

Title	Cancellation of finitely generated modules over regular rings
Author(s)	Goodearl, K. R.; Moncasi, J.
Citation	Osaka Journal of Mathematics. 26(3) P.679-P.685
Issue Date	1989
Text Version	publisher
URL	https://doi.org/10.18910/5514
DOI	10.18910/5514
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CANCELLATION OF FINITELY GENERATED MODULES OVER REGULAR RINGS

K.R. GOODEARL AND J. MONCASI

(Received August 12, 1988)

Introduction

In [3, Theorem 2] Evans proved that an R -module M cancels from direct sums if the stable range of the endomorphism ring $\text{End}_R(M)$ is 1. By using this result it follows that finitely generated projective modules over unit-regular rings cancel from direct sums [4, Corollary 4.7 and Proposition 4.12]. However, as the first author has shown [4, Example 5.13], there exists a unit-regular ring R with a cyclic module M which is directly infinite, that is, $M \cong M \oplus A$ for some nonzero R -module A , and so in particular M cannot be cancelled. Later, Menal [7, Theorem D] proved that if R is a regular ring whose primitive factor rings are all artinian, then the stable range of $\text{End}_R(M)$ is 1 for every finitely generated R -module M and so again Evans' theorem implies cancellation for M . However, the converse of Menal's result is not true for arbitrary regular rings. For instance, it suffices to take any regular locally finite-dimensional algebra over a field whose primitive factor rings are not artinian. For more information on this subject we refer the reader to [8].

In this note we will show that if R is a regular ring which is N^* -complete, or right or left \mathfrak{N}_0 -continuous, or left \mathfrak{N}_0 -injective, then every finitely generated right R -module cancels from direct sums if and only if R has bounded index of nilpotence (and so all primitive factor rings of R are artinian). As an application we also obtain a characterization of those polynomial rings $R[x]$ that are semihereditary in the case that R is either N^* -complete or right or left \mathfrak{N}_0 -continuous.

Cancellation

All rings considered in this paper are associative with 1. A ring R is (*von Neumann*) *regular* if for any $x \in R$ there exists a $y \in R$ such that $x = xyx$. If we can always choose y to be a unit, then R is called *unit-regular*. If X is a subset

of R , then we denote by $l(X)$ and $r(X)$ its left and right annihilators. By a *subring* we always mean a *unital* subring.

The example mentioned above, of a unit-regular ring which has a directly infinite cyclic module, is just $\prod_{n=1}^{\infty} M_n(F_n)$, where F_1, F_2, \dots are fields. Our method is to look for subrings of this form inside regular rings of index ∞ , using N^* -completeness, or \aleph_0 -continuity, or \aleph_0 -injectivity to build a complete direct product.

Lemma 1. *If R is a nonzero regular ring, then there is a nonzero central idempotent $e \in R$ such that eR is an algebra over a field.*

Proof. If R is not a \mathbf{Q} -algebra, then there is a prime p such that $pR \neq R$. Since p is central, $l(pR) = eR$ for some nonzero central idempotent $e \in R$. Now eR is an algebra over $\mathbf{Z}/p\mathbf{Z}$.

Recall that the *index of nilpotence* of an ideal I in a ring R is the supremum of the indices of all nilpotent elements in I . If this supremum is finite, we say I has *bounded index (of nilpotence)*, while if this supremum is infinite we say I has *index ∞* . The next lemma is a key result in this paper.

Lemma 2. *Let R be a regular ring and let I be a two-sided ideal of R of index ∞ . Then there exist nonzero orthogonal idempotents $e_1, e_2, \dots \in I$ such that $e_n R e_n$ has a subring isomorphic to the $n \times n$ matrix algebra over a field.*

