.12), we easily get the equalities

\begin{align}
(f, m^{p-1}u[u]^{p-2}) &= (\lambda mu + Lu, m^{p-1}u[u]^{p-2}) \\
&= \lambda ||Mu||_p^p + (K_0 u, u[u]^{p-2}) + (p-1) \left( m^{p-2} \sum_{i,j=1}^{n} a_{i,j} D_{x_i} m D_{x_j} u, u[u]^{p-2} \right).
\end{align}

(3.13)

Taking the real and imaginary parts in (3.13) and using (2.4) with $L - a_0$ replaced by $K = K_0 - m^{p-1} a_0$, we easily deduce the inequalities

\begin{align}
\text{Re} \lambda ||Mu||_p^p + \gamma \int_{\Omega} m^{p-1} |u|^p \, dx + \text{Re}((K_0 - m^{p-1} a_0) u, u[u]^{p-2}) \\
&\leq \left| (f, m^{p-1}u[u]^{p-2}) \right| + (p-1) \left| \left( m^{p-2} \sum_{i,j=1}^{n} a_{i,j} D_{x_i} m D_{x_j} u, u[u]^{p-2} \right) \right|, \\
|\text{Im} \lambda ||Mu||_p^p| &\leq |\text{Im}((K_0 - m^{p-1} a_0) u, u[u]^{p-2})| \\
&+ \left| (f, m^{p-1}u[u]^{p-2}) \right| + (p-1) \left| \left( m^{p-2} \sum_{i,j=1}^{n} a_{i,j} D_{x_i} m D_{x_j} u, u[u]^{p-2} \right) \right|.
\end{align}

(3.14)
\[ \leq \frac{|p - 2|}{2\sqrt{p - 1}} \Re((K_0 - m^{p-1}a_0)u, u|u|^{p-2}) + |(f, m^{p-1}u|u|^{p-2})| \]

(3.15) \hspace{1cm} + (p - 1) \left| \left( m^{p-2} \sum_{i,j=1}^n a_{i,j} D_{x_i} D_{x_j} u, u|u|^{p-2} \right) \right| .

Multiply now by \( k_1(p) \) (cf. (2.11)) the first and last sides in (3.15) and add to the first and last sides in (3.14). We get the estimate

\[
\begin{align*}
&[\Re \lambda + k_1(p)] |\Im \lambda| + \gamma |m|_\infty^{-1} |M_\mu|^p_p \\
&+ \left( 1 - k_1(p) \frac{|p - 2|}{2\sqrt{p - 1}} \right) \Re((K_0 - m^{p-1}a_0)u, u|u|^{p-2}) \\
&\leq [1 + k_1(p)] \left\{ |(f, m^{p-1}u|u|^{p-2})| + (p - 1) \left| \left( m^{p-2} \sum_{i,j=1}^n a_{i,j} D_{x_i} D_{x_j} u, u|u|^{p-2} \right) \right| \right\},
\end{align*}
\]

(3.16)

where we have made use of the elementary inequality

\[ m(x)^p \leq |m|_\infty |m(x)|^{p-1}, \quad x \in \Omega. \]

We now estimate the last term in (3.16) with the aid of (1.9). Using twice Hölder’s inequality, we get

\[
\left| \left( m^{p-2} \sum_{i,j=1}^n a_{i,j} D_{x_i} D_{x_j} u, u|u|^{p-2} \right) \right| \leq \int_{\Omega} m^{p-2}|u|^{p-1} \left| \sum_{i,j=1}^n a_{i,j} D_{x_i} D_{x_j} u \right| \, dx \\
\leq \int_{\Omega} m^{p-2}|u|^{p-1} \left| \sum_{i,j=1}^n a_{i,j} D_{x_i} m D_{x_j} u \right|^{1/2} \left| \sum_{i,j=1}^n a_{i,j} D_{x_i} u D_{x_j} u \right|^{1/2} \, dx \\
\leq C_7 \int_{\Omega} m^{p-2+\rho}|u|^{p-1} |\nabla u| \, dx = C_7 \int_{\Omega} m^{\rho/2} |u|^{p/2} m^{(p-2)(2-\rho)/2} |u|^{-1+\rho/2} |\nabla u| \, dx \\
\leq C_7 \left( \int_{\Omega} m^{\rho/2} |u|^{p/2} |u|^{1-\rho} \, dx \right)^{1/2} \left( \int_{\Omega} m^{(p-2)(2-\rho)/2} |u|^{p-2} |\nabla u|^2 \, dx \right)^{1/2} \\
\leq C_7 |M_\mu|_p^{\rho/2} |u|_p^{(1-\rho)p/2} |m|_\infty^{(p-2)(2-\rho)/2} \left( \int_{\Omega} |u|^{p-2} |\nabla u|^2 \, dx \right)^{1/2}. 
\]

(3.17)

On account of (2.3), (2.14) and (2.15), we easily observe the estimate

\[ \int_{\Omega} |u|^{p-2} |\nabla u|^2 \, dx \leq C_6(p) \|f\|_p^2 . \]

(3.18)
From (2.15), (3.17) and (3.18) we finally deduce the estimates

$$\left( m^{p-2} \sum_{i=1}^{n} a_{i,j} D_{x_i} m D_{x_j} u, u |u|^{p-2} \right) \leq C_0(p) \|f\|_p^{(p-2)(p/2)} \|Mu\|_p^{p/2}. \tag{3.19}$$

Moreover, we have

$$[(f, m^{p-1} u |u|^{p-2})] \leq \|f\|_p \|Mu\|_p^{p-1}. \tag{3.20}$$

Finally, from (3.16), (3.19), (3.20) and Lemma 3.2 with $K = K_0 - m^{p-1} a_0$ (which makes use of the assumption $p \in [2, +\infty)$) we deduce the inequality

$$\left[ \text{Re} \lambda + \frac{k_1(p)}{2} |\text{Im} \lambda| + \frac{\gamma |m|_\infty^{-1}}{2} \|Mu\|_p \right]$$

$$+ \left( 1 - \frac{k_1(p)}{2} \frac{|p - 2|}{\sqrt{2}} \right) \text{Re}((K_0 - m^{p-1} a_0) u, u |u|^{p-2})$$

$$\leq C_{10}(p) \|f\|_p \|Mu\|_p^{p-1} + \|f\|_p^{(p-2)(p/2)} \|Mu\|_p^{p/2}, \quad \lambda \in \Sigma_1. \tag{3.21}$$

We now introduce the sector

$$\Sigma_2 = \left\{ \lambda \in \mathbb{C} : \text{Re} \lambda + \frac{k_1(p)}{2} |\text{Im} \lambda| + \frac{\gamma}{2|m|_\infty} \geq 0 \right\}. \tag{3.24}$$

Since $h_1(p) \in (0, 1)$, (cf. (2.11)), we immediately deduce the inclusion $\Sigma_2 \subset \Sigma_1$ (see the definition of $\Sigma_2$).

