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ON RINGS WITH SELF-INJECTIVE DIMENSION $\leq 1$

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Let $R$ be a ring with an identity and, for a left $R$-module $R^M$, $\text{pd}(M)$ and $\text{id}(M)$ denote the projective and injective dimension of $R^M$, respectively. A (left and right) noether ring $R$ is called $n$-Gorenstein if $\text{id}(R^R) \leq n$ and $\text{id}(R_R) \leq n$ for $n \geq 0$, and Gorenstein means $n$-Gorenstein for some $n$. This is slightly different from the well known definition in the commutative case unless a ring is local (see Bass [5]) and, as a generalization to the non-commutative case, there is another one by Auslander [1]. However, when we want to consider many interesting properties about a quasi-Frobenius ring and an hereditary ring in more general situation, we cannot conclude yet which definition is best. So, in this paper, we follow the above definition of a Gorenstein ring and try to generalize some interesting properties for a quasi-Frobenius ring. On the other hand, for a 1-Gorenstein ring, a few papers have appeared, for instance, Jans [12], Bass [4] and recently Sumioka [18], Sato [17] and, for a Gorenstein ring with squarezero radical, Zaks [19].

As the typical examples of 1-Gorenstein rings which are neither hereditary nor quasi-Frobenius, we have

1) Gorenstein orders, especially the group ring $\mathbb{Z}[G]$ where $\mathbb{Z}$ the ring of rational integers, $G$ a finite group. (See Drozd-Kirichenko-Roiter [7], Roggenkamp [16] and Eilenberg-Nakayama [8].)

2) Triangular matrix rings over non-semisimple quasi-Frobenius rings. (See Sumioka [18] and Zaks [19].)

In §1, we shall show that for a 1-Gorenstein ring $R$, $E(R^R) \oplus E(R_R^R)/R$ is an injective cogenerator (Theorem 1) and as this corollary, an artin 1-Gorenstein ring which is $QF-1$ must be quasi-Frobenius (Corollary 3). This should compare with that for a quasi-Frobenius ring $R$, $R_R^R$ itself is an injective cogenerator. Next, as a generalization of "projectivity=injectivity" for modules over a quasi-Frobenius ring, we obtain that over a certain $n$-Gorenstein ring, finiteness of the projective dimension, projective dimension $\leq n$, finiteness of the injective dimension and injective dimension $\leq n$ for modules are all equivalent (Theorem 5).

In §2, first we attend to Nakayama's theorem [15] that a ring $R$ is uniserial if and only if any homomorphic image of $R$ is quasi-Frobenius, and replace
"quasi-Frobenius" with "1-Gorenstein." Then we have three classes of rings, i.e. a uniserial ring, an hereditary ring with square-zero radical and a quasi-Frobenius ring with square-zero radical (Theorem 10). Moreover, as an application, we can classify a semiprimary ring whose proper homomorphic images are artin 1-Gorenstein (Theorem 12) and generalize [11, Theorem 1]. Also, in prime noether case, it will be shown that a restricted Gorenstein ring in the sense of Zaks [20] is equivalent to a restricted uniserial ring under certain hypothesis which always holds for commutative rings (Proposition 11).

Finally, Kaplansky’s book [13] is suitable for looking at the recent development of commutative Gorenstein rings. In the present study about non-commutative Gorenstein rings, we should generalize the results described in [13] to the non-commutative case in appropriate form.

Notations. For a ring $R$ and an $R$-module $M$, we denote
- $n(R) =$ the number of non-isomorphic simple left $R$-modules,
- $\text{Rad } R =$ the radical of $R$,
- $\text{Soc}(M) =$ the left socle of $R$,
- $E(M) =$ the injective hull of $_RM$,
- $|M| =$ the composition length of $_RM$.

A noether (artin) ring stands for left and right noetherian (artinian) and an ideal means twosided. Further, we say a non-zero ideal twosided simple if it contains no non-trivial ideal.

1. An injective cogenerator over a Gorenstein ring

In this section, first we consider which module is an injective cogenerator over a 1-Gorenstein ring, and next show the equivalence of the finiteness of projective dimension and injective dimension for modules over an $n$-Gorenstein ring which has a cogenerator with projective dimension $\leq n$. These are well known for quasi-Frobenius rings, i.e. $n=0$.

**Theorem 1.** Let $R$ be a 1-Gorenstein ring, then $E(R) \oplus E(R)/R$ is an injective cogenerator.

**Proof.** It is enough to show that any simple left $R$-module is monomorphic to $E(R) \oplus E(R)/R$. Otherwise, and suppose a simple left module $S$ is not monomorphic to it, then

$$\text{Hom}_R(S, R) = 0 = \text{Ext}_R^1(S, R).$$

Now represent $S$ as

$$0 \to _RM \xrightarrow{i} _RR \to _RS \to 0$$

where $M$ is a maximal left ideal and $i$ is an inclusion map. If we denote $X^*=$
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Hom$_R(X, R)$ for an $R$-module $X$, we obtain an exact sequence:

$$S^*_k \rightarrow R^*_k \rightarrow M^*_k \rightarrow \operatorname{Ext}_k(S, R)$$

and so, by the assumption,

$$i^*: R^*_k \rightarrow M^*_k$$

with $i^*(r^*) = (m \rightarrow mr)$ for $r \in R, m \in M$

is an isomorphism. Hence

$$i^{**}: \_k M^{**} \rightarrow \_k R^{**} \simeq \_k R$$

with $i^{**}(f) = fi^*(1)$ for $f \in M^{**}$

is an isomorphism, too. On the other hand, by Jans [12],

$$\sigma: \_k M \rightarrow \_k M^{**}$$

with $\sigma(m) = (f \rightarrow f(m))$ for $m \in M, f \in M^*$

is also an isomorphism and therefore so is

$$i^{**} \sigma: \_k M \rightarrow \_k R.$$  

However $i^{**} \sigma$ is an inclusion which contradicts $M \neq R$.

