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KÄHLER C-SPACES AND $k$-SYMMETRIC SPACES

KOJI TOJO

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0. Introduction

Let $(M, J, g)$ be a compact, simply connected homogeneous Kählerian manifold (we call the space a Kähler C-space). In [10] we have proved that there is a positive integer $n$ such that the $n$-th covariant derivative of $(1,0)$-type of the curvature tensor of $(M, J, g)$ is identically zero (we call the least integer with above property the degree of $(M, J, g)$). It is clear that a compact Hermitian symmetric space is characterized as a Kähler C-space with degree one. Moreover we classified the spaces with degree $n$ ($n \leq 3$).

In this paper we shall prove explicitly that every Kähler C-space has a $k$-symmetric structure (see also Burstall and Rawnsley [1], p.52 and Pasiencier [9], Lemma 4.3). In [2] Gray showed that each Riemannian 3-symmetric space is a homogeneous almost Hermitian manifold with the canonical almost complex structure. He also proved that a Riemannian 3-symmetric space with the canonical almost complex structure is Kählerian if and only if it is a Hermitian symmetric space. In this paper we also show that the degree of a Kähler C-space equals three if and only if it is a compact Kähler manifold with a 3-symmetric structure which is not isometric to a Hermitian symmetric space (Theorem 2.4).

It is known that a Riemannian manifold $(M, g)$ with a $k$-symmetric structure is homogeneous, that is, $(M, g)$ has an expression $(M, g) = G/K$. For an irreducible Riemannian symmetric space the expression as a symmetric pair is unique as is well-known. In section 3 we shall show an analogous theorem on symmetric pair hold for a compact simply connected irreducible Riemannian 3-symmetric space which is not isometric to a Riemannian symmetric space (Theorem 3.6).

1. Preliminaries

In this section we recall notions and (some) properties of $k$-symmetric spaces ($k \in \mathbb{N}$) and Kähler C-spaces.

Let $(M, g)$ be a Riemannian manifold. For $x \in M$, an isometry of $(M, g)$ with an isolated fixed point $x$ is called a symmetry of $(M, g)$ at $x$. Assume that $(M, g)$ admits at least one symmetry at each point, and let $\{s_x : x \in M\}$ be the set of symmetries. Then it is known that $(M, g)$ is a Riemannian homogeneous space.
Moreover, if we denote by $\text{Cl}(\{s_x\})$ the closure of the group generated by the set $\{s_x : x \in M\}$ in the isometry group $I(M, g)$ of $(M, g)$, then $\text{Cl}(\{s_x\})$ acts transitively on $(M, g)$. (cf. Kowalski [7].)

Again, suppose that $(M, g)$ admits a set $\{s_x : x \in M\}$ of symmetries. We call $\{s_x : x \in M\}$ a Riemannian $k$-symmetric structure on $(M, g)$ if for $x, y \in M$

$$s_x \circ s_y = s_z \circ s_x, \quad (z = s_x(y)),$$

$$(s_x)^k = \text{id}, \quad (s_x)^l \neq \text{id}, \quad (l < k).$$

We note that $\{s_x : x \in M\}$ depends only on $s_p$ for a fixed $p \in M$. Furthermore $(M, g)$ with a Riemannian $k$-symmetric structure is said to be a Riemannian $k$-symmetric space.

Let $(M, g)$ be a Riemannian homogeneous space, i.e., there exits a group $G$ of isometries of $(M, g)$ such that $M = G/H$ ($H$ is a closed subgroup of $G$). Let $\pi : G \rightarrow G/H$ be the canonical projection and put $o = \pi(H)$. For an automorphism $\sigma$ of $G$ let $G^\sigma$ be the fixed point set and $(G^\sigma)_0$ the identity component of $G^\sigma$, respectively. Then the following is known (cf. [7]).

**Proposition 1.1.** Suppose that there exists an automorphism $\sigma$ of $G$ such that

(i) $(G^\sigma)_0 \subset H \subset G^\sigma$,

(ii) $\sigma^k = 1$ and $\sigma^l \neq 1$ for any $l < k$,

(iii) let $s$ be the transformation of $M$ defined by $\pi \circ \sigma = s \circ \pi$. Then $s$ preserves the metric at $o$.

Then $\{s_x = g \circ s \circ g^{-1} : x = g \cdot o \in M\}$ defines a Riemannian $k$-symmetric structure on $(M, g)$.

Next, we construct Kähler $C$-spaces. (for example, see Itoh [5] and Matsushima [8])

A compact simply connected homogeneous space with an invariant complex structure is called a $C$-space. Moreover, a $C$-space with an invariant Kähler metric is called a Kähler $C$-space. Let $G$ be a compact Lie group and $K$ a centralizer of a toral subgroup of $G$. Then $G/K$ admits a $G$-invariant Kähler structure. Conversely, every Kähler $C$-space can be obtained in this way.

In the following we describe an irreducible Kähler $C$-space in terms of a root system.

Let $G$ be a compact simple Lie group and $K$ a centralizer of a toral subgroup of $G$. $g$ and $k$ denote the Lie algebras of $G$ and $K$, respectively. $g_C$ and $k_C$ denote the complexification of $g$ and $k$. Then $k$ contains a maximal abelian subalgebra $h$ of $g$. Let $\Delta$ and $\Delta_0$ denote the set of nonzero roots of $g_C$ and $k_C$, respectively, with respect to $h_C$. We choose fundamental root systems $\Pi_0$ of $\Delta_0$ and $\Pi$ of $\Delta$ for some lexicographic ordering of $\Delta$ so that $\Pi_0 \subset \Pi$. Set $\Pi = \{\alpha_1, \ldots, \alpha_l\}$. For $\Pi_0$ and $\Pi$ we denote the positive root sets by $\Delta_0^+$ and $\Delta^+$, respectively. Then $\Delta_0^+ \subset \Delta^+$. 
Since the Killing form $B$ of $\mathfrak{g}_C$ is non-degenerate, we can define $H_\alpha \in \mathfrak{h}_C$ ($\alpha \in \Delta$) by

$$B(H, H_\alpha) = \alpha(H) \quad (H \in \mathfrak{h}_C).$$

We choose root vectors $\{E_\alpha\} (\alpha \in \Delta)$ so that for $\alpha, \beta \in \Delta$

$$B(E_\alpha, E_{-\alpha}) = 1, \quad [E_\alpha, E_\beta] = N_{\alpha, \beta}E_{\alpha + \beta}, \quad N_{\alpha, \beta} = -N_{-\alpha, -\beta} \in \mathbb{R}.$$

As is well-known, the following $\mathfrak{g}_u$ is a compact real form of $\mathfrak{g}_C$:

$$\mathfrak{g}_u = \sum_{\alpha \in \Delta^+} \mathbb{R}\sqrt{1}H_\alpha + \sum_{\alpha \in \Delta^+} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha),$$

where $A_\alpha = E_\alpha - E_{-\alpha}$ and $B_\alpha = \sqrt{-1}(E_\alpha + E_{-\alpha})$. Now we may identify $\mathfrak{g}$ with $\mathfrak{g}_u$. So we have

$$\mathfrak{k} = \sum_{\alpha \in \Delta^+} \mathbb{R}\sqrt{1}H_\alpha + \sum_{\alpha \in \Delta^+} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha).$$

Put $\Phi = \Pi \setminus \Pi_0 = \{\alpha_1, \ldots, \alpha_r\}$ and let $\Delta^+ (\Phi)$ be the set $\Delta^+ \setminus \Delta_0^+$. Moreover set

$$p = \sum_{\alpha \in \Delta^+ (\Phi)} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha).$$

Then $\mathfrak{g} = \mathfrak{k} + p$ (direct sum) and the tangent space $T_o(G/K)$ of $G/K$ at $o = \{K\}$ is identified with $p$. We define a linear mapping $J : p \rightarrow p$ as

$$J(A_\alpha) = B_\alpha, \quad J(B_\alpha) = -A_\alpha \quad (\alpha \in \Delta^+ (\Phi)).$$