Then, recalling that $\text{Re}((K_0 - m^{p-1} a_0) u, u |u|^{p-2})$ is non-negative (cf. Lemma 3.2) and applying Proposition 2.1, we obtain

$$((|\lambda| + 1) \|Mu\|_p^{p-1} \leq \gamma \int_{\Omega} m^{p-1} |u|^{p} \text{d}x + \text{Re}((K_0 - m^{p-1} a_0) u, u |u|^{p-2})$$

$$\leq C_{11}(p) \|f\|_p \|Mu\|_p^{p-1} + \|f\|_p^{(p-2)(p/2)} \|Mu\|_p^{p/2}, \quad \lambda \in \Sigma_2. \tag{3.22}$$

Consequently, since $\|u\|_p \leq C_{12}(p) \|f\|_p$ (cf. (2.15)), (3.15) and (3.22) imply

$$((|\lambda| + 1) \|Mu\|_p^{(p-2)(p/2)}$$

$$\leq C_{13}(p) \|f\|_p \|Mu\|_p^{p-1} \|f\|_p^{(p-2)(p/2)} + \|f\|_p^{(p-2)(p/2)}, \quad \lambda \in \Sigma_2. \tag{3.23}$$

By Proposition 2.2, it is verified that $\lambda M + L$ is surjective on $L^p(\Omega)$. Hence, estimate (1.5) holds with $\alpha = 1$ and $\beta = 2[p(2 - \rho)]^{-1}$.

We can summarize the results in this section in Theorem 3.1.

**Theorem 3.1.** Let $L$ and $M$ be the linear operators defined by (1.7) and (1.8), the coefficients $a_{i,j}$ $i, j = 1, \ldots, n$, $a_0$ enjoying properties (2.1) and (2.2) and $m$ being
a non-negative function satisfying (1.9). Then the spectral equation $\lambda Mu + Lu = f$, with $f \in L^p(\Omega)$, admits, for any $\lambda \in \Sigma_2 = \{ \mu \in \mathbb{C} : \text{Re} \mu + (k_1(p)/2)|\text{Im} \mu| + (\gamma/2||m||_\infty) \geq 0 \}$ and $p \in [2, +\infty)$, a unique solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ satisfying the estimates

$$
||u||_p \leq C_{14}(p)||f||_p, \quad ||Mu||_p \leq C_{15}(p)|\lambda|^{-2/(p^2-\rho)}||f||_p, \quad \lambda \in \Sigma_2, \\
||Lu||_p \leq C_{16}(p)(1 + |\lambda|^{(p^2-\rho)/2})||f||_p, \quad \lambda \in \Sigma_2.
$$

**Example 3.1.** Let $\Omega$ be a bounded domain and let $x_0$ be a fixed point in $\partial \Omega$. Define then $r = \max_{x \in \Omega} |x - x_0|$ and choose

$$
m(x) = \max_{x \in \Omega} |x - x_0| = |x - x_0| - r_1| q, \quad q \in (1, +\infty).
$$

An elementary computation shows that

$$
|\nabla m(x)| = q| |x - x_0| - r_1| \leq qr \max_{x \in \Omega} (x)^{q-1}/q, \quad x \in \Omega.
$$

Consequently, function $m$ satisfies condition (1.9).

We notice that for any open interval $\Omega \subset \mathbb{R}$ we have $r = \text{length}(\Omega)$.

**4. The case when $p \in (1, 2)$**

In this section we are going to considering the case $p \in (1, 2)$. From (2.4) we immediately deduce that the estimate

$$
|\text{Im}(Lu, u|u|^{p-2})| + \gamma ||u||_p^p \leq |\text{Im}(Lu, u|u|^{p-2})| + \int_\Omega a_0(x)|u(x)|^p \, dx \\
\leq \frac{|p - 2|}{2\sqrt{p - 1}} \text{Re}(Lu, u|u|^{p-2}), \quad u \in \mathcal{D}(L)
$$

holds true for any $p \in (1, +\infty)$.

Consider again the spectral problem

$$
u \in \mathcal{D}(L), \quad \lambda Mu + Lu = f \in L^p(\Omega).
$$

Multiplying both sides in (4.2) by $u|u|^{p-2}$ and integrating over $\Omega$, we get

$$
\lambda ||m^{1/p}u||_p^p + (Lu, u|u|^{p-2}) = (f, u|u|^{p-2}).
$$

Taking the real and imaginary parts, from (4.3) we deduce

$$
\text{Re} \lambda ||m^{1/p}u||_p^p + \text{Re}(Lu, u|u|^{p-2}) + (a_0u, u|u|^{p-2}) = \text{Re}(f, u|u|^{p-2}),
$$

$$
\text{Im} \lambda ||m^{1/p}u||_p^p + \text{Im}(Lu, u|u|^{p-2}) = \text{Im}(f, u|u|^{p-2}),
$$

