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## CANCELLATION OF FINITELY GENERATED MODULES OVER REGULAR RINGS

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### Introduction

In [3, Theorem 2] Evans proved that an  $R$ -module  $M$  cancels from direct sums if the stable range of the endomorphism ring  $\text{End}_R(M)$  is 1. By using this result it follows that finitely generated projective modules over unit-regular rings cancel from direct sums [4, Corollary 4.7 and Proposition 4.12]. However, as the first author has shown [4, Example 5.13], there exists a unit-regular ring  $R$  with a cyclic module  $M$  which is directly infinite, that is,  $M \cong M \oplus A$  for some nonzero  $R$ -module  $A$ , and so in particular  $M$  cannot be cancelled. Later, Menal [7, Theorem D] proved that if  $R$  is a regular ring whose primitive factor rings are all artinian, then the stable range of  $\text{End}_R(M)$  is 1 for every finitely generated  $R$ -module  $M$  and so again Evans' theorem implies cancellation for  $M$ . However, the converse of Menal's result is not true for arbitrary regular rings. For instance, it suffices to take any regular locally finite-dimensional algebra over a field whose primitive factor rings are not artinian. For more information on this subject we refer the reader to [8].

In this note we will show that if  $R$  is a regular ring which is  $N^*$ -complete, or right or left  $\aleph_0$ -continuous, or left  $\aleph_0$ -injective, then every finitely generated right  $R$ -module cancels from direct sums if and only if  $R$  has bounded index of nilpotence (and so all primitive factor rings of  $R$  are artinian). As an application we also obtain a characterization of those polynomial rings  $R[x]$  that are semihereditary in the case that  $R$  is either  $N^*$ -complete or right or left  $\aleph_0$ -continuous.

### Cancellation

All rings considered in this paper are associative with 1. A ring  $R$  is (*von Neumann*) *regular* if for any  $x \in R$  there exists a  $y \in R$  such that  $x = xyx$ . If we can always choose  $y$  to be a unit, then  $R$  is called *unit-regular*. If  $X$  is a subset

of  $R$ , then we denote by  $l(X)$  and  $r(X)$  its left and right annihilators. By a *subring* we always mean a *unital* subring.

The example mentioned above, of a unit-regular ring which has a directly infinite cyclic module, is just  $\prod_{n=1}^{\infty} M_n(F_n)$ , where  $F_1, F_2, \dots$  are fields. Our method is to look for subrings of this form inside regular rings of index  $\infty$ , using  $N^*$ -completeness, or  $\aleph_0$ -continuity, or  $\aleph_0$ -injectivity to build a complete direct product.

**Lemma 1.** *If  $R$  is a nonzero regular ring, then there is a nonzero central idempotent  $e \in R$  such that  $eR$  is an algebra over a field.*

Proof. If  $R$  is not a  $\mathcal{Q}$ -algebra, then there is a prime  $p$  such that  $pR \neq R$ . Since  $p$  is central,  $l(pR) = eR$  for some nonzero central idempotent  $e \in R$ . Now  $eR$  is an algebra over  $\mathbb{Z}/p\mathbb{Z}$ .

Recall that the *index of nilpotence* of an ideal  $I$  in a ring  $R$  is the supremum of the indices of all nilpotent elements in  $I$ . If this supremum is finite, we say  $I$  has *bounded index (of nilpotence)*, while if this supremum is infinite we say  $I$  has *index  $\infty$* . The next lemma is a key result in this paper.

**Lemma 2.** *Let  $R$  be a regular ring and let  $I$  be a two-sided ideal of  $R$  of index  $\infty$ . Then there exist nonzero orthogonal idempotents  $e_1, e_2, \dots \in I$  such that  $e_n R e_n$  has a subring isomorphic to the  $n \times n$  matrix algebra over a field.*

Proof. We first observe that it suffices to prove (1): There exist independent right ideals  $A_1, A_2, \dots$  contained in  $I$  such that  $A_n$  contains a direct sum of  $n$  nonzero pairwise isomorphic right ideals. For suppose (1) holds. Each  $A_n$  contains a right ideal  $B_n$  which is a direct sum of  $n$  nonzero pairwise isomorphic principal right ideals, and then  $B_n$  is principal. By [4, Proposition 2.13] there exist orthogonal idempotents  $f_1, f_2, \dots$  such that

$$f_1 R \oplus \dots \oplus f_n R = B_1 \oplus \dots \oplus B_n$$

for all  $n$ . Then  $f_n \in I$  and  $f_n R \cong B_n$  for each  $n$ , so  $f_n R$  is a direct sum of  $n$  nonzero pairwise isomorphic right ideals, whence  $f_n R f_n$  is isomorphic to an  $n \times n$  matrix ring. By Lemma 1, there is a nonzero central idempotent  $e_n$  in the ring  $f_n R f_n$  such that  $e_n R e_n = e_n f_n R f_n$  is an algebra over a field. Then  $e_n \in I$  and  $e_n R e_n$  has a subring isomorphic to the  $n \times n$  matrix algebra over a field. Since the  $f$ 's are orthogonal, so are the  $e$ 's.

