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Osaka J. Math.
34 (1997), 803–820

KÄHLER C-SPACES AND k-SYMMETRIC SPACES

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(Received June 7, 1995)

0. Introduction

Let (M, J, g) be a compact, simply connected homogeneous Kählerian manifold (we call the space a *Kähler C-space*). In [10] we have proved that there is a positive integer n such that the n-th covariant derivative of $(1,0)$-type of the curvature tensor of (M, J, g) is identically zero (we call the least integer with above property the *degree* of (M, J, g)). It is clear that a compact Hermitian symmetric space is characterized as a Kähler C-space with degree one. Moreover we classified the spaces with degree n $(n \leq 3)$.

In this paper we shall prove explicitly that every Kähler C-space has a k-symmetric structure (see also Burstall and Rawnsley [1], p.52 and Pasiencier [9], Lemma 4.3). In [2] Gray showed that each Riemannian 3-symmetric space is a homogeneous almost Hermitian manifold with the canonical almost complex structure. He also proved that a Riemannian 3-symmetric space with the canonical almost complex structure is Kählerian if and only if it is a Hermitian symmetric space. In this paper we also show that the degree of a Kähler C-space equals three if and only if it is a compact Kähler manifold with a 3-symmetric structure which is not isometric to a Hermitian symmetric space (Theorem 2.4).

It is known that a Riemannian manifold (M, g) with a k-symmetric structure is homogeneous, that is, (M, g) has an expression $(M, g) = G/K$. For an irreducible Riemannian symmetric space the expression as a symmetric pair is unique as is well-known. In section 3 we shall show an analogous theorem on symmetric pair hold for a compact simply connected irreducible Riemannian 3-symmetric space which is not isometric to a Riemannian symmetric space (Theorem 3.6).

1. Preliminaries

In this section we recall notions and (some) properties of k-symmetric spaces $(k \in \mathbb{N})$ and Kähler C-spaces.

Let (M, g) be a Riemannian manifold. For $x \in M$, an isometry of (M, g) with an isolated fixed point x is called a *symmetry* of (M, g) at x. Assume that (M, g) admits at least one symmetry at each point, and let $\{s_x : x \in M\}$ be the set of symmetries. Then it is known that (M, g) is a Riemannian homogeneous space.

Moreover, if we denote by $\mathrm{Cl}(\{s_x\})$ the closure of the group generated by the set $\{s_x : x \in M\}$ in the isometry group $I(M, g)$ of (M, g), then $\mathrm{Cl}(\{s_x\})$ acts transitively on (M, g). (cf. Kowalski [7].)

Again, suppose that (M, g) admits a set $\{s_x : x \in M\}$ of symmetries. We call $\{s_x : x \in M\}$ a *Riemannian k-symmetric structure* on (M, g) if for $x, y \in M$

$$(1.1) \qquad \begin{aligned} &s_x \circ s_y = s_z \circ s_x, \quad (z = s_x(y)), \\ &(s_x)^k = \mathrm{id}, \quad (s_x)^l \neq \mathrm{id}, \quad (l < k). \end{aligned}$$

We note that $\{s_x : x \in M\}$ depends only on s_p for a fixed $p \in M$. Furthermore (M, g) with a Riemannian k-symmetric structure is said to be a *Riemannian k-symmetric space*.

Let (M, g) be a Riemannian homogeneous space, i.e., there exits a group G of isometries of (M, g) such that $M = G/H$ (H is a closed subgroup of G). Let $\pi : G \to G/H$ be the canonical projection and put $o = \pi(H)$. For an automorphism σ of G let G^σ be the fixed point set and $(G^\sigma)_0$ the identity component of G^σ, respectively. Then the following is known (cf. [7]).

Proposition 1.1. *Suppose that there exists an automorphism σ of G such that*
(i) $(G^\sigma)_0 \subset H \subset G^\sigma$,
(ii) $\sigma^k = 1$ *and* $\sigma^l \neq 1$ *for any* $l < k$,
(iii) *let s be the transformation of M defined by $\pi \circ \sigma = s \circ \pi$. Then s preserves the metric at o.*

Then $\{s_x = g \circ s \circ g^{-1} : x = g \cdot o \in M\}$ defines a Riemannian k-symmetric structure on (M, g).

Next, we construct Kähler C-spaces. (for example, see Itoh [5] and Matsushima [8])

A compact simply connected homogeneous space with an invariant complex structure is called a C-space. Moreover, a C-space with an invariant Kähler metric is called a Kähler C-space. Let G be a compact Lie group and K a centralizer of a toral subgroup of G. Then G/K admits a G-invariant Kähler structure. Conversely, every Kähler C-space can be obtained in this way.

In the following we describe an irreducible Kähler C-space in terms of a root syetem.

Let G be a compact simple Lie group and K a centralizer of a toral subgroup of G. $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K, respectively. $\mathfrak{g}_\mathbb{C}$ and $\mathfrak{k}_\mathbb{C}$ denote the complexification of $\mathfrak{g}$ and $\mathfrak{k}$. Then $\mathfrak{k}$ contains a maximal abelian subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Let Δ and Δ_0 denote the set of nonzero roots of $\mathfrak{g}_\mathbb{C}$ and $\mathfrak{k}_\mathbb{C}$, respectively, with respect to $\mathfrak{h}_\mathbb{C}$. We choose fundamental root systems Π_0 of Δ_0 and Π of Δ for some lexicographic ordering of Δ so that $\Pi_0 \subset \Pi$. Set $\Pi = \{\alpha_1, \cdots, \alpha_l\}$. For Π_0 and Π we denote the positive root sets by ${\Delta_0}^+$ and Δ^+, respectively. Then ${\Delta_0}^+ \subset \Delta^+$.

Since the Killing form B of $\mathfrak{g}_{\mathbb{C}}$ is non-degenerate, we can define $H_\alpha \in \mathfrak{h}_{\mathbb{C}}$ $(\alpha \in \Delta)$ by

$$B(H, H_\alpha) = \alpha(H) \quad (H \in \mathfrak{h}_{\mathbb{C}}).$$

We choose root vectors $\{E_\alpha\}$ $(\alpha \in \Delta)$ so that for α, $\beta \in \Delta$

$$\begin{aligned} &B(E_\alpha, E_{-\alpha}) = 1, \\ &[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}, \quad N_{\alpha,\beta} = -N_{-\alpha,-\beta} \in \mathbb{R}. \end{aligned} \tag{1.2}$$

As is well-known, the following $\mathfrak{g}_u$ is a compact real form of $\mathfrak{g}_{\mathbb{C}}$:

$$\mathfrak{g}_u = \sum_{\alpha \in \Delta^+} \mathbb{R}\sqrt{-1} H_\alpha + \sum_{\alpha \in \Delta^+} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha),$$

where $A_\alpha = E_\alpha - E_{-\alpha}$ and $B_\alpha = \sqrt{-1}(E_\alpha + E_{-\alpha})$. Now we may identify $\mathfrak{g}$ with $\mathfrak{g}_u$. So we have

$$\mathfrak{k} = \sum_{\alpha \in \Delta^+} \mathbb{R}\sqrt{-1} H_\alpha + \sum_{\alpha \in \Delta_0{}^+} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha). \tag{1.3}$$

Put $\Phi = \Pi \setminus \Pi_0 = \{\alpha_{i_1}, \cdots, \alpha_{i_r}\}$ and let $\Delta^+(\Phi)$ be the set $\Delta^+ \setminus \Delta_0{}^+$. Moreover set

$$\mathfrak{p} = \sum_{\alpha \in \Delta^+(\Phi)} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha). \tag{1.4}$$

Then $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ (direct sum) and the tangent space $T_o(G/K)$ of G/K at $o = \{K\}$ is identified with $\mathfrak{p}$. We define a linear mapping $J : \mathfrak{p} \to \mathfrak{p}$ as

$$J(A_\alpha) = B_\alpha, \quad J(B_\alpha) = -A_\alpha \quad (\alpha \in \Delta^+(\Phi)). \tag{1.5}$$

Then J can be extended to a G-invariant complex structure on G/K. $\mathfrak{p}^{\pm}$ denote the eigenspaces of J corresponing with the eigenvalues $\pm\sqrt{-1}$, that is

$$\mathfrak{p}^{\pm} = \sum_{\alpha \in \Delta^+(\Phi)} \mathbb{C}E_{\pm\alpha}.$$

It is known that any G-invariant Kähler metric g is given at o by

$$g|_{\mathfrak{g}_\alpha \times \mathfrak{g}_\alpha} = -(\sum_{j=1}^{r} c_j n_{i_j}) B \quad (\alpha = \sum_{i=1}^{l} n_i \alpha_i \in \Delta^+(\Phi)). \tag{1.6}$$

Here c_j are positive numbers and $\mathfrak{g}_\alpha = \mathbb{R}A_\alpha + \mathbb{R}B_\alpha$. Conversely, any bilinear form defined by (1.6) on $\mathfrak{p}^{\mathbb{C}} \times \mathfrak{p}^{\mathbb{C}}$ can be extended to a G-invariant metric on G/K (see [5]). We have thus obtained a Kähler C-space $(G/K, g)$. In the remaining part of this paper we denote this Kähler C-space by $M(\mathfrak{g}, \Pi, \Phi, g)$.