Then $J$ can be extended to a $G$-invariant complex structure on $G/K$. $p^\pm$ denote the eigenspaces of $J$ correponding with the eigenvalues $\pm \sqrt{-1}$, that is

$$p^\pm = \sum_{\alpha \in \Delta^+ (\Phi)} CE_{\pm \alpha}.$$

It is known that any $G$-invariant Kähler metric $g$ is given at $o$ by

$$g|_{\mathfrak{g}_\alpha \times \mathfrak{g}_\alpha} = -(\sum_{j=1}^r c_j n_{ij})B \quad (\alpha = \sum_{i=1}^l n_i \alpha_i \in \Delta^+ (\Phi)).$$

Here $c_j$ are positive numbers and $\mathfrak{g}_\alpha = \mathbb{R}A_\alpha + \mathbb{R}B_\alpha$. Conversely, any bilinear form defined by (1.6) on $\mathfrak{p}_C \times \mathfrak{p}_C$ can be extended to a $G$-invariant metric on $G/K$ (see [5]). We have thus obtained a Kähler $C$-space $(G/K, g)$. In the remaining part of this paper we denote this Kähler $C$-space by $M(g, \Pi, \Phi, g)$. 
2. Symmetries of Kähler C-spaces

Let $G$ be a compact Lie group and $K$ a centralizer of a toral subgroup of $G$. Then the homogeneous space $G/K$ is called a generalized flag manifold. It is known that $G/K$ with $G$-invariant metric $(\cdot, \cdot)$ admits a Riemannian $m$-symmetric structure (cf. [1] and [9]). For later use we shall prove this fact in the case where $g$ is simple. As in section 1, we set

$$g = \sum_{\alpha \in \Delta^+} \mathbb{R}\sqrt{-1}H_\alpha + \sum_{\alpha \in \Delta^+} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha),$$

$$\Pi = \{\alpha_1, \cdots, \alpha_l\}, \quad \Phi = \{\alpha_{i_1}, \cdots, \alpha_{i_r}\}.$$

Let $\delta = \sum_{i=1}^l n_i \alpha_i$ be the highest root of $\Delta$ with respect to $\Pi$. For positive integers $m_i$ ($i = 0, \cdots, r$) put $m = m_0 + \sum_{j=1}^r n_j m_j$. Set

$$\sigma(E_{\pm \alpha_i}) = \xi^{\pm m_j} E_{\pm \alpha_i} \quad (\alpha_i \in \Phi),$$

$$\sigma(E_{\pm \delta}) = \xi^{\pm m_0} E_{\pm \delta}, \quad \sigma(E_{\alpha_i}) = E_{\alpha_i} \quad (\alpha_i \in \Phi_0).$$

Here $\xi$ denotes a primitive $m$-th root of unity. Then $\sigma$ can be extended to an inner automorphism of order $m$ of $g_C$. Conversely, every inner automorphism of finite order of $g_C$ is obtained in this way (cf. Helgason [4].)

**Lemma 2.1.** Let $\sigma$ be an inner automorphism of finite order of $g_C$. Then there exist a fundamental root system $\Pi = \{\alpha_1, \cdots, \alpha_l\}$ (with respect to a certain Cartan subalgebra $\mathfrak{h}$) and nonnegative integers $(m_0, m_1, \cdots, m_l)$ without nontrivial common factor such that $\sigma$ satisfies the following:

$$\sigma(E_{\pm \alpha_i}) = \xi^{\pm m_i} E_{\pm \alpha_i}, \quad \sigma(E_{\pm \delta}) = \xi^{\pm m_0} E_{\pm \delta},$$

where $\delta = \sum_{i=1}^l n_i \alpha_i$ denotes the highest root, $m = m_0 + \sum_{i=1}^l n_i m_i$ and $\xi$ a primitive $m$-th root of unity. Moreover $\sigma$ has the form

$$\sigma = e^{\text{ad}H} \quad \text{for some} \ H \in \mathfrak{h}.$$

Since $\sigma^m = 1$, we can see that $H \in \sum_{\alpha} \mathbb{R}\sqrt{-1}H_\alpha$. Therefore we can regard $\sigma$ as an inner automorphism of order $m$ of $g$. We can easily check that $g^\sigma = \mathfrak{k}$, where $g^\sigma$ is the fixed point set of $\sigma$. Set $\phi = (1 + \sigma + \cdots + \sigma^{m-1})$. Then $\phi$ is a linear map of $g$ and $\mathfrak{k} = \text{Im} \phi$. Moreover we have

$$\ker \phi = \sum_{\alpha \in \Delta^+ \Phi} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha) \quad (= \mathfrak{p}).$$

Therefore $g = \mathfrak{k} + \mathfrak{p}$ and $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$.
Let \( \langle \ , \rangle \) be a \( G \)-invariant Riemannian metric on \( G/K \). Then \( \langle \ , \rangle \) is identified with an \( \text{Ad}(K) \)-invariant scalar product on \( p \) (denoted by the same symbol \( \langle \ , \rangle \)). Hence by (2.2) the restriction of \( \sigma \) to \( p \) preserves \( \langle \ , \rangle \).

We denote the inner automorphism of \( G \) corresponding to \( \sigma \) by the same symbol \( \sigma \). Let \( \pi : G \to G/K \) be the canonical projection. Define a transformation \( s \) of \( G/K \) by \( s \circ \pi = \pi \circ \sigma \). Then the differential map of \( s \) at \( o = \{ K \} \) coincides with the restriction of \( \sigma \) to \( p \). Consequently, from Proposition 1.1, \((G/K,\langle \ , \rangle)\) admits a Riemannian \( m \)-symmetric structure.

Let \((M,J,g)\) be a Hermitian manifold with a complex structure \( J \). Suppose that \((M,g)\) admits a Riemannian \( m \)-symmetric structure \( \{ s_x : x \in M \} \). We call \( \{ s_x : x \in M \} \) a Hermitian \( m \)-symmetric structure if each \( s_x (x \in M) \) is a holomorphic isometry of \((M,J,g)\). In particular, if \((M,J,g)\) is Kählerian, then Hermitian \( m \)-symmetric structure is said to be Kählerian. It is known that a Hermitian symmetric space has a Kählerian \( m \)-symmetric structure for any \( m \geq 2 \).

**Proposition 2.2.** Let \( G/K \) be a generalized flag manifold, where \( G \) is simple. Then \( G/K \) admits a \( G \)-invariant complex structure \( J \) such that \((G/K,J,\langle \ , \rangle)\) has a Hermitian \( m \)-symmetric structure for any \( G \)-invariant Riemannian metric \( \langle \ , \rangle \). In particular, a Kähler \( C \)-space admits a Kählerian \( m \)-symmetric structure for some integer \( m \).

**Proof.** We define a \( G \)-invariant complex structure \( J \) by (1.5). Since \( \sum_{\alpha \in \Delta} \mathbb{R} \sqrt{-1} H_\alpha \) is contained in \( \mathfrak{k} \), each metric \( \langle \ , \rangle \) at \( o \) satisfies the following.

\[
\langle A_\alpha, A_\alpha \rangle = \langle B_\alpha, B_\alpha \rangle, \quad \langle A_\alpha, B_\beta \rangle = 0 \\
\left( \mathbb{R} A_\alpha + \mathbb{R} B_\alpha \right) \perp \left( \mathbb{R} A_\beta + \mathbb{R} B_\beta \right), \quad (\alpha, \beta \in \Delta^+(\Phi), \alpha \neq \beta).
\]

Hence \( \langle \ , \rangle \) is a Hermitian metric with respect to \( J \).