For each  $n$ , let  $J_n$  be the sum of all ideals contained in  $I$  of index at most  $n$ . Then  $J_1 \subseteq J_2 \subseteq \dots$ , and by [4, Corollary 7.8] each  $J_n$  has index at most  $n$ . Set  $B_n = J_{n+1} \cap l(J_n)$  for each  $n$ . Observe that  $(B_n \cap J_n)^2 \subseteq B_n J_n = 0$ , and so  $B_n \cap J_n = 0$ . Hence, the  $B$ 's are independent. When  $B_n \neq 0$ , then since  $B_n \subseteq J_{n+1}$  and  $B_n \cap J_n = 0$  we see that  $B_n$  has index  $n+1$ . In this case, [4, Theorem 7.2] implies

that  $B_n$  contains a direct sum of  $n+1$  nonzero pairwise isomorphic right ideals. If infinitely many  $B$ 's are nonzero, say  $B_{n(1)}, B_{n(2)}, \dots$  where  $n(1) < n(2) < \dots$ , then each  $B_{n(k)}$  contains a direct sum of  $k$  nonzero pairwise isomorphic right ideals, and (1) is proved. Thus we may assume that only finitely many  $B$ 's are not zero, and so for some  $t \in \mathbf{N}$  we have  $B_n = 0$  for all  $n \geq t$ . Then for  $n \geq t$  we have  $J_{n+1} \cap I(J_n) = 0$ , and it follows that  $J_n \leq_e J_{n+1}$  as right ideals of  $R$ . From [4, Corollary 7.5] the index of  $J_{n+1}$  is now at most  $n$ , whence  $J_{n+1} = J_n$ . Thus  $J_n = J_t$  for  $n \geq t$ .

Set  $A = I(J_t) \cap I$ , and observe that  $A \cap J_t = 0$ . Since  $I$  has index  $\infty$ , [4, Corollary 7.5] shows that  $J_t$  is not essential in  $I_R$ , and so  $A \neq 0$ . Now  $A \cap J_n = 0$  for all  $n$ . Thus  $A$  and all nonzero ideals contained in  $A$  have index  $\infty$ . If  $A$  contains an independent sequence of nonzero ideals  $A_1, A_2, \dots$ , then by [4, Theorem 7.2] each  $A_n$  contains a direct sum of  $n$  nonzero pairwise isomorphic right ideals and (1) is proved. Thus we may assume that  $A$  does not contain an infinite sequence of independent nonzero ideals.

Now  $A$  must contain a nonzero ideal  $B$  such that any two nonzero ideals contained in  $B$  have nonzero intersection. It follows that any two nonzero right ideals  $K, L \subseteq B$  must contain nonzero isomorphic right ideals. For taking nonzero elements  $x \in K$  and  $y \in L$ , we have  $RxR \cap RyR \neq 0$ , whence  $xay \neq 0$  for some  $a \in R$ , and  $xayR$  is isomorphic to a right ideal contained in  $yR$ . By induction, it follows that whenever  $B_1, \dots, B_k$  are nonzero right ideals contained in  $B$ , there exist nonzero right ideals  $C_i \subseteq B_i$  for  $i=1, \dots, k$  such that  $C_i \cong C_j$  for all  $i, j$ .

As  $B \neq 0$ , it has infinite index, so it cannot be artinian. Thus  $B$  contains an infinite direct sum of nonzero right ideals. Grouping finitely many of these together at a time, we obtain nonzero independent right ideals  $A_1, A_2, \dots \subseteq B$  such that each  $A_n$  is a direct sum of  $n$  nonzero right ideals. Invoking the result of the previous paragraph, we conclude that each  $A_n$  contains a direct sum of  $n$  nonzero pairwise isomorphic right ideals. Therefore (1) holds in this case too.

Recall that a *pseudo-rank function* on a regular ring  $R$  is a map  $P: R \rightarrow [0, 1]$  such that

- (a)  $P(1) = 1$ ;
- (b)  $P(xy) \leq P(x), P(y)$  for all  $x, y \in R$ ;
- (c)  $P(e+f) = P(e) + P(f)$  for all orthogonal idempotents  $e, f \in R$ .