2. Symmetries of Kähler C-spaces

Let G be a compact Lie group and K a centralizer of a toral subgroup of G. Then the homogeneous space G/K is called a generalized flag manifold. It is known that G/K with G-invariant metric $\langle\ ,\ \rangle$ admits a Riemannian m-symmetric structure (cf. [1] and [9]). For later use we shall prove this fact in the case where $\mathfrak{g}$ is simple.

As in section 1, we set

$$\mathfrak{g} = \sum_{\alpha\in\Delta^+} \mathbb{R}\sqrt{-1}H_\alpha + \sum_{\alpha\in\Delta^+} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha),$$
$$\Pi = \{\alpha_1, \cdots, \alpha_l\}, \quad \Phi = \{\alpha_{i_1}, \cdots, \alpha_{i_r}\}.$$

Let $\delta = \sum_{i=1}^{l} n_i\alpha_i$ be the highest root of Δ with respect to Π. For positive integers m_i ($i = 0, \cdots, r$) put $m = m_0 + \sum_{j=1}^{r} n_{i_j} m_j$. Set

$$\begin{aligned} \sigma(E_{\pm\alpha_{i_j}}) &= \xi^{\pm m_j} E_{\pm\alpha_{i_j}} \quad (\alpha_{i_j} \in \Phi), \\ \sigma(E_{\pm\delta}) &= \xi^{\mp m_0} E_{\pm\delta}, \quad \sigma(E_{\alpha_i}) = E_{\alpha_i} \quad (\alpha_i \in \Phi_0). \end{aligned} \tag{2.1}$$

Here ξ denotes a primitive m-th root of unity. Then σ can be extended to an inner automorphism of order m of $\mathfrak{g}_{\mathbb{C}}$. Conversely, every inner automorphism of finite order of $\mathfrak{g}_{\mathbb{C}}$ is obtained in this way (cf. Helgason [4].)

Lemma 2.1. *Let σ be an inner automorphism of finite order of $\mathfrak{g}_{\mathbb{C}}$. Then there exist a fundamental root system $\Pi = \{\alpha_1, \cdots, \alpha_l\}$ (with respect to a certain Cartan subalgebra $\mathfrak{h}$) and nonnegative integers $(m_0, m_1, \cdots, m_l)$ without nontrivial common factor such that σ satisfies the following:*

$$\sigma(E_{\pm\alpha_i}) = \xi^{\pm m_i} E_{\pm\alpha_i}, \quad \sigma(E_{\pm\delta}) = \xi^{\mp m_0} E_{\pm\delta},$$

where $\delta = \sum_{i=1}^{l} n_i\alpha_i$ denotes the highest root, $m = m_0 + \sum_{i=1}^{l} n_i m_i$ and ξ a primitive m-th root of unity. Moreover σ has the form

$$\sigma = e^{\operatorname{ad} H} \quad \textit{for some } H \in \mathfrak{h}. \tag{2.2}$$

Since $\sigma^m = 1$, we can see that $H \in \sum_\alpha \mathbb{R}\sqrt{-1}H_\alpha$. Therefore we can regard σ as an inner automorphism of order m of $\mathfrak{g}$. We can easily check that $\mathfrak{g}^\sigma = \mathfrak{k}$, where $\mathfrak{g}^\sigma$ is the fixed point set of σ. Set $\phi = (1 + \sigma + \cdots + \sigma^{m-1})$. Then ϕ is a linear map of $\mathfrak{g}$ and $\mathfrak{k} = \operatorname{Im}\phi$. Moreover we have

$$\ker\phi = \sum_{\alpha\in\Delta^+(\Phi)} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha) \quad (= \mathfrak{p}).$$

Therefore $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$.

Let $\langle\ ,\ \rangle$ be a G-invariant Riemannian metric on G/K. Then $\langle\ ,\ \rangle$ is identified with an Ad(K)-invariant scalar product on $\mathfrak{p}$ (denoted by the same symbol $\langle\ ,\ \rangle$). Hence by (2.2) the restriction of σ to $\mathfrak{p}$ preserves $\langle\ ,\ \rangle$.

We denote the inner automorphism of G corresponding to σ by the same symbol σ. Let $\pi : G \to G/K$ be the canonical projection. Define a transformation s of G/K by $s \circ \pi = \pi \circ \sigma$. Then the differential map of s at $o = \{K\}$ coincides with the restriction of σ to $\mathfrak{p}$. Consequently, from Proposition 1.1, $(G/K, \langle\ ,\ \rangle)$ admits a Riemannian m-symmetric structure.

Let (M, J, g) be a Hermitian manifold with a complex structure J. Suppose that (M, g) admits a Riemannian m-symmetric structure $\{s_x : x \in M\}$. We call $\{s_x : x \in M\}$ a ***Hermitian m-symmetric structure*** if each s_x $(x \in M)$ is a holomorphic isometry of (M, J, g). In particular, if (M, J, g) is Kählerian, then Hermitian m-symmetric structure is said to be ***Kählerian***. It is known that a Hermitian symmetric space has a Kählerian m-symmetric structure for any $m \geq 2$.

Proposition 2.2. *Let G/K be a generalized flag manifold, where G is simple. Then G/K admits a G-invariant complex structure J such that $(G/K, J, \langle\ ,\ \rangle)$ has a Hermitian m-symmetric structure for any G-invariant Riemannian metric $\langle\ ,\ \rangle$. In particular, a Kähler C-space admits a Kählerian m-symmetric structure for some integer m.*

Proof. We define a G-invariant complex structure J by (1.5). Since $\sum_{\alpha \in \Delta} \mathbb{R}\sqrt{-1}H_\alpha$ is contained in $\mathfrak{k}$, each metric $\langle\ ,\ \rangle$ at o satisfies the following.

$$\begin{aligned}&\langle A_\alpha, A_\alpha \rangle = \langle B_\alpha, B_\alpha \rangle, \quad \langle A_\alpha, B_\alpha \rangle = 0 \\ &(\mathbb{R}A_\alpha + \mathbb{R}B_\alpha) \perp (\mathbb{R}A_\beta + \mathbb{R}B_\beta), \quad (\alpha, \beta \in \Delta^+(\Phi), \alpha \neq \beta).\end{aligned}$$

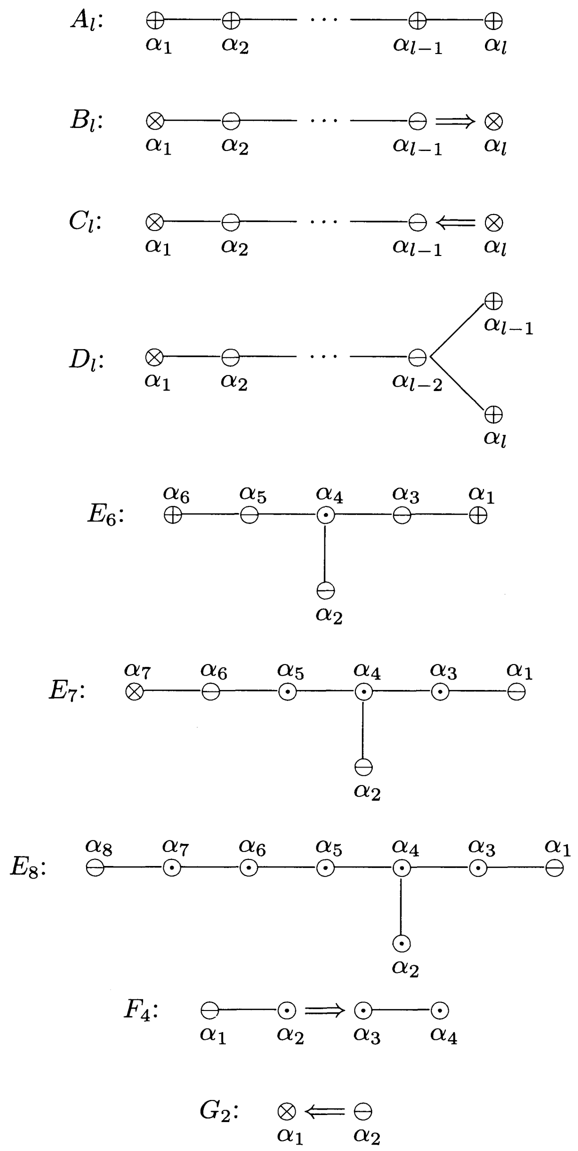
Hence $\langle\ ,\ \rangle$ is a Hermitian metric with respect to J.