(4.4) and (4.5) respectively.
where we have set
\[(4.6) \quad L_0 = L - a_0.\]
Then from Okazawa [4, p.703] we get
\[(4.7) \quad (L_0 u, u|u|^{p-2}) = \lim_{\delta \to 0^+} I_p(u, \delta),\]
where \(\delta > 0\) and
\[(4.8) \quad I_p(u, \delta) = -\int_\Omega \left( |u(x)|^2 + \delta \right)^{(p-2)/2} \frac{\nabla u(x)}{|u(x)|^2} \sum_{j,k=1}^n D_{jk}[a_j(x)D_{jk}u(x)] \, dx.\]
As mentioned at the beginning of Section 2, we have
\[(4.9) \quad \Re(L_0 u, u|u|^{p-2}) \geq c_0 \int_\Omega |u(x)||u|^{p-2} |\nabla u(x)|^2 \, dx, \quad \text{if } p \in [2, +\infty),\]
\[(4.10) \quad \Re(L_0 u, u|u|^{p-2}) \geq c_0 (p-1) \int_\Omega \left( |u(x)|^2 + \delta \right)^{(p-2)/2} |\nabla u(x)|^2 \, dx, \quad \text{if } p \in (1, 2).\]
From (4.1) and (4.5) we deduce the inequalities
\[(4.11) \quad \Im \lambda \|m^{1/p} u\|_p^p \leq \|f\|_p \|u\|_p^{p-1}.\]
Multiply then both sides in (4.11) by a positive constant \(\nu\) and add the obtained inequality to equation (4.4) to get (cf. (2.2))
\[(4.12) \quad (\Re \lambda + \nu |\Im \lambda|)\|m^{1/p} u\|_p^p + \left( 1 - \nu \frac{|p-2|}{2\sqrt{p-1}} \right) \Re(L_0 u, u|u|^{p-2}) + \frac{\nu}{2} \|u\|_p^p + \frac{\nu}{2} \|u\|_p^{p-1}
\leq (\Re \lambda + \nu |\Im \lambda|)\|m^{1/p} u\|_p^p + \left( 1 - \nu \frac{|p-2|}{2\sqrt{p-1}} \right) \Re(L_0 u, u|u|^{p-2}) + (a_0 u, u|u|^{p-2})
\leq (f, u|u|^{p-2}) + \nu \|f\|_p \|u\|_p^{p-1} \leq (1 + \nu) \|f\|_p \|u\|_p^{p-1}.\]
Choose now \(\nu = \nu(p)\) so small as to satisfy
\[(4.13) \quad \nu_1(p) := 1 - \nu(p) \frac{|p-2|}{2\sqrt{p-1}} > 0, \quad \forall p \in (1, +\infty).\]
On the other hand, since \(m \in L^\infty(\Omega), \|u\|_p \geq \|m\|_\infty^{1/p} \|m^{1/p} u\|_p.\) Then (4.12) and (4.13) imply
\[(\frac{\nu}{2\|m\|_\infty} + \Re \lambda + \nu |\Im \lambda|)\|m^{1/p} u\|_p^p + \nu_1(p) \Re(L_0 u, u|u|^{p-2}) + \frac{\nu}{2} \|u\|_p^p\]
Multiplying both sides in (4.2) by $(\cdot)^p$
From (4.3) we deduce the estimates
$$
\left(\frac{\gamma}{2\|m\|_{\infty}} + \Re \lambda + \nu |\Im \lambda|\right) \|m^{1/p}u\|_p^p + C_{18} \Re(L_0u, u|u|^{p-2}) + \frac{\gamma}{2} \|u\|_p^p
\leq C_{19}\|f\|_p\|u\|_p^{p-1}, \quad \lambda \in \Sigma,
$$
the sector $\Sigma$ being defined by
$$
\Sigma = \{ \lambda \in C : \Re \lambda + \nu |\Im \lambda| + \frac{\gamma}{2\|m\|_{\infty}} \geq \varepsilon_0 > 0 \}.
$$
Notice that (4.7), (4.8), (4.10), (4.15) yield, in particular, the basic bounds
$$
\|u\|_p \leq \frac{2}{\gamma} C_{19}\|f\|_p, \quad \Re(L_0u, u|u|^{p-2}) \leq C_{20}\|f\|_p,
$$
and
$$
(p - 1)\varepsilon_0 \lim_{\delta \to 0^+} \int_{\Omega} \left( |\mu(x)|^2 + \delta \right)^{(p-2)/2} \|\nabla u(x)\|^2 \, dx
\leq \lim_{\delta \to 0^+} \Re I_p(u, \delta) \leq C_{20}\|f\|_p\|u\|_p^{p-1}, \quad \lambda \in \Sigma.
$$
From (4.3) we deduce the estimates
$$
|\lambda|\|m^{1/p}u\|_p^p \leq |(L_0u, u|u|^{p-2})| + \|f\|_p\|u\|_p^{p-1}
\leq \left( 1 + \frac{|p - 2|}{2\sqrt{p - 1}} \right) \Re(L_0u, u|u|^{p-2}) + \|f\|_p\|u\|_p^{p-1}
\leq C_{21}\|f\|_p\|u\|_p^{p-1} \leq C_{22}\|f\|_p^p.
$$
Consequently, (4.18) immediately yields
$$
|\lambda|^{1/p}\|Mu\|_p \leq C_{23}\|f\|_p.
$$
This, in turn, implies that (1.5) holds with $\alpha = 1$ and $\beta = 1/p$ and provides a different proof to (1.5).

Now we focus our attention to the case when $m \in C^1(\overline{\Omega})$ satisfies inequality (1.9) with
$$
\rho \in (2 - p, 1).
$$
Multiplying both sides in (4.2) by $m(x)^{p-1}|u(x)|^{p-2}$ and integrating over $\Omega$, we
easily get
\[
\lambda \|Mu\|_p^p - \lim_{\delta \to 0^+} \int_\Omega m(x)^{p-1}\overline{u(x)}(|u(x)|^2 + \delta)^{(p-2)/2} \sum_{j,k=1}^n D_{j,\delta} [a_{j,k}(x)D_{\delta}u(x)] \, dx
\]
(4.21) \[ + \int_{\overline{\Omega}} a_0(x)m(x)^{p-1}|u(x)|^p \, dx = \int_{\Omega} f(x)m(x)^{p-1}\overline{u(x)}|u(x)|^{p-2} \, dx. \]