Denote by  $\mathbf{P}(R)$  the set of all pseudo-rank functions on  $R$ , and let

$$N^*(x) = \sup \{P(x) : P \in \mathbf{P}(R)\}$$

for all  $x \in R$ . Then the rule  $\delta(x, y) = N^*(x-y)$  defines a pseudo-metric on  $R$ ; we say  $R$  is  *$N^*$ -complete* if  $\delta$  is a metric and  $R$  is complete with respect to it.

**Lemma 3.** *Let  $R$  be an  $N^*$ -complete regular ring of index  $\infty$ . Then  $R$*

has a subring isomorphic to  $T \times \prod_{n=1}^{\infty} M_n(F_n)$  for some regular ring  $T$  and some fields  $F_1, F_2, \dots$ .

Proof. From the idempotents given in Lemma 2 we can choose non-zero orthogonal idempotents  $f_1, f_2, \dots \in R$  such that each  $f_n R f_n$  has a subring isomorphic to the  $n2^n \times n2^n$  matrix algebra over a field  $F_n$ . Then  $f_n$  is a sum of orthogonal idempotents  $g_{ni}$  (for  $i=1, \dots, n2^n$ ) such that  $g_{ni} R \cong g_{nj} R$  for all  $i, j$  and  $F_n$  is isomorphic to a subring of  $g_{n1} R g_{n1}$ . Since  $n2^n(g_{ni} R) \cong f_n R \subseteq R_R$ , we have  $N^*(g_{ni}) \leq 1/n2^n$ . Set  $h_n = g_{n1} + \dots + g_{nn}$ , and observe that

$$N^*(h_n) \leq N^*(g_{n1}) + \dots + N^*(g_{nn}) \leq n/n2^n = 1/2^n.$$

Also  $h_n R h_n \cong M_n(g_{n1} R g_{n1})$ , and so  $M_n(F_n)$  is isomorphic to a subring of  $h_n R h_n$ .

Given any sequence  $x = (x_n) \in \prod_{n=1}^{\infty} h_n R h_n$ , we have  $N^*(x_n) \leq N^*(h_n) \leq 1/2^n$  for all  $n$ , so the partial sums of  $\sum x_n$  are Cauchy with respect to  $N^*$ . Hence  $\sum x_n$  converges to some  $\phi(x) \in R$ . In particular  $\sum h_n$  converges to an idempotent  $h \in R$ . Then  $\phi$  gives a ring isomorphism of  $\prod_{n=1}^{\infty} h_n R h_n$  onto a subring of  $h R h$ , and so  $\prod_{n=1}^{\infty} M_n(F_n)$  is isomorphic to a subring of  $h R h$ .

Taking  $T = (1-h)R(1-h)$ , the proof is complete.

We say that a regular ring  $R$  is *right  $\aleph_0$ -continuous* if the lattice of principal right ideals  $L(R_R)$  is *upper  $\aleph_0$ -continuous*, that is, every countable subset of  $L(R_R)$  has a supremum in  $L(R_R)$  and  $A \wedge (\bigvee B_n) = \bigvee (A \wedge B_n)$  for all  $A$  and all countable linearly ordered subsets  $\{B_n\}$  in  $L(R_R)$ . For example, any right self-injective regular ring is right  $\aleph_0$ -continuous [4, Corollary 13.5].

Recall that a ring  $R$  is called *right (left)  $\aleph_0$ -injective* provided every homomorphism from a countably generated right (left) ideal into  $R$  is given by left (right) multiplication by an element of  $R$ .

**Lemma 4.** *Let  $R$  be a regular ring of index  $\infty$  which is either right or left  $\aleph_0$ -continuous. Then  $R$  has a subring isomorphic to  $T \times \prod_{n=1}^{\infty} M_n(F_n)$  for some regular ring  $T$  and some fields  $F_1, F_2, \dots$ .*

Proof. By symmetry, we may assume that  $R$  is right  $\aleph_0$ -continuous. By Lemma 2, there exist nonzero orthogonal idempotents  $e_1, e_2, \dots \in R$  such that each  $e_n R e_n$  has a subring isomorphic to the  $n \times n$  matrix algebra over a field  $F_n$ . By [4, Corollary 14.4] there is an idempotent  $e \in R$  such that  $\bigoplus_{n=2}^{\infty} e_n R \leq_e e R$ , and it suffices to show that  $\prod_{n=2}^{\infty} M_n(F_n)$  is isomorphic to a subring of  $e R e$  (since it is clear how to find a subring of  $(1-e)R(1-e)$  isomorphic to  $T \times F_1$ ).