Let $\{s_x : x \in M\}$ be the Riemannian m-symmetric structure corresponding with σ. Since σ has the form $e^{\mathrm{ad}H}$ for some $H \in \mathfrak{k}$, we can see that $s (= s_o)$ is holomorphic. Therefore, since J is G-invariant, $s_x = g \cdot s \cdot g^{-1}$ $(g \cdot o = x)$ is holomorphic. □

Let R and ∇ be the curvature tensor and the Levi-Civita connection, respectively, of a Kähler C-space $M(\mathfrak{g}, \Pi, \Phi, \langle\ ,\ \rangle)$. We denote by $\hat{\nabla}$ the covariant derivative in the derection of $\mathfrak{p}^+$. According to [10] there exists positive integer n such that

$$\hat{\nabla}^n R = 0 \quad \text{and} \quad \hat{\nabla}^{n-1} R \neq 0.$$

We call the integer n ***the degree*** of $M(\mathfrak{g}, \Pi, \Phi, \langle\ ,\ \rangle)$. Then the degree of a Kähler C-space is equal to one if and only if it is a Hermitian symmetric space. Moreover the following holds (see [10]).



Proposition 2.3. *There exists no Kähler C-space with degree two.*

Let α_a be any of the simple roots designed by the symbol $\ominus$ and α_i, α_j two of the simple roots designed by the symbol $\oplus$ in the above Dynkin diagrams. Then an irreducible Kähler C-space with degree three is one of $M(\mathfrak{g}, \Pi, \{\alpha_a\}, \langle\ ,\ \rangle)$ and $M(\mathfrak{g}, \Pi, \{\alpha_i, \alpha_j\}, \langle\ ,\ \rangle)$. (In the diagrams, for α_p corresponding to $\oplus$ or $\otimes$, a Kähler C-space $M(\mathfrak{g}, \Pi, \{\alpha_p\}, \langle\ ,\ \rangle)$ is a Hermitian symmetric space ([5]).*)*

Let $M(\mathfrak{g}, \Pi, \Phi, \langle\ ,\ \rangle)$ be an irreducible Kähler C-space with degree three and $\delta = \sum_{i=1}^{l} n_i\alpha_i$ the highest root. Then by Proposition 2.3 it is easy to see that $\Phi = \{\alpha_a\}$ or $\Phi = \{\alpha_j, \alpha_k\}$ with $n_a = 2$ and $n_j = n_k = 1$. Hence $M(\mathfrak{g}, \Pi, \Phi, \langle\ ,\ \rangle)$ has a Kählerian 3-symmetric structure. In fact, take 1 as m_a, m_j and m_k, and 0 as the other m_p (see the early part of this section and Proposition 2.2). More precisely, the following holds.

Theorem 2.4. *The degree of an irreducible Kähler C-space is three if and only if it is a compact irreducible simply connected Kählerian 3-symmetric space which is not isometric to a Hermitian symmetric space.*

Proof. Let $(M, J, \langle\ ,\ \rangle)$ be a compact irreducible Kählerian 3-symmetric space and $\{s_x : x \in M\}$ a Kählerian 3-symmetric structure of $(M, J, \langle\ ,\ \rangle)$. Let $\mathrm{Cl}(\{s_x\})$ be the closure of the group generated by the set $\{s_x : x \in M\}$ in the isometry group of (M, g). Then $\mathrm{Cl}(\{s_x\})$ is a closed subgroup of the holomorphic isometry group of $(M, J, \langle\ ,\ \rangle)$ and acts transitively on M. Thus $(M, J, \langle\ ,\ \rangle)$ is a Kähler C-space.

Let G be the identity component of $\mathrm{Cl}(\{s_x\})$ and K be the isotropy subgroup of G at a point $o \in M$. Then K is a centralizer of a toral subgroup of G since $(M, J, \langle\ ,\ \rangle)$ is Kähler C-space. Define an automorphism σ of order three of G as follows :

$$\sigma(g) = s_o \circ g \circ {s_o}^{-1}. \tag{2.3}$$

Since $s_o \circ k = k \circ s_o$ for $k \in K$ (see [7]) and o is an isolated fixed point of s_o, we have

$$(G^\sigma)_0 \subset K \subset G^\sigma, \quad \text{and} \quad \mathfrak{g}^\sigma = \mathfrak{k}.$$

Since $\mathfrak{k}$ contains a maximal abelian subalgebra of $\mathfrak{g}$ and σ leaves $\mathfrak{k}$ pointwise fixed, we can see that σ is inner. We set $\delta = \sum_{i=1} n_i\alpha_i$, $\alpha_0 = -\delta$ and $n_0 = 1$. (In other wards α_i and n_i $(0 \leq i \leq l)$ are the vertices and corresponding coefficients in the extended Dynkin diagram (cf. [4])). Then, by Lemma 2.1, the possibilities of $(m_0, m_1, \cdots, m_l)$ are the following :

(i) $m_i = m_j = m_k = 1$ and others are zero. In this case $n_i = n_j = n_k = 1$.
(ii) $m_i = m_j = 1$ and others are zero. In this case $n_i = 1, n_j = 2$.
(iii) $m_i = 1$ and others are zero. In this case $n_i = 3$.

However, case (iii) is not possible since $\mathfrak{k}$ must have a nonzero center (in the case, $\mathfrak{g}^\sigma$ is semisimple).

If σ is of the form (i), then the degree of $(M, J, \langle\ ,\ \rangle) = G/K$ equals three (if necessary, substitute $-\alpha_0$ for α_i). Similarly, if σ is of the form (ii), then the degree of $(M, J, \langle\ ,\ \rangle) = G/K$ is equal to three.

We have thus proved the theorem. □

REMARK 2.5. According to Koda [6], except for compact irreducible Kählerian 3-symmetric spaces, compact irreducible 3-symmetric spaces admit no (possibly not invariant) Kählerian structures because their second cohomology groups vanish.

REMARK 2.6. Let $M(\mathfrak{g}, \Pi, \Phi, \langle\ ,\ \rangle)$ be a Kähler C-space and set $\Phi = \{\alpha_{i_1}, \cdots, \alpha_{i_r}\}$. Let $\delta = \sum_{i=1}^{l} m_i \alpha_i$ be the highest root of $\mathfrak{g}$ and put $m = \sum_{j=1}^{r} m_{i_j}$. By the above argument we can see that the space has a Riemannian $(m+1)$-symmetric structure. Moreover, in [10], we implicitly proved that the degree of $M(\mathfrak{g}, \Pi, \Phi, \langle\ ,\ \rangle)$ is at most $(2m-1)$.

3. Isometry groups of Riemannian 3-symmetric spaces

In this section we examine the isometry groups of Riemannian 3-symmetric spaces.

Let $(M, \langle\ ,\ \rangle)$ be a Riemannian m-symmetric space $(m > 2)$ and $\{s_x : x \in M\}$ a Riemannian m-symmetric structure of $(M, \langle\ ,\ \rangle)$. Let G be the identity component of $\mathrm{Cl}(\{s_x\})$ and K be the isotropy subgroup of G at a point $o \in M$. As stated in Section 2, $\sigma(g) = s_o \circ g \circ {s_o}^{-1}$ $(g \in G)$ is an automorphism of order m of G. Moreover it follow that

$$(G^\sigma)_0 \subset K \subset G^\sigma. \tag{3.1}$$

Now we shall show the following proposition.

Proposition 3.1. *Let G be a compact, connected, simple Lie group and K a closed subgroup of G such that G/K is simply connected and G acts effectively on G/K. Let σ be an inner automorphism of order three of G such that (3.1) is satisfied. Suppose that G/K is not Riemannian symmetric for a G-invariant metric $\langle\ ,\ \rangle$. Then G coincides with the identity component of the isometry group of $(G/K, \langle\ ,\ \rangle)$.*

Proof. Let $\tilde{G}$ be the identity component of the isometry group of $(G/K, \langle\ ,\ \rangle)$ and $\tilde{K}$ the isotropy subgroup of $\tilde{G}$ at a point $o = \{K\}$. Since G acts effectively on G/K, the group G is a closed subgroup of $\tilde{G}$ and $K \subset \tilde{K}$. Let $\mathfrak{g}$, $\mathfrak{k}$, $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{k}}$ be the Lie algebras of G, K, $\tilde{G}$ and $\tilde{K}$, respectively.

We denote the differential map of σ by the same symbol σ. Set $\mathfrak{p} = \ker(1+\sigma+\sigma^2)$ $(\subset \mathfrak{g})$. Then $\mathfrak{k} = \mathrm{Im}(1+\sigma+\sigma^2)$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$. Since σ is inner, the restriction of σ to $\mathfrak{p}$ preserves $\langle\ ,\ \rangle$. Thus by Proposition 1.1 the space $M = (G/K, \langle\ ,\ \rangle)$ has a Riemannian 3-symmetric structure $\{s_x : x \in M\}$. Moreover

$$s_o \circ \pi = \pi \circ \sigma, \quad s_x = g \circ s_o \circ g^{-1} \quad (g \in G, g \cdot o = x),$$

where $\pi : G \to G/K$ be the canonical projection. We note that $s_o \in K$. Hence the automorphism $\tilde{\sigma}$ of $\tilde{G}$ defined by $\tilde{\sigma}(g) = s_o \circ g \circ {s_o}^{-1}$ is inner and of order three.