Proof. We first observe that it suffices to prove (1): There exist independent right ideals A_1, A_2, \dots contained in I such that A_n contains a direct sum of n nonzero pairwise isomorphic right ideals. For suppose (1) holds. Each A_n contains a right ideal B_n which is a direct sum of n nonzero pairwise isomorphic principal right ideals, and then B_n is principal. By [4, Proposition 2.13] there exist orthogonal idempotents f_1, f_2, \dots such that

$$f_1 R \oplus \dots \oplus f_n R = B_1 \oplus \dots \oplus B_n$$

for all n . Then $f_n \in I$ and $f_n R \cong B_n$ for each n , so $f_n R$ is a direct sum of n nonzero pairwise isomorphic right ideals, whence $f_n R f_n$ is isomorphic to an $n \times n$ matrix ring. By Lemma 1, there is a nonzero central idempotent e_n in the ring $f_n R f_n$ such that $e_n R e_n = e_n f_n R f_n$ is an algebra over a field. Then $e_n \in I$ and $e_n R e_n$ has a subring isomorphic to the $n \times n$ matrix algebra over a field. Since the f 's are orthogonal, so are the e 's.

For each n , let J_n be the sum of all ideals contained in I of index at most n . Then $J_1 \subseteq J_2 \subseteq \dots$, and by [4, Corollary 7.8] each J_n has index at most n . Set $B_n = J_{n+1} \cap l(J_n)$ for each n . Observe that $(B_n \cap J_n)^2 \subseteq B_n J_n = 0$, and so $B_n \cap J_n = 0$. Hence, the B 's are independent. When $B_n \neq 0$, then since $B_n \subseteq J_{n+1}$ and $B_n \cap J_n = 0$ we see that B_n has index $n+1$. In this case, [4, Theorem 7.2] implies

that B_n contains a direct sum of $n+1$ nonzero pairwise isomorphic right ideals. If infinitely many B 's are nonzero, say $B_{n(1)}, B_{n(2)}, \dots$ where $n(1) < n(2) < \dots$, then each $B_{n(k)}$ contains a direct sum of k nonzero pairwise isomorphic right ideals, and (1) is proved. Thus we may assume that only finitely many B 's are nonzero, and so for some $t \in \mathbf{N}$ we have $B_n = 0$ for all $n \geq t$. Then for $n \geq t$ we have $J_{n+1} \cap l(J_n) = 0$, and it follows that $J_n \leq_e J_{n+1}$ as right ideals of R . From [4, Corollary 7.5] the index of J_{n+1} is now at most n , whence $J_{n+1} = J_n$. Thus $J_n = J_t$ for $n \geq t$.

Set $A = l(J_t) \cap I$, and observe that $A \cap J_t = 0$. Since I has index ∞ , [4, Corollary 7.5] shows that J_t is not essential in I_R , and so $A \neq 0$. Now $A \cap J_n = 0$ for all n . Thus A and all nonzero ideals contained in A have index ∞ . If A contains an independent sequence of nonzero ideals A_1, A_2, \dots , then by [4, Theorem 7.2] each A_n contains a direct sum of n nonzero pairwise isomorphic right ideals and (1) is proved. Thus we may assume that A does not contain an infinite sequence of independent nonzero ideals.

Now A must contain a nonzero ideal B such that any two nonzero ideals contained in B have nonzero intersection. It follows that any two nonzero right ideals $K, L \subseteq B$ must contain nonzero isomorphic right ideals. For taking nonzero elements $x \in K$ and $y \in L$, we have $RxR \cap RyR \neq 0$, whence $xay \neq 0$ for some $a \in R$, and $xayR$ is isomorphic to a right ideal contained in yR . By induction, it follows that whenever B_1, \dots, B_k are nonzero right ideals contained in B , there exist nonzero right ideals $C_i \subseteq B_i$ for $i=1, \dots, k$ such that $C_i \cong C_j$ for all i, j .

As $B \neq 0$, it has infinite index, so it cannot be artinian. Thus B contains an infinite direct sum of nonzero right ideals. Grouping finitely many of these together at a time, we obtain nonzero independent right ideals $A_1, A_2, \dots \subseteq B$ such that each A_n is a direct sum of n nonzero right ideals. Invoking the result of the previous paragraph, we conclude that each A_n contains a direct sum of n nonzero pairwise isomorphic right ideals. Therefore (1) holds in this case too.

Recall that a *pseudo-rank function* on a regular ring R is a map $P: R \rightarrow [0, 1]$ such that

- (a) $P(1) = 1$;
- (b) $P(xy) \leq P(x), P(y)$ for all $x, y \in R$;
- (c) $P(e+f) = P(e) + P(f)$ for all orthogonal idempotents $e, f \in R$.

