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# SOME RESULTS ON THE WELL-POSEDNESS FOR SECOND ORDER LINEAR EQUATIONS

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**Abstract**

We investigate the Cauchy problem for second order hyperbolic equations of complete form, and we prove an extension of a classical result of Oleĭnik [10] concerning the well-posedness for equations in which are absent the terms with mixed time-space derivatives. Then, in space dimension $n = 1$, we compare our results with those in [8] for equations with analytic coefficients, and those of [7] and [11] for homogeneous equations with coefficients depending only either on $t$ or on $x$.

Moreover we exhibit, in space dimension $n \geq 2$, an equation of the form

$$u_{tt} - \sum_{i,j=1}^{n} (a_{ij}(t,x)u_{x_j})_{x_i} = 0, \quad \text{with} \quad \sum a_{ij}\xi_i\xi_j \geq 0,$$

where the coefficients are analytic functions, for which the Cauchy problem is ill-posed.

Finally, we present a sufficient condition for the well-posedness of $2 \times 2$ systems.

## Introduction

We consider the Cauchy problem

$$\begin{cases} \mathrm{L}\, u = f(t,x), \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \end{cases} \tag{1}$$

where L is a differential, second order operator which we write *in variational form*:

$$\mathrm{L}\, u = u_{tt} - \sum_{i,j=1}^{n} (a_{ij}(t,x)u_{x_j})_{x_i} + \sum_{i=1}^{n} [(b_i u_{x_i})_t + (b_i u_t)_{x_i}] + cu_t + \sum_{i=1}^{n} d_i u_{x_i} + eu. \tag{2}$$

With no loss of generality, we can assume that $a_{ij} = a_{ji}$. Note that the polynomial

$$p^{\mathrm{s}}(t,x,\tau,\xi) = c(t,x)\tau + \sum_{i=1}^{n} d_i(t,x)\xi_i \tag{3}$$

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is the *sub-principal symbol* of L. We can also write L in the non variational form

$$\mathrm{L}\,u = u_{tt} - \sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + 2\sum_{i=1}^{n} b_i u_{t x_i} + (c + c^\sharp) u_t + \sum_{i=1}^{n} (d_i + d_i^\sharp) u_{x_i} + e u, \tag{4}$$

where $c^\sharp = \sum_j \partial_{x_j} b_j$ and $d_i^\sharp = \partial_t b_i - \sum_j \partial_{x_j} a_{ij}$.

Trough this paper, all the coefficients of L are assumed to be real-valued $\mathcal{C}^\infty$ functions, bounded together with all their derivatives in the strip

$$G_T := [0, T] \times \mathbb{R}^n,$$

i.e. which belong to the class $\mathcal{B}^\infty(G_T, \mathbb{R})$.

We say that (1) is *well-posed* (in $\mathcal{C}^\infty$) when, for each $\overline{x} \in \mathbb{R}^n$, there is some nbd. $V$ of $(0, \overline{x})$ in $G_T$ such that there is a unique solution $u \in \mathcal{C}^\infty(V)$ for all $u_0, u_1 \in \mathcal{C}^\infty(\mathbb{R}^n)$ and all $f \in \mathcal{C}^\infty(G_T)$. If, moreover, we can take $V = G_T$, then we say that (1) is *globally well-posed*.

A *necessary condition* for the well-posedness is the *hyperbolicity* of the operator L, which means that, for all $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$,

$$\left( \sum_{i=1}^{n} b_i(t, x) \xi_i \right)^2 + \sum_{i,j=1}^{n} a_{ij}(t, x) \xi_i \xi_j \geq 0. \tag{5}$$

However, such a condition is far to be *sufficient*. Thus, we look for some additional conditions which can ensure the well-posedness of (1).

A very simple condition was found in 1970 by Oleĭnik, limited to the operators L which do not contain terms with mixed derivatives $\partial_t \partial_{x_i}$, i.e. of the form

$$\mathrm{L}\,u = u_{tt} - \sum_{i,j=1}^{n} (a_{ij}(t, x) u_{x_j})_{x_i} + c(t, x) u_t + \sum_{i=1}^{n} d_i(t, x) u_{x_i} + e(t, x) u. \tag{6}$$

In the following we shall refer to the operators of type (6) as to the *incomplete* operators.

**Theorem** (O.A. Oleĭnik, [10]). *The Cauchy problem for any hyperbolic operator of type* (6) *is globally well-posed if there exist two positive constants* $C$, $A$ *for which*

$$t \left[ \sum_{i=1}^{n} d_i(t, x) \xi_i \right]^2 \leq C \left\{ A \sum_{i,j=1}^{n} a_{ij}(t, x) \xi_i \xi_j + \sum_{i,j=1}^{n} \partial_t a_{ij}(t, x) \xi_i \xi_j \right\}. \tag{7}$$

We notice that (7) is fulfilled whenever the $d_i$'s vanish identically and the $a_{ij}$'s are not depending on $t$.

In order to extend this theorem to any operator of type (2), a special role is played by the operators such that for each pair $i, j \in \{1, \dots, n\}$, one at least of the following alternatives holds:

$$\text{either} \quad b_i \equiv 0 \quad \text{or} \quad \partial_{x_i} b_j \equiv 0. \tag{8}$$

The condition (8) means that each one of the coefficients $b_j(t, x)$ is depending, besides the variable $t$, only on those spatial variables $x_i$ for which $b_i \equiv 0$. In particular (8) holds true whenever all the $b_j \equiv b_j(t)$ are depending only on $t$. If $n = 1$, (8) simply means that $b \equiv b(t)$. If $n = 2$, (8) holds true only in the following cases:

- when $b_1 \equiv b_1(t, x_1)$ and $b_2 \equiv 0$,
- when $b_1 \equiv 0$ and $b_2 \equiv b_2(t, x_1)$,
- when $b_1 \equiv b_1(t)$ and $b_2 \equiv b_2(t)$.

Finally, we put

$$\Delta_{ij} = b_i b_j + a_{ij} \quad (i, j = 1, \dots, n),$$

so that the condition of hyperbolicity (5) reads

$$\sum_{i,j=1}^{n} \Delta_{ij}(t, x)\xi_i \xi_j \geq 0. \tag{9}$$

**Theorem 1.** *We distinguish two cases*:

- *For any hyperbolic operator of type* (2) *satisfying the condition* (8), *the Cauchy problem is globally well-posed if, for some constants* $C, A > 0$, *one has*

$$t\left[\sum_{i=1}^{n}(d_i - cb_i)\xi_i\right]^2 \leq C\Phi_A(t, x, \xi), \tag{10}$$

*where*

$$\Phi_A = \sum_{i,j=1}^{n}\left\{A\Delta_{ij} + \partial_t \Delta_{ij} + \sum_{h=1}^{n}(b_h \partial_{x_h}\Delta_{ij} - 2\Delta_{ih}\partial_{x_h} b_j)\right\}\xi_i \xi_j. \tag{11}$$

- *In absence of the condition* (8), *i.e. for a hyperbolic operator of the general type* (2), *in order to get the well-posedness we must replace* (10) *with the stronger condition*

$$t\left\{\left[\sum_{i=1}^{n}(d_i - cb_i)\xi_i\right]^2 + \rho(t, x, \xi)\right\} \leq C\Phi_A(t, x, \xi), \tag{12}$$

*where*

$$\rho = \sum_{i=1}^{n} \Delta_{ii}(t, x)\xi_i^2.$$

*Moreover, the well-posedness is no longer global, in general.*

In the special case of operators of type (6), the condition (8) is trivially fulfilled and our condition (10) coincides with the Oleĭnik's condition (7), since we have

$$\Delta_{ij} = a_{ij} \quad (i, j = 1, \dots, n).$$

REMARK 1. If the *sub-principal symbol* $p^s$ in (3) vanishes identically then our condition (10) (resp. (12)) reduces to:

$$0 \le \Phi_A \quad (\text{resp. } \rho \le C\Phi_A).$$

In this case the operator (2) can be written in the simpler form:

$$\mathrm{L}\, u = e(t, x)u - \sum_{i,j=0}^{n} (a_{ij}(t, x)u_{x_j})_{x_i}, \tag{13}$$

where we used the notation $x_0 = t$ and we put

$$a_{00} = -1, \quad a_{i0} = -b_i = a_{0i} \quad (i = 1, \dots, n).$$

REMARK 2. If the leading coefficients $a_{ij}$'s and $b_i$'s are not depending on $x$, then (8) holds true and

$$\Phi_A = A \sum_{i,j=1}^{n} \Delta_{ij}(t)\xi_i\xi_j + \sum_{i,j=1}^{n} \partial_t \Delta_{ij}(t)\xi_i\xi_j.$$

In the one dimensional case, i.e. for the operator

$$\mathrm{L}\, u = u_{tt} - (au_x)_x + [(bu_x)_t + (bu_t)_x] + cu_t + du_x + eu, \tag{14}$$

with

$$\Delta(t, x) \equiv (b(t, x))^2 + a(t, x) \ge 0,$$

our condition (12) takes a much simpler form:

**Corollary 1.** *Let $n = 1$. The Cauchy problem for a hyperbolic operator of the general type* (14) *is well-posed if, for some constants $C$, $A > 0$:*

$$t(d - cb)^2 \leq C\{A\Delta + \Delta_t + b\Delta_x\}. \tag{15}$$

*Moreover, when $b \equiv b(t)$ the well-posedness is global.*

In particular, the homogeneous operator:

$$\mathrm{L} = \mathrm{L}_2 := \partial_t^2 - a(t, x)\partial_x^2 + 2b(t, x)\partial_t\partial_x, \tag{16}$$

can be written in the form (14) with:

$$c = -c^\sharp = -b_x, \tag{17}$$

$$d = -d^\sharp = -b_t + a_x. \tag{18}$$

In such a case our condition (15), i.e.

$$t(-b_t + a_x + bb_x)^2 \leq C\{A\Delta + \Delta_t + b\Delta_x\},$$

reduces to:

$$t(b_t + bb_x)^2 \leq C'\{A\Delta + \Delta_t + b\Delta_x\}. \tag{19}$$

Indeed, we can apply the estimate

$$\Delta_x^2 \leq C''\Delta,$$

for some constant $C'' > 0$, thanks to the following well-known result:

**Lemma 1** (Glaeser's inequality [5]). *If $f \in \mathcal{B}^2(\mathbb{R}, \mathbb{R})$, $f(x) \geq 0$, then the following holds true*:

$$(f'(x))^2 \leq \sup_{y\in\mathbb{R}} \| f''(y)\| f(x) \leq Cf(x), \tag{20}$$

*for any $x \in \mathbb{R}$ and for some constant $C > 0$ not depending on $x$.*

In particular, the Oleĭnik's condition (7) for the *incomplete* homogeneous operator, that is

$$\mathrm{L} = \partial_t^2 - a(t, x)\partial_x^2, \tag{21}$$

becomes

$$0 \leq Aa + a_t.$$

As a matter of fact, to prove Theorem 1 we use a local change of variables, leaving the lines $t = const$ invariant, that transforms an operator L of the type (2) into an *incomplete* operator of the type (6), to which we apply the Oleĭnik's theorem. The new space variables

$$y_i = g_i(t, x) \quad (i = 1, \dots, n)$$

are implicitly defined as the (unique) solution of the Cauchy problem:

$$\begin{cases} \left(\partial_t + \sum_{j=1}^{n} b_j(t, x)\partial_{x_j}\right) g_i(t, x) = 0, \\ g_i(0, x) = x_i, \end{cases} \tag{22}$$

In particular, if L satisfies (8) this change of variables is global and explicit.

**Plan of the work.** For the reader's convenience, in §1 we give a direct proof of the Corollary 1 while §2 is devoted to the proof of Theorem 1.

In §3 we assume that the coefficients of L are analytic (at the origin). We recall the result of Nishitani [8] in space dimension $n = 1$, and we compare it with our Corollary 1. In particular, the Nishitani's Theorem states that, if the *sub-principal symbol* is identically zero, then the Cauchy problem (1) for L is well posed (at the origin). It is well-known that in space dimension $n \geq 2$ there exists some operators with analytic coefficients and *sub-principal symbol* zero for which the Cauchy problem is ill-posed. However, one could ask if the Cauchy problem for an *incomplete* operator with analytic coefficients is always well-posed. We prove that the answer to this question is negative by exhibiting a counter-example.

In §4 we consider the *complete operator* in space dimension $n = 1$ with the additional hypothesis:

$$b^2(t, x) \leq C\Delta(t, x). \tag{23}$$

In particular, Spagnolo and Taglialatela [11] proved that (23) is a sufficient condition for the well-posedness of the Cauchy problem for the homogeneous operator with coefficients not depending on $t$. We show that in this case our condition (12) is more general than (23). Moreover, we present a corollary of Theorem 1 that extends the result in [11] to space dimension $n \geq 2$.

In §5 we present a sufficient condition for the well-posedness of $2 \times 2$ systems in space dimension $n = 1$. Again, the proof relies on a change of variables.

## 1. Proof of Corollary 1

First of all, we notice that Corollary 1 can be easily derived from Theorem 1. Indeed, in space dimension $n = 1$, condition (10) and (12) become:

$$t(d - cb)^2\xi^2 \leq C\Phi_A(t, x, \xi), \tag{24}$$

$$t\{(d - cb)^2 + \Delta\}\xi^2 \leq C\Phi_A(t, x, \xi), \tag{25}$$

where

$$\Phi_A = \{A\Delta + \Delta_t + b\Delta_x - 2b_x\Delta\}\xi^2. \tag{26}$$

Both conditions (24) and (25) are trivially equivalent to (15). Indeed, we have:

$$2|b_x|\Delta \leq 2\|b_x\|_\infty\Delta \leq C'\Delta.$$

Now we give a direct proof of Corollary 1.

For the sake of brevity and the ease of reading, we introduce the following:

NOTATION. We write $f \lesssim g$ to mean that there is some constant $C > 0$ such that

$$f(t, x) \leq Cg(t, x). \tag{27}$$

We write $f \approx g$ to mean that $f \lesssim g$ and $g \lesssim f$.

The Cauchy problem (22) in space dimension $n = 1$ becomes:

$$\begin{cases}(\partial_t + b(t, x)\partial_x)g(t, x) = 0,\\ g(0, x) = x,\end{cases} \tag{28}$$

that is trivially well-posed. Let $g \in \mathcal{C}^\infty(U, \mathbb{R})$ be the (unique) solution of (28) in some nbd. $U$ of the initial line $\{t = 0\}$ in $G_T$.

We define the vector-valued function:

$$\mathcal{G} := (\pi_0, g)\colon (t, x) \in U \mapsto (s, y) := (t, g(t, x)) \in G_T,$$

where $\pi_0$ is the projection on the time-axis. We remark that $g_x(0, \cdot) \equiv 1$, hence we can take a nbd. $V \subset U$ of $\{t = 0\}$, such that

$$\frac{1}{2} \leq g_x(t, x) \leq 2. \tag{29}$$

Consequently

$$\mathcal{G}\colon V \to W := \mathcal{G}(V)$$

is a smooth change of variables, since:

$$\det \nabla_{t,x}\mathcal{G} = \det \begin{pmatrix} 1 & 0 \\ g_t & g_x \end{pmatrix} = g_x.$$

Let $\mathcal{H} = (\pi_0, h)\colon W \to V$ be the inverse of $\mathcal{G}$.

For the sake of simplicity, we introduce the following notation:

NOTATION. For any function $u \equiv u(t, x)$ or $v \equiv v(s, y)$ we define:

$$\tilde{u}(s, y) := u \circ \mathcal{H}(s, y),$$
$$\hat{v}(t, x) := v \circ \mathcal{G}(t, x).$$

We can write explicitly some relations between $g$ and $h$. Indeed $h \circ \mathcal{G}(t, x) = x$, hence we have:

$$0 \equiv (h \circ \mathcal{G})_t = \hat{h}_s(\pi_0)_t + \hat{h}_y g_t = \hat{h}_s + \hat{h}_y g_t, \tag{30}$$

$$1 \equiv (h \circ \mathcal{G})_x = \hat{h}_s(\pi_0)_x + \hat{h}_y g_x = \hat{h}_y g_x. \tag{31}$$

From (31) we get:

$$h_y = \frac{1}{g_x} \circ \mathcal{H}, \tag{32}$$

hence

$$\frac{1}{2} \le h_y(s, y) \le 2$$

in $W$ thanks to (29). On the other hand, thanks to (30) and using (28) and (32) we obtain

$$h_s = -\tilde{g}_t h_y = -\frac{g_t}{g_x} \circ \mathcal{H} = b \circ \mathcal{H}. \tag{33}$$

Therefore the function $h$ solves the following Cauchy problem in $W$:

$$\begin{cases} \partial_s h(s, y) - b(s, h(s, y)) = 0, \\ h(0, y) = y. \end{cases} \tag{34}$$

REMARK 3. If (8) holds true, that is if $b \equiv b(t)$, then we can write explicitly the solution $g(t, x)$ to (28):

$$y = g(t, x) = x - \int_0^t b(\tau)\, d\tau. \tag{35}$$

We notice that $g_x \equiv 1$, hence (29) is trivially satisfied in $G_T$.