Let  $S$  be the maximal right  $\aleph_0$ -quotient ring of  $R$  (see [4, Chapter 14]), and note that  $\bigoplus_{n=2}^{\infty} e_n S \leq_e e S$ . Any sequence  $x = (x_n) \in \prod_{n=2}^{\infty} e_n S e_n$  induces a homomorphism

$$\bigoplus_{n=2}^{\infty} e_n S \rightarrow \bigoplus_{n=2}^{\infty} e_n S \subseteq eS$$

which extends uniquely to a homomorphism  $eS \rightarrow eS$  because  $S$  is right  $\aleph_0$ -injective [4, Theorem 14.12], and this homomorphism is left multiplication by some unique element  $\phi(x) \in eSe$ . We observe that  $\phi$  is a unital ring map from  $\prod_{n=2}^{\infty} e_n Se_n$  into  $eSe$ , and that  $\phi$  is injective.

Now  $eSe$  has a subring  $S' \cong \prod_{n=2}^{\infty} M_n(F_n)$ . Since  $S'$  is regular and right self-injective with no nonzero abelian central idempotents,  $S'$  is generated as a ring by its idempotents [4, Theorem 13.16]. But all idempotents of  $S$  lie in  $R$  [4, Theorem 14.12]. Therefore  $S'$  is a subring of  $eRe$ .

For  $\aleph_0$ -injective regular rings, we have a weaker version of Lemmas 3 and 4, which is not left-right symmetric.

**Lemma 5.** *Let  $R$  be a left  $\aleph_0$ -injective regular ring of index  $\infty$ . Then  $R$  has a subring  $S$  with a two-sided ideal  $H$  such that  $(S/H)_S$  is flat and  $S/H \cong \prod_{n=1}^{\infty} M_n(F_n)$  for some fields  $F_1, F_2, \dots$ .*

*Proof.* By Lemma 2, there exist nonzero orthogonal idempotents  $e_1, e_2, \dots \in R$  such that  $e_n Re_n$  has a subring isomorphic to  $M_n(F_n)$  for some field  $F_n$ . Let  $J = \bigoplus_{n=1}^{\infty} Re_n$  and let  $I(J)$  be the idealizer of  $J$  in  $R$  and observe that the right annihilator  $r(J)$  is a two-sided ideal of  $I(J)$ .

Any sequence  $x = (x_n) \in \prod_{n=1}^{\infty} e_n Re_n$  induces a homomorphism  $J \rightarrow J \subseteq_R R$  which must be right multiplication by some  $\phi(x) \in I(J)$ , because  $R$  is left  $\aleph_0$ -injective. Although  $\phi(x)$  is not uniquely determined by  $x$ , it is unique modulo  $r(J)$ . Thus the rule  $x \mapsto \phi(x) + r(J)$  defines a unital ring map from  $\prod_{n=1}^{\infty} e_n Re_n$  into  $I(J)/r(J)$ , and this map is injective. Therefore  $I(J)$  has a subring  $S$  such that  $S \supseteq r(J)$  and  $S/r(J) \cong \prod_{n=1}^{\infty} M_n(F_n)$ .

Since  $r(J)$  is a right ideal of  $R$ , it is a directed union of right ideals  $eR$  where  $e$  is an idempotent. For any such  $e$ , observe that  $S = eR \oplus (1-e)S$ , so that  $eR$  is a direct summand of  $S_S$ . Therefore  $S/r(J)$  is a flat right  $S$ -module.

A module  $M$  is called *directly finite* provided  $M$  is not isomorphic to any proper direct summand of itself, that is,  $M \not\cong M \oplus A$  for all nonzero modules  $A$ . If  $M$  is not directly finite then  $M$  is called *directly infinite*.

Parts of the following theorem are due to Evans [3] and Menal [7].

**Theorem 6.** *Let  $R$  be a regular ring which is  $N^*$ -complete, or right or left  $\aleph_0$ -continuous, or left  $\aleph_0$ -injective. Then the following conditions are equivalent:*

- (a)  *$R$  has bounded index of nilpotence.*
- (b) *All primitive factor rings of  $R$  are artinian.*
- (c) *The endomorphism ring of every finitely generated right  $R$ -module has stable range 1.*
- (d) *All finitely generated right  $R$ -modules cancel from direct sums.*
- (e) *All finitely generated right  $R$ -modules are directly finite.*

Proof. (a) $\Rightarrow$ (b) is [4, Corollary 7.10]. (b) $\Rightarrow$ (c) is by [7, Theorem D]. (c) $\Rightarrow$ (d) is [3, Theorem 2]. (d) $\Rightarrow$ (e) is clear.