Let $\tilde{\mathfrak{g}}^{\tilde{\sigma}}$ be the fixed point set of $\tilde{\sigma}$ in $\tilde{\mathfrak{g}}$. Since o is an isolated fixed point of s_o, we have

$$\mathfrak{k} \subset \tilde{\mathfrak{g}}^{\tilde{\sigma}} \subset \tilde{\mathfrak{k}}. \tag{3.2}$$

Therefore $\tilde{\mathfrak{g}}$ is semisimple, since $\tilde{G}$ is compact and acts effectively on M. Moreover, $\mathfrak{k}$ contains a maximal abelian subalgebra of $\mathfrak{g}$ because σ is inner. Thus $M = (G/K, \langle\ ,\ \rangle)$ is an irreducible Riemannian manifold (see the proof of Theorem 5 in [3]). Also $\tilde{\mathfrak{k}}$ contains a maximal abelian subalgebra of $\tilde{\mathfrak{g}}$ because $\tilde{\sigma}$ is inner. Therefore $\tilde{\mathfrak{g}}$ must be simple. In fact, if not, then we have the decomposition

$$\tilde{\mathfrak{g}} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r, \quad \tilde{\mathfrak{k}} = \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_r,$$

where $\mathfrak{g}_i$ is an ideal of $\tilde{\mathfrak{g}}$ and $\mathfrak{k}_i \subset \mathfrak{g}_i$. This contradicts the irreducibility of M.

Using a similar method as in the proof of Theorem 2.4 we shall see that $\tilde{\mathfrak{g}}^{\tilde{\sigma}}$ coincides with $\tilde{\mathfrak{k}}$.

Since $\tilde{\mathfrak{g}}$ is simple and $\tilde{\sigma}$ is an inner automorphism of order three, $\tilde{\mathfrak{g}}^{\tilde{\sigma}}$ contains a maximal abelian subalgebra $\mathfrak{h}$ of $\tilde{\mathfrak{g}}$. Furthermore, by Lemma 2.1, there exists a fundamental root system $\Pi = \{\alpha_1, \cdots, \alpha_l\}$ with respect to $\mathfrak{h}_{\mathbb{C}}$ of $\tilde{\mathfrak{g}}_{\mathbb{C}}$ such that the possibilities of $(m_0, m_1, \cdots, m_l)$ are the following :
(i) $m_i = m_j = m_k = 1$ and others are zero. In this case $n_i = n_j = n_k = 1$.
(ii) $m_i = m_j = 1$ and others are zero. In this case $n_i = 1$, $n_j = 2$.
(iii) $m_i = 1$ and others are zero. In this case $n_i = 3$.

Here $-\alpha_0 = \sum_{i=1} n_i \alpha_i$ is the highest root and we set $n_0 = 1$. Let Δ^+ be the set of positive roots with respect to Π. For a subset $\Phi = \{\alpha_{i_1}, \cdots, \alpha_{i_r}\}$ of Π we set

$$\Delta^+(\Phi) = \left\{ \alpha = \sum_{p=1}^{l} k_p \alpha_p \in \Delta^+ : k_{i_j} > 0 \quad \text{for some } j \right\}.$$

Now we shall see that $\tilde{\mathfrak{k}} = \tilde{\mathfrak{g}}^{\tilde{\sigma}}$.

CASE (i) As mentioned in the proof of Theorem 2.4, we may assume that $\alpha_k = \alpha_0$ ($-\alpha_0$: the highest root). Set $\Phi = \{\alpha_i, \alpha_j\}$. Suppose that there is a root $\alpha \in \Delta^+(\Phi)$ such that

$$\mathfrak{g}_\alpha = (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha) \subset \tilde{\mathfrak{k}} \setminus \tilde{\mathfrak{g}}^{\tilde{\sigma}}.$$

If $k_i = 0$ and $k_j = 1$ ($\alpha = \sum_{p=1}^{l} k_p \alpha_p$), then, since $\mathfrak{g}_{\alpha_p}$ is contained in $\tilde{\mathfrak{g}}^{\tilde{\sigma}}$ ($p \neq i, j$), we see that $\mathfrak{g}_{\alpha_j}$ is contained in $\tilde{\mathfrak{k}}$. In this case the pair $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{k}})$ is symmetric (take an involutive automorphism so that $m_i = m_0 = 1$ and the others are zero).

If $k_i = k_j = 1$ ($\alpha = \sum_{p=1}^{l} k_p \alpha_p$), then the same argument as above implies that $\mathfrak{g}_{\alpha_0}$ is contained in $\tilde{\mathfrak{k}}$. Moreover $\mathfrak{g}_{\alpha_i}$ and $\mathfrak{g}_{\alpha_j}$ are not contained in $\tilde{\mathfrak{k}}$, since we assume

$\tilde{\mathfrak{k}} \neq \tilde{\mathfrak{g}}$. Then $\tilde{\mathfrak{k}}$ coincides with $\tilde{\mathfrak{g}}^{\tau}$, where τ is the inner automorphism of order two of $\tilde{\mathfrak{g}}$ defined by the relation $m_i = m_j = 1$ and $m_k = 0$ $(k \neq i,\ j,\ 0 \leq k \leq l)$. Hence $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{k}})$ is a symmetric pair.

Consequently, in this case, $\tilde{\mathfrak{g}}^{\tilde{\sigma}} = \tilde{\mathfrak{k}}$, since we assume that $M = G/K = \tilde{G}/\tilde{K}$ is not symmetric.

Case (ii) As in the Case (i) we assume $i = 0$. Suppose that there is a root $\alpha = \sum_{p=1}^{l} k_p \alpha_p$ in $\Delta^+(\alpha_j)$ such that

$$\mathfrak{g}_\alpha = (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha) \subset \tilde{\mathfrak{k}} \setminus \tilde{\mathfrak{g}}^{\tilde{\sigma}}.$$

It is clear that $k_j = 1$ or 2. If $k_j = 1$, then $\mathfrak{g}_{\alpha_j} \subset \tilde{\mathfrak{k}}$, that is, $\tilde{\mathfrak{k}} = \tilde{\mathfrak{g}}$. This is a contradiction.

If $k_j = 2$, then $\mathfrak{g}_{\alpha_0} \subset \tilde{\mathfrak{k}}$. Then $\tilde{\mathfrak{k}}$ coincides with $\tilde{\mathfrak{g}}^{\tau}$, where τ is the inner automorphism of order two of $\tilde{\mathfrak{g}}$ defined by the relation $m_j = 1$ and $m_k = 0$ $(k \neq j,\ 0 \leq k \leq l)$. Hence the pair $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{k}})$ is symmetric.

Case (iii) In this case we can see that $\mathfrak{g}_{\alpha_j} \subset \tilde{\mathfrak{g}}^{\tilde{\sigma}}$ for $j \neq i$ $(0 \leq j \leq l)$. Suppose that there is a root $\alpha = \sum_{p=1}^{l} k_p \alpha_p$ in $\Delta^+(\alpha_i)$ such that $\mathfrak{g}_\alpha \subset \tilde{\mathfrak{k}} \setminus \tilde{\mathfrak{g}}^{\tilde{\sigma}}$. Then $k_i = 1$ or 2 because $\mathfrak{g}_{\alpha_0} \subset \tilde{\mathfrak{g}}^{\tilde{\sigma}}$. If $k_i = 1$, then $\tilde{\mathfrak{k}}$ must be equal to $\tilde{\mathfrak{g}}$. If $k_i = 2$, then since $\mathfrak{g}_{\alpha_0} \subset \tilde{\mathfrak{k}}$ there is a root β in $\Delta^+(\alpha_i)$ such that $\mathfrak{g}_\beta \subset \tilde{\mathfrak{k}}$ and $h_i = 1$ $(\beta = \sum_{j=1}^{l} h_j \alpha_j)$. Therefore $\tilde{\mathfrak{k}} = \tilde{\mathfrak{g}}$.

We have thus $\tilde{\mathfrak{k}} = \tilde{\mathfrak{g}}^{\tilde{\sigma}}$.

Consequently, $\tilde{\mathfrak{k}}$ must be equal to $\tilde{\mathfrak{g}}^{\tilde{\sigma}}$.