An integration by parts in the integral appearing in the limit, which takes into account (4.20) and (4.21), easily yields
\[
- \int_{\Omega} m(x)^{p-1}\overline{u(x)}(|u(x)|^2 + \delta)^{(p-2)/2} \sum_{j,k=1}^n D_{j,\delta} [a_{j,k}(x)D_{\delta}u(x)] \, dx
\]
\[
= \int_{\Omega} (|u(x)|^2 + \delta)^{(p-2)/2} m(x)^{p-1} \sum_{j,k=1}^n a_{j,k}(x)D_{j,\delta}\overline{u(x)}D_{\delta}u(x) \, dx
\]
\[
+ (p-1) \int_{\Omega} \overline{u(x)}(|u(x)|^2 + \delta)^{(p-2)/2} \sum_{j,k=1}^n m(x)^{p-2} D_{j,\delta}m(x)a_{j,k}(x)D_{\delta}u(x) \, dx
\]
\[
+ (p-2) \int_{\Omega} m(x)^{p-1}(|u(x)|^2 + \delta)^{(p-1)/2} \sum_{j,k=1}^n a_{j,k}(x) \Re \left( \overline{u(x)}D_{j,\delta}u(x) \right) \overline{u(x)}D_{\delta}u(x) \, dx
\]
\[
= I_1(\delta) + (p-1)I_2(\delta) - (2-p)I_3(\delta).
\]
(4.22)

We have made use here of the following Proposition 4.1 whose proof is postponed to Section 6.

**Proposition 4.1.** Let \( m \) satisfy property (1.9). Then for any \( \beta \in (1 - \rho, 1) \), the function \( m(\cdot)^{\beta} \) belongs to \( C^1(\overline{\Omega}) \) and \( \nabla [m(\cdot)^{\beta}](x) = m_1(x) \) for any \( x \in \overline{\Omega} \), where
\[
m_1(x) = \begin{cases} 0, & x \in Z(m), \\ \beta m(x)^{\beta-1} \nabla m(x), & x \notin Z(m), \end{cases}
\]
and \( Z(m) \) denotes the zero-set of \( m \). Moreover,
\[
|\nabla [m(\cdot)^{\beta}](x)| \leq C m(x)^{\beta-1+\rho}, \quad x \in \overline{\Omega}.
\]

Since the matrix \( (a_{j,k}(x))_{j,k=1,\ldots,n} \) is real-valued and positive definite, from (4.22) we immediately deduce that
\[
I_1(\delta) \quad \text{and} \quad \Re I_3(\delta) \quad \text{are positive for any} \quad \delta \in \mathbb{R}_+.
\]
(4.24)
Then we observe that \( I_2(\delta) \) has a limit as \( \delta \to 0^+ \) and

\[
\lim_{\delta \to 0^+} I_2(\delta) = \int_{\Omega} \overline{u(x)}|u(x)|^{p-2} \sum_{j,k=1}^n m(x)^{p-2} D_{x_j}m(x) a_{j,k}(x) D_{x_k}u(x) \, dx.
\]

Note that the integral in the right-hand side is well-defined on the whole of \( W^{1,p}(\Omega) \) since \( \overline{u(x)}\|u\|^{p-2} \in L^p(\Omega), m^{p-2} D_{x_j}m \in L^\infty(\Omega) \) and \( D_{x_j}u \in L^p(\Omega) \).

Further, (4.25) implies that there exists also \( \lim_{\delta \to 0^+} [I_1(\delta) - (2 - p)I_3(\delta)]. \)

From (4.24) we deduce that there exist the limits

\[
\lim_{\delta \to 0^+} \text{Im} I_3(\delta) \quad \text{and} \quad \lim_{\delta \to 0^+} [I_1(\delta) - (2 - p) \text{Re} I_3(\delta)].
\]

We can now prove the following Lemma 4.1.

**Lemma 4.1.** The following estimates hold for any \( \delta \in \mathbb{R}_+, p \in (1,2) \) and \( \eta \in (0, 2(p-1)(2-p)^{-1}) \):

\[
I_1(\delta) - (2 - p) \text{Re} I_3(\delta) - \eta(2 - p)|\text{Im} I_3(\delta)| \geq 0,
\]

\[
I_1(\delta) + (p - 1) \text{Re} I_2(\delta) - (2 - p) \text{Re} I_3(\delta)
\]

\[
\lim_{\delta \to 0^+} \left[ I_1(\delta) + (p - 1) \text{Re} I_2(\delta) - (2 - p) \text{Re} I_3(\delta) \right]
\]

\[
- \eta \lim_{\delta \to 0^+} \left| (p - 1) \text{Im} I_3(\delta) - (2 - p) \text{Im} I_3(\delta) \right|
\]

\[
\geq -C_{24} \|f\|^{p/2}_p \|\bar{m}u\|^{p-2+p}_p \|u\|^{2-p-p/2}_p,
\]

\( C_{24} \) being a suitable positive constant.

**Proof.** Since the matrix \( (a_{j,k}(x))_{j,k=1,\ldots,n} \) is real-valued and positive definite, we immediately deduce the equality

\[
\sum_{j,k=1}^n a_{j,k}(x) \bar{\xi}_j \xi_k = \sum_{j,k=1}^n a_{j,k}(x)[\text{Re}(\xi_j) \text{Re}(\xi_k) + \text{Im}(\xi_j) \text{Im}(\xi_k)], \quad \forall \xi \in \mathbb{C}^n.
\]

