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NON COMMUTATIVE KRULL RINGS

Dedicated to Professor Kiiti Morita for his 60th birthday

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A commutative integral domain D is said to be a Krull domain provided there is a family $\{V_i\}_{i\in I}$ of discrete rank one valuation overrings of D such that

(1) $D=\cap_{i\in I}V_i$.

(2) each V_i is essential for D.

(3) Given x in D, $x\neq 0$, there is at most a finite number of i in I such that x is a non unit in V_i.

Using noetherian local Asano orders and noetherian simple rings instead of discrete rank one valuation rings we will introduce non commutative Krull rings and generalize some elementary results on commutative Krull domains to the case of non commutative Krull rings.

In §1, we will define non commutative Krull rings and study the relations between a prime Goldie ring R and noetherian local Asano orders containing R. We will introduce, in §2, the concept of divisor classes on bounded Krull rings and show that the divisor class of a non commutative Krull ring becomes an abelian group under some conditions. In §3, we will study orders over a commutative Krull domain $\mathfrak{o}$. Maximal $\mathfrak{o}$-orders are bounded Krull rings. Furthermore we will generalize the approximation theorem for commutative Krull domains to the case of maximal $\mathfrak{o}$-orders (Theorem 3.5). In §4 we will define the w and v-operations on one-sided R-ideals of prime Goldie rings in the same fashion as for commutative domains. We will show that these operations coincide on noetherian bounded Krull rings and maximal $\mathfrak{o}$-orders. Further we will show that every class of right v-ideals of maximal $\mathfrak{o}$-orders contains a right ideal generated by two regular elements. Several examples of non commutative Krull rings will be given in the final section.

Throughout this paper R will denote a prime Goldie ring[1] with identity element which is not artinian, and Q will denote the simple artinian quotient ring of R.

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1) Conditions assumed on rings will always be assumed to hold on both sides; for example, a Goldie ring always means a right and left Goldie ring.

1. Definitions and localizations

Let R be a prime Goldie ring with quotient ring Q. A right R-submodule I of Q is called a *right R-ideal* (*fractional*) provided I contains a regular element of Q and there is a regular element b of Q such that $bI \subseteqq R$. If $I \subseteqq R$, then we say that I is integral. We call a ring R an *Asano order* if its R-ideals form a group under multiplication. R is said to be *local* if its Jacobson radical J is the unique maximal ideal and R/J is artinian. Let R be a noetherian local Asano order. Then, by Proposition 1.3 of [5], R is hereditary, a principal right and left ideal ring. Let F be a right additive topology. We denote by R_F the ring of quotients of R with respect to F (cf. [10]). A subring of Q containing R is called an *overring* of R. An overring R' of R is said to be *right essential* if there is a perfect right additive topology F such that $R'=R_F$ (cf. p. 74 of [10] for the definition of perfect topologies). By the results of §13 of [10], R_F is a right essential overring of R if and only if the inclusion map $R \to R_F$ is an epimorphism and R_F is a flat left R-module. Further, if R_F is a right essential overring of R, then F consists of all right ideals I such that $IR_F=R_F$. In a similar way, we define the concept of left essential overrings of R. An overring R' of R is said to be *essential* if it satisfies the following two conditions:

(1) R' is a right and left essential overring of R, that is, $R'=R_F=R_{F_l}$, where $F(F_l)$ is a perfect right (left) additive topology.

(2) If $I \in F(J \in F_l)$, then $R'I=R'(JR'=R')$.

If A is an ideal of R, then we denote by $C(A)$ those elements of R which are regular in R/A. If R satisfies the Ore condition with respect to $C(P)$, where P is a prime ideal of R, then we denote by R_P the local ring of R with respect to P. Let A, B be subsets of Q. We use the notation: $(A:B)_r=\{q \in Q \mid Bq \subseteqq A\}$, $(A:B)_l=\{q \in Q \mid qB \subseteqq A\}$. We denote by $F_r(R)$ the set of right R-ideals of R, and by $F_l(R)$ the set of left R-ideals of R. We set $F(R)=F_r(R) \cap F_l(R)$.

A prime Goldie ring R is said to be a *Krull ring* if there are families $\{R_i\}_{i \in I}$ and $\{S_j\}_{j \in J}$ of essential overrings of R such that

(K1) $R=\cap_i R_i \cap \cap_j S_j$ $(i \in \boldsymbol{I}, j \in \boldsymbol{J})$.

(K2) Each R_i is a noetherian local Asano order, each S_j is a noetherian simple ring and the cardinal number of $\boldsymbol{J}$ is finite.

(K3) For every regular element c in R we have $cR_i \neq R_i (R_i c \neq R_i)$ for finitely many i only.

If $\boldsymbol{J}=\phi$, then we say that R is *bounded*.

Proposition 1.1. *Let R' be an essential overring of a prime Goldie ring R and let R' be a noetherian local Asano order with unique maximal ideal P'. Then*

(1) *$P'=PR'=R'P$, where $P=R \cap P'$.*

(2) *P is a prime ideal of R.*

(3) *R'/P' is the quotient ring of R/P.*

(4) *R satisfies the Ore condition with respect to $C(P)$ and $R'=R_P$.*

Proof. There is a perfect right (left) additive topology $F(F_l)$ such that $R'=R_F=R_{F_l}$. For any $I\in F$, we have $IR'=R'$ so that $IQ=Q$. This implies that I is an essential right ideal of R by Theorem 3.9 of [4]. Hence we easily obtain that $R'=\cup(R:I)_l$ $(I\in F)$. Similarly $R'=\cup(R:J)_r$ $(J\in F_l)$.

(1) The containment $PR'\subseteq P'$ is clear. Let q be any element of P'. Then there is an element $I\in F$ such that $qI\subseteq P'\cap R=P$. So we have $q\in qR'=qIR'\subseteq PR'$. Hence $P'=PR'$. Similarly $P'=R'P$.

(2) Assume that $AB\subseteq P$ and $B\not\subseteq P$, where A, B, are ideals of R. Then $R'BR'\not\subseteq P'$. Since the ideals of R' are only the powers of P', we have $R'BR'=R'$. Write $1=\sum t_ib_is_i$, where t_i, $s_i\in R'$ and $b_i\in B$. Then there are $I\in F$ and $J\in F_l$ such that $s_iI\subseteq R$ and $Jt_i\subseteq R$. So we have $JI\subseteq B$. Hence $A\subseteq AR'=AJIR'\subseteq P'$ and so $A\subseteq P$.

(3) Let q be any element of R' such that $q\notin P'$. Then $qI\subseteq R$ for some $I\in F$. If $qI\subseteq P$, then $q\in qR'=qIR'\subseteq PR'=P'$, a contradiction. Hence $qI\not\subseteq P$. This implies that R'/P' is an essential extension of R/P as right R/P-modules. Since R'/P' is a simple artinian ring and R/P is a prime ring, we obtain that R'/P' is the right quotient ring of R/P. Similarly R'/P' is the left quotient ring of R/P.