Let \( \{ s_x : x \in M \} \) be the Riemannian \( m \)-symmetric structure corresponding with \( \sigma \). Since \( \sigma \) has the form \( e^{\text{ad}H} \) for some \( H \in \mathfrak{k} \), we can see that \( s(= s_o) \) is holomorphic. Therefore, since \( J \) is \( G \)-invariant, \( s_x = g \cdot s \cdot g^{-1} \) \((g \cdot o = x)\) is holomorphic. \( \square \)

Let \( R \) and \( \nabla \) be the curvature tensor and the Levi-Civita connection, respectively, of a Kähler \( C \)-space \( M(g,\Pi,\Phi,\langle \ , \rangle) \). We denote by \( \nabla \) the covariant derivative in the direction of \( p^+ \). According to [10] there exists positive integer \( n \) such that

\[ \nabla^n R = 0 \quad \text{and} \quad \nabla^{n-1} R \neq 0. \]

We call the integer \( n \) the degree of \( M(g,\Pi,\Phi,\langle \ , \rangle) \). Then the degree of a Kähler \( C \)-space is equal to one if and only if it is a Hermitian symmetric space. Moreover the following holds (see [10]).
Proposition 2.3. There exists no Kähler $C$-space with degree two.

Let $\alpha_a$ be any of the simple roots designed by the symbol $\oplus$ and $\alpha_i, \alpha_j$ two of the simple roots designed by the symbol $\otimes$ in the above Dynkin diagrams. Then an irreducible Kähler $C$-space with degree three is one of $M(\mathfrak{g}, \Pi, \{\alpha_a\}, \langle, \rangle)$ and $M(\mathfrak{g}, \Pi, \{\alpha_i, \alpha_j\}, \langle, \rangle)$. (In the diagrams, for $\alpha_p$ corresponding to $\oplus$ or $\otimes$, a Kähler $C$-space $M(\mathfrak{g}, \Pi, \{\alpha_p\}, \langle, \rangle)$ is a Hermitian symmetric space ([5]).)
Let \( M(\mathfrak{g}, \Pi, \Phi, \langle \ , \ \rangle) \) be an irreducible Kähler \( C \)-space with degree three and \( \delta = \sum_{i=1}^{l} n_i \alpha_i \) the highest root. Then by Proposition 2.3 it is easy to see that \( \Phi = \{ \alpha_0 \} \) or \( \Phi = \{ \alpha_j, \alpha_k \} \) with \( n_j = 2 \) and \( n_k = 1 \). Hence \( M(\mathfrak{g}, \Pi, \Phi, \langle \ , \ \rangle) \) has a Kählerian 3-symmetric structure. In fact, take 1 as \( m_a \), \( m_j \) and \( m_k \), and 0 as the other \( m_p \) (see the early part of this section and Proposition 2.2). More precisely, the following holds.

**Theorem 2.4.** The degree of an irreducible Kähler \( C \)-space is three if and only if it is a compact irreducible simply connected Kählerian 3-symmetric space which is not isometric to a Hermitian symmetric space.

**Proof.** Let \( (M, J, \langle \ , \ \rangle) \) be a compact irreducible Kählerian 3-symmetric space and \( \{ s_x : x \in M \} \) a Kählerian 3-symmetric structure of \( (M, J, \langle \ , \ \rangle) \). Let \( \text{Cl}(\{ s_x \}) \) be the closure of the group generated by the set \( \{ s_x : x \in M \} \) in the isometry group of \( (M, g) \). Then \( \text{Cl}(\{ s_x \}) \) is a closed subgroup of the holomorphic isometry group of \( (M, J, \langle \ , \ \rangle) \) and acts transitively on \( M \). Thus \( (M, J, \langle \ , \ \rangle) \) is a Kähler \( C \)-space.

Let \( G \) be the identity component of \( \text{Cl}(\{ s_x \}) \) and \( K \) be the isotropy subgroup of \( G \) at a point \( o \in M \). Then \( K \) is a centralizer of a toral subgroup of \( G \) since \( (M, J, \langle \ , \ \rangle) \) is Kähler \( C \)-space. Define an automorphism \( \sigma \) of order three of \( G \) as follows:

\[
(2.3) \quad \sigma(g) = s_o \circ g \circ s_o^{-1}.
\]

Since \( s_o \circ k = k \circ s_o \) for \( k \in K \) (see [7]) and \( o \) is an isolated fixed point of \( s_o \), we have

\[
(G^\sigma)_0 \subset K \subset G^\sigma, \quad \text{and} \quad g^\sigma = \mathfrak{k}.
\]

Since \( \mathfrak{k} \) contains a maximal abelian subalgebra of \( \mathfrak{g} \) and \( \sigma \) leaves \( \mathfrak{k} \) pointwise fixed, we can see that \( \sigma \) is inner. We set \( \delta = \sum_{i=1}^{l} n_i \alpha_i, \alpha_0 = -\delta \) and \( n_0 = 1 \). (In other words \( \alpha_i \) and \( n_i \) \( 0 \leq i \leq l \) are the vertices and corresponding coefficients in the extended Dynkin diagram (cf. [4])). Then, by Lemma 2.1, the possibilities of \( (m_0, m_1, \ldots, m_l) \) are the following:

(i) \( m_i = m_j = m_k = 1 \) and others are zero. In this case \( n_i = n_j = n_k = 1 \).

(ii) \( m_i = m_j = 1 \) and others are zero. In this case \( n_i = 1, n_j = 2 \).

(iii) \( m_i = 1 \) and others are zero. In this case \( n_i = 3 \).

However, case (iii) is not possible since \( \mathfrak{k} \) must have a nonzero center (in the case, \( g^\sigma \) is semisimple).

If \( \sigma \) is of the form (i), then the degree of \( (M, J, \langle \ , \ \rangle) = G/K \) equals three (if necessary, substitute \(-\alpha_0 \) for \( \alpha_i \)). Similarly, if \( \sigma \) is of the form (ii), then the degree of \( (M, J, \langle \ , \ \rangle) = G/K \) is equal to three.

We have thus proved the theorem. \( \square \)
Remark 2.5. According to Koda [6], except for compact irreducible Kählerian 3-symmetric spaces, compact irreducible 3-symmetric spaces admit no (possibly not invariant) Kählerian structures because their second cohomology groups vanish.

Remark 2.6. Let \( M(g, \Pi, \Phi, \langle , \rangle) \) be a Kähler \( C \)-space and set \( \Phi = \{ \alpha_i, \ldots, \alpha_r \} \). Let \( \delta = \sum_{i=1}^{l} m_i \alpha_i \) be the highest root of \( g \) and put \( m = \sum_{j=1}^{r} m_j \). By the above argument we can see that the space has a Riemannian \((m + 1)\)-symmetric structure. Moreover, in [10], we implicitly proved that the degree of \( M(g, \Pi, \Phi, \langle , \rangle) \) is at most \((2m - 1)\).

3. Isometry groups of Riemannian 3-symmetric spaces

In this section we examine the isometry groups of Riemannian 3-symmetric spaces.

Let \((M, \langle , \rangle)\) be a Riemannian \(m\)-symmetric space \((m > 2)\) and \(\{s_x : x \in M\}\) a Riemannian \(m\)-symmetric structure of \((M, \langle , \rangle)\). Let \(G\) be the identity component of \(\text{Cl}(\{s_x\})\) and \(K\) be the isotropy subgroup of \(G\) at a point \(o \in M\). As stated in Section 2, \(\sigma(g) = s_o \circ g \circ s_o^{-1} \quad (g \in G)\) is an automorphism of order \(m\) of \(G\). Moreover it follow that

\[(G^o)_0 \subset K \subset G^o.\]

Now we shall show the following proposition.

Proposition 3.1. Let \(G\) be a compact, connected, simple Lie group and \(K\) a closed subgroup of \(G\) such that \(G/K\) is simply connected and \(G\) acts effectively on \(G/K\). Let \(\sigma\) be an inner automorphism of order three of \(G\) such that (3.1) is satisfied. Suppose that \(G/K\) is not Riemannian symmetric for a \(G\)-invariant metric \(\langle , \rangle\). Then \(G\) coincides with the identity component of the isometry group of \((G/K, \langle , \rangle)\).