Therefore we can take $V = G_T$ and $W = \mathcal{G}(G_T) = G_T$. For any $(s, y) \in G_T$ we can write explicitly:

$$h(s, y) := x = y + \int_0^s b(\sigma)\, d\sigma. \tag{36}$$

**Lemma 2.** *Let* L *be a complete operator of type* (2). *Therefore*

$$\mathrm{L}\, u = [\mathrm{L}^{\mathcal{H}}(u \circ \mathcal{H})] \circ \mathcal{G} \tag{37}$$

*in* $V$, *where*

$$\mathrm{L}^{\mathcal{H}}(s, y, \partial_s, \partial_y)v = v_{ss} - (\Delta^{\mathcal{H}} v_y)_y + c^{\mathcal{H}} v_s + d^{\mathcal{H}} v_y + e^{\mathcal{H}} v,$$

*is an incomplete operator with*:

$$\begin{aligned}
\Delta^{\mathcal{H}}(s, y) &:= [g_x^2 \Delta] \circ \mathcal{H}(s, y),\\
c^{\mathcal{H}}(s, y) &:= [c + b_x] \circ \mathcal{H}(s, y),\\
d^{\mathcal{H}}(s, y) &:= [(d - cb + g_{xx}(h_y \circ \mathcal{G})\Delta)g_x] \circ \mathcal{H}(s, y),\\
e^{\mathcal{H}}(s, y) &:= e \circ \mathcal{H}(s, y).
\end{aligned}$$

We remark that the coefficients of $\mathrm{L}^{\mathcal{H}}$ belong to $\mathcal{C}^\infty(W, \mathbb{R})$.

Proof. We immediately obtain, for any function $v \in \mathcal{C}^\infty(W, \mathbb{R})$, that:

$$\begin{aligned}
\partial_t \hat{v} &= \hat{v}_s + g_t \hat{v}_y = \hat{v}_s - b g_x \hat{v}_y,\\
\partial_x \hat{v} &= g_x \hat{v}_y,
\end{aligned}$$

hence

$$\begin{aligned}
\partial_t^2 \hat{v} + \partial_t(b \partial_x \hat{v}) &= \partial_t \hat{v}_s - \partial_t(b g_x \hat{v}_y) + \partial_t(b g_x \hat{v}_y)\\
&= \partial_t \hat{v}_s = \hat{v}_{ss} - b g_x \hat{v}_{sy},
\end{aligned}$$

whereas

$$\begin{aligned}
\partial_x(b \partial_t \hat{v}) - \partial_x(a \partial_x \hat{v}) &= \partial_x(b \hat{v}_s) - \partial_x(b^2 g_x \hat{v}_y) - \partial_x(a g_x \hat{v}_y)\\
&= \partial_x(b \hat{v}_s) - \partial_x(\Delta g_x \hat{v}_y)\\
&= b_x \hat{v}_s + b g_x \hat{v}_{sy} - \partial_x(\Delta g_x \hat{v}_y).
\end{aligned}$$

We have that:

$$\begin{aligned}
\partial_x(\Delta g_x \hat{v}_y) &= \partial_x[(\tilde{\Delta} \tilde{g}_x v_y) \circ \mathcal{G}] = g_x[(\tilde{\Delta} \tilde{g}_x v_y)_y \circ \mathcal{G}]\\
&= [(\tilde{g}_x \tilde{\Delta} \tilde{g}_x v_y)_y] \circ \mathcal{G} - [(\partial_y \tilde{g}_x) \tilde{\Delta} \tilde{g}_x v_y] \circ \mathcal{G}\\
&= [(\tilde{g}_x^2 \tilde{\Delta} v_y)_y] \circ \mathcal{G} - [h_y \tilde{g}_{xx} \tilde{\Delta} \tilde{g}_x v_y] \circ \mathcal{G},
\end{aligned}$$

thus we get:

$$\partial_t^2 \hat{v} + \partial_t(b\partial_x \hat{v}) + \partial_x(b\partial_t \hat{v}) - \partial_x(a\partial_x \hat{v}) = \hat{v}_{ss} + b_x \hat{v}_s - \partial_x(\Delta g_x \hat{v}_y).$$

Analogously we have

$$c\partial_t \hat{v} + d\partial_x \hat{v} = c\hat{v}_s + (d - cb)g_x \hat{v}_y,$$

hence we get

$$\mathrm{L}(v \circ \mathcal{G}) = (\mathrm{L}^{\mathcal{H}} v) \circ \mathcal{G},$$

that is (37) with $u := v \circ \mathcal{G}$. □

REMARK 4. If the condition (8) holds true then we can write explicitly:

$$\begin{aligned}
\Delta^{\mathcal{H}}(s, y) &= \Delta(s, h(s, y)),\\
c^{\mathcal{H}}(s, y) &= c(s, h(s, y)),\\
d^{\mathcal{H}}(s, y) &= d(s, h(s, y)) - c(s, h(s, y))b(s),\\
e^{\mathcal{H}}(s, y) &= e(s, h(s, y)),
\end{aligned}$$

since $g_x \equiv 1$ and $g_{xx} \equiv 0$ (see Remark 3). We remark that $\mathrm{L}^{\mathcal{H}}$ has coefficients in $\mathcal{B}^\infty(G_T, \mathbb{R})$.

Direct Proof of Corollary 1. In order to apply the Oleĭnik's theorem to the transformed operator $\mathrm{L}^{\mathcal{H}}$, we prove the Oleĭnik's condition (7), that is:

$$s(d^{\mathcal{H}})^2 \lesssim A\Delta^{\mathcal{H}} + \partial_s \Delta^{\mathcal{H}}, \tag{38}$$

in $W = \mathcal{G}(V)$. Thanks to (34) we can develop the last term in the right-hand side of (38):

$$\begin{aligned}
(\partial_s \Delta^{\mathcal{H}}) \circ \mathcal{G} &= \partial_t(g_x^2 \Delta) + b\partial_x(g_x^2 \Delta)\\
&= g_x^2(\Delta_t + b\Delta_x) + 2g_x \Delta(g_{tx} + bg_{xx})\\
&= g_x^2(\Delta_t + b\Delta_x) + 2g_x \Delta[\partial_x(g_t + bg_x) - b_x g_x]\\
&= g_x^2(\Delta_t + b\Delta_x - 2\Delta b_x).
\end{aligned} \tag{39}$$

Now we compose (38) on the right-hand with $\mathcal{G}$ and we replace the coefficients of $\mathrm{L}^{\mathcal{H}}$ with their explicit expressions, thus (38) becomes:

$$t[(d - cb + g_{xx}(h_y \circ \mathcal{G})\Delta)g_x]^2 \lesssim (A\Delta + \Delta_t + b\Delta_x - 2\Delta b_x)g_x^2. \tag{40}$$

Thanks to (29), we can divide both the left-hand term and the right-hand term of (40) by $g_x^2$. We can estimate

$$(g_{xx}(h_y \circ \mathcal{G})\Delta)^2 \lesssim \Delta \quad \text{and} \quad 2|b_x|\Delta \lesssim \Delta,$$

hence (40) is equivalent to our condition (15).

Hence we have proved the Oleĭnik's condition (7) for the operator $\mathrm{L}^{\mathcal{H}}$ in $W$. Now we distinguish two cases.

If the condition (8) holds true then we take $V = G_T = W$ and (7) holds globally for $\mathrm{L}^{\mathcal{H}}$. We apply the Oleĭnik's theorem: there exists a unique global solution $v$ of the Cauchy problem for $\mathrm{L}^{\mathcal{H}}$. Therefore $u := v \circ \mathcal{G}$ is the unique global solution of the Cauchy problem (1) for the operator L and this concludes the proof.

Now we prove the local well-posedness in absence of the condition (8). We fix arbitrarily $\overline{x} \in \mathbb{R}$ and we prove the well-posedness of (1) at $(0, \overline{x})$ for L. We take $T_1 \in (0, T]$ and $\varepsilon > 0$ in such a way that we have

$$W_1 := [0, T_1] \times B_{\varepsilon/2} \subset K := [0, T_1] \times \overline{B_\varepsilon} \subset W,$$

where $B_r := \{|x - \overline{x}| < r\}$.

By compactness arguments, the transformed operator $\mathrm{L}^{\mathcal{H}}$ has coefficients in $\mathcal{B}^\infty(K, \mathbb{R})$ hence we can extend the coefficients of $\mathrm{L}^{\mathcal{H}}$ from $\mathcal{B}^\infty(K, \mathbb{R})$ to $\mathcal{B}^\infty(G_{T_1}, \mathbb{R})$. We take $\phi \in \mathcal{B}^\infty(\mathbb{R}, \mathbb{R}_+)$ in such a way that:

$$\phi|_{B_{\varepsilon/2}} \equiv 0, \quad \phi|_{G_{T_1} \setminus \overline{B_\varepsilon}} \equiv C + 1,$$

where $C$ is a suitable positive constant such that $\Delta^{\mathcal{H}} \geq -C$ in $G_{T_1}$ (we notice that the extended operator $\mathrm{L}^{\mathcal{H}}$ may be no longer hyperbolic in $G_{T_1}$). Now the operator

$$\mathrm{M}\, v := \mathrm{L}^{\mathcal{H}}\, v - (\phi(y) v_y)_y$$

satisfies (7) in $G_{T_1}$, hence we can apply the Oleĭnik's theorem to M.

We have a (unique) solution $v \in \mathcal{C}^\infty(G_{T_1}, \mathbb{R})$ of the Cauchy problem for M in $G_{T_1}$; in particular, $v$ is the (unique) solution of the Cauchy problem for $\mathrm{L}^{\mathcal{H}}$ in $W_1$. Therefore $u := v \circ \mathcal{G}$ is the unique solution of the Cauchy problem (1) for L in $V_1 := \mathcal{H}(W_1)$, nbd. of $(0, \overline{x})$. This concludes the proof. □

## 2. Proof of Theorem 1

Through this section we assume space dimension $n \geq 2$. For the sake of brevity and the ease of reading, we introduce the following notation.