Remark. In the theorem above, the assumption for $R$ noetherian is necessary. For instance, let $R=\prod_a K_a$ be a direct product of infinitely many fields $K_a$, then $R$ is self-injective but $\_k R$ is not a cogenerator.

Next, we shall examine when only $E(\_k R)$ or $E(\_k R)/R$ is an injective cogenerator. A ring $R$ is called a right $S$-ring if $E(\_k R)$ is a cogenerator and see Bass [3] or Morita [14] for details. In the latter case, we have the next result.

Corollary 2. Let $R$ be a 1-Gorenstein ring, then $E(\_k R)/R$ is a cogenerator if and only if $\operatorname{Soc}(\_k R)=0$.

Proof. “Only if”: Suppose a simple left module $S$ is monomorphic to $\_k R$, then from the exact sequence

$$0 \rightarrow \_k S \rightarrow \_k R \rightarrow \_k C \rightarrow 0,$$

we have an exact sequence

$$\operatorname{Ext}^1_k(R, R) \rightarrow \operatorname{Ext}^1_k(S, R) \rightarrow \operatorname{Ext}^2_k(C, R).$$

Here, $\operatorname{Ext}^1_k(R, R)=0$ and $\operatorname{Ext}^2_k(C, R)=0$ since $id(\_k R)\leq 1$, so $\operatorname{Ext}^1_k(S, R)=0$ which contradicts that $E(\_k R)/R$ is a cogenerator.

“If”: Since $E(\_k R) \oplus E(\_k R)/R$ is a cogenerator, for any simple left module $\_k S$, $S$ is either monomorphic to $E(\_k R)$ or $E(\_k R)/R$. However, from $\operatorname{Soc}(\_k R)=0$, $\_k S$ must be monomorphic to $E(\_k R)/R$.

As an example of a ring $R$ such that $E(\_k R)/R$ is a cogenerator, we obtain
the following: Let \( R \) be an indecomposable semiprime 1-Gorenstein ring, then \( E(\_R)/R \) is a cogenerator unless \( R \) is artinian. More concretely, \( R=\mathbb{Z}[G] \) is an example satisfying above assumption. Therefore Theorem 1 and Corollary 2 generalize Sato [17, Corollaries 3.3, 3.4 and Proposition 3.5].

As a second corollary of Theorem 1, we obtain a result about \( QF-1 \) rings. We recall a ring \( R \) is left \( QF-1 \) if every faithful \( R \)-module has the double centralizer property.

**Corollary 3.** Let \( R \) be an artin 1-Gorenstein ring. If \( R \) is its own maximal left quotient ring, \( R \) is quasi-Frobenius. Hence an artin 1-Gorenstein ring which is left \( QF-1 \) is quasi-Frobenius.

Proof. Since \( R \) is its own maximal left quotient ring, \( E(\_R)/R \) is monomorphic to a direct product of copies of \( E(\_R) \) and so \( E(\_R) \) is a cogenerator and, for any simple left module \( \_S \), we have an exact sequence:

\[
0 \to \_S \to \_R \to \_C \to 0,
\]

which induces \( \text{Ext}^1(\_S, R)=0 \) similarly to the proof of Corollary 2. Therefore \( \_R \) is injective, i.e. \( R \) is quasi-Frobenius.

If \( R \) is left \( QF-1 \), \( E(\_R) \) has the double centralizer property and hence \( R \) is its own maximal left quotient by Lambek’s result.

**Remark.** Now, we have a further investigation about \( QF-1 \) rings, that is, we consider hereditary \( QF-1 \) rings. We have the following: “A left non-singular left \( QF-1 \) ring is semisimple (artinian).” In fact, if \( R \) is left non-singular, its maximal left quotient ring \( Q \) is semiprimitive. Furthermore, if \( R \) is left \( QF-1 \), \( Q\cong R \) by Lambek’s result and hence \( R \) is semisimple by Camillo [6, Proposition 5].

As a consequence, for a ring \( R \) the following are equivalent:

1. \( R \) is left hereditary left \( QF-1 \),
2. \( R \) is right hereditary right \( QF-1 \),
3. \( R \) is semisimple (artinian).

To investigate the latter problem in the beginning of this section, we require the next lemma.

**Lemma 4.** For an exact sequence of modules over a ring \( R \):

\[
0 \to \_A \to \_B \to \_C \to 0,
\]

\[
(1) \quad \text{id}(\_A), \text{id}(\_B) \leq n \text{ implies } \text{id}(\_C) \leq n;
\]

\[
(2) \quad \text{pd}(\_B), \text{pd}(\_C) \leq n \text{ implies } \text{pd}(\_A) \leq n.
\]

Proof. (1) For any \( R \)-module \( \_X \), we have
\[ \text{Ext}^1_R(X, B) \to \text{Ext}^1_R(X, C) \to \text{Ext}^2_R(X, A) \] (exact).

Now, \( \text{Ext}^1_R(X, B) = \text{Ext}^2_R(X, A) \) by the assumption, so \( \text{Ext}^1_R(X, C) = 0 \), i.e. \( \text{id}(C) \leq n \).

(2) is dual to (1)

**Theorem 5.** Let \( R \) be an artin \( n \)-Gorenstein ring and suppose there exists a cogenerator \( _RW \) with \( \text{pd}(W) \leq n \). Then the following are equivalent for a left \( R \)-module \( _RM \):

1. \( \text{pd}(M) < \infty \),
2. \( \text{pd}(M) \leq n \),
3. \( \text{id}(M) < \infty \),
4. \( \text{id}(M) \leq n \).