Set $\tilde{\mathfrak{p}} = \ker(1 + \tilde{\sigma} + \tilde{\sigma}^2)$. Then since $\tilde{\mathfrak{k}} = \operatorname{Im}(1 + \tilde{\sigma} + \tilde{\sigma}^2)$, we have

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{k}} \oplus \tilde{\mathfrak{p}}, \quad [\tilde{\mathfrak{k}}, \tilde{\mathfrak{p}}] \subset \tilde{\mathfrak{p}}. \tag{3.3}$$

Then $\mathfrak{p} = \tilde{\mathfrak{p}}$ because $\mathfrak{p} \subset \tilde{\mathfrak{p}}$ and $\dim \mathfrak{p} = \dim M = \dim \tilde{\mathfrak{p}}$. On the other hand, $\mathfrak{g} = \mathfrak{p} + [\mathfrak{p}, \mathfrak{p}]$ and $\tilde{\mathfrak{g}} = \tilde{\mathfrak{p}} + [\tilde{\mathfrak{p}}, \tilde{\mathfrak{p}}]$ since $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ are simple Lie algebras. Finally, we have $\mathfrak{g} = \tilde{\mathfrak{g}}$. □

We consider the similar problem in other cases. Let (G, K) be one of the following:

(i) $(\mathrm{Spin}(8), \mathrm{SU}(3)/\mathbb{Z}_3)$,

(ii) $(\mathrm{Spin}(8), \mathrm{G}_2)$,

(iii) $(\{L \times L \times L\}/\delta Z, \delta L/\delta Z)$,

where L and Z denote the compact, simply connected, simple Lie group and its center, respectively. Moreover $\delta(g) = (g, g, g)$ $(g \in L)$. Let $\mathfrak{l}$ be the Lie algebra of L. Then the Lie algebra $\delta\mathfrak{l}$ of δL is given by

$$\delta\mathfrak{l} = \{(X, X, X) : X \in \mathfrak{l}\}.$$

Moreover, the automorphism σ of order three of $\mathfrak{l} \oplus \mathfrak{l} \oplus \mathfrak{l}$ is given by $\sigma(X, Y, Z) = (Z, X, Y)$.

Now, we shall show that $\delta\mathfrak{l}$ is a maximal σ-invariant subalgebra of $\mathfrak{l} \oplus \mathfrak{l} \oplus \mathfrak{l}$.

Let $\mathfrak{k}$ be a σ-invariant Lie subalbegra of $\mathfrak{l} \oplus \mathfrak{l} \oplus \mathfrak{l}$ such that $\delta\mathfrak{l} \subset \mathfrak{k}$. At first, we shall see that there is $X \in \mathfrak{l}$ such that $(0, 0, X) \in \mathfrak{k}$ if $\mathfrak{k} \neq \delta\mathfrak{l}$.

We may assume that there exist X, $Y \in \mathfrak{l}$ $(X \neq Y)$ such that $(0, X, Y) \in \mathfrak{k}$. If $[X, Y] \neq 0$, then $(0, 0, [X, Y]) \in \mathfrak{k}$ because $(X, X, X) \in \mathfrak{k}$. Thus we suppose that $[X, Y] = 0$. Then there exists a maximal abelian subalgebra $\mathfrak{h}$ of $\mathfrak{l}$ such that X, $Y \in \mathfrak{h}$. Let Δ be the set of nonzero roots of $\mathfrak{l}_{\mathbb{C}}$ with respect to $\mathfrak{h}_{\mathbb{C}}$ and choose a Weyl basis $\{E_\alpha, H_\alpha\}$ $(\alpha \in \Delta)$ so that for any $\alpha \in \Delta$

$$A_\alpha = (E_\alpha - E_{-\alpha}) \in \mathfrak{l}, \quad B_\alpha = \sqrt{-1}(E_\alpha + E_{-\alpha}) \in \mathfrak{l}, \quad \sqrt{-1}H_\alpha \in \mathfrak{l}$$

(see Section 1). Set $X = \sqrt{-1}H$ and $Y = \sqrt{-1}H'$ $(H, H' \in \mathfrak{h})$. Then

$$[(0, \sqrt{-1}H, \sqrt{-1}H'), (A_\alpha, A_\alpha, A_\alpha)] = (0, \alpha(H)B_\alpha, \alpha(H')B_\alpha) \in \mathfrak{k}.$$

Similarly, $(0, \alpha(H)B_\alpha, \alpha(H')B_\alpha) \in \mathfrak{k}$ from which we have

$$\begin{aligned}&[(0, \alpha(H)A_\alpha, \alpha(H')A_\alpha), (0, \alpha(H)B_\alpha, \alpha(H')B_\alpha)]\\ &= (0, 2\alpha(H)^2\sqrt{-1}H_\alpha, 2\alpha(H')^2\sqrt{-1}H_\alpha) \in \mathfrak{k}.\end{aligned}$$

Now, we may assume $\alpha(H) \neq 0$ since $\mathfrak{l}$ is simple. If $\alpha(H)^2 = \alpha(H')^2$, then we obtain

$$(\alpha(H)^2\sqrt{-1}H_\alpha, 0, 0) \in \mathfrak{k}$$

since $\alpha(H)^2(\sqrt{-1}H_\alpha, \sqrt{-1}H_\alpha, \sqrt{-1}H_\alpha)$ and $(0, \alpha(H)^2\sqrt{-1}H_\alpha, \alpha(H)^2\sqrt{-1}H_\alpha)$ are in $\mathfrak{k}$. Thus $(0, 0, \sqrt{-1}H_\alpha) \in \mathfrak{k}$ because $\mathfrak{k}$ is σ-invariant.

We suppose that $\alpha(H)^2 \neq \alpha(H')^2$. Then there exist $\alpha \in \Delta$ and nonnegative number c such that $(0, \sqrt{-1}H_\alpha, c\sqrt{-1}H_\alpha) \in \mathfrak{k}$. Since $\mathfrak{k}$ is σ-invariant, we have

$$(c\sqrt{-1}H_\alpha, 0, \sqrt{-1}H_\alpha), \quad (\sqrt{-1}H_\alpha, c\sqrt{-1}H_\alpha, 0) \in \mathfrak{k}.$$

Hence $(0, -c^2\sqrt{-1}H_\alpha, \sqrt{-1}H_\alpha) \in \mathfrak{k}$. Then it is easy to see that $(0, (1+c^3)\sqrt{-1}H_\alpha, 0)$ is in $\mathfrak{k}$. Thus $(0, 0, \sqrt{-1}H_\alpha)$ is in $\mathfrak{k}$.

From the above argument, we assume that there is $\alpha \in \Delta$ such that $(0, 0, \sqrt{-1}H_\alpha) \in \mathfrak{k}$. Let $\{\alpha_1, \cdots, \alpha_l\}$ be a fundamental root system of Δ with respect to some lexicographic ordering. Then there is i $(1 \leq i \leq l)$ such that $\alpha_i(H_\alpha) \neq 0$. By a similar method as above, we can see that

$$(0, 0, \mathbb{R}A_{\alpha_i} \oplus \mathbb{R}B_{\alpha_i} \oplus \mathbb{R}\sqrt{-1}H_{\alpha_i}) \subset \mathfrak{k}. \tag{3.4}$$

Next, choose j $(j \neq i)$ so that $\alpha_j(H_{\alpha_i}) \neq 0$. Then

$$(0, 0, \mathbb{R}A_{\alpha_j} \oplus \mathbb{R}B_{\alpha_j} \oplus \mathbb{R}\sqrt{-1}H_{\alpha_j}) \subset \mathfrak{k}.$$

By induction, (3.4) holds for all i $(1 \leq i \leq l)$, since $\mathfrak{l}$ is simple. Therefore $(0, 0, \mathfrak{l}) \subset \mathfrak{k}$, and $\mathfrak{k}$ coincides with $\mathfrak{l} \oplus \mathfrak{l} \oplus \mathfrak{l}$. We have thus proved the following.