NOTATION. For any $(t, x, \xi) \in G_T \times \mathbb{R}^n$, we put:

$$a(t, x, \xi) := \sum_{i,j=1}^{n} a_{ij}(t, x)\xi_i\xi_j, \tag{41}$$

$$\Delta(t, x, \xi) := \sum_{i,j=1}^{n} \Delta_{ij}(t, x)\xi_i\xi_j, \tag{42}$$

$$b(t, x, \xi) := \sum_{i=1}^{n} b_i(t, x)\xi_i, \tag{43}$$

$$d(t, x, \xi) := \sum_{i=1}^{n} d_i(t, x)\xi_i. \tag{44}$$

Moreover, we define:

$$\gamma(t, x, \xi) := \sum_{i,j=1}^{n} \gamma_{ij}(t, x)\xi_i\xi_j, \quad \text{with} \quad \gamma_{ij} := -2\sum_{k=1}^{n} \Delta_{ik}\partial_{x_k} b_j.$$

NOTATION. We write $f \lesssim g$ to mean that there is some constant $C > 0$ such that we have:

$$f(t, x, \xi) \le Cg(t, x, \xi). \tag{45}$$

We write $f \approx g$ to mean that $f \lesssim g$ and $g \lesssim f$.

With this notation, our conditions (10) and (12) become:

$$t(d - cb)^2 \lesssim \Phi_A, \tag{46}$$

$$t((d - cb)^2 + \rho) \lesssim \Phi_A. \tag{47}$$

NOTATION. We write $\mathbf{b}(t, x)$ to mean the vector-valued function

$$\mathbf{b} = (b_i)_{i=1,\dots,n} : G_T \to \mathbb{R}^n, \tag{48}$$

and similarly we define $\mathbf{d}(t, x)$.

With this notation we have:

$$b(t, x, \xi) = \mathbf{b}(t, x) \cdot \xi, \quad \text{and} \quad d(t, x, \xi) = \mathbf{d}(t, x) \cdot \xi,$$

where $\cdot$ denotes the scalar product in $\mathbb{R}^n$. Moreover, (11) reads:

$$\Phi_A = A\Delta + \Delta_t + \mathbf{b} \cdot \nabla_x \Delta + \gamma. \tag{49}$$

As in §1, let

$$g_i : (t, x) \in V \mapsto y \in \mathbb{R} \quad (i = 1, \dots, n),$$

be the the unique $\mathcal{C}^\infty$ solution of the Cauchy problem (22) in $V$, nbd. of $\{t = 0\}$. We put

$$\mathcal{G} = (\pi_0, \mathbf{g}), \quad \text{where} \quad \mathbf{g} = (g_i)_{i=1,\dots,n}.$$

We can take $V$ in a such way that:

$$\inf_V \det \nabla_x \mathbf{g}(t, x) > 0, \tag{50}$$

since $\nabla_x \mathbf{g}(0, \cdot) \equiv \mathrm{I}$. Consequently

$$\mathcal{G} \colon V \to W := \mathcal{G}(V),$$

is a smooth change of variables. Let

$$\mathcal{H} = (\pi_0, \mathbf{h}) \colon W \to V, \quad \text{where} \quad \mathbf{h} = (h_i)_{i=1,\dots,n},$$

be the inverse of $\mathcal{G}$.

Proceeding as in Lemma 2 we get (37) where now

$$\mathrm{L}^{\mathcal{H}}(s, y, \partial_s, \nabla_y)v = v_{ss} - \sum_{i,j=1}^{n} (\Delta_{ij}^{\mathcal{H}} v_{y_j})_{y_i} + c^{\mathcal{H}} v_s + \sum_{i=1}^{n} d_i^{\mathcal{H}} v_{y_i} + e^{\mathcal{H}} v,$$

is an *incomplete* operator with

$$\begin{aligned}
\Delta_{ij}^{\mathcal{H}}(s, y) &:= \left[ \sum_{k,l=1}^{n} (\partial_{x_k} g_i) \Delta_{kl} (\partial_{x_l} g_j) \right] \circ \mathcal{H}(s, y), \\
c^{\mathcal{H}}(s, y) &:= \left[ c + \sum_{k=1}^{n} \partial_{x_k} b_k \right] \circ \mathcal{H}(s, y), \\
d_i^{\mathcal{H}}(s, y) &:= \left[ \sum_{k=1}^{n} ((d_k - c b_k) + r_k)(\partial_{x_k} g_i) \right] \circ \mathcal{H}(s, y), \\
e^{\mathcal{H}}(s, y) &:= e \circ \mathcal{H}(s, y),
\end{aligned}$$

and

$$r_k(t, x) := \sum_{l,p,q=1}^{n} \left( (\partial_{x_l} \partial_{x_p} g_q) (\widehat{\partial_{y_q} h_p}) \right) \Delta_{lk}(t, x).$$

Moreover the vector-valued function $\mathbf{h}$ solves in $W$ the Cauchy problem:

$$\begin{cases} \partial_s \mathbf{h}(s, y) = \mathbf{b}(s, \mathbf{h}(s, y)), \\ \mathbf{h}(0, y) = y. \end{cases} \tag{51}$$

**Proposition 1.** *If* L *satisfies the condition* (8) *then we can write explicitly*

$$\mathbf{g}(t, x) = x - \int_0^t \mathbf{b}(\tau, x)\, d\tau, \quad (t, x) \in G_T, \tag{52}$$

$$\mathbf{h}(s, y) = y + \int_0^s \mathbf{b}(\sigma, y)\, d\sigma, \quad (s, y) \in G_T. \tag{53}$$

*In particular,* (50) *holds trivially true since we have*:

$$\det \nabla_x \mathbf{g} \equiv 1. \tag{54}$$

*Moreover we can write*:

$$\Delta_{ij}^{\mathcal{H}}(s, y) = \left[ b_i b_j + \sum_{k,l=1}^{n} (\partial_{x_k} g_i) a_{kl} (\partial_{x_l} g_j) \right] \circ \mathcal{H}(s, y), \tag{55}$$

$$c^{\mathcal{H}}(s, y) = c \circ \mathcal{H}(s, y), \tag{56}$$

$$d_i^{\mathcal{H}}(s, y) = \left[ \sum_{k=1}^{n} d_k (\partial_{x_k} g_i) - c b_i \right] \circ \mathcal{H}(s, y). \tag{57}$$

We remark that the coefficients of $\mathrm{L}^{\mathcal{H}}$ are in $\mathcal{B}^\infty(G_T, \mathbb{R})$, since $\nabla_x \mathbf{g} \in \mathcal{B}^\infty(G_T, M_n(\mathbb{R}))$.

Proof. Condition (8) implies immediately (56).

It is easy to check that each function $g_i$ solves the Cauchy problem (22). Indeed, by (8) we have:

$$\partial_t g_i(t, x) + \sum_{j=1}^{n} b_j(t, x) \partial_{x_j} g_i(t, x) = - \int_0^t \sum_{j=1}^{n} b_j(t, x) \partial_{x_j} b_i(\tau, x)\, d\tau = 0.$$

We can assume with no loss of generality that

$$b_i \equiv 0 \quad (i = 1, \dots, m),$$
$$\partial_{x_i} \mathbf{b} \equiv 0 \quad (i = m+1, \dots, n),$$

for some $m \in \{0, \dots, n\}$. We put:

$$x \equiv (x', x''), \quad x' = (x_i)_{i=1,\dots,m}, \quad x'' = (x_i)_{i=m+1,\dots,n},$$
$$y \equiv (y', y''), \quad y' = (y_i)_{i=1,\dots,m}, \quad y'' = (y_i)_{i=m+1,\dots,n},$$

$$\mathbf{b} \equiv (\mathbf{b}', \mathbf{b}''), \quad \mathbf{b}' = (b_i)_{i=1,\dots,m}, \quad \mathbf{b}'' = (b_i)_{i=m+1,\dots,n},$$
$$\mathbf{g} \equiv (\mathbf{g}', \mathbf{g}''), \quad \mathbf{g}' = (g_i)_{i=1,\dots,m}, \quad \mathbf{g}'' = (g_i)_{i=m+1,\dots,n},$$
$$\mathbf{h} \equiv (\mathbf{h}', \mathbf{h}''), \quad \mathbf{h}' = (h_i)_{i=1,\dots,m}, \quad \mathbf{h}'' = (h_i)_{i=m+1,\dots,n}.$$

We remark that $\mathbf{b}' \equiv 0$. We get immediately:

$$\nabla_x \mathbf{g} = \begin{pmatrix} \nabla_{x'}\mathbf{g}' & \nabla_{x''}\mathbf{g}' \\ \nabla_{x'}\mathbf{g}'' & \nabla_{x''}\mathbf{g}'' \end{pmatrix} = \begin{pmatrix} \mathrm{I}_m & 0 \\ -\displaystyle\int_0^t \nabla_{x'}\mathbf{b}''(\tau, x')\,d\tau & \mathrm{I}_{n-m} \end{pmatrix}, \tag{58}$$

that implies trivially (54).