Proof. (1)\(\to\)(2): Say \( \text{pd}(M) = m < \infty \), there is a left module \( _RX \) such that \( \text{Ext}^n_R(M, X) \neq 0 \). Represent \( X \) as

\[ 0 \to \_RK \to \_RF \to \_RX \to 0 \] (exact), \( _RF \) free

then this induces

\[ \text{Ext}^n_R(M, F) \to \text{Ext}^n_R(M, X) \to \text{Ext}^{n+1}_R(M, K) \] (exact).

Hence, \( \text{Ext}^{n+1}_R(M, R) = 0 \) implies \( \text{Ext}^n_R(M, F) \neq 0 \), from which we have \( \text{id}(F) \geq m \).

Now, \( \text{id}(F) = \text{id}(R) \leq n \) and hence \( \text{pd}(M) = m \leq n \).

(2)\(\to\)(3): Let

\[ 0 \to P_n \xrightarrow{f_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0 \]

be a projective resolution of \( M \) and \( K_i = \text{Ker}(f_i) \) \( 0 \leq i \leq n - 1 \), \( K_{n-1} = M \), then first in an exact sequence:

\[ 0 \to P_n \to P_{n-1} \to K_{n-2} \to 0, \]

\( \text{id}(P_n), \text{id}(P_{n-1}) \leq \text{id}(R) \leq n \) implies \( \text{id}(K_{n-2}) \leq n \) by Lemma 4 (1). For general \( i \), in an exact sequence:

\[ 0 \to K_i \to P_i \to K_{i-1} \to 0, \]

if \( \text{id}(K_i) \leq n \), then \( \text{id}(K_{i-1}) \leq n \) again by Lemma 4 (1). Therefore by the induction, \( \text{id}(M) = \text{id}(K_{n-1}) \leq n \).

(3)\(\to\)(4): Say \( \text{id}(M) = m < \infty \), then there is a left module \( _RX \) such that \( \text{Ext}^n_R(X, M) \neq 0 \). Let

\[ 0 \to _RX \to _RE \to _RC \to 0 \] with \( _RE \) injective

be an injective presentation of \( X \), then we have \( \text{Ext}^n_R(E, M) \neq 0 \) from an exact sequence;

\[ \text{Ext}^n_R(M, E) \to \text{Ext}^n_R(X, M) \to \text{Ext}^{n+1}_R(C, M) \]
and so \(pd(E) \geq m\). On the one hand, as \(E\) is isomorphic to a direct summand of a direct product \(\prod W\) of copies of \(\mathcal{R} W\), \(pd(E) \leq pd(\prod W) = pd(W) \leq n\) whence \(id(M) = m \leq n\).

(4) \(\rightarrow\) (1): Let

\[
0 \rightarrow M \xrightarrow{f_0} E_0 \xrightarrow{f_1} E_1 \rightarrow \cdots \rightarrow E_{n-1} \xrightarrow{f_n} E_n \rightarrow 0
\]

be an injective resolution of \(\mathcal{R} M\) and \(C_i = \text{Cok}(f_i)\) \(0 \leq i \leq n-1\), \(C_{n-1} = M\), then an exact sequence:

\[
0 \rightarrow C_{n-2} \rightarrow E_{n-1} \rightarrow E_n \rightarrow 0
\]

and \(pd(E_{n-1}), pd(E_n) \leq pd(W) \leq n\) imply \(pd(C_{n-2}) \leq n\) by Lemma 4 (2). By the same discussion as the proof (2) \(\rightarrow\) (3), we obtain \(pd(M) \leq n\).

As a corollary of Theorems 1 and 5 we have the following where we recall a ring \(R\) is left \(QF-3\) if \(E(R)\) is projective.

**Corollary 6.** Let \(R\) be a \(1\)-Gorenstein ring which is left \(QF-3\), then the following are equivalent for a left \(R\)-module \(M\):

(1) \(pd(M) < \infty\), (2) \(pd(M) \leq 1\), (3) \(id(M) < \infty\), (4) \(id(M) \leq 1\).

Proof. By Theorem 1, \(\mathcal{R} W = E(R) \oplus E(R)/R\) is a cogenerator with \(pd(W) \leq 1\) because

\[
0 \rightarrow E(R) \xrightarrow{j} E(R) \oplus E(R) \rightarrow \mathcal{R} W \rightarrow 0
\]

with \(j(x) = (0, x)\) for \(x \in R\) is a projective resolution of \(\mathcal{R} W\). Further, it is well known a noetherian left \(QF-3\) ring is artinian, so we may apply Theorem 5 in case \(n = 1\).

**Remark.** (1) For any \(n > 0\), there exists a non-quasi-Frobenius ring satisfying the hypothesis in Theorem 5. For instance, let \(R\) be a serial (=generalized uniserial) ring with admissible sequence: \(1, 2, \cdots, 2\) (2 are \(n\) times), then \(id(R) = id(R) = gl. \dim R = n\) and \(\mathcal{R} W = \prod_{i=0}^{n} E_i\) is an injective cogenerator with \(pd(W) = n\) where \(0 \rightarrow \mathcal{R} R \rightarrow E_0 \rightarrow E_1 \rightarrow \cdots \rightarrow E_n \rightarrow 0\) is the minimal injective resolution of \(\mathcal{R} R\). (See [10] for details of serial rings.)

More generally, an \(n\)-Gorenstein ring \(R\) with \(dom. \dim R \geq n\) has an injective cogenerator \(\mathcal{R} W = \prod_{i=0}^{n} E_i\) with \(pd(W) \leq n\) where \(0 \rightarrow \mathcal{R} R \rightarrow \{E_i; 0 \leq i \leq n\}\) is the minimal injective resolution.