(4) First we shall prove that each element of $C(P)$ is a unit in R'. Since P' is the Jacobson radical of R' and R'/P' is a simple artinian ring, an element q of R' is a unit in R' if and only if it is an element in $C(P')$. Hence it suffices to prove that $C(P)\subseteq C(P')$. To prove this, we assume that $cq\in P'$, where $c\in C(P)$ and $q\in R'$, then $cqI\subseteq P$ and $qI\subseteq R$ for some $I\in F$. Hence $qI\subseteq P$ so that $q\in P'$. Consequently $c\in C(P')$. Let F_P be the set of right ideals I of R such that $r^{-1}I=\{x\in R\,|\,rx\in I\}$ meets $C(P)$ for all $r\in R$. Then we shall prove that $F=F_P$. Since any element of $C(P)$ is a unit in R', the containment $F_P\subseteq F$ is clear. To prove the converse inclusion let I be any element of F and let r be any element of R. Since $r^{-1}I\in F$, we have $(r^{-1}I)R'=R'$. This implies that $[(r^{-1}I+P)/P](R'/P')=R'/P'$. Hence, by (3) and Theorem 3.9. of [4], $(r^{-1}I+P)/P$ is an essential right ideal of R/P so that $(r^{-1}I+P)\cap C(P)\neq\phi$. Write $c=d+p$, where $c\in C(P)$, $d\in r^{-1}I$ and $p\in P$. Then $d\in r^{-1}I\cap C(P)$ and thus $I\in F_P$. Hence $F_P=F$, as desired. Consequently for any element q of R' we have $qc\in R$ for some $c\in C(P)$. Now take any $r\in R$ and $c\in C(P)$. Then $c^{-1}r\in R'$ so that $c^{-1}rd=s\in R$ for some $d\in C(P)$, that is, $rd=cs$. Therefore R satisfies the Ore condition with respect to $C(P)$ and $R'=R_P$.

2. The divisor classes

Throughout this section we assume that R is a bounded Krull ring, that is, $R=\cap R_i\,(i\in I)$ and each R_i is a noetherian local Asano order with unique maximal

ideal P_i'. We let $P_i=P_i'\cap R$. Then P_i is a prime ideal of R by Proposition 1.1. Further we assume, in this section, the following condition:

(K4) For any $i, j\in I$, $i\neq j$, $P_i\not\supseteq P_j$ and $P_i\not\subseteq P_j$.

This condition is equivalent to the following one.

(K4)′ For any $i, j\in \boldsymbol{I}$, $i\neq j$, $P_iR_j=R_j=R_jP_i$.

Lemma 2.1. *Let R be a bounded Krull ring which satisfies (K4). Let I_i' be any element of $F_r(R_i)$ such that $I_i'=R_i$ for almost all $i\in \boldsymbol{I}$. Then $I=\cap I_i'\in F_r(R)$ and $IR_i=I_i'$ for all $i\in \boldsymbol{I}$.*

Proof. It is clear that $I\in F_r(R)$. First assume that $I_i'\subseteq R_i$ for all $i\in \boldsymbol{I}$. We put $I_i=I_i'\cap R$. Then $I=I_1\cap\cdots\cap I_k$. Since R_i is bounded, there are natural numbers n_i such that $I_i'\supseteq P_i'^{n_i}$ and so $I_i\supseteq P_i^{n_i}$. Since R_i is flat, we have $IR_i=I_iR_i=I_i'$ for all $i\in \boldsymbol{I}$ by (K4)′. In general case there is a regular element c of R such that $cI_i'\subseteq R_i$ for all $i\in \boldsymbol{I}$. Hence $cIR_i=cI_i'$ so that $IR_i=I_i'$ for all $i\in \boldsymbol{I}$.

Lemma 2.2 (Robson [7]). *Let S be a prime Goldie ring with quotient ring $Q(S)$. Then any right S-ideal of S is generated by the units in $Q(S)$ which it contains.*

Proof. Let I be any element of $F_r(S)$. Then there is a regular element c of S such that $cI\subseteq S$. By Theorem 5.5 of [7], $cI=\sum b_iS$, where the set $\{b_i\}$ is the regular elements of S contained in cI. Hence $I=\sum(c^{-1}b_i)S$.

For any $A\in F(R)$ we denote by A_d the (R, R)-bimodule $\cap R_iAR_i(i\in \boldsymbol{I})$. There are only finitely many $R_i(1\leqq i\leqq k)$ such that $R_iAR_i\neq R_i$. Since $A\in F(R)$, we have $cA\subseteq R$ for some regular element c of R. Thus $R_iAR_i\subseteq R_ic^{-1}R_i$. Since R_i is a bounded Aasano order, $R_ic^{-1}R_i\in F(R_i)$ so that $R_iAR_i\in F(R_i)$. Therefore $R_iAR_i=P_i'^{n_i}(1\leqq i\leqq k)$ and $R_jAR_j=R_j$ for $j\in \boldsymbol{I}, j\notin\{1, \cdots, k\}$. Hence we get $A_d=P_1'^{n_1}\cap\cdots\cap P_k'^{n_k}\cap {}_jR_j$ and $A_d\in F(R)$.

We obtain immediately

(i) $A\subseteq A_d$.

(ii) If $A\subseteq B$, then $A_d\subseteq B_d$.

(iii) $A_{dd}=A_d$.

If $A=A_d$, then it is said to be *d-ideal*. If we define an equivalence relation on $F(R)$ by saying that $A\sim B$ if and only if $A_d=B_d$. For any $A\in F(R)$, we denote by $\bar{A}$ the equivalence class determined by A. Each such equivalence class $\bar{A}$ contains a unique d-ideal A_d. The set $D(R)$ of all such equivalence classes forms a semi-group under the multiplication "*" defined by $\bar{A}*\bar{B}=\overline{(A_dB_d)}$.

Theorem 2.3. *If a bounded Krull ring R satisfies (K4), then $D(R)$ forms an abelian group and it is a direct product of infinite cyclic subgroups $\{(\bar{P}_i)\}_{i\in I}$.*

Proof. If A is d-ideal, then $A=\cap P_i'^{n_i}$, where n_i are integers and $n_i=0$

for almost all $i\in I$. We let $B=\cap P_i'^{-n_i}$. Then we get the following;

(i) $B_d=B$.

(ii) $B=(R:A)_l=(R:A)_r$, and so we denote B by A^{-1}.

(iii) $(A^{-1})^{-1}=A$.