For any $(t, y) \in G_T$ there exists $x \in \mathbb{R}^n$ such that $\mathbf{g}(t, x) = y$. Hence, using (52), we get:

$$\mathbf{h}'(t, y) = x' = y'. \tag{59}$$

Now we can use (52) and (59) to have:

$$\mathbf{h}''(t, y) = x'' = y'' + \int_0^t \mathbf{b}''(\tau, x')\,d\tau = y'' + \int_0^t \mathbf{b}''(\tau, y')\,d\tau. \tag{60}$$

This proves (53).

On the other hand, using (52) and (53), we get

$$\begin{aligned} &\sum_{q=1}^n \partial_{x_l}\partial_{x_p} g_q(t, x)\,\partial_{y_q} h_p(t, y) \\ &= \sum_{q=1}^n \partial_{x_l}\partial_{x_p} g_q(t, x)\,\partial_{y_q} y_p \\ &= \partial_{x_l}\partial_{x_p} g_p(t, x) = \partial_{x_l} 1 = 0 \quad (p = 1, \dots, m,\ l = 1, \dots, n), \end{aligned}$$

and

$$\begin{aligned} &\sum_{q=1}^n \partial_{x_l}\partial_{x_p} g_q(t, x)\,\partial_{y_q} h_p(t, y) \\ &= \sum_{q=1}^n \partial_{x_l}(\partial_{x_p} x_q - 0)\,\partial_{y_q} h_p(t, y) \\ &= \partial_{x_l} 1\ \partial_{y_p} h_p(t, y) = 0 \quad (p = m+1, \dots, n,\ l = 1, \dots, n). \end{aligned}$$

This proves that:

$$r_k \equiv 0\ (k = 1, \dots, n), \quad \text{hence} \quad d_i^{\mathcal{H}} = \left[\sum_{k=1}^n (d_k - cb_k)(\partial_{x_k} g_i)\right] \circ \mathcal{H}.$$

We notice that:

$$\sum_{k=1}^{n} b_k(\partial_{x_k} g_i) = \sum_{k=1}^{n} \left( b_k \partial_{x_k} x_i - b_k \int_0^t \partial_{x_k} b_i \right) = b_i,$$

thanks to (8). Hence we have proved (57). By the same way we prove (55). □

Proof of Theorem 1. By the same arguments used in §1, we get:

$$(\partial_s \Delta_{ij}^{\mathcal{H}}) \circ \mathcal{G} = \sum_{k,l=1}^{n} (\partial_{x_k} g_i) \left( \partial_t \Delta_{kl} + \sum_{m=1}^{n} b_m \partial_{x_m} \Delta_{kl} + \gamma_{kl} \right) (\partial_{x_l} g_j).$$

We put:

$$\eta_k := \sum_{i=1}^{n} (\partial_{x_k} g_i) \xi_i \quad (k = 1, \dots, n),$$

for any $\xi \in \mathbb{R}^n$. We remark that for any $(t, x) \in V$ the linear map

$$\xi \in \mathbb{R}^n \mapsto \eta := (\nabla_x \mathbf{g}(t, x)) \cdot \xi \in \mathbb{R}^n,$$

is bijective, thanks to (50). Therefore the Oleĭnik's condition (7) for $\mathrm{L}^{\mathcal{H}}$ is verified if:

$$t(d - cb + r)^2 \lesssim A\Delta + \Delta_t + \mathbf{b} \cdot \nabla_x \Delta + \gamma, \tag{61}$$

where $r(t, x, \xi) := \sum_{i=1}^{n} r_i(t, x)\xi_i$.

If the condition (8) holds true, then $r \equiv 0$ hence (61) reduces to our condition (10). On the other hand, in absence of the condition (8) we can prove that $r^2 \lesssim \rho$.

By the positivity of $\Delta$ it follows that:

$$0 \le \Delta(t, x, \lambda e_l + e_k) = \lambda^2 \Delta_{ll}(t, x) + 2\lambda \Delta_{lk}(t, x) + \Delta_{kk}(t, x) \quad (l, k = 1, \dots, n),$$

for any $\lambda \in \mathbb{R}$, hence:

$$(\Delta_{lk}(t, x))^2 \le \Delta_{ll}(t, x)\Delta_{kk}(t, x) \quad (l, k = 1, \dots, n). \tag{62}$$

Thanks to (62), we can estimate:

$$r_k^2(t, x) \lesssim \sum_{l=1}^{n} \Delta_{lk}^2(t, x) \le \left( \sum_{l=1}^{n} \Delta_{ll}(t, x) \right) \Delta_{kk}(t, x) \lesssim \Delta_{kk}(t, x) \quad (k = 1, \dots, n). \tag{63}$$

Using (63) we have:

$$(r(t, x, \xi))^2 \lesssim \sum_{k=1}^{n} r_k^2(t, x)\xi_k^2 \lesssim \sum_{k=1}^{n} \Delta_{kk}(t, x)\xi_k^2 = \rho(t, x, \xi).$$

Therefore (61) reduces to our condition (12).

The conclusion of the proof is the same as in Corollary 1. □

Our condition (12) is far to be equivalent to (61), but the last one does contain the term $r(t, x, \xi)$ that depends explicitly on the solution of (22). On the other hand, under assumption (8), (10) and (61) are equivalent. We can obtain $r \equiv 0$ for some operator also in absence of the condition (8). In these cases, condition (61) reduces to our condition (10).

EXAMPLE 1. Let L be some operator of type (2) with $\mathbf{b} = x$. Therefore:

$$g_i(t, x) = x_i e^{-t} \quad (i = 1, \dots, n),$$

solves the Cauchy problem (22). Moreover, we have that:

$$h_i(s, y) = y_i e^{s} \quad (i = 1, \dots, n).$$

The condition (8) is not satisfied. Nevertheless, it is easy to notice that $r \equiv 0$ and we can write explicitly the coefficients of $\mathrm{L}^{\mathcal{H}}$:

$$\begin{aligned}
\Delta_{ij}^{\mathcal{H}}(s, y) &= \Delta_{ij}(s, ye^s)e^{-2s} = y_i y_j + a_{ij}(s, ye^s)e^{-2s},\\
c^{\mathcal{H}}(s, y) &= c(s, ye^s) + n,\\
d_i^{\mathcal{H}}(s, y) &= d_i(s, ye^s)e^{-s} - y_i c(s, ye^s),\\
e^{\mathcal{H}}(s, y) &= e(s, ye^s).
\end{aligned}$$

## 3. The case of analytic coefficients

Through this section, we assume that the coefficients of L are analytic (at the origin). In space dimension $n = 1$ we recall the following well-known result.

**Lemma** (T. Nishitani [8]). *Let* L *be a hyperbolic operator with coefficients in* $\mathcal{C}^{\omega}(U, \mathbb{C})$ *where* $U$ *is a nbd. of the origin in* $\mathbb{R}^2$.

*Then we can find another nbd.* $V$ *of the origin, in a such way that the characteristic roots* $\tau_1$ *and* $\tau_2$ *of* L *can be taken continuous in* $V$ *and analytic in* $V \setminus \{(0, 0)\}$.

**Theorem** (T. Nishitani [8]). *Let* L, $V$ *and* $\tau_1$, $\tau_2$ *be as described in the previous lemma. Assume that there are two constants* $A, B > 0$ *such that, for any* $(t, x) \in W$, *where* $W \subset V$ *is a nbd. of the origin, and for any* $\xi \in \mathbb{R}$, *we have*:

$$|p^s(t, x, \tau_1(t, x)\xi, \xi)| \le A|\{\tau - \tau_1\xi, \tau - \tau_2\xi\}| + B|\xi(\tau_1 - \tau_2)|. \tag{64}$$

*Here* $\{f, g\}$ *is the Poisson bracket, i.e.*

$$\{f, g\} := (\partial_\tau f \partial_t g + \partial_\xi f \partial_x g) - (\partial_t f \partial_\tau g + \partial_x f \partial_\xi g).$$

*Then the Cauchy problem* (1) *for* L *is well-posed in* $\mathcal{C}^\infty$ (*at the origin*).

We remark that the assumption of analyticity in $t$ of the coefficients of L is essential. Indeed there exists an operator

$$\mathrm{L} = \partial_t^2 - a(t)\partial_x^2, \quad \text{with} \quad a(t) \in \mathcal{C}^\infty([0, T]) \quad a(t) \geq 0, \tag{65}$$

for which the Cauchy problem is ill-posed [3].

From the results in [6] it follows that, if $\Delta(t, x)$ can be written in the form

$$\Delta(t, x) = (\Phi(t, x))^N \Psi(t, x), \tag{66}$$

for some integer $N \in \mathbb{N}$ and some smooth functions $\Phi$ and $\Psi$ satisfying

$$\Phi(0, 0) = 0, \quad (\Phi_t(0, 0), \Phi_x(0, 0)) \neq (0, 0), \quad \Psi(0, 0) \neq 0,$$

then (64) is also a necessary condition for the well-posedness in $\mathcal{C}^\infty$. In particular, if either $\Delta \equiv \Delta(t)$ or $\Delta \equiv \Delta(x)$ then (66) holds true. On the other hand, in the general case $\Delta \equiv \Delta(t, x)$, Nishitani extended (64) to a necessary and sufficient condition in [9].