(2) We may construct an \(n\)-Gorenstein ring \(R_n\) with \(gl. \dim R_n = \infty\) for any \(n \geq 0\) in the following way. Let \(R_0\) be a non-semisimple quasi-Frobenius ring, and for any \(n > 0\), \(R_n\) the triangular matrix ring over \(R_{n-1}\), i.e. \(R_n = (R_{n-1} 0\)

\[
\begin{pmatrix}
R_{n-1} & 0 \\
R_{n-2} & R_{n-1}
\end{pmatrix}
\]
2. Rings whose homomorphic images are Gorenstein

In [19, §2], Zaks showed that, for a semiprimary ring $R$ with square-zero radical, $\text{id}(\pi R) \leq 1$ if and only if $R$ is a direct product of a quasi-Frobenius ring and an hereditary ring, and hence $\text{id}(\pi R) \leq 1$ is equivalent to $\text{id}(R_\pi) \leq 1$. Such a decomposition theorem no longer holds unless the square of its radical is zero. For example, let $Q$ be a local quasi-Frobenius ring with $(\text{Rad } Q)^2 = 0$ and $R$ the triangular matrix ring over $Q$, then $R$ is artin 1-Gorenstein and indecomposable but is neither quasi-Frobenius nor hereditary.

Now, for a serial ring, we have a decomposition theorem as above.

**Proposition 7.** Let $R$ be a serial ring, then the following are equivalent:

1. $\text{id}(R_\pi) \leq 1$,
2. $\text{id}(\pi R) \leq 1$,
3. $R$ is a direct product of a quasi-Frobenius ring and a hereditary ring.

**Proof.** Without loss of generality, we may assume that $R$ is self-basic (twosided) indecomposable, and decompose $\pi R$ as $R = R_e_1 \oplus \cdots \oplus R_e_n$ such that $\{e_1, \ldots, e_n\}$ is the Kupisch series. If $R$ is not quasi-Frobenius, $Re_i$ is non-injective for some $i$ ($1 \leq i \leq n$) and then, from $|Re_{i+1}| \leq |Re_i| + 1$ for $1 \leq j < n$, we obtain that if $i < n$, $|Re_{i+1}| = |Re_i| + 1$, $Re_i$ is monomorphic to $Re_{i+1}$ and $E(Re_i) \simeq Re_j$ for some $j$ ($i < j \leq n$) by [10, 1.1]. Now, let the number $i$ be the smallest one with $Re_i$ non-injective and $Re_{i+1}$ injective. Here, we may suppose $i < n$ because, in case of $Ne_i = 0$, $Re_i$ is monomorphic to $Re_i$ and if $Ne_i = 0$, by permuting $\{1, \ldots, n\}$, it is possible for $Re_1$ to be non-injective and $Re_2$ injective. Therefore we have

$$E(Re_i) \simeq Re_{i+1} \text{ and } |Re_i| + 1 = |Re_{i+1}|.$$  

So, saying $N = \text{Rad } R$,

$$E(Re_i)/Re_i \simeq Re_{i+1}/Ne_{i+1}$$

is simple injective and from that $Re_{i+1}$ is epimorphic to $Ne_{i+2}$ if $i + 1 < n$,

$$Re_{i+1}/Ne_{i+1} \simeq Ne_{i+2}/N^2e_{i+2} \subseteq Re_{i+2}/N^2e_{i+2}$$

induces $Ne_{i+2} = 0$ since $Re_{i+2}/N^2e_{i+2}$ is indecomposable. This contradicts $|Re_j| \geq 2$ for $j = 2, \ldots, n$ and so $i + 1 = n$ and $|Re_{i+1}| = |Re_i| + 1$ for $1 \leq i \leq n$. Hence $Re_{i} \simeq Ne_{i+1}$ for $1 \leq i \leq n - 1$, i.e. $Ne_i$ ($i = 2, \ldots, n$) are projective and $R$ is hereditary.

Applying this proposition we classify the rings all of which homomorphic images are artin 1-Gorenstein. Before proceeding, we need two lemmas.

**Lemma 8** (Bass [3]). For a right perfect, right $S$-ring $R$, $\text{id}(\pi R)$ is finite if and only if $\pi R$ is injective.
Proof. Say, \( id(R) = n < \infty \), then there exists a simple left module \( _RS \) with \( \text{Ext}^n_R(S, R) \neq 0 \). Now, since \( R \) is a right \( S \)-ring, we have an exact sequence:

\[
0 \rightarrow _RS \rightarrow _SR \rightarrow _SC \rightarrow 0
\]

which induces

\[
\text{Ext}^n_R(R, R) \rightarrow \text{Ext}^n_R(S, R) \rightarrow \text{Ext}^{n+1}_R(C, R) \quad \text{(exact)}.
\]

Here, \( \text{Ext}^{n+1}_R(C, R) = 0 \) from \( id(_SR) = n \), so \( \text{Ext}^n_R(R, R) = 0 \) and \( n = 0 \), i.e. \( _SR \) is injective.

**Lemma 9.** Let \( I \) be a (twosided) ideal in any ring \( R \) and \( R/I^n \) a left hereditary ring for some \( n > 1 \). Then \( I^n = I^{n+1} \). Hence, if we assume \( R/N = \text{Rad} R \) is finitely generated (or nilpotent) and \( R/N^n \) is left hereditary for \( n > 1 \), then \( N^n = 0 \) and so \( R \) itself left hereditary.