Now we prove (e) $\Rightarrow$ (a). Suppose that  $R$  has index  $\infty$ . By Lemma 3, 4, or 5,  $R$  has a subring  $S$  with a two-sided ideal  $H$  such that  $(S/H)_S$  is flat and  $S/H \cong \prod_{n=1}^{\infty} M_n(F_n)$  for some fields  $F_1, F_2, \dots$ . By [4, Example 5.13],  $S/H$  has a cyclic right module  $M$  which is directly infinite.

Then  $M \cong M \oplus A$  for some nonzero right  $(S/H)$ -module  $A$ , and consequently

$$M \otimes_S R \cong (M \otimes_S R) \oplus (A \otimes_S R).$$

Since  $S/H$  is regular,  $A$  is flat as an  $(S/H)$ -module, and then since  $(S/H)_S$  is flat,  $A$  is flat as an  $S$ -module. Hence, the natural map  $A \otimes_S S \rightarrow A \otimes_S R$  is injective, and so  $A \otimes_S R \neq 0$ . But then  $M \otimes_S R$  is a directly infinite cyclic right  $R$ -module, contradicting (e). Therefore  $R$  has bounded index.

In case the ring  $R$  in Theorem 6 is either  $N^*$ -complete or right or left  $\aleph_0$ -continuous, the given conditions are also equivalent to the corresponding left module versions of conditions (c), (d), (e) (because conditions (a), (b) are left-right symmetric). We do not know whether Theorem 6 holds for right  $\aleph_0$ -injective regular rings.

### Semihereditary Polynomial Rings

By applying Lemmas 3 and 4 we will obtain a result on semihereditary polynomial rings. First we need a relatively well-known lemma.

**Lemma 7.** *Let  $S$  be a regular subring of a ring  $R$ . If  $R[x]$  is right semihereditary, then so is  $S[x]$ .*

Proof. Since  $S$  is regular,  ${}_S R$  is faithfully flat, and then [1, Lemma 3] shows that  ${}_{S[x]} R[x]$  is faithfully flat. Then  $S[x]$  is right coherent by [6, Corollary 2.1]. As  $S$  is regular,  $S[x]$  has weak global dimension 1, and therefore  $S[x]$  must be right semihereditary.

Recall that a ring  $R$  is *strongly  $\pi$ -regular* if for each element  $a \in R$  there is a positive integer  $n$  such that  $a^n R = a^{n+1} R$ . That this condition is left-right symmetric was proved by Dischinger [2, Théorème 1].

**Theorem 8.** *Let  $R$  be a regular ring which is either  $N^*$ -complete or right or left  $\aleph_0$ -continuous. Then the following conditions are equivalent:*

- (a)  $R[x]$  is right semihereditary.
- (b)  $R[x]$  is left semihereditary.
- (c)  $R$  has bounded index of nilpotence
- (d)  $R$  is strongly  $\pi$ -regular.

Proof. (c) $\Rightarrow$ (a) and (b) by [4, Corollary 7.10] and [5, Corollaire].

(a) or (b) $\Rightarrow$ (c): If  $R$  has index  $\infty$ , then by Lemma 3 or 4  $R$  has a subring  $S \cong T \times \prod_{n=1}^{\infty} M_n(F_n)$  for some regular ring  $T$  and some fields  $F_1, F_2, \dots$ . Then [5, Proposition 11] implies that  $S[x]$  is neither right nor left semihereditary. But in view of Lemma 7 this contradicts (a) and (b). Therefore  $R$  must have bounded index.

(c) $\Rightarrow$ (d) by [4, Theorem 7.15].

(d) $\Rightarrow$ (c): If  $R$  has index  $\infty$ , then by Lemma 3 or 4,  $R$  has a subring  $S \cong T \times \prod_{n=1}^{\infty} M_n(F_n)$  as before. Choose matrices  $a_n \in M_n(F_n)$  such that  $a_n$  is nilpotent of index  $n$ . Then  $(0, a_1, a_2, \dots)$  corresponds to an element  $a \in S$  such that  $l_S(a^n) \neq l_S(a^{n+1})$  for all  $n=1, 2, \dots$ . But then  $l_R(a^n) \neq l_R(a^{n+1})$  and so  $a^n R \neq a^{n+1} R$  for all  $n$ , contradicting (d). Therefore  $R$  has bounded index.

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