Proof. Let \(\tilde{G}\) be the identity component of the isometry group of \((G/K, \langle , \rangle)\) and \(\tilde{K}\) the isotropy subgroup of \(\tilde{G}\) at a point \(o = \{K\}\). Since \(G\) acts effectively on \(G/K\), the group \(G\) is a closed subgroup of \(\tilde{G}\) and \(K \subset \tilde{K}\). Let \(g, \tilde{g}, \tilde{\alpha}\) and \(\tilde{\varepsilon}\) be the Lie algebras of \(G, K, \tilde{G}\) and \(\tilde{K}\), respectively.

We denote the differential map of \(\sigma\) by the same symbol \(\sigma\). Set \(p = \ker(1 + \sigma + \sigma^2) \quad (\subset g)\). Then \(\varepsilon = \text{Im}(1 + \sigma + \sigma^2), g = \varepsilon \oplus p\) and \([\varepsilon, p] \subset p\). Since \(\sigma\) is inner, the restriction of \(\sigma\) to \(p\) preserves \(\langle , \rangle\). Thus by Proposition 1.1 the space \(M = (G/K, \langle , \rangle)\) has a Riemannian 3-symmetric structure \(\{s_x : x \in M\}\). Moreover

\[s_o \circ \pi = \pi \circ s_o, \quad s_x = g \circ s_o \circ g^{-1} \quad (g \in G, g \circ o = x),\]

where \(\pi : G \to G/K\) be the canonical projection. We note that \(s_o \in K\). Hence the automorphism \(\tilde{\sigma}\) of \(\tilde{G}\) defined by \(\tilde{\sigma}(g) = s_o \circ g \circ s_o^{-1}\) is inner and of order three.
Let $\tilde{g}^\sigma$ be the fixed point set of $\bar{\sigma}$ in $\bar{g}$. Since $o$ is an isolated fixed point of $s_o$, we have

\begin{equation}
\mathfrak{k} \subset \tilde{g}^\sigma \subset \tilde{\mathfrak{k}}.
\end{equation}

Therefore $\tilde{g}$ is semisimple, since $\tilde{G}$ is compact and acts effectively on $M$. Moreover, $\mathfrak{k}$ contains a maximal abelian subalgebra of $\mathfrak{g}$ because $\sigma$ is inner. Thus $M = (G/K, \langle , \rangle)$ is an irreducible Riemannian manifold (see the proof of Theorem 5 in [3]). Also $\tilde{\mathfrak{k}}$ contains a maximal abelian subalgebra of $\tilde{g}$ because $\bar{\sigma}$ is inner. Therefore $\tilde{g}$ must be simple. In fact, if not, then we have the decomposition

$$\tilde{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r, \quad \tilde{\mathfrak{k}} = \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_r,$$

where $\mathfrak{g}_i$ is an ideal of $\bar{g}$ and $\mathfrak{k}_i \subset \mathfrak{g}_i$. This contradicts the irreducibility of $M$.

Using a similar method as in the proof of Theorem 2.4 we shall see that $\tilde{g}^\sigma$ coincides with $\tilde{\mathfrak{k}}$.

Since $\bar{g}$ is simple and $\bar{\sigma}$ is an inner automorphism of order three, $\bar{g}^\sigma$ contains a maximal abelian subalgebra $\mathfrak{h}$ of $\bar{g}$. Furthermore, by Lemma 2.1, there exists a fundamental root system $\Pi = \{\alpha_1, \cdots, \alpha_l\}$ with respect to $\mathfrak{h}_C$ of $\bar{g}_C$ such that the possibilities of $(m_0, m_1, \cdots, m_l)$ are the following:

(i) $m_i = m_j = m_k = 1$ and others are zero. In this case $n_i = n_j = n_k = 1$.  
(ii) $m_i = m_j = 1$ and others are zero. In this case $n_i = 1, n_j = 2$.  
(iii) $m_i = 1$ and others are zero. In this case $n_i = 3$.

Here $-\alpha_0 = \sum_{i=1}^{l} n_i \alpha_i$ is the highest root and we set $n_0 = 1$. Let $\Delta^+$ be the set of positive roots with respect to $\Pi$. For a subset $\Phi = \{\alpha_i, \cdots, \alpha_l\}$ of $\Pi$ we set

$$\Delta^+(\Phi) = \left\{ \alpha \in \Delta^+ : k_{ij} > 0 \text{ for some } j \right\}.$$

Now we shall see that $\tilde{\mathfrak{k}} = \tilde{g}^\sigma$.

**Case (i)** As mentioned in the proof of Theorem 2.4, we may assume that $\alpha_k = \alpha_0$ ($-\alpha_0$ : the highest root). Set $\Phi = \{\alpha_i, \alpha_j\}$. Suppose that there is a root $\alpha \in \Delta^+(\Phi)$ such that

$$\mathfrak{g}_\alpha = (\mathbb{R} A_\alpha + \mathbb{R} B_\alpha) \subset \tilde{\mathfrak{k}} \setminus \tilde{g}^\sigma.$$

If $k_i = 0$ and $k_j = 1$ ($\alpha = \sum_{p=1}^{l} k_p \alpha_p$), then, since $\mathfrak{g}_{\alpha_p}$ is contained in $\tilde{g}^\sigma$ ($p \neq i, j$), we see that $\mathfrak{g}_{\alpha_i}$ is contained in $\tilde{\mathfrak{k}}$. In this case the pair $(\tilde{g}, \tilde{\mathfrak{k}})$ is symmetric (take an involutive automorphism so that $m_i = m_0 = 1$ and the others are zero).

If $k_i = k_j = 1$ ($\alpha = \sum_{p=1}^{l} k_p \alpha_p$), then the same argument as above implies that $\mathfrak{g}_{\alpha_0}$ is contained in $\tilde{\mathfrak{k}}$. Moreover $\mathfrak{g}_{\alpha_i}$ and $\mathfrak{g}_{\alpha_j}$ are not contained in $\tilde{\mathfrak{k}}$, since we assume
Then \( \tilde{\mathfrak{g}} \) coincides with \( \mathfrak{g}^\tau \), where \( \tau \) is the inner automorphism of order two of \( \mathfrak{g} \) defined by the relation \( m_i = m_j = 1 \) and \( m_k = 0 \) (\( k \neq i, j, 0 \leq k \leq l \)). Hence \((\tilde{\mathfrak{g}}, \tilde{\mathfrak{f}})\) is a symmetric pair.

Consequently, in this case, \( \tilde{\mathfrak{g}}^\sigma = \tilde{\mathfrak{f}} \), since we assume that \( M = G/K = \tilde{G}/\tilde{K} \) is not symmetric.

**Case (ii)** As in the Case (i) we assume \( i = 0 \). Suppose that there is a root \( \alpha = \sum_{p=1}^l k_p \alpha_p \) in \( \Delta^+(\alpha_j) \) such that

\[
\mathfrak{g}_\alpha = (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha) \subset \tilde{\mathfrak{f}} \setminus \tilde{\mathfrak{g}}.
\]

It is clear that \( k_j = 1 \) or 2. If \( k_j = 1 \), then \( \mathfrak{g}_{\alpha_j} \subset \tilde{\mathfrak{f}} \), that is, \( \tilde{\mathfrak{f}} = \tilde{\mathfrak{g}} \). This is a contradiction.

If \( k_j = 2 \), then \( \mathfrak{g}_{\alpha_0} \subset \tilde{\mathfrak{f}} \). Then \( \tilde{\mathfrak{f}} \) coincides with \( \mathfrak{g}^\tau \), where \( \tau \) is the inner automorphism of order two of \( \mathfrak{g} \) defined by the relation \( m_j = 1 \) and \( m_k = 0 \) (\( k \neq j, 0 \leq k \leq l \)). Hence the pair \((\tilde{\mathfrak{g}}, \tilde{\mathfrak{f}})\) is symmetric.