**Proof.** Since \( I^{n-1}/I^n \) is an ideal in \( R/I^n \), it is \( R/I^n \)-projective and the exact sequence of \( R/I^n \)-modules:

\[
0 \rightarrow I^n/I^{n+1} \rightarrow I^{n-1}/I^{n+1} \rightarrow I^{n-1}/I^n \rightarrow 0
\]

splits, i.e.

\[
I^{n-1}/I^{n+1} \cong I^{n-1}/I^n \oplus I^n/I^{n+1}
\]

as \( R/I^n \)-modules. However, \( I \cdot (I^{n-1}/I^n \oplus I^n/I^{n+1}) = 0 \), so \( I \cdot (I^{n-1}/I^{n+1}) = 0 \), i.e. \( I^n = I^{n+1} \).

**Theorem 10.** For an indecomposable semiprimary ring \( R \), the following are equivalent:

1. For any homomorphic image \( T \) of \( R \), \( id(T) \leq 1 \),
2. For any homomorphic image \( T \) of \( R \), \( id(T_R) \leq 1 \),
3. \( R \) is one of the following:
   i. \( R \) is uniserial,
   ii. \( R \) is hereditary with \( (\text{Rad} R)^2 = 0 \),
   iii. \( R \) is quasi-Frobenius with \( (\text{Rad} R)^2 = 0 \) and \( n(R) = 2 \).

**Proof.** (3) is left-right symmetry, so we prove only the equivalence of (1) and (3).

(1) \( \rightarrow \) (3): Say, \( N = \text{Rad} R \), since \( R/N^2 \) is also indecomposable, \( R/N^2 \) is either hereditary or quasi-Frobenius by Zaks [19]. In case of hereditary, \( N^2 = 0 \) by Lemma 9 and hence \( R \) is of type (ii). In another case, \( R/N^2 \) is a serial ring, so \( R \) is artinian and serial, too whence \( R \) is either hereditary or quasi-Frobenius by Proposition 7. If \( R \) is hereditary, \( \text{gl.dim} R/N^2 < \infty \) by Eilenberg-Nagao-Nakayama [9, Theorem 8] and hence by Bass [4, Proposition 4.3], \( \text{gl.dim} R/N^2 = id(R/N^2 R/N^2) \leq 1 \), i.e. \( R/N^2 \) is hereditary, so \( N^2 = 0 \) and \( R \) is hereditary again by Lemma 9.
Thus, let \( R \) be serial quasi-Frobenius and \( n(R) = n(R/N^2) = n \). Further, \( \bar{R} = R/N^2 \) also satisfies (1) and since (1) is Morita-invariant, we may assume \( \bar{R} \) is self-basic and decompose \( \bar{R} \) as \( \bar{R} = \bar{R}e_1 \oplus \cdots \oplus \bar{R}e_n \) with \( \{e_1, \ldots, e_n\} \) Kupisch series. If \( n > 2 \), \( Je_1 = e_n Je_1 \) (\( J = \mathrm{Rad} \, \bar{R} \)) is an ideal of \( \bar{R} \) and the ring:

\[
T = \bar{R}/Je_1 = T\bar{e}_1 \oplus \cdots \oplus T\bar{e}_n \quad \text{where} \quad \bar{e}_i = e_i + Je_1 \in T
\]

satisfies \( \mathrm{id}(T) \leq 1 \). Hence, from \( Je_2 \cong Re_1/Je_1 \),

\[
E(T\bar{e}_1)/T\bar{e}_1 \cong T\bar{e}_2/J\bar{e}_2 \quad (J = \mathrm{Rad} \, T)
\]

is \( T \)-injective. However, \( e_2 \bar{e}_2 \neq 0 \), i.e. \( T\bar{e}_2/J\bar{e}_2 \cong T\bar{e}_2 \supseteq T\bar{e}_2 \) which contradicts the indecomposability of \( T\bar{e}_2 \), so \( n \leq 2 \). Then, since \( R \) is uniserial if \( n = 1 \), let \( n = 2 \), i.e. we may represent \( R = Re_1 \oplus Re_2 \) with \( \{e_1, e_2\} \) Kupisch series because \( R \) is self-basic, too. Furthermore, if \( N^2 \neq 0 \), then \( N^2 e_1 \) and \( N^2 e_2 \neq 0 \) as \( R \) is quasi-Frobenius and the homomorphic image \( T = R/(N^3 e \oplus N^2 e) = T\bar{e}_1 \oplus T\bar{e}_2 \) where \( \bar{e}_i = e_i + (N^3 e_1 \oplus N^2 e) \in T \) satisfies \( \mathrm{id}(T) \leq 1 \). Now, from \( E(T\bar{e}_2) \cong T\bar{e}_1 \),

\[
E(T\bar{e}_2)/T\bar{e}_2 \cong T\bar{e}_1/J\bar{e}_1 \quad (J = \mathrm{Rad} \, T)
\]

is \( T \)-injective. However,

\[
J^2 \bar{e}_1 = N^2 e_1 \mid N^3 e_1 \cong Re_1/Ne_1 \cong T\bar{e}_1/J\bar{e}_1
\]

is \( T \)-injective which contradicts that \( \tau T\bar{e}_1 \) is indecomposable. Hence \( N^2 = 0 \).

(3) \( \rightarrow \) (1): In any case of (i) — (iii), \( R \) may be assumed self-basic. It is well known that a uniserial ring is characterized as a ring all of which homomorphic images are quasi-Frobenius.