Consider now the formulae

\[
I_1(\delta) = \int_{\Omega} (|u(x)|^2 + \delta)^{(p-4)/2} m(x)^{p-1} \sum_{j,k=1}^n a_{j,k}(x) \overline{u(x)} D_{x_j}u(x) \overline{u(x)} D_{x_k}u(x) \, dx
\]

\[
+ \delta \int_{\Omega} (|u(x)|^2 + \delta)^{(p-4)/2} m(x)^{p-1} \sum_{j,k=1}^n a_{j,k}(x) |D_{x_j}u(x)| D_{x_k}u(x) \, dx
\]

\[
= \int_{\Omega} (|u(x)|^2 + \delta)^{(p-4)/2} m(x)^{p-1} \left\{ \sum_{j,k=1}^n a_{j,k}(x) \text{Re} \left[ \overline{u(x)} D_{x_j}u(x) \right] \text{Re} \left[ \overline{u(x)} D_{x_k}u(x) \right] \right\}
\]

\[
\text{and} \quad I_3(\delta) = \int_{\Omega} (|u(x)|^2 + \delta)^{(p-4)/2} m(x)^{p-1} \left\{ \sum_{j,k=1}^n a_{j,k}(x) \text{Im} \left[ \overline{u(x)} D_{x_j}u(x) \right] \text{Im} \left[ \overline{u(x)} D_{x_k}u(x) \right] \right\}
\].
\[ + \sum_{j,k=1}^{n} a_{j,k}(x) \text{Im} \left[ \overline{u(x)} D_{x_j} u(x) \right] \text{Im} \left[ \overline{u(x)} D_{x_k} u(x) \right] \text{d}x \]

\[ \int_{\Omega} (|u(x)|^2 + \delta)^{(p-4)/2} m(x)^{p-1} \sum_{j,k=1}^{n} a_{j,k}(x) D_{x_j} u(x) \overline{D_{x_k} u(x)} \text{d}x, \quad \forall \delta \in \mathbb{R}_+ , \]

\[ I_1(\delta) = (2 - p) \text{Re} I_3(\delta) = \int_{\Omega} (|u(x)|^2 + \delta)^{(p-4)/2} m(x)^{p-1} \]

\[ \times \left\{ (p - 1) \sum_{j,k=1}^{n} a_{j,k}(x) \text{Re} \left[ \overline{u(x)} D_{x_j} u(x) \right] \text{Re} \left[ \overline{u(x)} D_{x_k} u(x) \right] \right\} \]

\[ + \sum_{j,k=1}^{n} a_{j,k}(x) \text{Im} \left[ \overline{u(x)} D_{x_j} u(x) \right] \text{Im} \left[ \overline{u(x)} D_{x_k} u(x) \right] \text{d}x \]

\[ \int_{\Omega} (|u(x)|^2 + \delta)^{(p-4)/2} m(x)^{p-1} \sum_{j,k=1}^{n} a_{j,k}(x) D_{x_j} u(x) \overline{D_{x_k} u(x)} \text{d}x, \quad \forall \delta \in \mathbb{R}_+ , \]

\[ \text{Im} I_3(\delta) \]

\[ = \left| \int_{\Omega} (|u(x)|^2 + \delta)^{(p-4)/2} m(x)^{p-1} \sum_{j,k=1}^{n} a_{j,k}(x) \text{Re} \left[ \overline{u(x)} D_{x_j} u(x) \right] \text{Im} \left[ \overline{u(x)} D_{x_k} u(x) \right] \text{d}x \right| \]

\[ \leq \frac{1}{2} \int_{\Omega} (|u(x)|^2 + \delta)^{(p-4)/2} m(x)^{p-1} \sum_{j,k=1}^{n} a_{j,k}(x) \left\{ \text{Re} \left[ \overline{u(x)} D_{x_j} u(x) \right] \text{Re} \left[ \overline{u(x)} D_{x_k} u(x) \right] \right\} \]

\[ + \text{Im} \left[ \overline{u(x)} D_{x_j} u(x) \right] \text{Im} \left[ \overline{u(x)} D_{x_k} u(x) \right] \right\} \text{d}x, \quad \forall \delta \in \mathbb{R}_+ . \]

We have here used the Cauchy–Schwarz inequality and the geometric-arithmetic mean, i.e.

\[ \left| \sum_{j,k=1}^{n} a_{j,k}(x) \xi_j \eta_k \right| \leq \left( \sum_{j,k=1}^{n} a_{j,k}(x) \xi_j \xi_k \right)^{1/2} \left( \sum_{j,k=1}^{n} a_{j,k}(x) \eta_j \eta_k \right)^{1/2} \]

\[ \leq \frac{1}{2} \left( \sum_{j,k=1}^{n} a_{j,k}(x) \xi_j \xi_k + \sum_{j,k=1}^{n} a_{j,k}(x) \eta_j \eta_k \right) = \frac{1}{2} \sum_{j,k=1}^{n} a_{j,k}(x) [\xi_j \xi_k + \eta_j \eta_k], \quad \forall \xi, \eta \in \mathbb{R}^n . \]

From (4.24) and (4.31) we deduce the following inequality, where we take advantage of the membership \( \eta \in (0, 2(p-1)(2 - p)^{-1}) \):

\[ I_1(\delta) = (2 - p) \text{Re} I_3(\delta) - \eta(2 - p) \text{Im} I_3(\delta) = \int_{\Omega} (|u(x)|^2 + \delta)^{(p-4)/2} m(x)^{p-1} \]

\[ \times \left\{ \left[ p - 1 - \frac{1}{2} \eta(2 - p) \right] \sum_{j,k=1}^{n} a_{j,k}(x) \text{Re} \left[ \overline{u(x)} D_{x_j} u(x) \right] \text{Re} \left[ \overline{u(x)} D_{x_k} u(x) \right] \right\} \]
\[+ \left[ 1 - \frac{1}{2} \eta(2 - p) \right] \sum_{j,k=1}^{n} a_{j,k}(x) \text{Im} \left[ \overline{u(x)} \overline{D_{x,j} u(x)} \right] \text{Im} \left[ u(x) D_{x,k} u(x) \right] \right] \, dx \]

\[(4.32) \quad + \delta \int_{\Omega} |u(x)|^2 + \delta^{(p-1)/2} m(x)^{p-1} \sum_{j,k=1}^{n} a_{j,k}(x) D_{x,j} u(x) \overline{D_{x,k} u(x)} \, dx \geq 0, \quad \forall \delta \in \mathbb{R}_+.\]

We have thus proved (4.26).