By Lemma 2.1, (i) immediately follows. (ii); we have $BA\subseteqq P_i'^{-n_i}P_i'^{n_i}=R_i$ for all $i\in I$. So $BA\subseteqq R$, and thus $B\subseteqq(R:A)_l$. To prove the converse let c be a unit in Q contained in $(R:A)_l$, that is, $cA\subseteqq R$ and so $cAR_i=cP_i'^{n_i}\subseteqq R_i$ by Lemma 2.1. Hence $c\in P_i'^{-n_i}$ for all i and therefore $c\in B$. Consequently $B\supseteqq(R:A)_l$ by Lemma 2.2 and the equality holds. Similarly $B=(R:A)_r$. (iii) is clear from (ii). By Lemma 2.1 we get $(AA^{-1})R_i=AP_i'^{-n_i}=P_i'^{n_i}P_i'^{-n_i}=R_i$ for all $i\in I$. Hence $R=\cap(AA^{-1})R_i\subseteqq(AA^{-1})_d\subseteqq R$ so that $\bar{R}=\bar{A}*\bar{A}^{-1}$. Therefore $D(R)$ forms a group. Let $A=P_1'^{n_1}\cap\cdots\cap P_k'^{n_k}\cap{}_jR_j$ be any d-ideal of R. Then we obtain immediately that $A=(P_1^{n_1}\cdots P_k^{n_k})_d=(P_1^{n_1}\cap\cdots\cap P_k^{n_k})_d$. Hence $\bar{A}=\overline{P_1^{n_1}\cdots P_k^{n_k}}=\prod_{i=1}^{k}*\bar{P}_i^{n_i}$ so that $D(R)$ is an abelian group generated by $\{\bar{P}_i\}_{i\in I}$. If $\prod_{i=1}^{k}*\bar{P}_i^{n_i}=1$, then $(\prod_{i=1}^{k}P_i^{n_i})_d=R$. This implies that $R_i=P_i'^{n_i}$ and so $n_i=0$ for all $i\in I$. Therefore $D(R)$ is a direct product of infinite cyclic subgroups $\{(\bar{P}_i)\}_{i\in I}$.

3. Maximal orders over commutative Krull domains

In the remainder of this paper, $\mathfrak{o}$ denotes a commutative Krull domains, K denotes the quotient field of $\mathfrak{o}$, and Σ a fixed central simple K-algebra with finite dimension over K. Let $\boldsymbol{P}$ be the set of all minimal prime ideals of $\mathfrak{o}$. Then $\mathfrak{o}=\cap\mathfrak{o}_\mathfrak{z}(\mathfrak{z}\in\boldsymbol{P})$ and $\mathfrak{o}_\mathfrak{z}$ is a discrete rank one valuation overring of $\mathfrak{o}$.

Following [2], A subring Λ of Σ is said to be $\mathfrak{o}$-*order* if the following conditions are satisfied:

(i) $\mathfrak{o}\subseteqq\Lambda$.

(ii) $K\Lambda=\Sigma$.

(iii) Each element of Λ is integral over $\mathfrak{o}$.

If Λ is an $\mathfrak{o}$-order, then, by Proposition 1.1 of [2], there is a finitely generated $\mathfrak{o}$-free submodule F of Σ such that $F\supseteqq\Lambda$. Further if Λ is maximal $\mathfrak{o}$-order, then $\Lambda\otimes\mathfrak{o}_\mathfrak{z}$ is also a maximal $\mathfrak{o}_\mathfrak{z}$-order by Proposition 1.3 of [2]. Therefore if Λ is a maximal $\mathfrak{o}$-order, then $\Lambda\otimes\mathfrak{o}_\mathfrak{z}$ is a noetherian local Asano order and $\Lambda=\cap\Lambda\otimes\mathfrak{o}_\mathfrak{z}(\mathfrak{z}\in\boldsymbol{P})$. For any $\mathfrak{z}\in\boldsymbol{P}$, we denote by P' the unique maximal ideal of $\Lambda\otimes\mathfrak{o}_\mathfrak{z}$, and denote by P the contracted prime ideal $P'\cap\Lambda$.

Proposition 3.1. *Let Λ be a maximal $\mathfrak{o}$-order. Then*

(1) *Λ is a bounded Krull ring and satisfies* (K4).

(2) $D(\Lambda)\cong D(\mathfrak{o})$.

Proof. For any $\mathfrak{z}\in\boldsymbol{P}$, we denote by $F(F_l)$ the set of all right (left) ideals

$I(J)$ of Λ such that $I(J)$ meets $\mathfrak{o}-\mathfrak{z}$. Then it follows that $F(F_l)$ is a perfect right (left) additive topology and that $\Lambda_F=\Lambda\otimes\mathfrak{o}_\mathfrak{z}=\Lambda_{F_l}$. Hence $\Lambda\otimes\mathfrak{o}_\mathfrak{z}$ is an essential overring of Λ. Let x be any regular element of Λ. Since x is algebric over K, we get; $a_t x^t+\cdots+a_0=0$ for some $a_i\in\mathfrak{o}$. We assume that $a_0\neq 0$. Put $y=-a_t x^{t-1}-\cdots-a_1$. Then we have $a_0=xy=yx$ and $y\in\Lambda$. So $(xy)\mathfrak{o}_\mathfrak{z}=\mathfrak{o}_\mathfrak{z}$ for almost all $\mathfrak{z}\in\boldsymbol{P}$. Therefore $x(\Lambda\otimes\mathfrak{o}_\mathfrak{z})=\Lambda\otimes\mathfrak{o}_\mathfrak{z}$ for almost all $\mathfrak{z}\in\boldsymbol{P}$. It is evident that $P\cap\mathfrak{o}=\mathfrak{z}$ for any $\mathfrak{z}\in\boldsymbol{P}$ and that $\mathfrak{o}$ satisfies (K4). Therefore Λ also satisfies (K4). (2) is clear from (1) and Theorem 2.3.

Proposition 3.2. *Let Λ be a maximal $\mathfrak{o}$-order and let $\mathfrak{z}\in\boldsymbol{P}$. Then the set of prime ideals of Λ lying over $\mathfrak{z}$ is only $\{P\}$. In particular, P is a minimal prime ideal of Λ.*

Proof. Assume that P_0 is a non zero prime ideal of Λ such that $P_0\cap\mathfrak{o}=\mathfrak{z}$ and $P_0\neq P$. We set $\bar{\Lambda}=\Lambda/P_0$, $\bar{\mathfrak{o}}=\mathfrak{o}/\mathfrak{z}$ and $\bar{K}=\mathfrak{o}_\mathfrak{z}/\mathfrak{z}\mathfrak{o}_\mathfrak{z}$. First we shall prove that $\bar{\Lambda}$ is a prime Goldie ring with quotient ring $\bar{\Lambda}\otimes\bar{K}$. It is evident that $\bar{\Lambda}$ is $\bar{\mathfrak{o}}$-torsion-free. Hence the natural map $\bar{\Lambda}\to\bar{\Lambda}\otimes\bar{K}$ is a monomorphism and $\bar{\Lambda}\otimes\bar{K}$ is an essential extension of $\bar{\Lambda}$ as $\bar{\Lambda}$-modules. Since $\Lambda\otimes\mathfrak{o}_\mathfrak{z}$ is finitely generated as $\mathfrak{o}_\mathfrak{z}$-modules, $\bar{\Lambda}\otimes\bar{K}$ is a $\bar{K}$-algebra with finite dimension over $\bar{K}$, so that $\bar{\Lambda}\otimes\bar{K}$ is a simple artinian ring. Therefore $\bar{\Lambda}\otimes\bar{K}$ is a quotient ring of $\bar{\Lambda}$, and so $\bar{\Lambda}$ is a prime Goldie ring.