In order to compare the Nishitani's theorem with our Corollary 1, we re-state (64) in the following form.

**Proposition 2.** *We define*

$$\delta(t, x) := \frac{1}{2}(\tau_1(t, x) - \tau_2(t, x)),$$

*that is continuous in* $V$ *and analytic in* $V \setminus \{(0, 0)\}$, *thanks to the Nishitani's lemma* [8]. *Now the Nishitani's condition* (64) *is equivalent to*:

$$|d - cb| \lesssim |\delta_t + b\delta_x| + |\delta|. \tag{67}$$

We remark that $\delta^2(t, x) = \Delta(t, x)$.

Proof. We notice that:

$$b(t, x) = -\frac{1}{2}(\tau_1(t, x) + \tau_2(t, x)),$$

hence we can compute:

$$\begin{aligned}
\{\tau - \tau_1, \tau - \tau_2\} &= \{\tau - (\delta - b)\xi, \tau + (\delta + b)\xi\} \\
&= ((\delta + b)_t\xi - (\delta - b)(\delta + b)_x\xi) \\
&\quad - (-(\delta - b)_t\xi - (\delta + b)(\delta - b)_x\xi) \\
&= 2\xi(\delta_t + b\delta_x - b_x\delta),
\end{aligned}$$

By estimating $|b_x\delta| \lesssim |\delta|$, we obtain:

$$A|\{\tau - \tau_1\xi,\ \tau - \tau_2\xi\}| + B|\xi(\tau_1 - \tau_2)| \approx |\xi|(A'|\delta| + |\delta_t + b\delta_x|).$$

In order to conclude our proof, it is sufficient to notice that:

$$|p^s(t, x, \tau_1\xi, \xi)| = |\xi|\,|d + c(\delta - b)| \le |\xi|(|d - cb| + |c\delta|) \lesssim |\xi|(|d - cb| + |\delta|). \qquad \square$$

**Proposition 3.** *If $\Delta \equiv \Delta(t)$ is depending only on $t$ and belongs to $\mathcal{C}^\omega((-T, T), \mathbb{R})$ for some $T > 0$, then our condition* (15) *is locally equivalent to* (67).

In particular, Proposition 3 proves that (15) is a necessary and sufficient condition for the well-posedness, provided that the coefficients are depending only on the time variable and are analytic at the origin.

Proof. If L is strictly hyperbolic, that is $\Delta(0) > 0$, then both (15) and (67) hold locally true, hence we can assume $\Delta(0) = 0$ in the following.

Thanks to the positive analyticity of $\Delta(t)$ we can write for some integer $\kappa \ge 1$:

$$\Delta(t) = \alpha_{2\kappa} t^{2\kappa} + \mathcal{O}(t^{2\kappa+1}), \quad \text{with} \quad \alpha_{2\kappa} > 0,$$

near to the origin. Hence (15) is locally equivalent to

$$t(d - cb)^2 \lesssim t^{2\kappa-1}, \tag{68}$$

whereas (67) is locally equivalent to

$$|d - cb| \lesssim |t^{\kappa-1}|. \tag{69}$$

We notice that (68) and (69) are both equivalent to

$$(d - cb)^2 \lesssim t^{2(\kappa-1)}.$$

This concludes the proof. $\square$

**Proposition 4.** *If the leading coefficients $a \equiv a(x)$ and $b \equiv b(x)$ are depending only on $x$ and belong to $\mathcal{C}^\omega((-\varepsilon, \varepsilon), \mathbb{R})$ for some $\varepsilon > 0$, then our condition* (15) *is locally equivalent to* (67), *provided that $a(0) = b(0) = 0$.*

In particular, Proposition 4 proves that (15) is a necessary and sufficient condition for the well-posedness of L, provided that the coefficients are depending only on the space variable and are analytic at the origin, and that $a(0) = b(0) = 0$.

Proof of Proposition 4. Thanks to the analyticity of the coefficients, we have $b(x) = \mathcal{O}(x)$ and

$$\Delta(x) = \alpha_{2\kappa} x^{2\kappa} + \mathcal{O}(x^{2\kappa+1}), \quad \text{with} \quad \alpha_{2\kappa} > 0,$$

for some integer $\kappa \in \mathbb{N}^*$. Therefore:

$$A\Delta + b\Delta_x \approx x^{2k},$$

in (15) whereas

$$|b\delta_x| + |\delta| \approx x^{\kappa},$$

in (67). Consequently both the conditions (15) and (67) are locally equivalent to:

$$(d - cb)^2 \lesssim x^{2k}. \qquad \square$$

We remark that the Nishitani's condition (64) holds trivially true whenever the subprincipal symbol vanishes identically. On the other hand, condition (67) for an homogeneous operator (16) become:

$$|b_t + bb_x| \lesssim |\delta_t + b\delta_x| + |\delta|, \tag{70}$$

since we can estimate

$$|2bb_x + a_x| = |\Delta_x| = 2|\delta\delta_x| \lesssim |\delta|.$$

We notice that (70) does not necessarily hold true: the homogeneous *complete* operator

$$\mathrm{L} = \partial_t^2 + 2t\partial_t\partial_x + t^2\partial_x^2 \tag{71}$$

does not satisfy (70). Indeed $b_t \equiv 1$ whereas the right-hand term vanishes identically, since $\Delta \equiv 0$. In facts it is well known that the Cauchy problem (1) for the operator (71) is ill-posed. Indeed, via the change of variables

$$\begin{cases} s = t, \\ y = x - \dfrac{1}{2}t^2, \end{cases}$$

the operator (71) is transformed into:

$$\mathrm{L}^{\mathcal{H}} = \partial_s^2 - \partial_y,$$

for which the Cauchy problem is ill-posed.

On the other hand, (70) trivially holds true for an *incomplete* homogeneous operator (21) since the left-hand term vanishes.

In space dimension $n \geq 2$, the Cauchy problem for operators with identically vanishing sub-principal symbol is not necessarily well-posed. In particular, the Cauchy problem (1) for the *complete* operator

$$\mathrm{L} = \partial_t^2 + 2x_2\partial_t\partial_{x_1} + x_2^3\partial_{x_1}^2 - \partial_{x_2}^2, \tag{72}$$

is ill-posed at the origin [1]. We notice that such an operator has analytic (in facts polynomial) coefficients and that its sub-principal symbol is identically zero. Moreover, (72) is homogeneous.

However, one could ask if the Cauchy problem for an *incomplete* operator with analytic coefficients is always well-posed. We prove that the answer to this question is negative by exhibiting a counter-example obtained by applying the change of variables in §2 to (72):

**Theorem 2.** *The Cauchy problem* (1) *for the incomplete operator*

$$\mathrm{L}^{\mathcal{H}} = \partial_s^2 - (y_2^2(1-y_2) + s^2)\partial_{y_1}^2 + 2s\partial_{y_1}\partial_{y_2} - \partial_{y_2}^2, \tag{73}$$

*is ill-posed at the origin. We notice that such an operator has analytic* (*in facts polynomial*) *coefficients and that its sub-principal symbol is identically zero. Moreover,* (73) *is homogeneous.*

Proof. By applying the change of variables

$$\begin{cases} s = t, \\ y_1 = x_1 - tx_2, \\ y_2 = x_2, \end{cases}$$

to (72), we get the transformed operator (73). Indeed:

$$\begin{aligned} \Delta_{11}^{\mathcal{H}}(s, y) &= [(\partial_{x_1} g_1)^2 \Delta_{11} + (\partial_{x_2} g_1)^2 \Delta_{22}] \circ \mathcal{H}(s, y) \\ &= [x_2^2(1-x_2) + t^2] \circ \mathcal{H}(s, y) \\ &= y_2^2(1-y_2) + s^2, \\ \Delta_{12}^{\mathcal{H}}(s, y) &= [(\partial_{x_2} g_1)(\partial_{x_2} g_2)\Delta_{22}] \circ \mathcal{H}(s, y) = -s, \\ \Delta_{22}^{\mathcal{H}}(s, y) &= [(\partial_{x_2} g_2)^2 \Delta_{22}] \circ \mathcal{H}(s, y) = 1. \end{aligned}$$

Therefore the Cauchy problem for the *complete* operator (73) and the Cauchy problem for the *incomplete* operator (72) are equivalent: both of them are ill-posed (at the origin). This concludes the proof. □

## 4. On the Mizohata's condition

Let space dimension be $n = 1$. We compare our condition (15) with the Mizohata's condition (23):

$$b^2(t, x) \lesssim \Delta(t, x).$$

We introduce the following:

DEFINITION 1. We say that $f(t)$, smooth, has finite degeneracy (at the origin) when there is some integer $\kappa \in \mathbb{N}$ such that we have

$$\partial_t^\kappa f(0) \neq 0.$$

**Theorem** (S. Mizohata [7]). *We assume that the homogeneous operator* (16) *has coefficients depending only on* $t$, *i.e.*

$$\mathrm{L} = \partial_t^2 - a(t)\partial_x^2 + 2b(t)\partial_t\partial_x.$$

*We also assume that* $a(0) = b(0) = 0$ *and that* $a(t)$ *or* $b(t)$ *has finite degeneracy. Therefore the Cauchy problem* (1) *is well-posed if and only if* L *satisfies* (23).