Denote by $\mathbf{P}(R)$ the set of all pseudo-rank functions on R , and let

$$N^*(x) = \sup \{P(x) : P \in \mathbf{P}(R)\}$$

for all $x \in R$. Then the rule $\delta(x, y) = N^*(x-y)$ defines a pseudo-metric on R ; we say R is *N^* -complete* if δ is a metric and R is complete with respect to it.

Lemma 3. *Let R be an N^* -complete regular ring of index ∞ . Then R*

has a subring isomorphic to $T \times \prod_{n=1}^{\infty} M_n(F_n)$ for some regular ring T and some fields F_1, F_2, \dots .

Proof. From the idempotents given in Lemma 2 we can choose non-zero orthogonal idempotents $f_1, f_2, \dots \in R$ such that each $f_n R f_n$ has a subring isomorphic to the $n2^n \times n2^n$ matrix algebra over a field F_n . Then f_n is a sum of orthogonal idempotents g_{ni} (for $i=1, \dots, n2^n$) such that $g_{ni} R \cong g_{nj} R$ for all i, j and F_n is isomorphic to a subring of $g_{n1} R g_{n1}$. Since $n2^n(g_{ni} R) \cong f_n R \subseteq R$, we have $N^*(g_{ni}) \leq 1/n2^n$. Set $h_n = g_{n1} + \dots + g_{nn}$, and observe that

$$N^*(h_n) \leq N^*(g_{n1}) + \dots + N^*(g_{nn}) \leq n/n2^n = 1/2^n.$$

Also $h_n R h_n \cong M_n(g_{n1} R g_{n1})$, and so $M_n(F_n)$ is isomorphic to a subring of $h_n R h_n$.

Given any sequence $x = (x_n) \in \prod_{n=1}^{\infty} h_n R h_n$, we have $N^*(x_n) \leq N^*(h_n) \leq 1/2^n$ for all n , so the partial sums of $\sum x_n$ are Cauchy with respect to N^* . Hence $\sum x_n$ converges to some $\phi(x) \in R$. In particular $\sum h_n$ converges to an idempotent $h \in R$. Then ϕ gives a ring isomorphism of $\prod_{n=1}^{\infty} h_n R h_n$ onto a subring of $h R h$, and so $\prod_{n=1}^{\infty} M_n(F_n)$ is isomorphic to a subring of $h R h$.

Taking $T = (1-h)R(1-h)$, the proof is complete.

We say that a regular ring R is *right \aleph_0 -continuous* if the lattice of principal right ideals $L(R_R)$ is *upper \aleph_0 -continuous*, that is, every countable subset of $L(R_R)$ has a supremum in $L(R_R)$ and $A \wedge (\bigvee B_n) = \bigvee (A \wedge B_n)$ for all A and all countable linearly ordered subsets $\{B_n\}$ in $L(R_R)$. For example, any right self-injective regular ring is right \aleph_0 -continuous [4, Corollary 13.5].

Recall that a ring R is called *right (left) \aleph_0 -injective* provided every homomorphism from a countably generated right (left) ideal into R is given by left (right) multiplication by an element of R .

Lemma 4. *Let R be a regular ring of index ∞ which is either right or left \aleph_0 -continuous. Then R has a subring isomorphic to $T \times \prod_{n=1}^{\infty} M_n(F_n)$ for some regular ring T and some fields F_1, F_2, \dots .*

Proof. By symmetry, we may assume that R is right \aleph_0 -continuous. By Lemma 2, there exist nonzero orthogonal idempotents $e_1, e_2, \dots \in R$ such that each $e_n R e_n$ has a subring isomorphic to the $n \times n$ matrix algebra over a field F_n . By [4, Corollary 14.4] there is an idempotent $e \in R$ such that $\bigoplus_{n=2}^{\infty} e_n R \leq_e e R$, and it suffices to show that $\prod_{n=2}^{\infty} M_n(F_n)$ is isomorphic to a subring of $e R e$ (since it is clear how to find a subring of $(1-e)R(1-e)$ isomorphic to $T \times F_1$).