**Case (iii)** In this case we can see that \( \mathfrak{g}_{\alpha_j} \subset \tilde{\mathfrak{g}}^\sigma \) for \( j \neq i \) (\( 0 \leq j \leq l \)). Suppose that there is a root \( \alpha = \sum_{p=1}^l k_p \alpha_p \) in \( \Delta^+(\alpha_j) \) such that \( \mathfrak{g}_\alpha \subset \tilde{\mathfrak{f}} \setminus \tilde{\mathfrak{g}} \). Then \( k_i = 1 \) or 2 because \( \mathfrak{g}_{\alpha_0} \subset \tilde{\mathfrak{g}}^\sigma \). If \( k_i = 1 \), then \( \tilde{\mathfrak{f}} \) must be equal to \( \tilde{\mathfrak{g}} \). If \( k_i = 2 \), then since \( \mathfrak{g}_{\alpha_0} \subset \tilde{\mathfrak{f}} \) there is a root \( \beta \) in \( \Delta^+(\alpha_i) \) such that \( \mathfrak{g}_\beta \subset \tilde{\mathfrak{f}} \) and \( h_i = 1 \) (\( \beta = \sum_{j=1}^l h_j \alpha_j \)). Therefore \( \tilde{\mathfrak{f}} = \tilde{\mathfrak{g}} \).

We have thus \( \tilde{\mathfrak{f}} = \tilde{\mathfrak{g}}^\sigma \).

Consequently, \( \tilde{\mathfrak{f}} \) must be equal to \( \tilde{\mathfrak{g}}^\sigma \).

Set \( \tilde{\mathfrak{p}} = \ker(1 + \tilde{\sigma} + \tilde{\sigma}^2) \). Then since \( \tilde{\mathfrak{f}} = \text{Im}(1 + \tilde{\sigma} + \tilde{\sigma}^2) \), we have

\[
\tilde{\mathfrak{g}} = \tilde{\mathfrak{f}} \oplus \tilde{\mathfrak{p}}, \quad [\tilde{\mathfrak{f}}, \tilde{\mathfrak{p}}] \subset \tilde{\mathfrak{p}}.
\]

Then \( \mathfrak{p} = \tilde{\mathfrak{p}} \) because \( \mathfrak{p} \subset \tilde{\mathfrak{p}} \) and \( \dim \mathfrak{p} = \dim M = \dim \tilde{\mathfrak{p}} \). On the other hand, \( \mathfrak{g} = \mathfrak{p} + [\mathfrak{p}, \mathfrak{p}] \) and \( \tilde{\mathfrak{g}} = \tilde{\mathfrak{p}} + [\tilde{\mathfrak{p}}, \tilde{\mathfrak{p}}] \) since \( \mathfrak{g} \) and \( \tilde{\mathfrak{g}} \) are simple Lie algebras. Finally, we have \( \tilde{\mathfrak{g}} = \tilde{\mathfrak{g}} \).

We consider the similar problem in other cases. Let \((G, K)\) be one of the following:

(i) \((\text{Spin}(8), \text{SU}(3)/\mathbb{Z}_3)\),
(ii) \((\text{Spin}(8), G_2)\),
(iii) \((\{L \times L \times L\}/\delta Z, \delta L/\delta Z)\),

where \( L \) and \( Z \) denote the compact, simply connected, simple Lie group and its center, respectively. Moreover \( \delta(g) = (g, g, g) \) (\( g \in L \)). Let \( \mathfrak{l} \) be the Lie algebra of \( L \). Then the Lie algebra \( \delta \mathfrak{l} \) of \( \delta L \) is given by

\[
\delta \mathfrak{l} = \{(X, X, X) : X \in \mathfrak{l}\}.\]
Moreover, the automorphism $\sigma$ of order three of $l \oplus l \oplus l$ is given by $\sigma(X, Y, Z) = (Z, X, Y)$.

Now, we shall show that $\delta l$ is a maximal $\sigma$-invariant subalgebra of $l \oplus l \oplus l$.

Let $\mathfrak{k}$ be a $\sigma$-invariant Lie subalgebra of $l \oplus l \oplus l$ such that $\delta l \subset \mathfrak{k}$. At first, we shall see that there is $X \in l$ such that $(0, 0, X) \in \mathfrak{k}$ if $\mathfrak{k} \neq \delta l$.

We may assume that there exist $X, Y \in l$ ($X \neq Y$) such that $(0, X, Y) \in \mathfrak{k}$. If $[X, Y] \neq 0$, then $(0, 0, [X, Y]) \in \mathfrak{k}$ because $(X, X, X) \in \mathfrak{k}$. Thus we suppose that $[X, Y] = 0$. Then there exists a maximal abelian subalgebra $\mathfrak{h}$ of $l$ such that $X, Y \in \mathfrak{h}$. Let $\Delta$ be the set of nonzero roots of $\mathfrak{g}_C$ with respect to $\mathfrak{h}_C$ and choose a Weyl basis $\{E_\alpha, H_\alpha\}$ ($\alpha \in \Delta$) so that for any $\alpha \in \Delta$,

$$A_\alpha = (E_\alpha - E_{-\alpha}) \in l, \quad B_\alpha = \sqrt{-1}(E_\alpha + E_{-\alpha}) \in l, \quad \sqrt{-1}H_\alpha \in l$$

(see Section 1). Set $X = \sqrt{-1}H$ and $Y = \sqrt{-1}H'$ ($H, H' \in \mathfrak{h}$). Then

$$[[0, \sqrt{-1}H, \sqrt{-1}H'], (A_\alpha, A_\alpha, A_\alpha)] = (0, \alpha(H)B_\alpha, \alpha(H')B_\alpha) \in \mathfrak{k}.$$

Similarly, $(0, \alpha(H)B_\alpha, \alpha(H')B_\alpha) \in \mathfrak{k}$ from which we have

$$[(0, \alpha(H)A_\alpha, \alpha(H')A_\alpha), (0, \alpha(H)B_\alpha, \alpha(H')B_\alpha)] = (0, 2\alpha(H)^2\sqrt{-1}H_\alpha, 2\alpha(H')^2\sqrt{-1}H_\alpha) \in \mathfrak{k}.$$

Now, we may assume $\alpha(H) \neq 0$ since $l$ is simple. If $\alpha(H)^2 = \alpha(H')^2$, then we obtain

$$(\alpha(H)^2\sqrt{-1}H_\alpha, 0, 0) \in \mathfrak{k}$$

since $\alpha(H)^2(\sqrt{-1}H_\alpha, \sqrt{-1}H_\alpha, \sqrt{-1}H_\alpha)$ and $(0, \alpha(H)^2\sqrt{-1}H_\alpha, \alpha(H')^2\sqrt{-1}H_\alpha)$ are in $\mathfrak{k}$. Thus $(0, 0, \sqrt{-1}H_\alpha) \in \mathfrak{k}$ because $\mathfrak{k}$ is $\sigma$-invariant.

We suppose that $\alpha(H)^2 \neq \alpha(H')^2$. Then there exist $\alpha \in \Delta$ and nonnegative number $c$ such that $(0, \sqrt{-1}H_\alpha, c\sqrt{-1}H_\alpha) \in \mathfrak{k}$. Since $\mathfrak{k}$ is $\sigma$-invariant, we have

$$(c\sqrt{-1}H_\alpha, 0, \sqrt{-1}H_\alpha) \in \mathfrak{k}.$$
By induction, (3.4) holds for all $i$ ($1 \leq i \leq l$), since $\mathfrak{l}$ is simple. Therefore $(0, 0, l) \subset \mathfrak{k}$, and $\mathfrak{k}$ coincides with $\mathfrak{l} \oplus \mathfrak{l} \oplus \mathfrak{l}$. We have thus proved the following.

**Lemma 3.2.** $\delta \mathfrak{l}$ is a maximal $\sigma$-invariant subalgebra of $\mathfrak{l} \oplus \mathfrak{l} \oplus \mathfrak{l}$.