The Mizohata's condition (23) trivially holds true whenever L is *incomplete*.

REMARK 5. If we assume the Mizohata's condition (23), then our condition (15) for *complete* operators (14) in space dimension $n = 1$ reduces to the Oleĭnik's condition (7). Indeed, in (15) the left-hand term $(cb)^2$ can be easily estimated by $A\Delta$, whereas by Glaeser's inequality (20) it follows that:

$$|b\Delta_x| \lesssim b^2 + (\Delta_x)^2 \lesssim \Delta.$$

In particular, if the homogeneous *complete* operator (16) verifies (23), then our condition (19) reduces to:

$$t(b_t(t, x))^2 \lesssim A\Delta(t, x) + \Delta_t(t, x). \tag{74}$$

Consequently, the Cauchy problem for an homogeneous *complete* operator with coefficients depending only on $x$, is well-posed, provided Mizohata's condition (23) be fulfilled.

REMARK 6. If L is homogeneous and has analytic coefficients depending only on $t$, then our condition (19) and the Nishitani's condition (70) are locally equivalent

to the Mizohata's condition (23), provided that $a(0) = b(0) = 0$. Indeed we have:

$$b(t) = \beta_l t^l + \mathcal{O}(t^{l+1}), \quad \text{with} \quad \beta_l \neq 0,$$
$$\Delta(t) = \alpha_{2\kappa} t^{2\kappa} + \mathcal{O}(t^{2\kappa+1}), \quad \text{with} \quad \alpha_{2\kappa} > 0,$$

for some integer $\kappa, l \in \mathbb{N}^*$. Hence (19), (23) and (70) are locally equivalent to ask that $l \geq \kappa$.

Colombini and Orrú [2] proved, under a finite degeneracy assumption, that:

$$\mu^2 + \lambda^2 \lesssim (\mu - \lambda)^2, \quad \text{for any pair of } \textit{characteristic roots} \quad \mu, \lambda, \tag{75}$$

is a necessary and sufficient condition for the well-posedness of the Cauchy problem for higher order homogeneous operators with $\mathcal{C}^\infty([0, T], \mathbb{R})$ coefficients depending only on $t$. If the operator has order $m = 2$, then the Colombini-Orrú's condition (75) is equivalent to the Mizohata's condition (23).

On the other hand, Spagnolo and Taglialatela [11] proved that (75) is a sufficient condition for the well-posedness of the Cauchy problem for homogeneous operators of any order, with coefficients depending only on $x$.

Corollary 1 is more general than Theorem 1.1 in [11] for second-order homogeneous operators. Indeed, in this case the Mizohata's condition (23) implies our condition (19), since (74) holds trivially true. However (19) does not imply (23), as the following counter-example shows.

EXAMPLE 2. We consider the homogeneous operator

$$\mathrm{L} = \partial_t^2 + 2x^2 \partial_t \partial_x + x^4(1 - x^2)\partial_x^2. \tag{76}$$

We have $(b(x))^2 = x^4$ whereas $\Delta(x) = x^6$: in $[-\varepsilon, \varepsilon]$ the Mizohata's condition (23) does not hold true. Nevertheless we have the following estimates:

$$|b(x)\Delta_x(x)| = 6x^7 \leq 6\varepsilon x^6 = 6\varepsilon\Delta(x),$$
$$(b(x)b_x(x))^2 = 4x^6 = 4\Delta(x),$$

hence our condition (19) holds true.

In space dimension $n \geq 2$, we could consider the following generalization of the Mizohata's condition (23):

$$(b(t, x, \xi))^2 \lesssim \Delta(t, x, \xi), \tag{77}$$

but such a condition does not ensure the well-posedness. Indeed, the operator (72) fulfills (77) but the corresponding Cauchy problem is ill-posed [1]. The following corollary of Theorem 1 extends Theorem 1.1 in [11] for second order operators in space dimension $n \geq 2$.

**Corollary 2.** *Assume that* L *has coefficients depending only on* $x$ *and assume that*

$$\Delta(x,\xi)=\varphi(x)\Delta^{(0)}(x,\xi),\quad \textit{with}\quad \Delta^{(0)}(x,\xi)\approx|\xi|^2. \tag{78}$$

*If* L *satisfies* (77) *together with the Oleĭnik's condition* (7) (*that reduces to* $d^2\lesssim\Delta$), *then the Cauchy problem* (1) *for* L *is well-posed. Moreover, if* L *satisfies the condition* (8) *then* (1) *is globally well-posed.*

REMARK 7. In space dimension $n=1$, the condition (78) is trivially satisfied. The operator (72) does not satisfy (78) since $\Delta_{11}(x)=x_2^3$ whereas $\Delta_{22}\equiv 1$.

Proof. First we prove that

$$|\Delta_{ij}(x)|\lesssim\varphi(x)\quad(i,j=1,\dots,n). \tag{79}$$

Indeed we have:

$$0\le\Delta_{ii}(x)=\Delta(x,e_i)=\varphi(x)\Delta^{(0)}(x,e_i)\approx\varphi(x)\quad(i=1,\dots,n),$$

thanks to (78). On the other hand (see (62)):

$$|\Delta_{ij}|\le\sqrt{\Delta_{ii}\Delta_{jj}}\lesssim\varphi\quad(i,j=1,\dots,n).$$

Now we can prove that $|\gamma|\lesssim\Delta$. Indeed, thanks to (79) we have that:

$$\sum_{k=1}^{n}|\Delta_{ik}(x)|\,|\partial_{x_k}b_j(x)|\,|\xi_i\xi_j|\lesssim\varphi(x)|\xi|^2\approx\Delta(x,\xi)\quad(i,j=1,\dots,n).$$

Analogously $|\mathbf{b}\cdot\nabla_x\Delta|\lesssim\Delta$. Indeed, thanks to (77) and (78) we get:

$$\begin{aligned}|b_k(x)\Delta_{x_k}(x,\xi)|&\le|\xi|^2b_k^2(x)+|\xi|^{-2}(\Delta_{x_k}(x,\xi))^2\\&\lesssim|\xi|^2(b(x,e_k))^2+\Delta(x,\xi)\\&\lesssim|\xi|^2\Delta(x,e_k)+\Delta(x,\xi)\\&\approx\Delta(x,\xi)\quad(k=1,\dots,n).\end{aligned}$$

Here we applied Glaeser's inequality (20) to

$$x_k\mapsto\Delta(t,x,\xi),$$

which is a positive function in $\mathcal{B}^\infty(\mathbb{R},\mathbb{R})$ depending on the space variable $x_k$ and on the $(2n-1)$-dimensional parameter $(x',\xi)$ where $x':=(x_j)_{j\neq k}$. In facts:

$$(\Delta_{x_k}(x,\xi))^2\le\Delta(x,\xi)\sup_{x_k\in\mathbb{R}}|\partial_{x_k}^2\Delta(x,\xi)|\lesssim|\xi|^2\Delta(x,\xi).$$

Analogously, $|\rho| \lesssim \Delta$ thanks to (79).
Thus our conditions (10) and (12) reduce to

$$(d - cb)^2 \lesssim \Delta,$$

which holds true thanks to the Oleĭnik's condition (7) and to the generalized Mizohata's condition (77). To conclude the proof, we apply Theorem 1. □

## 5. The 2 x 2 first-order systems

Through this section we study the Cauchy problem for the $2 \times 2$ first-order systems:

$$\begin{cases} \mathrm{L}(t, x, \partial_t, \partial_x)U(t, x) = F(t, x), \\ U(0, x) = U_0(x), \end{cases} \tag{80}$$

in space dimension $n = 1$ with

$$\mathrm{L} = \mathrm{I}\partial_t + A(t, x)\partial_x + B(t, x). \tag{81}$$

We assume that $A, B \in \mathcal{B}^\infty(G_T, M_2(\mathbb{R}))$.

NOTATION. We define:

$$A := \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \text{and} \quad A^+ := \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix},$$

and we notice that

$$A + A^+ = (\operatorname{tr} A)\mathrm{I}, \quad \text{and} \quad AA^+ = (\det A)\mathrm{I}.$$

Moreover, we put:

$$\mathcal{A} := \frac{1}{2}(A - A^+) = \begin{pmatrix} \dfrac{A_{11} - A_{22}}{2} & A_{12} \\ A_{21} & \dfrac{A_{22} - A_{11}}{2} \end{pmatrix}.$$

We assume that L is hyperbolic, that is:

$$\Delta := \left(\frac{\operatorname{tr} A}{2}\right)^2 - \det A \equiv -\det \mathcal{A} \geq 0.$$

**Theorem 3.** *Assume that there is some constant $A > 0$ for which*:

$$t\{(\operatorname{tr}(A^+B) - (\operatorname{tr} B)b)^2 + \|\mathcal{A}_t + b\mathcal{A}_x\|^2\} \lesssim \Phi_A, \tag{82}$$

*where* $b := (1/2)\operatorname{tr} A$, $\|M\| := \max_{i,j=1,2}|M_{ij}|$ *and*

$$\Phi_A := A\Delta + \Delta_t + b\Delta_x. \tag{83}$$

*Hence the Cauchy problem* (80) *is well-posed in* $\mathcal{C}^\infty$.