(i) In case $P\supsetneqq P_0$. Let $P_0(\Lambda\otimes\mathfrak{o}_\mathfrak{z})=P'^n$ for some natural number n. We shall prove that $C(P_0)\subseteq C(P')$. By Lemma 2.3 of [5], $C(P'^n)=C(P')$, and so it suffices to prove that $C(P_0)\subseteq C(P'^n)$. If $cx\in P'^n$, where $c\in C(P_0)$, $x\in\Lambda\otimes\mathfrak{o}_\mathfrak{z}$, then there exists $m\in\mathfrak{o}-\mathfrak{z}$ such that $cxm\in P_0$ and $xm\in\Lambda$. Hence $xm\in P_0$ and so $x\in P'^n$. Therefore each element of $C(P_0)$ is a unit in $\Lambda\otimes\mathfrak{o}_\mathfrak{z}$. Since $P\not\subseteq P_0$ and $\bar{\Lambda}$ is a prime Goldeie ring, we have $P\cap C(P_0)\neq\phi$. This implies that $P(\Lambda\otimes\mathfrak{o}_\mathfrak{z})=\Lambda\otimes\mathfrak{o}_\mathfrak{z}$, a contradiction.

(ii) In case $P\not\supseteq P_0$. The family $F=\{I\mid x^{-1}I\cap C(P_0)=\phi$ for any $x\in\Lambda$, I: right ideal$\}$ is a topology (cf. Exer. 4 of [10, p. 18]). The F-torsion submodule $t(\Lambda)$ of Λ is an ideal and $t(\Lambda)\subseteq P_0$. We denote by $\tilde{\Lambda}$ the factor ring $\Lambda/t(\Lambda)$ and by Λ_F the ring of quotients of Λ, that is, $\Lambda_F=\varinjlim \mathrm{Hom}\,(I,\tilde{\Lambda})$ $(I\in F)$. Then $\tilde{\Lambda}$ is a subring of Λ_F(cf. Chap. 2 of [10]). We shall prove that $\tilde{P}_0\Lambda_F\subsetneqq\Lambda_F$, where $\tilde{P}_0=P_0/t(\Lambda)$. It suffices to prove that $\tilde{P}_0\Lambda_F\cap\tilde{\Lambda}=\tilde{P}_0$. Assume that $\tilde{T}=\tilde{P}_0\Lambda_F\cap\tilde{\Lambda}\supsetneqq\tilde{P}_0$, where T is an ideal of Λ. Then $T\supsetneqq P_0$ and so $T\cap C(P_0)\neq\phi$. Let c be any element of $T\cap C(P_0)$. Write $\tilde{c}=\sum\tilde{p}_i q_i$, where $p_i\in P_0$ and $q_i\in\Lambda_F$. There exists $I\in F$ such that $q_i I\subseteq\tilde{\Lambda}$. Hence $cI\subseteq P_0$ and thus $I\subseteq P_0$. This contradicts to $C(P_0)\cap I\neq\phi$. Therefore $\tilde{P}_0\Lambda_F\cap\tilde{\Lambda}=\tilde{P}_0$. It is evident that $m\Lambda\in F$ for every $m\in\mathfrak{o}-\mathfrak{z}$. Hence we may assume that $\tilde{\Lambda}\otimes\mathfrak{o}_\mathfrak{z}\subseteq\Lambda_F$. Since $P_0\not\subseteq P$, $C(P)\cap P_0\neq\phi$. This implies that $P_0(\Lambda\otimes\mathfrak{o}_\mathfrak{z})=\Lambda\otimes\mathfrak{o}_\mathfrak{z}$. Therefore $\tilde{P}_0\Lambda_F=\Lambda_F$, a contradiction.

Finally we shall prove that P is a minimal prime ideal of Λ. If $P\supseteq P_0\neq 0$

and P_0 is a prime ideal of Λ, then $P_0 \cap \mathfrak{o} \neq 0$ is a prime ideal of $\mathfrak{o}$ and $\mathfrak{z} \supseteq P_0 \cap \mathfrak{o}$. Hence $\mathfrak{z} = P_0 \cap \mathfrak{o}$. Therefore $P_0 = P$ by (i).

Lemma 3.3. *Let $\mathfrak{z}_i (1 \leqq i \leqq k)$ be any elements in $\boldsymbol{P}$. Then $\Lambda_1 = \bigcap_{i=1}^{k} \Lambda \otimes \mathfrak{o}_{\mathfrak{z}_i}$ is a boundeed Dedekind prime ring, and is a right and left principal ideal ring.*

Proof. The set $M = \mathfrak{o} - (\mathfrak{z}_1 \cup \cdots \cup \mathfrak{z}_k)$ is a multiplicative closed set and $\mathfrak{o}_M = \mathfrak{o}_{\mathfrak{z}_1} \cap \cdots \cap \mathfrak{o}_{\mathfrak{z}_k}$ is a Dedekind domain. Further it is a principal ideal ring. By Corollary 1.9 of [2], $\Lambda \otimes \mathfrak{o}_M = \bigcap_{i=1}^{k} \Lambda \otimes \mathfrak{o}_{\mathfrak{z}_i}$ is a maximal $\mathfrak{o}_M$-order. Hence it is a bounded Dedekind prime ring. Since the set of prime ideals of $\Lambda \otimes \mathfrak{o}_M$ is $\{P_i(\Lambda \otimes \mathfrak{o}_M) \mid 1 \leqq i \leqq k\}$, $\Lambda \otimes \mathfrak{o}_M$ is a right and left principal ideal ring.