Let \( R \) be of type (ii). For any ideal \( I \) contained in \( N \), since \( R\) is a direct summand of \( R\)N, \( R/I \) is also hereditary by Eilenberg-Nagao-Nakayama [9, Proposition 9]. If \( I \) is not contained in \( N \), \( I \) contains a primitive idempotent \( e \) with \( I = Re_1 \oplus (I \cap R(1-e)) \) and further, if \( I \cap R(1-e) \subseteq N \), choose a primitive idempotent \( e_2 \) orthogonal to \( e_1 \) in \( I \cap R(1-e) \). By repeating this method, we have

\[
I = Re_1 \oplus \cdots \oplus Re_n \oplus I'
\]

where \( e_i^2 = e_i \) is primitive and \( I' = I \cap R(1 - \sum_{i=1}^n e_i) \subseteq N \). Then, let \( e = 1 - (e_1 + \cdots + e_n) \), from \( I' \), \( eR(1-e) \subseteq N \),

\[
I'R = I'eR + I'eR(1-e) \subseteq I \cap Re = I',
\]

i.e. \( I' \) is an ideal. Hence \( T' = R/I' \) is an hereditary ring with

\[
\tau \mathrm{Rad} \, T' = N/I' \cong \tau \mathrm{Ne} \oplus \tau N(1-e)/I'
\]

and so \( N(1-e)/I' \) is \( T' \)-projective. On the other hand,
implies $\text{Rad } T = N(1-e)I'$ and, as $T'$ is epimorphic to $T$, $N(1-e)I'$ is $T$-projective, i.e. $T$ is hereditary.

Let $R$ be of type (iii) and $R = R\mathfrak{e}_1 \oplus R\mathfrak{e}_2$ where $\{\mathfrak{e}_1, \mathfrak{e}_2\}$ Kupisch series. For any ideal $I$ contained in $N$, $I$ is a direct summand and, as $N = N\mathfrak{e}_1 \oplus N\mathfrak{e}_2$ with $N\mathfrak{e}_i$ simple, $\mathfrak{r}I$ is isomorphic to $N\mathfrak{e}_i$ or $N\mathfrak{e}_2$ provided $I \neq 0$, $N$. If $I \cong N\mathfrak{e}_i$,

$$gI = gN\mathfrak{e}_1 = e_2N\mathfrak{e}_1 = e_2I$$

implies $I = e_2I$ and so, saying $N = I \oplus K$,

$$e_2I \oplus e_2K = e_2(I \oplus K) = e_2N = e_2N\mathfrak{e}_1.$$

Hence

$$I = e_2I = e_2N\mathfrak{e}_1 = e_2N = N\mathfrak{e}_1$$

and

$$\tau T = R/I \cong \tau R\mathfrak{e}_1/N\mathfrak{e}_1 \oplus \tau R\mathfrak{e}_2$$

which induces $\tau R\mathfrak{e}_1/N\mathfrak{e}_1$ projective. Now, let $J = \text{Rad } T$,

$$\tau R\mathfrak{e}_1/N\mathfrak{e}_1 \cong \tau N\mathfrak{e}_1 \cong \tau J(e_2 + I) = J,$$

so $\tau J$ is projective and $T$ is hereditary. In case of $I \cong N\mathfrak{e}_2$, we have the same discussion. Next, let $e_1 \in I$, then

$$2 = |R\mathfrak{e}_1| \leq |gI| \leq |gR| = 4.$$  

However, $|gI| = 2$ implies $I = R\mathfrak{e}_1$ and $N\mathfrak{e}_2 \subseteq R\mathfrak{e}_2 R \subseteq R\mathfrak{e}_1$ which is a contradiction. Therefore, we may take $|gI| = 3$ and then $|gR/I| = 1$, i.e. $R/I$ is a division ring. This completes the proof.

Finally, we investigate a ring whose proper homomorphic images are artin 1-Gorenstein, and here consider in two cases of a prime noether ring and a semiprimary ring.

For a prime noether case, we have a generalization of Zaks [20, Theorem 3]. Here an ideal $I$ is said to have the Artin-Rees property if for every left ideal $L$, there is an $n$ with $n \mathfrak{e}_1 \cap L \subseteq nL$.

**Proposition 11.** Let $R$ be a prime noether ring and assume every maximal ideal in $R$ has the Artin-Rees property. Then any proper homomorphic image of $R$ is artin Gorenstein if and only if $R$ is restricted uniserial.

Proof. "Only if": For any maximal ideal $M$ in $R$, $M = 0$ implies $R$ a simple ring, so we may suppose $M \neq 0$. Then $R/M^2$ is primary Gorenstein and hence quasi-Frobenius (in this case, uniserial) by Lemma 8. Thus let
Next, for any nonzero ideal $I$ in $R$, there exist maximal ideals $M_1, \cdots, M_n$ in $R$ with $M_1, \cdots, M_n \subseteq I$. Since $M_1, \cdots, M_n$ have the Artin-Rees property, there are integers $k_1, \cdots, k_n$ such that

$$M_1^{k_1} \cap \cdots \cap M_n^{k_n} \subseteq M_1 \cdots M_n \subseteq I.$$ 

Hence, we may suppose all $M_1, \cdots, M_n$ are distinct and, by the Chinese Remainder Theorem,

$$R/(M_1^{k_1} \cap \cdots \cap M_n^{k_n}) \cong R/M_1^{k_1} \oplus \cdots \oplus R/M_n^{k_n}$$

is uniserial. On the other hand, $R/(M_1^{k_1} \cap \cdots \cap M_n^{k_n})$ is epimorphic to $R/I$, so $R/I$ is uniserial too.

Now, we state the last theorem which is of a semiprimary case.