Lemma 3.2. *$\delta\mathfrak{l}$ is a maximal σ-invariant subalgebra of $\mathfrak{l} \oplus \mathfrak{l} \oplus \mathfrak{l}$.*

Next, let σ be an outer automorphism of order three on a compact simple Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}$ is of type D_4. Let $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be a fundamental root system (see Proposition 2.3). As before, we choose a Weyl basis $\{E_\alpha, H_\alpha; \alpha \in \Delta\}$ so that it satisfies (1.2). Let ξ be a primitive cube root of unity. Set

$$\begin{aligned}
a_\pm &= E_{\pm\alpha_1} + E_{\pm\alpha_3} + E_{\pm\alpha_4}, \quad a'_\pm = E_{\pm\alpha_1} + \xi E_{\pm\alpha_3} + \xi^2 E_{\pm\alpha_4}, \\
a''_\pm &= E_{\pm\alpha_1} + \xi^2 E_{\pm\alpha_3} + \xi E_{\pm\alpha_4}, \\
b_\pm &= E_{\pm(\alpha_1+\alpha_2)} + E_{\pm(\alpha_3+\alpha_2)} + E_{\pm(\alpha_4+\alpha_2)}, \\
b'_\pm &= E_{\pm(\alpha_1+\alpha_2)} + \xi E_{\pm(\alpha_3+\alpha_2)} + \xi^2 E_{\pm(\alpha_4+\alpha_2)}, \\
b''_\pm &= E_{\pm(\alpha_1+\alpha_2)} + \xi^2 E_{\pm(\alpha_3+\alpha_2)} + \xi E_{\pm(\alpha_4+\alpha_2)}, \\
c_\pm &= E_{\pm(\alpha_1+\alpha_2+\alpha_3)} + E_{\pm(\alpha_2+\alpha_3+\alpha_4)} + E_{\pm(\alpha_1+\alpha_2+\alpha_4)}, \\
c'_\pm &= E_{\pm(\alpha_1+\alpha_2+\alpha_3)} + \xi E_{\pm(\alpha_2+\alpha_3+\alpha_4)} + \xi^2 E_{\pm(\alpha_1+\alpha_2+\alpha_4)}, \\
c''_\pm &= E_{\pm(\alpha_1+\alpha_2+\alpha_3)} + \xi^2 E_{\pm(\alpha_2+\alpha_3+\alpha_4)} + \xi E_{\pm(\alpha_1+\alpha_2+\alpha_4)}.
\end{aligned}$$

Let $\mathfrak{g}(\sigma, \xi^i)$ be the complex eigenspace of σ with eigenvalue ξ^i $(i = 0, 1, 2)$. According to Wolf and Gray [11], σ is conjugate to τ_1 or τ_2, where τ_i $(i = 1, 2)$ are defined by the following :

$$\begin{aligned}
(3.5) \qquad &\mathfrak{g}(\tau_1, 1) : \{H_{\alpha_2}, H_{\alpha_1} + H_{\alpha_3} + H_{\alpha_4}, E_{\pm\alpha_2}, E_{\pm(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}, \\
&\qquad E_{\pm(\alpha_1+2\alpha_2+\alpha_3+\alpha_4)}, a_\pm, b_\pm, c_\pm\} \\
&\mathfrak{g}(\tau_1, \xi) : \{H_{\alpha_1} + \xi^2 H_{\alpha_3} + \xi H_{\alpha_4}, a''_\pm, b''_\pm, c''_\pm\} \\
&\mathfrak{g}(\tau_1, \xi^2) : \{H_{\alpha_1} + \xi H_{\alpha_3} + \xi^2 H_{\alpha_4}, a'_\pm, b'_\pm, c'_\pm\} \\
(3.6) \qquad &\mathfrak{g}(\tau_2, 1) : \{H_{\alpha_2}, H_{\alpha_1} + H_{\alpha_3} + H_{\alpha_4}, a_\pm, b'_+, b''_-, c'_+, c''_-\} \\
&\mathfrak{g}(\tau_2, \xi) : \{H_{\alpha_1} + \xi^2 H_{\alpha_3} + \xi H_{\alpha_4}, E_{\alpha_2}, E_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}, \\
&\qquad E_{-(\alpha_1+2\alpha_2+\alpha_3+\alpha_4)}, a''_\pm, b_+, b'_-, c_+, c'_-\} \\
&\mathfrak{g}(\tau_2, \xi^2) : \{H_{\alpha_1} + \xi H_{\alpha_3} + \xi^2 H_{\alpha_4}, E_{-\alpha_2}, E_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}, \\
&\qquad E_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}, a'_\pm, b_-, b''_+, c_-, c''_+\}
\end{aligned}$$

REMARK 3.3. By (3.5) and (3.6) we can see that there is no element X in $\mathfrak{g}(\tau_i, \xi) \oplus \mathfrak{g}(\tau_i, \xi^2)$ such that

$$[X, \mathfrak{g}(\tau_i, 1)] = \{0\}.$$

We note that $(\mathfrak{g}, \mathfrak{g}(\tau_1, 1))$ and $(\mathfrak{g}, \mathfrak{g}(\tau_2, 1))$ correspond to the cases (ii) and (i), respectively.

Let (G, K) be one of (i), (ii) and (iii). $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K, respectively. Let σ be an outer automorphism of order three of $\mathfrak{g}$ such that $\mathfrak{k} = \mathfrak{g}^{\sigma}$. As in Proposition 1.1 we define a transformation s of G/K corresponding to σ. Let $\langle\ ,\ \rangle$ be a G-invariant metric on G/K such that $\langle\ ,\ \rangle$ is preserved by s at the origin $o = \{K\}$. Then $(G/K, \langle\ ,\ \rangle)$ has a Riemannian 3-symmetric structure $\{s_x : x \in G/K\}$ associated with s. Let $\tilde{G}$ be the identity component of the isometry group of $(G/K, \langle\ ,\ \rangle)$ and $\tilde{\mathfrak{g}}$ its Lie algebra. Since $\tilde{G}$ is compact, the algebra $\tilde{\mathfrak{g}}$ has the following form :

$$\tilde{\mathfrak{g}} = \mathfrak{z} \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r. \tag{3.7}$$

Here $\mathfrak{z}$ is the center and $\mathfrak{g}_i$ $(i = 1, \cdots, r)$ are simple ideals of $\tilde{\mathfrak{g}}$ and $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}] = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$. Define an automorphism $\tilde{\sigma}$ of $\tilde{G}$ by $\tilde{\sigma}(g) = s \circ g \circ s^{-1}$. Let $\tilde{K}$ be the isotropy subgroup of $\tilde{G}$ at o and $\tilde{\mathfrak{k}}$ its Lie algebra. We also denote by $\tilde{\sigma}$ the differential map of $\tilde{\sigma}$ at the identity of $\tilde{G}$. Then, as before, we have $\tilde{\mathfrak{g}}^{\tilde{\sigma}} \subset \tilde{\mathfrak{k}}$. Moreover, since each $\mathfrak{g}_i$ in (3.7) is a simple ideal, it is easy to see that

$$\tilde{\sigma}(\mathfrak{z}) = \mathfrak{z}, \quad \tilde{\sigma}(\mathfrak{g}_i) = \mathfrak{g}_j,$$

for some i, j $(i, j = 1, \cdots, j)$. Therefore we may assume that

$$[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}] = \mathfrak{g}_{(1)} \oplus \cdots \oplus \mathfrak{g}_{(l)} \quad (\tilde{\sigma}\text{-invariant decomposition}),$$

where $\mathfrak{g}_{(i)}$ is a simple ideal or $\mathfrak{l} \oplus \mathfrak{l} \oplus \mathfrak{l}$. In the following we denote the restriction of $\tilde{\sigma}$ to $\mathfrak{g}_{(i)}$ by the same symbol $\tilde{\sigma}$.

Suppose that $\tilde{\mathfrak{k}} \neq \tilde{\mathfrak{g}}^{\tilde{\sigma}}$. Let $X = (Z, X_{(1)}, \cdots, X_{(l)})$ be an element of $\tilde{\mathfrak{k}} \setminus \tilde{\mathfrak{g}}^{\tilde{\sigma}}$. Assume that $X_{(1)} \neq 0$. Then it is easy to see that there exists $Y \in {\mathfrak{g}_{(1)}}^{\tilde{\sigma}}$ such that $[Y, X_{(1)}] \neq 0$. (In fact, if $\mathrm{rk}({\mathfrak{g}_{(1)}}^{\tilde{\sigma}}) = \mathrm{rk}(\mathfrak{g}_{(1)})$, then take Y from a maximal abelian subalgebra contained in ${\mathfrak{g}_{(1)}}^{\tilde{\sigma}}$. For the other cases, by Remark 3.3 we can see that such Y exists.) In particular, $\mathfrak{g}_{(1)}$ is a compact simple Lie algebra from Lemma 3.2. Then $[Y, X_{(1)}]$ is contained in $\tilde{\mathfrak{k}} \cap \mathfrak{g}_{(1)}$. Hence the subalgebra $\mathfrak{k}_{(1)}$ of $\mathfrak{g}_{(1)}$ generated by $[Y, X_{(1)}]$ and ${\mathfrak{g}_{(1)}}^{\tilde{\sigma}}$ is contained in $\tilde{\mathfrak{k}} \cap \mathfrak{g}_{(1)}$.

If $X_{(1)}$ is not in $\mathfrak{k}_{(1)}$, then we may assume that $X_{(1)}$ is perpendicular to $\mathfrak{k}_{(1)}$ with respect to the Killing form of $\mathfrak{g}_{(1)}$. Then $[X_{(1)}, \mathfrak{k}_{(1)}]$ is perpendicular to $\mathfrak{k}_{(1)}$. This contradicts the definition of $\mathfrak{k}_{(1)}$. Thus $X_{(1)}$ is contained in $\tilde{\mathfrak{k}}$. By a similar argument, if $Z \neq 0$, then Z is in $\mathfrak{k}$. However, this contradicts the effectivity of $\tilde{G}$. Therefore we have

$$\tilde{\mathfrak{k}} = \mathfrak{k}_1 \oplus \cdots \mathfrak{k}_l, \quad (\mathfrak{k}_i \subset \mathfrak{g}_{(i)}).$$

Since $(G/K, \langle\ ,\ \rangle)$ is simply connected and irreducible (cf. Gray [2]), the algebra $\tilde{\mathfrak{g}}$ is simple or $\mathfrak{l} \oplus \mathfrak{l} \oplus \mathfrak{l}$.