Lemma 3.4. *Let $\mathfrak{z}_i (1 \leqq i \leqq k)$ be any elements in $\boldsymbol{P}$ and let n be any positive integer. Then there is a regular element $x \in \Lambda$ such that $x(\Lambda \otimes \mathfrak{o}_{\mathfrak{z}_1}) = P_1'^{n} = (\Lambda \otimes \mathfrak{o}_{\mathfrak{z}_1})x$ and $x(\Lambda \otimes \mathfrak{o}_{\mathfrak{z}_i}) = \Lambda \otimes \mathfrak{o}_{\mathfrak{z}_i} (2 \leqq i \leqq k)$.*

Proof. We put $M = \mathfrak{o} - (\mathfrak{z}_1 \cup \cdots \cup \mathfrak{z}_k)$ and $P_i'' = (\Lambda \otimes \mathfrak{o}_M) \cap P_i'$. By Lemma 3.3, $P_1'' = y(\Lambda \otimes \mathfrak{o}_M) = (\Lambda \otimes \mathfrak{o}_M)y$ for some regular element $y \in \Lambda \otimes \mathfrak{o}_M$. Since $y = zm^{-1}$ ($z \in \Lambda$ and $m \in M$), we may assume that $y \in \Lambda \cap P_1'' = P_1$. We shall prove that $y \in C(P_i)$ $(2 \leqq i \leqq k)$. By Proposition 3.1, Λ satisfies (K4) and so $P_1 \cap C(P_i) \neq \phi (2 \leqq i \leqq k)$. Let $c = yw (w \in \Lambda \otimes \mathfrak{o}_M)$ be any element of $P_1 \cap C(P_i)$. If $zy \in P_i$, where $z \in \Lambda$, then $zyw = zc \in P_i'' \cap \Lambda = P_i$ so that $z \in P_i$. Hence y is a unit in $\Lambda \otimes \mathfrak{o}_{\mathfrak{z}_i} (2 \leqq i \leqq k)$. We put $x = y^n$. Then x satisfies the assertion of the lemma.

Theorem 3.5. *Let $\mathfrak{z}_1, \cdots, \mathfrak{z}_k \in \boldsymbol{P}$ and let $n_1, \cdots, n_k$ be any integers. Then there is a unit $x \in \Sigma$ such that $x(\Lambda \otimes \mathfrak{o}_{\mathfrak{z}_i}) = P_i'^{n_i} (1 \leqq i \leqq k)$ and $x \in \Lambda \otimes \mathfrak{o}_{\mathfrak{z}_\sigma}$ for all $\mathfrak{z}_\sigma \in \boldsymbol{P}$ with $\mathfrak{z}_\sigma \neq \mathfrak{z}_i$.*

Proof. It is enough to show that for any i, there exists a unit $x_i \in \Sigma$ such that $x_i(\Lambda \otimes \mathfrak{o}_{\mathfrak{z}_i}) = P_i'^{n_i}$, $x_i(\Lambda \otimes \mathfrak{o}_{\mathfrak{z}_j}) = \Lambda \otimes \mathfrak{o}_{\mathfrak{z}_j} (i \neq j,\ 1 \leqq j \leqq k)$ and $x_i \in \Lambda \otimes \mathfrak{o}_{\mathfrak{z}_\sigma}$ for all $\mathfrak{z}_\sigma \in \boldsymbol{P}$ with $\sigma \notin \{1, \cdots, k\}$. We will exhibit a unit x_1 in Σ such that $x_1(\Lambda \otimes \sigma_{\mathfrak{z}_1}) = P_1'^{n_1}$, $x_1(\Lambda \otimes \mathfrak{o}_{\mathfrak{z}_i}) = \Lambda \otimes \mathfrak{o}_{\mathfrak{z}_i} (2 \leqq i \leqq k)$ and $x_1 \in (\Lambda \otimes \mathfrak{o}_{\mathfrak{z}_\sigma})$ for $\sigma \notin \{1, \cdots, k\}$.

(i) If $n_1 > 0$, then the assertion follows from Lemma 3.4.

(ii) If $n_1 < 0$, then, by Lemma 3.4, there is a regular element y in Λ such that $y(\Lambda \otimes \mathfrak{o}_{\mathfrak{z}_1}) = (\Lambda \otimes \mathfrak{o}_{\mathfrak{z}_1})y = P_1'^{-n_1}$ and $y(\Lambda \otimes \mathfrak{o}_{\mathfrak{z}_i}) = \Lambda \otimes \mathfrak{o}_{\mathfrak{z}_i} (2 \leqq i \leqq k)$. There are only finitely many elements $\mathfrak{z}_{k+1}, \cdots, \mathfrak{z}_l \in \boldsymbol{P}$ such that $y(\Lambda \otimes \mathfrak{o}_{\mathfrak{z}_j}) \subsetneq \Lambda \otimes \mathfrak{o}_{\mathfrak{z}_j}$. For any $j (k+1 \leqq j \leqq l)$, there is a positive integer h_j such that $y(\Lambda \otimes \mathfrak{o}_{\mathfrak{z}_j}) \supseteq P_j'^{h_j}$, because $\Lambda \otimes \mathfrak{o}_{\mathfrak{z}_j}$ is bounded, that is, $y^{-1}P_j'^{h_j} \subseteq \Lambda \otimes \mathfrak{o}_{\mathfrak{z}_j}$. Again, by Lemma 3.4, there is a regular element z_j in Λ such that $z_j(\Lambda \otimes \mathfrak{o}_{\mathfrak{z}_j}) = (\Lambda \otimes \mathfrak{o}_{\mathfrak{z}_j})z_j = P_j'^{h_j}$ and $z_j(\Lambda \otimes \mathfrak{o}_{\mathfrak{z}_n}) = \Lambda \otimes \mathfrak{o}_{\mathfrak{z}_n}$ for any $j (k+1 \leqq j \leqq l)$ and any $n \neq j$, $n \in \{1, \cdots, k, \cdots, l\}$. Here we set $z = z_{k+1} \cdots z_l \in \Lambda$. Then we get: $z(\Lambda \otimes \mathfrak{o}_{\mathfrak{z}_i}) = \Lambda \otimes \mathfrak{o}_{\mathfrak{z}_i}$ for any $i (1 \leqq i \leqq k)$ and $z(\Lambda \otimes \mathfrak{o}_{\mathfrak{z}_j}) = P_j'^{h_j}$ for any $j (k+1 \leqq j \leqq l)$. If $x_1 = y^{-1}z$, then we obtain that

$x_1(\Lambda\otimes\mathfrak{o}_{\mathfrak{d}_1})=P_1'^{n_1}$, $x_1(\Lambda\otimes\mathfrak{o}_{\mathfrak{d}_i})=\Lambda\otimes\mathfrak{o}_{\mathfrak{d}_i}$ for any $i(2\leqq i\leqq k)$ and $x_1\in\Lambda\otimes\mathfrak{o}_{\mathfrak{d}_\sigma}$ for any $\sigma\notin\{1, \cdots, k\}$.

4. $*$-operations

Let R be a prime Goldie ring with quotient ring Q. Following [3], A mapping $*$: $I\to I^*$ of $F_r(R)$ into $F_r(R)$ is called a *$*$-operation* on R if the following conditions hold for any unit a in Q and any $I, J\in F_r(R)$:

(i) $(aR)^*=aR$, $(aI)^*=aI^*$.