Next, let $\sigma$ be an outer automorphism of order three on a compact simple Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}$ is of type $D_4$. Let $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be a fundamental root system (see Proposition 2.3). As before, we choose a Weyl basis $\{E_\alpha, H_\alpha; \alpha \in \Delta\}$ so that it satisfies (1.2). Let $\xi$ be a primitive cube root of unity. Set

$$
\begin{align*}
& a_\pm = E_{\pm \alpha_1} + E_{\pm \alpha_3} + E_{\pm \alpha_4}, \quad a'_\pm = E_{\pm \alpha_1} + \xi E_{\pm \alpha_3} + \xi^2 E_{\pm \alpha_4}, \\
& a''_\pm = E_{\pm \alpha_1} + \xi^2 E_{\pm \alpha_3} + \xi E_{\pm \alpha_4}, \\
& b_\pm = E_{\pm(\alpha_1 + \alpha_2)} + E_{\pm(\alpha_3 + \alpha_2)} + E_{\pm(\alpha_4 + \alpha_2)}, \\
& b'_\pm = E_{\pm(\alpha_1 + \alpha_2)} + \xi E_{\pm(\alpha_3 + \alpha_2)} + \xi^2 E_{\pm(\alpha_4 + \alpha_2)}, \\
& b''_\pm = E_{\pm(\alpha_1 + \alpha_2)} + \xi^2 E_{\pm(\alpha_3 + \alpha_2)} + \xi E_{\pm(\alpha_4 + \alpha_2)}, \\
& c_\pm = E_{\pm(\alpha_1 + \alpha_2 + \alpha_3)} + E_{\pm(\alpha_2 + \alpha_3 + \alpha_4)} + E_{\pm(\alpha_1 + \alpha_2 + \alpha_4)}, \\
& c'_\pm = E_{\pm(\alpha_1 + \alpha_2 + \alpha_3)} + \xi E_{\pm(\alpha_2 + \alpha_3 + \alpha_4)} + \xi^2 E_{\pm(\alpha_1 + \alpha_2 + \alpha_4)}, \\
& c''_\pm = E_{\pm(\alpha_1 + \alpha_2 + \alpha_3)} + \xi^2 E_{\pm(\alpha_2 + \alpha_3 + \alpha_4)} + \xi E_{\pm(\alpha_1 + \alpha_2 + \alpha_4)}.
\end{align*}
$$

Let $g(\sigma, \xi^i)$ be the complex eigenspace of $\sigma$ with eigenvalue $\xi^i$ ($i = 0, 1, 2$). According to Wolf and Gray [11], $\sigma$ is conjugate to $\tau_1$ or $\tau_2$, where $\tau_i$ ($i = 1, 2$) are defined by the following:

**Remark 3.3.** By (3.5) and (3.6) we can see that there is no element $X$ in $\mathfrak{g}(\tau_1, \xi) \oplus \mathfrak{g}(\tau_1, \xi^2)$ such that

$$
[X, \mathfrak{g}(\tau_1, 1)] = \{0\}.
$$

We note that $(\mathfrak{g}, \mathfrak{g}(\tau_1, 1))$ and $(\mathfrak{g}, \mathfrak{g}(\tau_2, 1))$ correspond to the cases (ii) and (i), respectively.
respectively.

Let \((G, K)\) be one of (i), (ii) and (iii). \(g\) and \(\mathfrak{k}\) denote the Lie algebras of \(G\) and \(K\), respectively. Let \(\sigma\) be an outer automorphism of order three of \(g\) such that \(\mathfrak{k} = g^\sigma\). As in Proposition 1.1 we define a transformation \(s\) of \(G/K\) corresponding to \(\sigma\). Let \((\langle , \rangle)\) be a \(G\)-invariant metric on \(G/K\) such that \((\langle , \rangle)\) is preserved by \(s\) at the origin \(o = \{K\}\). Then \((G/K, \langle , \rangle)\) has a Riemannian 3-symmetric structure \(\{s_x : x \in G/K\}\) associated with \(s\). Let \(\tilde{G}\) be the identity component of the isometry group of \((G/K, \langle , \rangle)\) and \(\tilde{g}\) its Lie algebra. Since \(\tilde{G}\) is compact, the algebra \(\tilde{g}\) has the following form:

\[(3.7) \quad \tilde{g} = \mathfrak{z} \oplus g_1 \oplus \cdots \oplus g_r.\]

Here \(\mathfrak{z}\) is the center and \(g_i (i = 1, \cdots, r)\) are simple ideals of \(\tilde{g}\) and \([\tilde{g}, \tilde{g}] = g_1 \oplus \cdots \oplus g_r\).

Define an automorphism \(\tilde{\sigma}\) of \(\tilde{G}\) by \(\tilde{\sigma}(g) = s g^\sigma s^{-1}\). Let \(\tilde{K}\) be the isotropy subgroup of \(\tilde{G}\) at \(o\) and \(\mathfrak{k}\) its Lie algebra. We also denote by \(\tilde{\sigma}\) the differential map of \(\tilde{\sigma}\) at the identity of \(\tilde{G}\). Then, as before, we have \(g^\tilde{\sigma} \subset \mathfrak{k}\). Moreover, since each \(g_i\) in (3.7) is a simple ideal, it is easy to see that

\[\tilde{\sigma}(\mathfrak{z}) = \mathfrak{z}, \quad \tilde{\sigma}(g_i) = g_j,\]

for some \(i, j (i, j = 1, \cdots, j)\). Therefore we may assume that

\n
\[
[\tilde{g}, \tilde{g}] = g_{(1)} \oplus \cdots \oplus g_{(l)} \quad (\tilde{\sigma}\text{-invariant decomposition}),
\]

where \(g_{(i)}\) is a simple ideal of \(\tilde{g}\) and \(g_{(i)} \subset g_{(i)} \oplus \mathfrak{l} \oplus \mathfrak{l}\). In the following we denote the restriction of \(\tilde{\sigma}\) to \(g_{(i)}\) by the same symbol \(\tilde{\sigma}\).

Suppose that \(\mathfrak{t} \neq g_{(i)}^\tilde{\sigma}\). Let \(X = (Z, X_{(1)}, \cdots, X_{(l)})\) be an element of \(\mathfrak{t} \setminus \tilde{g}_{(i)}^\tilde{\sigma}\). Assume that \(X_{(1)} \neq 0\). Then it is easy to see that there exists \(Y \in g_{(1)}^\tilde{\sigma}\) such that \([Y, X_{(1)}]\neq 0\). (In fact, if \(\text{rk}(g_{(1)}^\tilde{\sigma}) = \text{rk}(g_{(1)})\), then take \(Y\) from a maximal abelian subalgebra contained in \(g_{(1)}^\tilde{\sigma}\). For the other cases, by Remark 3.3 we can see that such \(Y\) exists.) In particular, \(g_{(1)}\) is a compact simple Lie algebra from Lemma 3.2. Then \([Y, X_{(1)}]\) is contained in \(\mathfrak{t} \cap g_{(1)}\). Hence the subalgebra \(\mathfrak{t}_{(1)}\) of \(g_{(1)}\) generated by \([Y, X_{(1)}]\) and \(g_{(1)}^\tilde{\sigma}\) is contained in \(\mathfrak{t} \cap g_{(1)}\).

If \(X_{(1)}\) is not in \(\mathfrak{t}_{(1)}\), then we may assume that \(X_{(1)}\) is perpendicular to \(\mathfrak{t}_{(1)}\) with respect to the Killing form of \(g_{(1)}\). Then \([X_{(1)}, \mathfrak{t}_{(1)}]\) is perpendicular to \(\mathfrak{t}_{(1)}\). This contradicts the definition of \(\mathfrak{t}_{(1)}\). Thus \(X_{(1)}\) is contained in \(\mathfrak{t}_{(1)}\). By a similar argument, if \(Z \neq 0\), then \(Z\) is in \(\mathfrak{t}_{(1)}\). However, this contradicts the effectivity of \(\tilde{G}\). Therefore we have

\[\tilde{\mathfrak{t}} = \mathfrak{t}_{1} \oplus \cdots \mathfrak{t}_{l}, \quad (\mathfrak{t}_{i} \subset g_{(i)}).\]

Since \((G/K, \langle , \rangle)\) is simply connected and irreducible (cf. Gray [2]), the algebra \(\tilde{g}\) is simple or \(\mathfrak{l} \oplus \mathfrak{l} \oplus \mathfrak{l}\).
CASE (i) Let $\tilde{G}$ be the identity component of the isometry group of the Riemannian 3-symmetric space

$$M = (\text{Spin}(8)/(\text{SU}(3)/\mathbb{Z}_3), \langle , \rangle).$$

Then $\tilde{\sigma}$ is an outer automorphism of $\tilde{G}$. (If not, then the Euler number of $M$ is nonzero.) Thus, by the above argument, $\tilde{G}$ is one of the following:

$$\text{Spin}(8), \quad \{L \times L \times L\}/\delta \mathbb{Z}.$$

If $\tilde{G} = \{L \times L \times L\}/\delta \mathbb{Z}$, Then by Lemma 3.2 we have

$$M = (\{L \times L \times L\}/\delta \mathbb{Z})/(\delta L/\delta \mathbb{Z}).$$

However, from [6] we can see that $\text{Spin}(8)/(\text{SU}(3)/\mathbb{Z}_3)$ is not diffeomorphic to it for any compact simple Lie group $L$. Thus $\text{Spin}(8)$ is the identity component of the isometry group.