**Theorem 12.** Let $R$ be an indecomposable semiprimary ring and $R_0$ the basic subring of $R$ with $N=\text{Rad } R_0$. Then any proper homomorphic image of $R$ is 1-Gorenstein if and only if $R$ is one of the following:

1. $R$ is uniserial;
2. $R$ is serial with admissible sequence 3, 2;
3. $R$ is hereditary with square-zero radical;
4. $n(R) \leq 2$, $(\text{Rad } R)^2 = 0$ and for any primitive idempotent $e$ in $R_0$, (a) $eNe=0$ provided $e \neq 1$, (b) If $Ne$ contains a nonzero ideal properly, it is a maximal left and right subideal in $Ne$ and $N(1-e)$ is a simple left and right ideal of $R_0$;
5. $n(R)=2$, $(\text{Rad } R)^2 = 0$ and $R_0$ has a primitive idempotent $e$ such that (a) $eNe$ is simple left and right ideal of $R_0$, (b) Either $(1-e)Ne=0$ or $N(1-e)=0$, (c) Each of $(1-e)Ne$ and $N(1-e)$ is two-sided simple unless it is zero and $N(1-e)=eN(1-e)$;
6. $R$ is triangular with $n(R)=3$, $(\text{Rad } R)^3 = 0$ and $Ne$ is two-sided simple for a primitive idempotent $e$ in $R_0$ provided $Ne \neq 0$.

Proof. Throughout the proof, we may assume $R$ self-basic and then $N=\text{Rad } R$.

"Only if." If $N^3 \neq 0$, $R/N^3$ is uniserial by Theorem 10 and so is $R$ by [15].

Let $N^3=0$ but $N^2 \neq 0$, then $R/N^2$ is quasi-Frobenius with $n(R/N^3)=2$ again by Theorem 10 and Lemma 9 and hence $R$ is serial with $n(R)=2$. Thus, let $\{e_1, e_2\}$ be a Kupisch series, then $Ne_1 \neq 0$. For, $Ne_1=0$ implies $N^2=0$ (contradiction) because $Re_1$ is epimorphic to $N_2$. So $Ne_1 \neq 0$ and $Re_2$ is epimorphic to $Ne_2$. If both $N^2e_1$ and $N^2e_2$ are nonzero, $R/N^2e_1$ is neither hereditary since $Ne_1/N^2e_1$ is not projective nor quasi-Frobenius since $R/N^2e_1$ has non-constant admissible sequence 2, 3. Therefore
In either case, \( R \) has the admissible sequence 2, 3; i.e. \( R \) is of type (2).

In the following, we may assume \( N^2=0, Ne=0 \) and \( R \) not hereditary because otherwise \( R \) is of type (3). Here, we remark that for a semiprimary ring \( R \) with square-zero radical \( N, R \) is hereditary if and only if any primitive idempotent \( e \) in \( R \) satisfies either \( eN=0 \) or \( Ne=0 \). Now, if \( n(R)=1 \), i.e. \( R \) is local and \( N \) contains a nonzero ideal \( I \), \( R/I \) must be quasi-Frobenius. Hence \( N|I, N/I \) are simple and \( R \) is of type (4).

Therefore, now suppose \( n(R)=2 \), then there exists a primitive idempotent \( e \) with \( eN \neq 0 \) and \( 1-e \) is primitive too. In case of \( eNe \neq 0 \), \( I=(1-e)Ne \) is indecomposable and \( R/I \cong eRe \oplus (1-e)R(1-e) \) as rings implies that \( eRe \) is quasi-Frobenius, so \( eNe, eNe_1 \) are simple. Next, if both \( (1-e)Ne \) and \( N(1-e) \) were nonzero, \( R/N(1-e) \) is indecomposable but neither hereditary nor quasi-Frobenius. Hence either \( (1-e)Ne=0 \) or \( N(1-e)=0 \) and each of them is twosided simple unless it is zero. Further, \( N(1-e)=eN(1-e) \) because \( R \) is indecomposable. These show that \( R \) is of type (5) in case of \( eNe\neq 0 \). So we assume \( eNe=0 \), in which case \( eN(1-e) \neq 0 \) as \( e \) was chosen with \( eN \neq 0 \). Then \( R/eN(1-e) \) must be hereditary and \( (1-e)N(1-e)=0 \). Here, if \( Ne \) contains a nonzero ideal \( I \), \( R/I \) has to be quasi-Frobenius whence both \( N(1-e)=eN(1-e) \) and \( Ne/I \) are simple. These also hold for a right side. On the one hand, if \( N(1-e) \) contains a proper nonzero ideal \( I \), by exchanging the idempotent \( e \) with \( 1-e \), the same argument as above holds. Hence \( R \) becomes of type (4).

Finally, suppose \( n(R)\geq 3 \). As \( N \) is not projective, there are primitive idempotents \( e, f \) with \( eNe \neq 0 \) and \( Nf \neq 0 \). Now, assume \( (1-e)Ne=0 \), then \( eNe \) is a nonzero ideal, \( n(R/eNe)=n(R)\geq 3 \) and \( R/eNe \) is indecomposable, so \( R/eNe \) must be hereditary by Theorem 10. Therefore there exists a primitive idempotent \( e' \neq e \) with \( eNe' \neq 0 \) by an indecomposability of \( R \) and then \( I=(1-e)Ne'+N(1-e-e') \) is a nonzero ideal since \( R \) is indecomposable and \( n(R)\geq 3 \). If we put \( \bar{R}=R/I, \bar{e}=e+I \) and \( \bar{e}'=e'+I, \bar{R}e\oplus \bar{R}e' \) is a block of \( R \) and not any of the ring stated in Theorem 10 (contradiction). Thus \( (1-e)Ne\neq 0 \), i.e. \( f \neq e \) and, by setting \( e_1=e, e_2=f, R \) is expressible as \( R=Re_1 \oplus Re_2 \oplus \cdots \oplus Re_n \) where \( n=n(R)\geq 3, e_i, 1 \leq i \leq n \) are primitive idempotents and either \( e_3Ne_3 \neq 0 \) or \( e_2Ne_2 \neq 0 \). If an ideal \( I=(1-e_2)Ne_1+(1-e_1-e_3)Ne_3+(1-e_2)Ne_3+\sum_{j>3} Re_j \) is nonzero, then \( R/I \) must be hereditary by Theorem 10 as \( R/I \) is indecomposable and \( n(R/|I|=3 \), and so we obtain that \( Ne_1=e_2Ne_1+1e_1, e_1Ne_2=0=e_3Ne_3 \) and \( Ne_3=e_2Ne_3+1e_3 \). In this case \( R/\sum_{j>3} Ne_j \) has to be quasi-Frobenius, which contradicts \( e_3Ne_3 \neq 0 \). Hence \( I=0 \) implies \( n=3, Ne_1=e_3Ne_1 \neq 0, Ne_2=e_1Ne_2+e_3Ne_3 \neq 0 \) and \( Ne_3=e_2Ne_3 \). Moreover, if \( Ne_2 \neq 0, e_2Ne_2=0=e_3Ne_3 \) for \( R/Ne_2 \) or \( R/Ne_3 \) is indecomposable but neither hereditary nor quasi-Frobenius according