(ii) $I\subseteqq I^*$, if $I\subseteqq J$, then $I^*\subseteqq J^*$.

(iii) $(I^*)^*=I^*$.

If $I=I^*(I\in F_r(R))$, then we say that I is a *right $*$-ideal*. In this section we shall define $*$-operations of two kinds and give sufficient conditions that these two $*$-operations coincide. Let I be any element of $F_r(R)$. We define the *v-operation* on R by $I_v=(R:(R:I)_l)_r$. If R is a Krull ring, that is, $R=\cap R_i\cap S_j$, then we define the *w-operation* on R by $I_w=\cap IR_i\cap IS_j$. If $\mathfrak{o}$ is a commutative Krull domain and if Λ is a maximal $\mathfrak{o}$-order, that is, $\Lambda=\cap\Lambda\otimes\mathfrak{o}_{\mathfrak{d}}(\mathfrak{d}\in\boldsymbol{P})$, then $A_w=A_d$ for any $A\in F(\Lambda)$. It is evident that the w-operation is a $*$-operation.

By using Lemma 2.2, the following proposition is proved by the same way as in commutative domains (cf. Theorem 28.1 of [3]).

Proposition 4.1. *Let R be a prime Goldie ring with quotient ring Q and let I be any element in $F_r(R)$. Then I_v is the intersection of the set of principal right R-ideals of R containing I.*

Corollary 4.2. (1) *The v-operation on R is a $*$-operation.*

(2) *If $I\in F_r(R)$, then $I^*\subseteqq I_v$ for any $*$-operation on R. In particular, any right v-ideal is a right $*$-ideal.*

(3) *If $J\in F_l(R)$, then $(R:J)_r$ is a right v-ideal.*

Proof. (1) and (2) are evident from Proposition 4.1 and the definitions. (3): An element x in Q is an element in $(R:J)_r$ if and only if $Jx\subseteqq R$. This condition is equivalent to $cx\in R$ for any unit c in Q which is contained in J. Hence $(R:J)_r=\cap c^{-1}R=\cap(c^{-1}R)_v=(\cap c^{-1}R)_v$, where c ranges over all units in Q which are contained in J. Therefore $(R:J)_r$ is a right v-ideal.

Proposition 4.3. *Let R be a noetherian Krull ring, that is, $R=\cap R_i\cap S_j$ $(i\in\boldsymbol{I}, j\in\boldsymbol{J})$ and let each S_j be hereditary. Then $I_w=I_v$ for any $I\in F_r(R)$.*

Proof. For a convenience, we denote R_i or S_j by T_k. We let $J=(R:I)_l$. First we shall prove that $(T_k:IT_k)_l=T_kJ$. Let x be any element in $(T_k:IT_k)_l$. Then $xI\subseteqq T_k$. Since R is noetherian, xI is finitely generated so that $J_0xI\subseteqq R$ for some $J_0\in F_{k_l}$, where F_{k_l} is the left additive topology on R such that $T_k=R_{F_{k_l}}$. Thus $J_0x\subseteqq J$. Consequently we have $x\in T_kx=T_kJ_0x\subseteqq T_kJ$ and so $(T_k:IT_k)_l$

$\subseteq T_kJ$. The converse containment is evident. Similarly we get $(T_k : T_kJ)_r = (R : J)_rT_k$. By Corollary 4.2, J is a left w-ideal so that $J = \cap T_kJ = \cap (T_k : IT_k)_l$. Similarly $(R : J)_r = \cap (T_k : T_kJ)_r$. By Corollary 4.6 of [7], simple rings are also maximal orders. So $(T_k : T_kJ)_r = (T_k : (T_k : IT_k)_l)_r = IT_k$ for all k. Hence we get $I_v = (R : J)_r = \cap (T_k : T_kJ)_r = \cap IT_k = I_w$, as desired.

Theorem 4.4. *Let $\mathfrak{o}$ be a commutative Krull domain and let Λ be a maximal $\mathfrak{o}$-order. Then $I_v = I_w$ for any $I \in F_r(\Lambda)$.*

Proof. (i) First we assume that I is integral. By Corollary 4.2 $I_w \subseteq I_v$. To prove the converse inclusion it suffices to prove that any regualr element c in $\Lambda - I_w$ is not contained in I_v by Lemma 2.2. We set $I_w = \cap I_i'$, where $I_i' = I(\Lambda \otimes \mathfrak{o}_{\mathfrak{P}_i})$ and $\mathfrak{P}_i \in \boldsymbol{P}$. There are only finitely many $I_1', \cdots, I_k'$ such that $I_i' \subsetneq \Lambda \otimes \mathfrak{o}_{\mathfrak{P}_i}$ $(1 \leqq i \leqq k)$. Put $M = \mathfrak{o} - (\mathfrak{P}_1 \cup \cdots \cup \mathfrak{P}_k)$. By Lemma 3.3 there is a regular element x in Λ such that $x(\Lambda \otimes \mathfrak{o}_M) = I_1' \cap \cdots \cap I_k' \cap (\Lambda \otimes \mathfrak{o}_M)$ and so $x(\Lambda \otimes \mathfrak{o}_{\mathfrak{P}_i}) = I_i'$ for $i (1 \leqq i \leqq k)$. There are only finitely many $k+1, \cdots, l$ such that $x(\Lambda \otimes \mathfrak{o}_{\mathfrak{P}_j}) \subsetneq \Lambda \otimes \mathfrak{o}_{\mathfrak{P}_j}$ $(k+1 \leqq j \leqq l)$. Since $\Lambda \otimes \mathfrak{o}_{\mathfrak{P}_j}$ is bounded, there are positive integers $n_{k+1}, \cdots, n_l$ such that $x(\Lambda \otimes \mathfrak{o}_{\mathfrak{P}_j}) \supseteq P_j'^{n_j}$. By Theorem 3.5, there is a unit y in Σ such that $y(\Lambda \otimes \mathfrak{o}_{\mathfrak{P}_i}) = \Lambda \otimes \mathfrak{o}_{\mathfrak{P}_i} (1 \leqq i \leqq k)$, $y(\Lambda \otimes \mathfrak{o}_{\mathfrak{P}_j}) = (\Lambda \otimes \mathfrak{o}_{\mathfrak{P}_j})y = P_j'^{n_j} (k+1 \leqq j \leqq l)$ and $y \in \Lambda \otimes \mathfrak{o}_{\mathfrak{P}_\sigma}$ for $\sigma \notin \{1, \cdots, l\}$. Here we put $z = xy^{-1}$. Then we get: $z(\Lambda \otimes \mathfrak{o}_{\mathfrak{P}_i}) = I_i' (1 \leqq i \leqq k)$, $z(\Lambda \otimes \mathfrak{o}_{\mathfrak{P}_\sigma}) \supseteq (\Lambda \otimes \mathfrak{o}_{\mathfrak{P}_\sigma})$ for $\sigma \notin \{1, \cdots, k\}$. Therefore $z\Lambda = \cap z(\Lambda \otimes \mathfrak{o}_{\mathfrak{P}}) \supseteq I_w$. If $c \in z\Lambda$, then $c \in I_1' \cap \cdots \cap I_k' \cap \Lambda = I_w$, a contradiction. So $c \notin z\Lambda$. This implies that $c \notin I_v$ by Proposition 4.1. Consequently we have $I_w = I_v$. (ii) If I is fractional, then cI is integral for some regular element c in Λ, then $cI_v = (cI)_v = (cI)_w = cI_w$ so that $I_v = I_w$.