Let S be the maximal right \aleph_0 -quotient ring of R (see [4, Chapter 14]), and note that $\bigoplus_{n=2}^{\infty} e_n S \leq_e e S$. Any sequence $x = (x_n) \in \prod_{n=2}^{\infty} e_n S e_n$ induces a homomorphism

$$\bigoplus_{n=2}^{\infty} e_n S \rightarrow \bigoplus_{n=2}^{\infty} e_n S \subseteq eS$$

which extends uniquely to a homomorphism $eS \rightarrow eS$ because S is right \aleph_0 -injective [4, Theorem 14.12], and this homomorphism is left multiplication by some unique element $\phi(x) \in eSe$. We observe that ϕ is a unital ring map from $\prod_{n=2}^{\infty} e_n Se_n$ into eSe , and that ϕ is injective.

Now eSe has a subring $S' \cong \prod_{n=2}^{\infty} M_n(F_n)$. Since S' is regular and right self-injective with no nonzero abelian central idempotents, S' is generated as a ring by its idempotents [4, Theorem 13.16]. But all idempotents of S lie in R [4, Theorem 14.12]. Therefore S' is a subring of eRe .

For \aleph_0 -injective regular rings, we have a weaker version of Lemmas 3 and 4, which is not left-right symmetric.

Lemma 5. *Let R be a left \aleph_0 -injective regular ring of index ∞ . Then R has a subring S with a two-sided ideal H such that $(S/H)_S$ is flat and $S/H \cong \prod_{n=1}^{\infty} M_n(F_n)$ for some fields F_1, F_2, \dots .*

Proof. By Lemma 2, there exist nonzero orthogonal idempotents $e_1, e_2, \dots \in R$ such that $e_n Re_n$ has a subring isomorphic to $M_n(F_n)$ for some field F_n . Let $J = \bigoplus_{n=1}^{\infty} Re_n$ and let $I(J)$ be the idealizer of J in R and observe that the right annihilator $r(J)$ is a two-sided ideal of $I(J)$.

Any sequence $x = (x_n) \in \prod_{n=1}^{\infty} e_n Re_n$ induces a homomorphism $J \rightarrow J \subseteq_R R$ which must be right multiplication by some $\phi(x) \in I(J)$, because R is left \aleph_0 -injective. Although $\phi(x)$ is not uniquely determined by x , it is unique modulo $r(J)$. Thus the rule $x \mapsto \phi(x) + r(J)$ defines a unital ring map from $\prod_{n=1}^{\infty} e_n Re_n$ into $I(J)/r(J)$, and this map is injective. Therefore $I(J)$ has a subring S such that $S \supseteq r(J)$ and $S/r(J) \cong \prod_{n=1}^{\infty} M_n(F_n)$.

Since $r(J)$ is a right ideal of R , it is a directed union of right ideals eR where e is an idempotent. For any such e , observe that $S = eR \oplus (1-e)S$, so that eR is a direct summand of S_S . Therefore $S/r(J)$ is a flat right S -module.

A module M is called *directly finite* provided M is not isomorphic to any proper direct summand of itself, that is, $M \not\cong M \oplus A$ for all nonzero modules A . If M is not directly finite then M is called *directly infinite*.

Parts of the following theorem are due to Evans [3] and Menal [7].

Theorem 6. *Let R be a regular ring which is N^* -complete, or right or left \aleph_0 -continuous, or left \aleph_0 -injective. Then the following conditions are equivalent :*

- (a) *R has bounded index of nilpotence.*
- (b) *All primitive factor rings of R are artinian.*
- (c) *The endomorphism ring of every finitely generated right R -module has stable range 1.*
- (d) *All finitely generated right R -modules cancel from direct sums.*
- (e) *All finitely generated right R -modules are directly finite.*

Proof. (a) \Rightarrow (b) is [4, Corollary 7.10]. (b) \Rightarrow (c) is by [7, Theorem D]. (c) \Rightarrow (d) is [3, Theorem 2]. (d) \Rightarrow (e) is clear.