Then we note that (4.27) is a consequence of (4.26):

\[I_1(\delta) + (p - 1) \text{Re} I_3(\delta) - (2 - p) \text{Re} I_3(\delta) - \eta |(p - 1) \text{Im} I_3(\delta) - (2 - p) \text{Im} I_3(\delta)|\]

\[\geq I_1(\delta) - (2 - p) \text{Re} I_3(\delta) - \eta (2 - p) |\text{Im} I_3(\delta)| + (p - 1) |\text{Re} I_2(\delta) - \eta |\text{Im} I_2(\delta)||\]

\[\geq -(p - 1)(1 + \eta^2)^{1/2} |I_2(\delta)|, \quad \forall \delta \in \mathbb{R}_+.\]

(4.33)

To conclude the proof of the lemma we take into account the relations

\[\lim_{\delta \to 0^+} \left[ I_1(\delta) + (p - 1) \text{Re} I_2(\delta) - (2 - p) \text{Re} I_2(\delta) \right] - \eta \lim_{\delta \to 0^+} |(p - 1) \text{Im} I_2(\delta) - (2 - p) \text{Im} I_2(\delta)|\]

\[\geq \lim_{\delta \to 0^+} \left\{ \text{Re} I_1(\delta) + (p - 1) \text{Re} I_2(\delta) - (2 - p) \text{Re} I_2(\delta) \right. \]

\[\left. - \eta |(p - 1) \text{Im} I_2(\delta) - (2 - p) \text{Im} I_2(\delta)| \right\}\]

\[\geq -(p - 1)(1 + \eta^2)^{1/2} \lim_{\delta \to 0^+} |I_2(\delta)|, \quad \forall \delta \in \mathbb{R}_+.\]

(4.34)

Next, consider the following chain of inequalities, which holds for any \(\delta \in \mathbb{R}_+\):

\[\lim_{\delta \to 0^+} |I_2(\delta)|\]

\[\leq \limsup_{\delta \to 0^+} \int_{\Omega} (|u(x)|^2 + \delta)^{(p-1)/2} \sum_{j,k=1}^{n} m(x)^{p-2} |D_{x,j} m(x)| |a_{j,k}(x) D_{x,k} u(x)| \, dx\]

\[\leq \limsup_{\delta \to 0^+} \int_{\Omega} (|u(x)|^2 + \delta)^{p/4} \]

\[\times (|u(x)|^2 + \delta)^{(p-2)/4} \sum_{j,k=1}^{n} m(x)^{p-2} |D_{x,j} m(x)| |a_{j,k}(x)| |D_{x,k} u(x)| \, dx\]

\[\leq C_1 \limsup_{\delta \to 0^+} \left[ \int_{\Omega} m(x)^{2(p-2+p)} (|u(x)|^2 + \delta)^{p/2} \, dx \right]^{1/2}\]

\[\times \limsup_{\delta \to 0^+} \left[ \int_{\Omega} \sum_{j,k=1}^{n} |a_{j,k}(x) D_{x,k} u(x)|^2 (|u(x)|^2 + \delta)^{(p-2)/2} \, dx \right]^{1/2}\]

(cf. (4.16), (4.17))
\[ C_{26} \left\| \int_{\Omega} (|u(x)|^{2(p-2+\rho)} + \delta \right) \left( (\nabla u(x))^2 \right) \right\|_{p/2}^{1/2} \leq C_{27} \left\| f \right\|_{p/2}^{2} \| Mu \|^2_{p-2+\rho} \| u \|^{2-p-\rho/2}_{p} \]

To derive the last inequality we have applied Hölder’s inequality with index \( q = p(2(p-2+\rho))^{-1} \) to the integral

\[ \int_{\Omega} [m(x)|u(x)|^{2(p-2+\rho)} |u(x)|^{-p+4-2\rho} \right] \, dx \]

From (4.34) and (4.35) we immediately conclude (4.28).

Taking now the real part and the modulus of the imaginary part in (4.21) and using (4.22), we easily derive the relations

\[ \text{Re} \lambda \| Mu \|^2_{p} + \lim_{\delta \to 0+} [I_1(\delta) + (p-1) \text{Re} I_2(\delta) - (2-p) \text{Re} I_3(\delta)] \]

\[ + \int_{\Omega} a_0(x)m(x)^{p-1}|u(x)|^p \, dx = \text{Re} \int_{\Omega} m(x)^{p-1} f(x)\bar{u}(x)|u(x)|^{p-2} \, dx, \]

\[ \text{Im} \lambda \| Mu \|^2_{p} \leq \lim_{\delta \to 0+} [(p-1) \text{Im} I_2(\delta) - (2-p) \text{Im} I_3(\delta)] \]

\[ + \text{Im} \int_{\Omega} m(x)^{p-1} f(x)\bar{u}(x)|u(x)|^{p-2} \, dx, \quad \forall \lambda \in \mathbb{C}. \]

Add now member by member (4.36) and (4.37) multiplied by \( \eta \in (0, 2\sqrt{p-1}(2-p)^{-1}) \) and use (4.28) and (2.2). We easily deduce the following estimate for any \( \lambda \in \Sigma = \{ \mu \in \mathbb{C} : \text{Re} \mu + |\text{Im} \mu| \geq 0 \} \):

\[ \left[ \text{Re} \lambda + \eta \text{Im} \lambda \right] \| Mu \|^2_{p} + \frac{\eta}{\| \lambda \|^2_{\infty}} \leq \left[ \lim_{\delta \to 0+} [I_1(\delta) + (p-1) \text{Re} I_2(\delta) - (2-p) \text{Re} I_3(\delta)] \right] \]

\[ - \eta \lim_{\delta \to 0+} [(p-1) \text{Im} I_2(\delta) - (2-p) \text{Im} I_3(\delta)] \]

\[ + \text{Re} \int_{\Omega} f(x)m(x)^{p-1}|u(x)|^p \, dx + \eta \text{Im} \int_{\Omega} f(x)m(x)^{p-1}|u(x)|^{p-2} \, dx \]

\[ \leq - \lim_{\delta \to 0+} [(p-1) \text{Re} I_2(\delta) - (2-p) \text{Re} I_3(\delta)] \]

\[ - \eta \| (p-1) \text{Im} I_2(\delta) - (2-p) \text{Im} I_3(\delta) \| \]
+ Re $\int_{\Omega} f(x)m(x)^{p-1}u(x)|u(x)|^{p-2}\,dx + \eta \left| \int_{\Omega} f(x)m(x)^{p-1}u(x)|u(x)|^{p-2}\,dx \right|$
\[ \leq C_{28} ||f||_{p}^{\frac{p}{2}} ||Mu||_{p}^{\frac{2-p}{2}} ||u||_{p}^{2-p/2} + (1 + \eta^2)^{1/2} ||f||_{p} ||Mu||_{p}^{1-p/2}. \]
(4.38)