REMARK. Let R be a non commutative Krull domain in the sense of [1]. Then, by the similar way as in Theorem 4.4, we get $I_v = I_w$ for any $I \in F_r(R)$.

Theorem 4.5. *Let $\mathfrak{o}$ be a commutative Krull domain and let Λ be a maximal $\mathfrak{o}$-order. If $I \in F_r(R)$, then there are units c, d in Σ such that $I_v = (c\Lambda + d\Lambda)_v$.*

Proof. It suffices to prove the theorem for any integral right R-ideal I. We write $I_v = I_1' \cap \cdots \cap I_k' \cap \Lambda$, where $I_i' = I(\Lambda \otimes \mathfrak{o}_{\mathfrak{P}_i})$. Set $M_1 = \mathfrak{o} - (\mathfrak{P}_1 \cup \cdots \cup \mathfrak{P}_k)$. By Lemma 3.3, $I_v(\Lambda \otimes \mathfrak{o}_{M_1}) = c(\Lambda \otimes \mathfrak{o}_{M_1})$ for some regular element c in Λ. By Lemma 2.1 we have $I_v(\Lambda \otimes \mathfrak{o}_{\mathfrak{P}_i}) = I_i' = c(\Lambda \otimes \mathfrak{o}_{\mathfrak{P}_i})$ $(1 \leqq i \leqq k)$. There are only finite numbers $k+1, \cdots, l$ such that $c(\Lambda \otimes \mathfrak{o}_{\mathfrak{P}_j}) \subsetneq (\Lambda \otimes \mathfrak{o}_{\mathfrak{P}_j})$ $(k+1 \leqq j \leqq l)$. Set $M_2 = \mathfrak{o} - (\mathfrak{P}_{k+1} \cup \cdots \cup \mathfrak{P}_l)$ and $N = M_1 \cap M_2$. Now for any I_i' we get $I_i' \supseteq P_i'^{n_i}$ for some positive number n_i and so $I_i = I_i' \cap \Lambda \supseteq P_i^{n_i}$. We put $A = P_1^{n_1} \cap \cdots \cap P_k^{n_k}$. Then $I_v \supseteq A$ and $A(\Lambda \otimes \mathfrak{o}_N) = d(\Lambda \otimes \mathfrak{o}_N)$ for some regular element d in Λ. Since Λ satisfies (K4) and $\Lambda \otimes \mathfrak{o}_{\mathfrak{P}_j}$ is Λ-flat, we have $d(\Lambda \otimes \mathfrak{o}_{\mathfrak{P}_j}) = \Lambda \otimes \mathfrak{o}_{\mathfrak{P}_j}$ for $j (k+1 \leqq j \leqq l)$. Therefore we get: $(c\Lambda + d\Lambda)(\Lambda \otimes \mathfrak{o}_{\mathfrak{P}_i}) = I_i' + P_i'^{n_i} = I_i' (1 \leqq i \leqq k)$ and $(c\Lambda + d\Lambda)(\Lambda \otimes \mathfrak{o}_{\mathfrak{P}_\sigma}) = \Lambda \otimes \mathfrak{o}_{\mathfrak{P}_\sigma}$ for any $\sigma \notin \{1, \cdots, k\}$. Thus we obtain $(c\Lambda + d\Lambda)_v = I_v$, as

desired.

5. Examples

In this section we shall give some examples of Krull rings.

(i) A commutative Krull domain $\mathfrak{o}$ and a maximal $\mathfrak{o}$-order are both bounded Krull rings (cf. §3).

(ii) Noetherian Asano orders are Krull rings (cf. [5]). In particular, bounded noetherian Asano orders are bounded Krull rings.

(iii) If R is Krull ring, then the complete matrix ring $(R)_n$ is also Krull. In particular, if R is bounded, then $(R)_n$ is also bounded.

(iv) If R is a Krull ring, then the polynomial ring $R[x]$ over R is also Krull.

In the remainder of this section, we shall give the proof of (iv).

Lemma 5.1. *Let R be a noetherian local Asano order with unique maximal ideal P, and let Q be the quotient ring of R. Then*

(1) *$R[x]$ satisfies the Ore condition with respect to $C(P[x])$ and $R[x]_{P[x]}$ is also a noetherian local Asano order.*

(2) $Q[x]\cap R[x]_{P[x]}=R[x]$.

Proof. It is well known that $R[x]$ is a noetherian prime ring. Since $P[x]$ is invertible, the proof of (1) is similar to one in §2 of [5].

(2) It is evident that $Q[x]\cap R[x]_{P[x]}\supseteq R[x]$. Let $f(x)=a_nx^n+\cdots+a_0$ be any element of $Q[x]\cap R[x]_{P[x]}$, where $a_i\in Q$ and let $f(x)=g(x)h(x)^{-1}$, where $g(x)$, $h(x)\in R[x]$ and $h(x)\in C(P[x])$. Since $\overline{h(x)}=[h(x)+P[x]]$ is a regular element in $R[x]/P[x]\cong\bar{R}[x]$, where $\bar{R}=R/P$, there is an element $r(x)$ in $R[x]$ such that $\overline{h(x)}\,\overline{r(x)}=\bar{c}_mx^m+\cdots+\bar{c}_0$ by Lemma 2 of [9], where $\bar{c}_m$ is a unit in $\bar{R}$. Hence c_m is a unit in R. So $\overline{f(x)}\overline{h(x)}\overline{r(x)}=\overline{g(x)}\overline{r(x)}\in\bar{R}[x]$ and $\overline{f(x)}\in Q/P[x]$. From this we have $\bar{a}_n\bar{c}_m\in\bar{R}$. Thus $\bar{a}_n\in\bar{R}$, that is, $a_n\in R$. By induction, we get $a_i\in R$ for all i and thus $f(x)\in R[x]$.