Case (i) Let $\tilde{G}$ be the identity component of the isometry group of the Riemannian 3-symmetric space

$$M = (\mathrm{Spin}(8)/(\mathrm{SU}(3)/\mathbb{Z}_3), \langle\ ,\ \rangle).$$

Then $\tilde{\sigma}$ is an outer automorphism of $\tilde{G}$. (If not, then the Euler number of M is nonzero.) Thus, by the above argument, $\tilde{G}$ is one of the following :

$$\mathrm{Spin}(8), \quad \{L \times L \times L\}/\delta Z.$$

If $\tilde{G} = \{L \times L \times L\}/\delta Z$, Then by Lemma 3.2 we have

$$M = (\{L \times L \times L\}/\delta Z)/(\delta L/\delta Z).$$

However, from [6] we can see that $\mathrm{Spin}(8)/(\mathrm{SU}(3)/\mathbb{Z}_3)$ is not diffeomorphic to it for any compact simple Lie group L. Thus Spin(8) is the identity component of the isometry group.

Case (ii) By similar argument as above, $\tilde{G}$ is one of the following :

$$\mathrm{Spin}(8), \quad \{L \times L \times L\}/\delta Z.$$

However, since there is no simple Lie algebra with dimension seven, the latter case is impossible. Thus Spin(8) coincides with the identity component of the isometry group.

Finally, we consider the case (iii).

We shall prove the following lemmas.

Lemma 3.4. *Let $\mathfrak{g} = D_4$. Then $\mathfrak{g}(\tau_2, 1)$ is a maximal subalgebra of $\mathfrak{g}$.*

Lemma 3.5. *Let $\mathfrak{g} = D_4$. Then B_3 and $\mathfrak{g}(\tau_1, 1)$ are only proper subalgebras containing $\mathfrak{g}(\tau_1, 1)$. Here the pair $(\mathfrak{g}, B_3)$ is symmetric.*

If the lemmas hold, then $\{L \times L \times L\}/\delta Z$ coincides with the identity component of the isometry group of

$$((\{L \times L \times L\}/\delta Z)/(\delta L/\delta Z), \langle\ ,\ \rangle).$$

In fact, if the Lie algebra of the isometry group coincides with D_4, then the Lie algebra of the isotropy subgroup must be equal to one of $\mathfrak{g}(\tau_1, 1)$, $\mathfrak{g}(\tau_2, 1)$ and B_3. However, this contradicts the above argument. (Since $\dim \mathfrak{g} - \dim B_3 = 7$, the last case is impossible.)

Proof of Lemma 3.4. In this case $\mathfrak{g}(\tau_2, 1)$ is isomorphic to A_2. Set

$$H_0 = H_{\alpha_1} + H_{\alpha_3} + H_{\alpha_4}, \quad H_1 = H_{\alpha_1} + \xi^2 H_{\alpha_3} + \xi H_{\alpha_4},$$
$$H_2 = H_{\alpha_1} + \xi H_{\alpha_3} + \xi^2 H_{\alpha_4}.$$

Then we note that

$$\sqrt{-1}H_0, \quad (H_1 - H_2), \quad \sqrt{-1}(H_1 + H_2), \quad (a_+ - a_-), \quad \sqrt{-1}(a_+ + a_-) \in \mathfrak{g}$$
$$(a'_+ - a''_-), \quad \sqrt{-1}(a'_+ + a''_-), \quad (a'_- - a''_+), \quad \sqrt{-1}(a'_- + a''_+) \in \mathfrak{g},$$
$$\cdots, \quad (c'_- - c''_+), \quad \sqrt{-1}(c'_- + c''_+) \in \mathfrak{g}.$$

Let $\mathfrak{k}$ be a subalgebra of $\mathfrak{g}$ such that $\mathfrak{g}(\tau_2, 1) \subset \mathfrak{k}$ and $\mathfrak{g}(\tau_2, 1) \neq \mathfrak{k}$. Let X be an element of $\mathfrak{k} \setminus \mathfrak{g}(\tau_2, 1)$. Since $\sqrt{-1}H_{\alpha_2}$ and $\sqrt{-1}H_0$ are contained in $\mathfrak{g}(\tau_2, 1)$, we may assume that X is contained in one of the following (see (3.6)) :

$$\mathbb{C}a'_\pm \oplus \mathbb{C}a''_\pm, \quad \mathbb{C}b_\pm \oplus \mathbb{C}b'_- \oplus \mathbb{C}b''_+,$$
$$\mathbb{C}c_\pm \oplus \mathbb{C}c'_- \oplus \mathbb{C}c''_+, \quad \mathbb{C}E_{\pm(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}$$
$$\mathbb{C}E_{\pm\alpha_2}, \quad \mathbb{C}E_{\pm(\alpha_1+2\alpha_2+\alpha_3+\alpha_4)}, \quad \mathbb{C}H_1 \oplus \mathbb{C}H_2.$$

(Consider $[\sqrt{-1}H, X]$ for some $H \in \mathbb{R}H_0 \oplus \mathbb{R}H_{\alpha_2}$.)

(1) The case $X \in \mathbb{C}E_{\pm\alpha_2}$.

In this case $Y = [\sqrt{-1}H_{\alpha_2}, X]$ is also in $\mathfrak{k}$. Hence we have $E_{\pm\alpha_2} \in \mathfrak{k}_\mathbb{C}$. On the other hand, it is known that E_{α_2}, a_+ and c''_- generate $\mathfrak{g}_\mathbb{C}$ (cf. chapter X of [4]). Thus $\mathfrak{k}_\mathbb{C} = \mathfrak{g}_\mathbb{C}$, that is, $\mathfrak{k} = \mathfrak{g}$.

(2) The case $X \in \mathbb{C}E_{\pm(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}$.

As in (1), we can see that $E_{\pm(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \in \mathfrak{k}_\mathbb{C}$. Then

$$[a_-, E_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}] \in \mathfrak{g}(\tau_2, \xi) \cap (\mathbb{C}E_{\alpha_1+\alpha_2+\alpha_3} + \mathbb{C}E_{\alpha_1+\alpha_2+\alpha_4} + \mathbb{C}E_{\alpha_2+\alpha_3+\alpha_4}).$$

Thus $c_+ \in \mathfrak{k}_\mathbb{C}$. Similarly we have $b_+ \in \mathfrak{k}_\mathbb{C}$ and $E_{\alpha_2} \in \mathfrak{k}_\mathbb{C}$. Hence, by the same reason as (1), it follows that $\mathfrak{k} = \mathfrak{g}$.

(3) The case $X \in \mathbb{C}E_{\pm(\alpha_1+2\alpha_2+\alpha_3+\alpha_4)}$.

As in (1), we can see that $E_{\pm(\alpha_1+2\alpha_2+\alpha_3+\alpha_4)} \in \mathfrak{k}_\mathbb{C}$. Then we get

$$[E_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}, b''_-](\neq 0) \in \mathfrak{g}(\tau_2, \xi^2)$$
$$\cap(\mathbb{C}E_{\alpha_1+\alpha_2+\alpha_3} + \mathbb{C}E_{\alpha_1+\alpha_2+\alpha_4} + \mathbb{C}E_{\alpha_2+\alpha_3+\alpha_4}).$$

Hence $c''_+ \in \mathfrak{k}_\mathbb{C}$. Similarly we can check that $c'_- \in \mathfrak{k}_\mathbb{C}$, $b''_+, b'_- \in \mathfrak{k}_\mathbb{C}$. Then

$$[b'_+, b'_-] = H_1 \in \mathfrak{k}_\mathbb{C}, \quad [c''_+, c''_-] = -\xi H_2 \in \mathfrak{k}_\mathbb{C}.$$

Then there is $H \in \sum_{i=0}^{3} \mathbb{C}H_i$ ($H_3 = H_{\alpha_2}$) such that $\alpha_2(H) = \alpha_3(H) = \alpha_4(H) = 0$ and $\alpha_1(H) \neq 0$. Thus we can see that $E_{\pm\alpha_1} \in \mathfrak{k}_\mathbb{C}$. Similar argument implies that $E_{\pm\alpha} \in \mathfrak{k}_\mathbb{C}$ for all $\alpha \in \Delta$. Therefore $\mathfrak{k} = \mathfrak{g}$.

(4) The case $X \in \mathbb{C}b_\pm \oplus \mathbb{C}b'_- \oplus \mathbb{C}b''_+$.