Take $\lambda$ in the sector
\[ \Sigma_{3} = \left\{ \mu \in \mathbb{C} : \Re \mu + \frac{\eta}{2} \Im \mu + \frac{\gamma}{2||m||_{\infty}} \geq 0 \right\}. \]
Then, since $||u||_{p} \leq C_{19} ||f||_{p}$ (cf. (2.11), (2.12) and our definition of $\eta$) and $2 - \rho - p/2 > 0$ (cf. (4.20)), by Proposition 2.1 we immediately derive the inequality
\[ (|\lambda| + 1)||Mu||_{p}^{1-p/2} \leq C_{24} [||f||_{p}^{2-p} + ||f||_{p} ||Mu||_{p}^{1-p}], \quad \text{if} \quad \lambda \in \Sigma_{3}. \]
(4.40)

Finally, $||Mu||_{p} \leq ||m||_{\infty} ||u||_{p} \leq C_{19} ||m||_{\infty} ||f||_{p}$ implies
\[ (|\lambda| + 1)||Mu||_{p}^{1-p/2} \leq C_{30} ||f||_{p}^{1-p}, \quad \text{if} \quad \lambda \in \Sigma_{3}. \]
(4.41)

We can now collect the result in this section in the following Theorem 4.1.

**Theorem 4.1.** Let $L$ and $M$ be the linear operators defined by (1.7) and (1.8), the coefficients $a_{i,j}$ $i, j = 1, \ldots, n$, $a_0$ enjoying properties (2.1) and (2.2) and $m$ being a non-negative function satisfying (1.9). Then the spectral equation $\lambda Mu + Lu = f$, with $f \in L^{p}(\Omega)$, admits, for any $\lambda \in \Sigma_{3}$ and $p \in (1, 2)$, $\rho \in [2 - p, 1]$, a unique solution $u \in W^{2,p}(\Omega) \cap W_{0}^{1,p}(\Omega)$ satisfying the estimates
\[ ||u||_{p} \leq C_{30} ||f||_{p}, \quad ||Mu||_{p} \leq C_{31}(p)||\lambda||^{(2-p)^{-\rho}} ||f||_{p}, \quad \lambda \in \Sigma_{3}, \]
\[ ||Lu||_{p} \leq C_{32}(1 + ||\lambda||^{(1-\rho)(2-p)^{-1}}) ||f||_{p}, \quad \lambda \in \Sigma_{3}. \]
(4.42)

**Example 4.1.** Let $n = 1$, $m(x) = x^{q}(1-x)^{q}$, $q \in (1, +\infty)$, $\Omega = (0, 1)$. Then
\[ m'(x) = q(1-2x)m(x)^{q-1}/q, \quad x \in (0, 1). \]
Hence (4.25) holds true for any $q \in (1, +\infty)$. If we have to deal with $L^{p}(0, 1)$ with $p \in (1, 2)$, to satisfy (4.20) we are forced to assume $q > (p - 1)^{-1}$.

**5. Solving problem (1.1)–(1.3)**

Taking the spectral Theorems 2.1, 3.1, 4.1 into account, from Theorem 3.26 in [3] we can easily derive our existence and uniqueness result. For this purpose we need to introduce the following interpolation space
\[ L_{0,\infty}^{p} = \left\{ g \in L^{p}(\Omega) : \sup_{t \geq 1} t^{p} ||L(tM + L)^{-1}||_{L^{p}(\Omega)} < +\infty \right\}. \]
In particular, any \( g = mh \) belongs to \( L^p_{\Omega,\infty} \) whenever \( m \in L^\infty(\Omega) \) and \( h \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \). Notice that \( L^p_{\Omega,\infty} \subset (X; D(LM^{-1})))_{\theta,\infty} \).

**Theorem 5.1.** Let \( p \in (1, +\infty) \), let \( m \in L^\infty(\Omega) \) be a non-negative function and let the coefficients \( a_{i,j} \), \( i,j = 1, \ldots, n \), \( a_0 \) enjoy properties (2.1) and (2.2). Then for any

\[
(5.2) \quad u_0 \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \quad f \in C^\theta([0, T]; L^p(\Omega)), \quad \theta \in (1 - \beta, 1),
\]

with \( \beta = 1/p \) and

\[
(5.3) \quad -A(x, D_x)u_0 + f(0, \cdot) = g_0, \quad g_0 \in L^p_{\Omega,\infty},
\]

problem (1.1)–(1.3) admits a unique solution

\[
(5.4) \quad mu \in C^{\theta+\beta}([0, T]; L^p(\Omega)), \quad u \in C^{\theta+\beta-1}([0, T]; W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)).
\]

Moreover, if \( m \) is a non-negative function satisfying (1.9) and \( \beta \) is defined by (1.10), the same result holds under assumptions (5.1) and (5.2) on \((u_0, f)\).

### 6. Proofs of the propositions

**Proof of Proposition 2.1.** Let \( \lambda \in \Sigma_{k,\varepsilon} \) and \( \Re \lambda \geq 0 \). Then it is clear that

\[ |\Re \lambda| \leq \Re \lambda + |k| \Im \lambda + \varepsilon. \]

On the other hand, if \( \lambda \in \Sigma_{k,\varepsilon} \) and \( \Re \lambda < 0 \), then

\[ |\Re \lambda| = \Re \lambda - (k/2) |\Im \lambda| + (\varepsilon/2) \leq \Re \lambda + k |\Im \lambda| + \varepsilon. \]