CASE (ii) By similar argument as above, $\tilde{G}$ is one of the following:

$$\text{Spin}(8), \quad \{L \times L \times L\}/\delta \mathbb{Z}.$$

However, since there is no simple Lie algebra with dimension seven, the latter case is impossible. Thus $\text{Spin}(8)$ coincides with the identity component of the isometry group.

Finally, we consider the case (iii). We shall prove the following lemmas.

**Lemma 3.4.** Let $g = D_4$. Then $g(\tau_2, 1)$ is a maximal subalgebra of $g$.

**Lemma 3.5.** Let $g = D_4$. Then $B_3$ and $g(\tau_1, 1)$ are only proper subalgebras containing $g(\tau_1, 1)$. Here the pair $(g, B_3)$ is symmetric.

If the lemmas hold, then $\{L \times L \times L\}/\delta \mathbb{Z}$ coincides with the identity component of the isometry group of

$$(((L \times L \times L)/\delta \mathbb{Z})/(\delta L/\delta \mathbb{Z}), \langle , \rangle).$$

In fact, if the Lie algebra of the isometry group coincides with $D_4$, then the Lie algebra of the isotropy subgroup must be equal to one of $g(\tau_1, 1)$, $g(\tau_2, 1)$ and $B_3$. However, this contradicts the above argument. (Since $\dim g - \dim B_3 = 7$, the last case is impossible.)
Proof of Lemma 3.4. In this case \( g(\tau_2, 1) \) is isomorphic to \( A_2 \). Set

\[
H_0 = H_{\alpha_1} + H_{\alpha_3} + H_{\alpha_4}, \quad H_1 = H_{\alpha_1} + \xi^2 H_{\alpha_3} + \xi H_{\alpha_4}, \\
H_2 = H_{\alpha_1} + \xi H_{\alpha_3} + \xi^2 H_{\alpha_4}.
\]

Then we note that

\[
\sqrt{-1}H_0, \quad (H_1 - H_2), \quad \sqrt{-1}(H_1 + H_2), \quad (a_+ - a_-), \quad \sqrt{-1}(a_+ + a_-) \in g
\]

\[
(a'_+ - a'_-), \quad \sqrt{-1}(a'_+ + a'_-), \quad (a'_- - a'_+), \quad \sqrt{-1}(a'_- + a'_+) \in g,
\]

\[
\cdots, \quad (c'_- - c'_+), \quad \sqrt{-1}(c'_- + c'_+) \in g.
\]

Let \( \mathfrak{k} \) be a subalgebra of \( g \) such that \( g(\tau_2, 1) \subset \mathfrak{k} \) and \( g(\tau_2, 1) \neq \mathfrak{k} \). Let \( X \) be an element of \( \mathfrak{k} \setminus g(\tau_2, 1) \). Since \( \sqrt{-1}H_{\alpha_2} \) and \( \sqrt{-1}H_0 \) are contained in \( g(\tau_2, 1) \), we may assume that \( X \) is contained in one of the following (see (3.6)):

\[
\begin{align*}
&\mathcal{C}a'_+ \oplus \mathcal{C}a''_+, \quad \mathcal{C}b'_+ \oplus \mathcal{C}b'_- \oplus \mathcal{C}b''_+,
&\mathcal{C}c'_+ \oplus \mathcal{C}c'_- \oplus \mathcal{C}c''_+, \quad \mathcal{C}E_{\pm(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}, \\
&\mathcal{C}E_{\pm\alpha_2}, \quad \mathcal{C}E_{\pm(\alpha_1+2\alpha_2+\alpha_3+\alpha_4)}, \quad \mathcal{C}H_1 \oplus \mathcal{C}H_2.
\end{align*}
\]

(Consider \( \sqrt{-1}H, X \) for some \( H \in \mathbb{R}H_0 \oplus \mathbb{R}H_{\alpha_2} \).)

1. The case \( X \in \mathcal{C}E_{\pm\alpha_2} \).

In this case \( Y = \sqrt{-1}H_{\alpha_2}X \) is also in \( \mathfrak{k} \). Hence we have \( E_{\pm\alpha_2} \in \mathfrak{g}_C \). On the other hand, it is known that \( E_{\alpha_2}, a_+ \) and \( c''_+ \) generate \( g_C \) (cf. chapter X of [4]).

Thus \( \mathfrak{g}_C = \mathfrak{g}_C \), that is, \( \mathfrak{k} = \mathfrak{g} \).

2. The case \( X \in \mathcal{C}E_{\pm(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \).

As in (1), we can see that \( E_{\pm(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \in \mathfrak{g}_C \). Then

\[
[a_-, E_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}] \in g(\tau_2, \xi) \cap (\mathcal{C}E_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} + \mathcal{C}E_{\alpha_1+\alpha_2+\alpha_4} + \mathcal{C}E_{\alpha_2+\alpha_3+\alpha_4}).
\]

Thus \( c_+ \in \mathfrak{g}_C \). Similarly we have \( b_+ \in \mathfrak{g}_C \) and \( E_{\alpha_2} \in \mathfrak{g}_C \). Hence, by the same reason as (1), it follows that \( \mathfrak{k} = \mathfrak{g} \).

3. The case \( X \in \mathcal{C}E_{\pm(\alpha_1+2\alpha_2+\alpha_3+\alpha_4)} \).

As in (1), we can see that \( E_{\pm(\alpha_1+2\alpha_2+\alpha_3+\alpha_4)} \in \mathfrak{g}_C \). Then we get

\[
[E_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}, b''_+|(\neq 0) \in g(\tau_2, \xi^2)
\]

\[
\cap (\mathcal{C}E_{\alpha_1+\alpha_2+\alpha_3} + \mathcal{C}E_{\alpha_1+\alpha_2+\alpha_4} + \mathcal{C}E_{\alpha_2+\alpha_3+\alpha_4}).
\]

Hence \( c''_+ \in \mathfrak{g}_C \). Similarly we can check that \( c'_- \in \mathfrak{g}_C, b''_+, b'_- \in \mathfrak{g}_C \). Then

\[
[b'_+, b'_-] = H_1 \in \mathfrak{g}_C, \quad [c''_+, c''_-] = -\xi H_2 \in \mathfrak{g}_C.
\]

Then there is \( H \in \sum_{i=0}^3 \mathcal{C}H_i \) (\( H_3 = H_{\alpha_2} \)) such that \( \alpha_2(H) = \alpha_3(H) = \alpha_4(H) = 0 \) and \( \alpha_1(H) \neq 0 \). Thus we can see that \( E_{\pm\alpha_1} \in \mathfrak{g}_C \). Similar argument implies that \( E_{\pm\alpha} \in \mathfrak{g}_C \) for all \( \alpha \in \Delta \). Therefore \( \mathfrak{k} = \mathfrak{g} \).
(4) The case \( X \in \mathbb{C}b_+ \oplus \mathbb{C}b' \oplus \mathbb{C}b'' \).