\[ N^2e_1=0, \ N^2e_2=0 \text{ or } N^2e_1=0, \ N^2e_2=0. \]
to \( e_1N_2 \neq 0 \) or \( e_2N_2 \neq 0 \), but it contradicts \( N_2 = 0 \). Therefore \( N_2 = 0 \) and \( e_2N_2 = 0 \) induces \( e_1N_2 = 0 \) since \( \text{gl. dim } R/e_1N_2 = 2 \), i.e. \( R \) is of type (6).

"If." Case (1): By Nakayama [15], \( R \) is uniserial if and only if any homomorphic image of \( R \) is quasi-Frobenius.

Case (2): Let \( R = Re_1 \oplus Re_2 \) where \( e_1 \), \( e_2 \) are primitive idempotents and \( |Re_1| = 3 \), \( |Re_2| = 2 \). Then, for any nonzero proper ideal \( I \) in \( R \\
0 \neq I \cap \text{Soc}_R(R) = I \cap (N^2e_1 \oplus Ne_2) = (I \cap N^2e_1) \oplus (I \cap Ne_2)
implies either \( I \cap N^2e_1 = 0 \) or \( I \cap Ne_2 = 0 \). In either case, we obtain \( N^2e_1 \subseteq I \).

Now, suppose \( N^2e_1 = I \), then \( R/I \) is quasi-Frobenius with the admissible sequence 2,2. Next, if \( N^2e_1 = I \), \( R/I \) is a proper homomorphic image of \( R/N^2e_1 \) and hence has the admissible sequence \{1, 2\}, \{1, 1\} or \{1\}. In all cases, \( R/I \) is hereditary.

Case (3): Any homomorphic image of \( R \) is hereditary by [9, Proposition 9].

Case (4): For any nonzero ideal \( I \) of \( R \), if \( I \subseteq N \), \( I = Ie \oplus I(1-e) \) with \( Ie \), \( I(1-e) \) ideals for a primitive idempotent \( e \) and \( R/I \cong \text{Re}e \oplus R(1-e)/I(1-e) \) is either hereditary or quasi-Frobenius by the property (b). If \( I \subseteq N \), \( I \) contains a primitive idempotent \( e \) and so \( R/I \) is isomorphic to \((1-e)R(1-e) \) or 0.

Case (5): For any nonzero ideal \( I \) of \( R \), if \( I \subseteq N \), \( I = eIe \oplus (1-e)Ie \oplus I(1-e) \) and these summands are all ideals. By the property (b), in case of \((1-e)Ne = 0 \), \( R/I \cong \text{Re}e \oplus R(1-e)/I(1-e) \) implies that \( R/I \) is hereditary or quasi-Frobenius according to \( eIe = 0 \) or \( I(1-e) = 0 \). In case of \( N(1-e) = 0 \), \( R/I \cong R/I \oplus R(1-e) \) shows that \( R/I \) is quasi-Frobenius (resp. hereditary) provided \((1-e)Ie = 0 \) (resp. \( eIe = 0 \)). Next, if \( I \) is not contained in \( N \), \( e \) or \( 1-e \) belongs to \( I \) and so \( I = Re \oplus (I \cap R(1-e)) \) or \( I = (I \cap Re) \oplus R(1-e) \) respectively. In the former case, we may assume \( I \cap R(1-e) \subseteq N \) and hence \( R/I \cong (1-e)R(1-e)/(1-e)N(1-e) \) is a division ring. Also, in the latter case, we have the same conclusion.

Case (6): \( R \) has a complete set \( e_1, e_2, e_3 \) of mutually orthogonal primitive idempotents satisfying \( e_iN_j = 0 \) if \( i \neq j \). Hence, for any nonzero ideal \( I \) of \( R \), if \( I \subseteq N \), \( I = Ie_1 \oplus Ie_2 \) with \( Ie_1 \), \( Ie_2 \) ideals and \( R/I \cong Re_1/Ie_1 \oplus Re_2/Ie_2 \) is hereditary since \( Ie_i = Ne_i \) or 0 (\( i = 1, 2 \)). If \( I \subseteq N \), some \( e_i \) for \( i = 1, 2, 3 \) is contained in \( I \) and we may show similarly that \( R/I \) is hereditary.

Remark. In [20], Zaks showed that, for a commutative noether ring \( R \), any (proper) homomorphic image of \( R \) is Gorenstein if and only if any (proper) homomorphic image of \( R \) is quasi-Frobenius. For a non-commutative case, however, we see it no longer holds by Theorems 10 and 12. In prime noether case (see Proposition 11), we don't know whether the hypothesis of the Artin-Ress property is superfluous or not.
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References