*Moreover, if* $\operatorname{tr} A \equiv \operatorname{tr} A(t)$ *is not depending on* $x$, *then* (80) *is globally well-posed.*

REMARK 8. If we assume that

$$(\operatorname{tr} A)^2 \lesssim \Delta, \tag{84}$$

then our condition (82) reduces to:

$$t\{(\operatorname{tr}(A^+B))^2 + \|\mathcal{A}_t\|^2\} \lesssim A\Delta + \Delta_t. \tag{85}$$

Moreover, if $\mathcal{A} = \mathcal{A}(x)$ then (85) reduces to:

$$(\operatorname{tr}(A^+B))^2 \lesssim \Delta. \tag{86}$$

We remark that (84) and (86) involve only matrices invariants.

In order to prove Theorem 3 we recall a result of Ebert [4] that extends the Oleĭnik's theorem [10] to $2 \times 2$ second-order systems with a scalar principal part.

**Theorem** (M. Ebert [4]). *The Cauchy problem*

$$\begin{cases} \mathcal{L}U(t,x) = F(t,x), \\ U(0,x) = U_0(x), \\ U_t(0,x) = U_1(x), \end{cases} \tag{87}$$

*for the second-order system*

$$\mathcal{L}U = U_{tt} - (\Delta(t,x)U_x)_x + C(t,x)U_t + D(t,x)U_x + E(t,x)U,$$

*with coefficients in* $\mathcal{B}^\infty(G_T, M_2(\mathbb{R}))$, *is globally well-posed in* $\mathcal{C}^\infty$ *if there is some constant* $A > 0$ *such that*

$$t\|D\|^2 \lesssim A\Delta + \Delta_t. \tag{88}$$

Proof of Theorem 3. We put:

$$a := \det A, \quad b := \frac{1}{2}\operatorname{tr} A, \quad c := \operatorname{tr} B, \quad d := \operatorname{tr}(A^+B),$$

and we define

$$\mathcal{L}(t, x, \partial_t, \partial_x)U := U_{tt} + (bU_x)_t + (bU_t)_x + (aU_x)_x + cU_t + dU_x.$$

We compose L on the left-hand with the operator

$$N^{(l)}(t, x, \partial_t, \partial_x) := \mathrm{L}^+(t, x, \partial_t, \partial_x) + A_x^+(t, x),$$

where

$$\mathrm{L}^+ = \mathrm{I}\partial_t + A^+\partial_x + B^+,$$

hence we have

$$\begin{aligned} N^{(l)}\,\mathrm{L}\,U &= U_{tt} + 2bU_{tx} + aU_{xx} + cU_t + dU_x \\ &\quad + (A_t + A^+A_x)U_x + A_x^+U_t + A_x^+AU_x + E^{(l)}U \\ &= \mathcal{L}U + (A_x^+ - b_x)U_t + (A_t + A^+A_x + A_x^+A - b_t - a_x)U_x + E^{(l)}U. \end{aligned}$$

By the identities $A_xA^+ + AA_x^+ = a_x\mathrm{I}$ and $A - b\mathrm{I} = \mathcal{A} = -(A^+ - b\mathrm{I})$, we get:

$$N^{(l)}\,\mathrm{L}\,U = \mathcal{L}U - \mathcal{A}_xU_t + \mathcal{A}_tU_x + E^{(l)}U.$$

As in §1 we take $V$, nbd. of the initial line, and $\mathcal{G} = (\pi_0, g)$, smooth change of variables on $V$, such that $N^{(l)}\,\mathrm{L}$ is equivalent to:

$$(N^{(l)}\,\mathrm{L})^{\mathcal{H}}V = V_{ss} - (\Delta^{\mathcal{H}}(s, y)V_y)_y + C^{\mathcal{H}}(s, y)V_s + D^{\mathcal{H}}(s, y)V_y + E^{\mathcal{H}}(s, y)V, \tag{89}$$

where:

$$\begin{aligned} \Delta^{\mathcal{H}} &= [g_x^2\Delta] \circ \mathcal{H}, \\ C^{\mathcal{H}} &= [(c - b_x)\mathrm{I} - \mathcal{A}_x] \circ \mathcal{H}, \\ D^{\mathcal{H}} &= [(\hat{h}_y g_{xx}\Delta + d - cb)\mathrm{I} + \mathcal{A}_t + b\mathcal{A}_x] \circ \mathcal{H}, \\ E^{\mathcal{H}} &= E^{(l)} \circ \mathcal{H}. \end{aligned}$$

The system

$$(N^{(l)}\,\mathrm{L}\,a)^{\mathcal{H}}V = [N^{(r)}\,\mathrm{L}(V \circ \mathcal{G})] \circ \mathcal{H},$$

verifies (88) in $W := \mathcal{G}(V)$, since

$$t\|(d - cb)\mathrm{I} + (\partial_t + b\partial_x)\mathcal{A}\|^2 \lesssim A\Delta + \Delta_t + b\Delta_x, \tag{90}$$

holds true thanks to condition (82).

We compose L on the right-hand with the operator

$$N^{(r)}(t, x, \partial_t, \partial_x) := \mathrm{L}^+(t, x, \partial_t, \partial_x) - A_x^+(t, x),$$

thus (here we use again $\mathcal{A} = -(A^+ - b\mathrm{I})$):

$$\begin{aligned} \mathrm{L}\, N^{(r)}U &= U_{tt} + 2bU_{tx} + aU_{xx} + cU_t + dU_x - A_x^+U_t + A_t^+U_x + E^{(r)}U \\ &= \mathcal{L}U + (\mathcal{A}_x - 2b_x)U_t - (\mathcal{A}_t + a_x)U_x + E^{(l)}U. \end{aligned}$$

The system $(\mathrm{L}\, N^{(r)})^{\mathcal{H}}V$ verifies (88) in $W$. Indeed by applying the Glaeser's inequality (20) to

$$(2bb_x - a_x)^2 = \Delta_x^2 \lesssim \Delta,$$

it follows that

$$t\|(d - a_x - cb + 2bb_x)\mathrm{I} - (\partial_t + b\partial_x)\mathcal{A}\|^2 \lesssim A\Delta + \Delta_t + b\Delta_x,$$

holds true thanks to condition (82).

Following the proof of Corollary 1 we can prove that the Cauchy problem (87) is well-posed for both the operators $N^{(l)}\,\mathrm{L}$ and $\mathrm{L}\, N^{(r)}$. Consequently, the Cauchy problem (80) for L is well-posed. Moreover, if $\operatorname{tr} A \equiv \operatorname{tr} A(t)$ then we can take $V = G_T = W$ and (87) is globally well-posed. □

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## References

[1] E. Bernardi and A. Bove: *On the Cauchy problem for some hyperbolic operator with double characteristics*; in Phase Space Analysis of Partial Differential Equations, Birkhäuser, Boston, MA, 2006, 29–44.

[2] F. Colombini and N. Orrú: *Well-posedness in $C^\infty$ for some weakly hyperbolic equations*, J. Math. Kyoto Univ. **39** (1999), 399–420.

[3] F. Colombini and S. Spagnolo: *An example of a weakly hyperbolic Cauchy problem not well posed in $C^\infty$*, Acta Math. **148** (1982), 243–253.

[4] M.R. Ebert: *On the Cauchy problem for $2 \times 2$ weakly hyperbolic systems*, Osaka J. Math. **42** (2005), 831–860.

[5] G. Glaeser: *Racine carrée d'une fonction différentiable*, Ann. Inst. Fourier (Grenoble) **13** (1963), 203–210.

[6] V.Ja. Ivrii and V.M. Petkov: *Necessary conditions for the Cauchy problem for non-strictly hyperbolic equations to be well-posed*, Russ. Math. Surveys. **29** (1974), 1–70.
[7] S. Mizohata: On the Cauchy Problem, Academic Press, Orlando, FL, 1985.
[8] T. Nishitani: *The Cauchy problem for weakly hyperbolic equations of second order*, Comm. Partial Differential Equations **5** (1980), 1273–1296.
[9] T. Nishitani: *A necessary and sufficient condition for the hyperbolicity of second order equations with two independent variables*, J. Math. Kyoto Univ. **24** (1984), 91–104.
[10] O.A. Oleĭnik: *On the Cauchy problem for weakly hyperbolic equations*, Comm. Pure Appl. Math. **23** (1970), 569–586.
[11] S. Spagnolo and G. Taglialatela: *Homogeneous hyperbolic equations with coefficients depending on one space variable*, J. Hyperbolic Differ. Equ. **4** (2007), 533–553.

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