Therefore, \( |\Re \lambda| \leq \Re \lambda + k |\Im \lambda| + \varepsilon \) for any \( \lambda \in \Sigma_{k,\varepsilon} \). In the meantime it is obvious that \( |\Im \lambda| + 1 \leq 2((1/k)+(1/\varepsilon))(\varepsilon/2) \leq 2(1/k+(1/\varepsilon))(\Re \lambda + k |\Im \lambda| + \varepsilon) \) for any \( \lambda \in \Sigma_{k,\varepsilon} \). Hence, we conclude that \( |\lambda| + 1 \leq |\Re \lambda| + |\Im \lambda| + 1 \leq [2/k+(2/\varepsilon)+1](\Re \lambda + k |\Im \lambda| + \varepsilon), \lambda \in \Sigma_{k,\varepsilon}. \]

**Proof of Proposition 2.2.** We consider the set \( J = \{ \theta \in [0, 1]; \mathcal{R}(A(\theta)) = X \} \) and shall prove that this set is an open and closed subset of the interval \([0, 1]\) under (2.17) and (2.18). In fact, let \( \theta \in J \); then, it follows from (2.17) that \( A(\theta)^{-1} \in \mathcal{L}(X) \) with \( \|A(\theta)^{-1}\| \leq \delta^{-1} \). Moreover, for any \( \theta' \in [0, 1] \), we have

\[ A(\theta') = [1 + A(\theta') - A(\theta)]A(\theta)^{-1}A(\theta). \]

Since \( \|(A(\theta') - A(\theta))A(\theta)^{-1}\| \leq N \delta^{-1} |\theta' - \theta| \), the operator \( 1 + [A(\theta') - A(\theta)]A(\theta)^{-1} \) is a linear isomorphism of \( X \) provided \( |\theta' - \theta| < N^{-1} \delta \). This then shows that \( \theta' \in J \) for any \( \theta' \) such that \( |\theta' - \theta| < N^{-1} \delta \); hence, \( J \) is an open set. Consider now a sequence \( \theta_n \in J \) and assume that \( \theta_n \to \theta \) as \( n \to +\infty \). Let \( f \in X \) be any vector; then, there exists a sequence \( u_n \in D \) such that \( A(\theta_n)u_n = f \). From (2.17) it follows that \( \|u_n\| \leq \delta^{-1} \|f\| \). Furthermore we observe that \( \|A(\theta)u_n - f\| \leq \|(A(\theta) - A(\theta_n))u_n\| \leq N \delta^{-1} |\theta - \theta_n| \|f\| \).
therefore, \( A(\bar{\theta})u_n \to f \) as \( n \to +\infty \). In the meantime, \( \delta\|u_m - u_n\| \leq \|A(\bar{\theta})(u_m - u_n)\| \leq \|A(\bar{\theta})u_m - f\| + \|f - A(\bar{\theta})u_n\| \to 0 \) as \( m, n \to +\infty \). So, \( u_n \) has a limit \( u \in X \) as \( n \to +\infty \). Since \( A(\bar{\theta}) \) is a closed operator, \( u \in \mathcal{D} \) and \( A(\bar{\theta})u = f \); hence, \( \bar{\theta} \in J \). That is, \( J \) is a closed set. As \( 1 \in J \neq \emptyset \), we conclude that \( J = [0, 1] \).

Proof of Proposition 4.1. According to (1.9), we have the inclusion \( Z(m) \subset Z(\nabla m) \). Moreover, formula (4.23) is trivial if \( x \notin Z(m) \). This therefore shows that we have to deal with the case \( x \in Z(m) \) only.

First we will consider the one-dimensional case \( (n = 1) \). For this purpose assume \( x_0 \in Z(m) \). Our starting point is the following formula:

\[
\lim_{x \to x_0} \frac{m(x)^\beta - m(x_0)^\beta}{x - x_0} = \lim_{x \to x_0} \frac{[m(x) + \varepsilon]^{\beta} - \varepsilon^{\beta}}{x - x_0} \\
= \lim_{x \to x_0} \left| \frac{\beta}{x - x_0} \int_{x_0}^x [m(t) + \varepsilon]^{\beta - 1} m'(t) \, dt \right|.
\]

(6.1)

We next notice that \( \lim_{x \to x_0^+} [m(t) + \varepsilon]^{\beta - 1} m'(t) = m_1(t) \) for any \( t \in \Omega \) and that

\[
\|[m(t) + \varepsilon]^{\beta - 1} m'(t)\| \leq C[m(t) + \varepsilon]^{\beta - 1} m(t)^\rho \\
= C \left[ \frac{m(t)}{m(t) + \varepsilon} \right]^{1-\beta} m(t)^{\beta - 1 + \rho} \leq C m(t)^{\beta - 1 + \rho}, \quad \forall t \in \Omega.
\]

By virtue of the dominated convergence theorem and by the bound \( |m_1(t)| \leq C m(t)^{\beta - 1 + \rho} \) for any \( t \in \Omega \), we deduce the following relations:

\[
\lim_{x \to x_0^+} \frac{m(x)^\beta - m(x_0)^\beta}{x - x_0} = \lim_{x \to x_0^+} \left| \frac{1}{x - x_0} \int_{x_0}^x m_1(t) \, dt \right| \\
(6.2)
\leq \lim_{x \to x_0^+} \left| \frac{1}{x - x_0} \int_{x_0}^x |m(t)| \, dt \right| \leq \lim_{x \to x_0^+} \frac{C}{x - x_0} \int_{x_0}^x m(t)^{\beta - 1 + \rho} \, dt = 0.
\]

Note here that \( m(\cdot)^{\beta - 1 + \rho} \) is continuous in \( \Omega \) and \( x_0 \in Z(m) \). An analogous argument holds for \( \lim_{x \to x_0^-} [m(x)^\beta - m(x_0)^\beta]/(x - x_0) \) also.

We have thus shown that there exists \( D_{\xi}[m(\cdot)^\beta](x_0) \) and coincides with \( 0 = m_1(x_0) \). Therefore the formula \( D_{\xi}[m(\cdot)^\beta](x) = m_1(x) \) holds for any \( x \in \Omega \). Since \( \beta \in (1 - \rho, 1) \), bound (1.9) and (4.23) immediately imply that \( m_1 \in C(\Omega) \). Consequently, \( m(\cdot)^\beta \in C(\Omega) \).

Finally, the multi-dimensional case is an immediate consequence of the case \( n = 1 \).
References


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