In this case we may assume that

$$\{(b_+ + pb''_+ + qb'_-), \quad (b_- + rb''_+ + sb'_-) \in \mathfrak{k}_\mathbb{C}\} \quad \text{or} \quad \{b''_+, \quad b'_- \in \mathfrak{k}_\mathbb{C}\},$$

for some p, q, r, $s \in \mathbb{C}$. If b''_+, $b'_- \in \mathfrak{k}_\mathbb{C}$, then $[b''_+, c'_+](\in \mathbb{C}E_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}) \subset \mathfrak{k}_\mathbb{C}$. Thus $E_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}$ (and $E_{-(\alpha_1+2\alpha_2+\alpha_3+\alpha_4)}$) is contained in $\mathfrak{k}_\mathbb{C}$. Hence this case is reduced to (3).

If $(b_+ + pb''_+ + qb'_-)$, $(b_- + rb''_+ + sb'_-) \in \mathfrak{k}_\mathbb{C}$, then

$$\begin{aligned}
&[a_+, (b_+ + pb''_+ + qb'_-)] \in \mathbb{C}c_+ \oplus \mathbb{C}c''_+ \oplus \{0\},\\
&[a_+, [a_+, (b_+ + pb''_+ + qb'_-)]] \in \mathbb{C}E_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \oplus \{0\} \oplus \{0\}.
\end{aligned}$$

Therefore we have $E_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \in \mathfrak{k}_\mathbb{C}$ (and $E_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \in \mathfrak{k}_\mathbb{C}$). This case is reduced to (2).

(5) The case $X \in \mathbb{C}a'_\pm \oplus \mathbb{C}a''_\pm$.

In this case we may assume that

$$\{(a''_+ + pa'_+ + qa''_-), \quad (a'_- + ra'_+ + sa''_-) \in \mathfrak{k}_\mathbb{C}\} \quad \text{or} \quad \{a'_+, \quad a''_- \in \mathfrak{k}_\mathbb{C}\},$$

for some p, q, r, $s \in \mathbb{C}$. If a'_+ and a''_- are in $\mathfrak{k}_\mathbb{C}$, then we have $[b''_-, a'_+] \in \mathbb{C}E_{-\alpha_2}$ and $[b'_+, a''_-] \in \mathbb{C}E_{\alpha_2}$. This case is reduced to (1).

If $(a''_+ + pa'_+ + qa''_-)$ and $(a'_- + ra'_+ + sa''_-)$ are in $\mathfrak{k}_\mathbb{C}$, then

$$\begin{aligned}
&[(a''_+ + pa'_+ + qa''_-), c'_+] \in \mathbb{C}E_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \oplus \{0\} \oplus \mathbb{C}b_+,\\
&[[(a''_+ + pa'_+ + qa''_-), c'_+], a_-] \in \mathbb{C}c_+ \oplus \{0\} \oplus \mathbb{C}E_{\alpha_2},\\
&[[[(a''_+ + pa'_+ + qa''_-), c'_+], a_-], a_-] \in \mathbb{C}E_{\alpha_2} \oplus \{0\} \oplus \{0\}.
\end{aligned}$$

Hence $E_{\alpha_2} \in \mathfrak{k}_\mathbb{C}$. Similarly we have $E_{-\alpha_2} \in \mathfrak{k}_\mathbb{C}$. This case is reduced to (1).

(6) The case $X \in \mathbb{C}c_\pm \oplus \mathbb{C}c'_- \oplus \mathbb{C}c''_+$.

In this case we may assume that

$$(c_+ + pc'_- + qc''_+), \quad (c_- + rc'_- + sc''_+) \in \mathfrak{k}_\mathbb{C} \quad \text{or} \quad c'_-, \quad c''_+ \in \mathfrak{k}_\mathbb{C},$$

for some p, q, r, $s \in \mathbb{C}$. If c'_- and c''_+ are in $\mathfrak{k}_\mathbb{C}$, then

$$[c''_+, b'_+](\in \mathbb{C}E_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}), \quad [c'_-, b''_-](\in \mathbb{C}E_{-(\alpha_1+2\alpha_2+\alpha_3+\alpha_4)})$$

are contained in $\mathfrak{k}_\mathbb{C}$. This case is reduced to (3).

If $(c_+ + pc'_- + qc''_+)$ and $(c_- + rc'_- + sc''_+)$ are in $\mathfrak{k}_\mathbb{C}$, then since

$$\begin{aligned}
&[c_+ + pc'_- + qc''_+, a_+] \in \mathbb{C}E_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \oplus \{0\} \oplus \{0\},\\
&[c_- + rc'_- + sc''_+, a_-] \in \mathbb{C}E_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \{0\} \oplus \{0\},
\end{aligned}$$

it follows that $E_{\pm(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \in \mathfrak{k}_{\mathbb{C}}$. Hence this case is reduced to (2).

(7) The case $X \in \mathbb{C}H_1 \oplus \mathbb{C}H_2$.

It is easy to see that $[X, a_\pm] \neq 0$ and $[X, a_\pm]$ are contained in $\mathbb{C}a'_\pm \oplus \mathbb{C}a''_\pm$. Thus this case is reduced to (5).

We have thus proved the lemma. □

Sketch of the proof of Lemma 3.5. Suppose that there exists a Lie subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ such that $\mathfrak{k}$ contains $\mathfrak{g}(\tau_1, 1)$. As above, we may assume that there is $X \in \mathfrak{k} \setminus \mathfrak{g}(\tau_1, 1)$ such that X is contained in one of the following (see (3.5)) :

$$(\mathbb{R}(H_1 - H_2) \oplus \mathbb{R}\sqrt{-1}(H_1 + H_2)), \quad (\mathbb{R}(a''_\pm - a'_\mp) \oplus \mathbb{R}\sqrt{-1}(a''_\pm + a'_\mp)),$$
$$(\mathbb{R}(b''_\pm - b'_\mp) \oplus \mathbb{R}\sqrt{-1}(b''_\pm + b'_\mp)), \quad (\mathbb{R}(c''_\pm - c'_\mp) \oplus \mathbb{R}\sqrt{-1}(c''_\pm + c'_\mp)).$$

In particular, we may suppose that there exists an element in $\mathbb{R}(H_1 - H_2) \oplus \mathbb{R}\sqrt{-1}\,(H_1 + H_2)$ such that it is contained in $\mathfrak{k}$. In fact, if X is in $\mathbb{R}(a''_\pm - a'_\mp) \oplus \mathbb{R}\sqrt{-1}(a''_\pm + a'_\mp)$, then

$$\begin{aligned}
&(a''_+ - a'_-) + p(a''_- - a'_+) + q\sqrt{-1}(a''_- + a'_+) \in \mathfrak{k}, \\
&\sqrt{-1}(a''_+ + a'_-) + r(a''_- - a'_+) + s\sqrt{-1}(a''_- + a'_+) \in \mathfrak{k}, \\
&\text{or} \quad (a''_- - a'_+), \quad \sqrt{-1}(a''_- + a'_+) \in \mathfrak{k}.
\end{aligned}$$

If $(a''_- - a'_+) \in \mathfrak{k}$, then we have

$$[a''_- - a'_+, \sqrt{-1}(a_+ + a_-)] \in \mathbb{R}(H_1 - H_2) \oplus \mathbb{R}\sqrt{-1}(H_1 + H_2) \subset \mathfrak{k}.$$

For the other cases, we can check that there exists an element in $\mathbb{R}(H_1 - H_2) \oplus \mathbb{R}\sqrt{-1}(H_1 + H_2))$ such that it is contained in $\mathfrak{k}$. Thus we assume that there exist $p, q \in \mathbb{R}$ such that

$$X = p(H_1 - H_2) + q\sqrt{-1}(H_1 + H_2) \in \mathfrak{k}.$$

Since $[X, \mathfrak{g}(\tau_1, 1)] \subset \mathfrak{k}_{\mathbb{C}}$ and $[X, [X, \mathfrak{g}(\tau_1, 1)]] \subset \mathfrak{k}_{\mathbb{C}}$, we can check that if $\mathfrak{k} \neq \mathfrak{g}$ then H_{α_i} ($i = 1$, 3 or 4) is in $\mathfrak{k}$. For any case we can see that $\mathfrak{k}$ is isomorphic to B_3 and the pair $(\mathfrak{g}, B_3)$ is symmetric. □

Finally we have the following.

Theorem 3.6. *Let $(M, \langle\ ,\ \rangle)$ be a compact irreducible simply connected Riemannian 3-symmetric space which is not isometric to a symmetric space. Then there exists a unique pair (G, K) of a compact connected Lie group G and a closed subgroup K of G satisfying* (3.1) *such that $(M, \langle\ ,\ \rangle) = G/K$ and G acts effectively on M. In particular, G is the identity component of the isometry group of $(M, \langle\ ,\ \rangle)$.*

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