Now we prove (e) \Rightarrow (a). Suppose that R has index ∞ . By Lemma 3, 4, or 5, R has a subring S with a two-sided ideal H such that $(S/H)_S$ is flat and $S/H \cong \prod_{n=1}^{\infty} M_n(F_n)$ for some fields F_1, F_2, \dots . By [4, Example 5.13], S/H has a cyclic right module M which is directly infinite.

Then $M \cong M \oplus A$ for some nonzero right (S/H) -module A , and consequently

$$M \otimes_S R \cong (M \otimes_S R) \oplus (A \otimes_S R).$$

Since S/H is regular, A is flat as an (S/H) -module, and then since $(S/H)_S$ is flat, A is flat as an S -module. Hence, the natural map $A \otimes_S R \rightarrow A \otimes_S R$ is injective, and so $A \otimes_S R \neq 0$. But then $M \otimes_S R$ is a directly infinite cyclic right R -module, contradicting (e). Therefore R has bounded index.

In case the ring R in Theorem 6 is either N^* -complete or right or left \mathfrak{N}_0 -continuous, the given conditions are also equivalent to the corresponding left module versions of conditions (c), (d), (e) (because conditions (a), (b) are left-right symmetric). We do not know whether Theorem 6 holds for right \mathfrak{N}_0 -injective regular rings.

Semihereditary Polynomial Rings

By applying Lemmas 3 and 4 we will obtain a result on semihereditary polynomial rings. First we need a relatively well-known lemma.

Lemma 7. *Let S be a regular subring of a ring R . If $R[x]$ is right semihereditary, then so is $S[x]$.*

Proof. Since S is regular, ${}_S R$ is faithfully flat, and then [1, Lemma 3] shows that ${}_{S[x]} R[x]$ is faithfully flat. Then $S[x]$ is right coherent by [6, Corollary 2.1]. As S is regular, $S[x]$ has weak global dimension 1, and therefore $S[x]$ must be right semihereditary.

Recall that a ring R is *strongly π -regular* if for each element $a \in R$ there is a positive integer n such that $a^n R = a^{n+1} R$. That this condition is left-right symmetric was proved by Dischinger [2, Théorème 1].

Theorem 8. *Let R be a regular ring which is either N^* -complete or right or left \mathfrak{N}_0 -continuous. Then the following conditions are equivalent:*

- (a) $R[x]$ is right semihereditary.
- (b) $R[x]$ is left semihereditary.
- (c) R has bounded index of nilpotence
- (d) R is strongly π -regular.

Proof. (c) \Rightarrow (a) and (b) by [4, Corollary 7.10] and [5, Corollaire].

(a) or (b) \Rightarrow (c): If R has index ∞ , then by Lemma 3 or 4 R has a subring $S \cong T \times \prod_{n=1}^{\infty} M_n(F_n)$ for some regular ring T and some fields F_1, F_2, \dots . Then [5, Proposition 11] implies that $S[x]$ is neither right nor left semihereditary. But in view of Lemma 7 this contradicts (a) and (b). Therefore R must have bounded index.

(c) \Rightarrow (d) by [4, Theorem 7.15].

(d) \Rightarrow (c): If R has index ∞ , then by Lemma 3 or 4, R has a subring $S \cong T \times \prod_{n=1}^{\infty} M_n(F_n)$ as before. Choose matrices $a_n \in M_n(F_n)$ such that a_n is nilpotent of index n . Then $(0, a_1, a_2, \dots)$ corresponds to an element $a \in S$ such that $l_S(a^n) \neq l_S(a^{n+1})$ for all $n=1, 2, \dots$. But then $l_R(a^n) \neq l_R(a^{n+1})$ and so $a^n R \neq a^{n+1} R$ for all n , contradicting (d). Therefore R has bounded index.

ACKNOWLEDGEMENT. The second author would like to express his thanks to Pere Menal for many conversations on the subject of this paper.

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K.R. Goodearl
 Department of Mathematics
 University of Utah
 Salt Lake City, Utah 84112
 U.S.A.

J. Moncasi
 Departament de Matemàtiques
 Universitat Autònoma de Barcelona
 Bellaterra, Barcelona
 Spain