In this case we may assume that
\[
\{(b_+ + pb''_+ + qb'_-), \quad (b_- + rb''_+ + sb'_-) \in \mathfrak{t}_C \} \quad \text{or} \quad \{(b''_+, \quad b'_-) \in \mathfrak{t}_C \},
\]
for some \( p, q, r, s \in \mathbb{C} \). If \( b''_+, b'_- \in \mathfrak{t}_C \), then \( [b''_+, c'_+] \in \mathbb{C}E_{-\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4} \subset \mathfrak{t}_C \).

Thus \( E_{\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4} \) (and \( E_{-(\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4)} \)) is contained in \( \mathfrak{t}_C \). Hence this case is reduced to (3).

If \( (b_+ + pb''_+ + qb'_-), \quad (b_- + rb''_+ + sb'_-) \in \mathfrak{t}_C \), then
\[
[a_+, (b_+ + pb''_+ + qb'_-)] \in Cc_+ \oplus Cc''_+ \oplus \{0\},
\]
\[
[a_+, [a_+, (b_+ + pb''_+ + qb'_-)]] \in CE_{\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4} \oplus \{0\} \oplus \{0\}.
\]

Therefore we have \( E_{\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4} \in \mathfrak{t}_C \) (and \( E_{-(\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4)} \in \mathfrak{t}_C \)). This case is reduced to (2).

(5) The case \( X \in \mathbb{C}c_+ \oplus \mathbb{C}c'_+ \).

In this case we may assume that
\[
\{(a''_+ + pa''_+ + qa''_+), \quad (a'_- + ra'_+ + sa''_+) \in \mathfrak{t}_C \} \quad \text{or} \quad \{a'_+, \quad a''_+ \in \mathfrak{t}_C \},
\]
for some \( p, q, r, s \in \mathbb{C} \). If \( c'_+ \) and \( c''_+ \) are in \( \mathfrak{t}_C \), then we have \( [b''_+, a'_+] \in CE_{-\alpha_2} \) and \( [b'_+, a''_+] \in CE_{\alpha_2} \). This case is reduced to (1).

If \( (a''_+ + pa''_+ + qa''_+) \) and \( (a'_- + ra'_+ + sa''_+) \) are in \( \mathfrak{t}_C \), then
\[
[[a''_+ + pa''_+ + qa''_+], c'_+] \in CE_{\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4} \oplus \{0\} \oplus \mathbb{C}b_+,
\]
\[
[[a''_+ + pa''_+ + qa''_+], c'_+, a_-] \in Cc_+ \oplus \{0\} \oplus CE_{\alpha_2},
\]
\[
[[[a''_+ + pa''_+ + qa''_+], c'_+, a_-], a_-] \in CE_{\alpha_2} \oplus \{0\} \oplus \{0\}.
\]

Hence \( E_{\alpha_2} \in \mathfrak{t}_C \). Similarly we have \( E_{-\alpha_2} \in \mathfrak{t}_C \). This case is reduced to (1).

(6) The case \( X \in \mathbb{C}c_+ \oplus \mathbb{C}c'_- \oplus \mathbb{C}c''_+ \).

In this case we may assume that
\[
(c_+ + pc'_- + qc''_+), \quad (c_- + rc'_- + sc''_+) \in \mathfrak{t}_C \quad \text{or} \quad c'_-, \quad c''_+ \in \mathfrak{t}_C,
\]
for some \( p, q, r, s \in \mathbb{C} \). If \( c'_- \) and \( c''_+ \) are in \( \mathfrak{t}_C \), then
\[
[c''_+, b'_+] \in CE_{\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4}, \quad [c'_-, b''_+] \in CE_{-(\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4)}
\]
are contained in \( \mathfrak{t}_C \). This case is reduced to (3).

If \( (c_+ + pc'_- +qc''_+) \) and \( (c_- + rc'_- + sc''_+) \) are in \( \mathfrak{t}_C \), then since
\[
[c_+ + pc'_- + qc''_+, a_+] \in CE_{\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4} \oplus \{0\} \oplus \{0\},
\]
\[
[c_- + rc'_- + sc''_+, a_-] \in CE_{-(\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4)} \oplus \{0\} \oplus \{0\},
\]
it follows that $E_{\pm}(\alpha_1+\alpha_2+\alpha_3+\alpha_4) \in \mathfrak{k}_c$. Hence this case is reduced to (2).

(7) The case $X \in \mathfrak{ch}_1 \oplus \mathfrak{ch}_2$.

It is easy to see that $[X, a_{\pm}] \neq 0$ and $[X, a_{\pm}]$ are contained in $\mathfrak{ca}'_+ \oplus \mathfrak{ca}'_-$. Thus this case is reduced to (5).

We have thus proved the lemma. □

Sketch of the proof of Lemma 3.5. Suppose that there exists a Lie subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ such that $\mathfrak{k}$ contains $\mathfrak{g}(r, 1)$. As above, we may assume that there is $X \in \mathfrak{k} \setminus \mathfrak{g}(r, 1)$ such that $X$ is contained in one of the following (see (3.5)):

- $(\mathbb{R}(H_1 - H_2) \oplus \mathbb{R}\sqrt{-1}(H_1 + H_2))$,
- $(\mathbb{R}(a'' - a'_+) \oplus \mathbb{R}\sqrt{-1}(a'_+ + a')$,
- $(\mathbb{R}(b'' - b'_+) \oplus \mathbb{R}\sqrt{-1}(b'_+ + b'))$,
- $(\mathbb{R}(c'' - c'_+) \oplus \mathbb{R}\sqrt{-1}(c'_+ + c'))$.

In particular, we may suppose that there exists an element in $\mathbb{R}(H_1 - H_2) \oplus \mathbb{R}\sqrt{-1}(H_1 + H_2)$ such that it is contained in $\mathfrak{k}$. In fact, if $X$ is in $\mathbb{R}(a'' - a'_+) \oplus \mathbb{R}\sqrt{-1}(a'_+ + a')$, then

$$\begin{align*}
(a'' - a'_+) + p(a'' - a'_+) + q\sqrt{-1}(a'' + a'_+) & \in \mathfrak{k}, \\
\sqrt{-1}(a'' + a'_+) + r(a'' - a'_+) + s\sqrt{-1}(a'' + a'_+) & \in \mathfrak{k},
\end{align*}$$

or $\begin{align*}
(a'' - a'_+) & \in \mathfrak{k}.
\end{align*}$

If $(a'' - a'_+) \in \mathfrak{k}$, then we have

$$[a'' - a'_+, \sqrt{-1}(a'_+ + a_+)] \in \mathbb{R}(H_1 - H_2) \oplus \mathbb{R}\sqrt{-1}(H_1 + H_2) \subset \mathfrak{k}.$$ 

For the other cases, we can check that there exists an element in $\mathbb{R}(H_1 - H_2) \oplus \mathbb{R}\sqrt{-1}(H_1 + H_2)$ such that it is contained in $\mathfrak{k}$. Thus we assume that there exist $p, q \in \mathbb{R}$ such that

$$X = p(H_1 - H_2) + q\sqrt{-1}(H_1 + H_2) \in \mathfrak{k}.$$

Since $[X, \mathfrak{g}(r, 1)] \subset \mathfrak{k}_c$ and $[X, [X, \mathfrak{g}(r, 1)]] \subset \mathfrak{k}_c$, we can check that if $\mathfrak{k} \neq \mathfrak{g}$ then $H_{\alpha_i}$ $(i = 1, 3$ or $4)$ is in $\mathfrak{k}$. For any case we can see that $\mathfrak{k}$ is isomorphic to $B_3$ and the pair $(\mathfrak{g}, B_3)$ is symmetric. □

Finally we have the following.

**Theorem 3.6.** Let $(M, \langle , \rangle)$ be a compact irreducible simply connected Riemannian $3$-symmetric space which is not isometric to a symmetric space. Then there exists a unique pair $(G, K)$ of a compact connected Lie group $G$ and a closed subgroup $K$ of $G$ satisfying (3.1) such that $(M, \langle , \rangle) = G/K$ and $G$ acts effectively on $M$. In particular, $G$ is the identity component of the isometry group of $(M, \langle , \rangle)$. 

References


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