Lemma 5.2. *Let R be a prime Goldie ring with quotient ring Q and let $S\subseteq T$ be overrings of R. If S is an essential overring of R and if T is an essential overring of S, then T is an essential overring of R.*

Proof. It is evident that T is a flat R-module. Because

$$T\cong T\otimes_S T\cong T\otimes_S(S\otimes_S T)\cong T\otimes_S[(S\otimes_R S)\otimes_S T]\cong$$
$$T\otimes_S[S\otimes_R(S\otimes_S T)]\cong T\otimes_S(S\otimes_R T)\cong T\otimes_R T,$$

the inclusion map: $R\to T$ is an eqimorphism. Therefore there is a perfect right (left) additive topology $F(F_l)$ such that $R_F=T=R_{F_l}$ by Theorem 13.10 of [10].

From the assumption, there are topologies $F_i, F_{i_l}(i=1,2)$ such that $R_{F_1}=S=R_{F_{1_l}}$ and $S_{F_2}=T=S_{F_{2_l}}$. Let I be any element of F. Then $T=IT=IST$ so that $IS\in F_2$. Hence $TIS=T$. Write $1=\Sigma t_i a_i s_i$, where $t_i\in T$, $a_i\in I$ and $s_i\in S$. There is an element A of F_1 such that $s_iA\subseteqq R$. Therefore we get $A=(\Sigma t_i a_i s_i)A \subseteqq \Sigma(t_i a_i)(s_iA)\subseteqq TI$. Since $SA=S$, we have $T\supseteqq TI\supseteqq TA=TSA=TS=T$ so that $T=TI$. Similarly, we have $JT=T$ for any $J\in F_l$. Hence T is an essential overring of R.

Lemma 5.3. *Let S be a (right) essential overring of R. Then $S[x]$ is a (right) essential overring of $R[x]$.*

Proof. There is a perfect right additive topology F such that $S=R_F$. We denote by F^* the set of right ideals of $R[x]$ which contain an element in F. We shall prove that F^* is a perfect right additive topology on $R[x]$ and that $S[x]=R[x]_{F^*}$. Let I^* be any element of F^* and let $f(x)=a_nx^n+\cdots+a_0$ be any element of $R[x]$. (i) We shall prove that $f(x)^{-1}I^*\in F^*$ by induction on degree $f(x)=n$. From the assumption there is an element I in F such that $I\subseteqq I^*$. If $n=0$, then $a_0^{-1}I\subseteqq a_0^{-1}I^*$ and $a_0^{-1}I\in F$ so that $f(x)^{-1}I^*\in F^*$. By the induction hypothesis, $f_1(x)^{-1}I^*\supseteqq I_1$, where $f_1(x)=a_{n-1}x^{n-1}+\cdots+a_0$ and $I_1\in F$. Since $a_nx^n(a_n^{-1}I)=x^na_n(a_n^{-1}I)\subseteqq x^nI=Ix^n$, we get $a_n^{-1}I\subseteqq(a_nx^n)^{-1}I^*$. Hence we obtain that $f(x)^{-1}I^*\supseteqq(a_nx^n)^{-1}I^*\cap f_1(x)^{-1}I^*\supseteqq I_1\cap a_n^{-1}I$. This implies that $f(x)^{-1}I^*\in F^*$. (ii) We shall prove that if J^* is a right ideal of $R[x]$ and if there is $I^*\in F^*$ such that $a^{*-1}J^*\in F^*$ for every $a^*\in I^*$, then $J^*\in F^*$. There exists $I\in F$ such that $I^*\supseteqq I$. For any $a\in I$, there exists $J_a\in F$ with $a^{-1}J^*\supseteqq J_a$. Set $J=\Sigma aJ_a(a\in I)$. Since $a^{-1}J\supseteqq J_a$ for any $a\in I$, we have $J\in F$. It is evident that $J^*\supseteqq J$ so that $J^*\in F^*$, as desired. Hence F^* is a right additive topology. (iii) We shall prove that F^* is perfect and $S[x]=R[x]_{F^*}$. Let I^* be any element of F^*. Then it is clear that $I^*S[x]=S[x]$. Conversely, let I^* be a right ideal of $R[x]$ with $I^*S[x]=S[x]$. Write $1=\Sigma a_iq_i$, where $a_i\in I^*$, $q_i\in S[x]$. There exists $I\in F$ such that $q_iI\subseteqq R[x]$. Hence $I=(\Sigma a_iq_i)I\subseteqq I^*$ so that $I^*\in F^*$. Thus, by Theorem 13.10 of [10], F^* is perfect and $S[x]=R[x]_{F^*}$. Consequently $S[x]$ is a right essential overring of $R[x]$. If S is an essential overring of R, then it is evident from the definition of F^* that $S[x]$ is an essential overring of $R[x]$.

Theorem 5.4. *If R is a Krull ring, then $R[x]$ is a Krull ring.*

Proof. Let $R=\cap R_i\cap S_j$, where $i\in \boldsymbol{I}, j\in \boldsymbol{J}$, the cardinal number of $\boldsymbol{J}$ is finite, each R_i is a noetherian local Asano order with uniqe maximal ideal P_i' and each S_j is a noetherian simple ring. By Lemma 5.1 $R_i[x]_{P_i'[x]}$ is a noetherian local Asano order, and $R_i[x]_{P_i'[x]}\cap Q[x]=R_i[x]$. By Example 6.1 of [8], $S_j[x]$ is a noetherian Asano order. Hence $S_j[x]=S_{j0}^*\cap S_{jk}^*(jk\in \boldsymbol{J}_j)$, where S_{j0}^* is a noetherian simple ring and S_{jk}^* are noetherian local Asano orders (cf. [5]). $Q[x]$ is a Dedekind prime ring by Example 6.3 of [8]. Hence $Q[x]=T_0\cap T_l(l\in \boldsymbol{L})$,

where T_0 is a simple hereditary ring and T_l are noetherian local Asano orders (cf. [6]). By Lemmas 5.2 and 5.3, all $R_i[x]_{P_i'[x]}$, S_{j0}^*, S_{jk}^*, T_0, and T_l are essential overrings of $R[x]$. Further we get:

$$R[x] = \cap R_i[x] \cap S_j[x] = \cap_i \{Q[x] \cap R_i[x]_{P_i'[x]}\} \cap_j \{S_{j0}^* \cap S_{jk}^*\}$$
$$= [\cap_i R_i[x]_{P_i'[x]} \cap_l T_l \cap_j \cap_k S_{jk}^*] \cap [\cap_j S_{j0}^* \cap T_0],$$

and the simple overrings are only T_0, $S_{j0}^*(j \in \boldsymbol{J})$ so that the number of these is finite.

Finally we shall prove that $R[x]$ satisfiies the axiom (K3). Let I^* be any essential right ideal of $R[x]$. Then there exists $f(x)=a_n x^n+\cdots+a_0$ in I^* such that a_n is a regular element in R by Lemma 2 of [9]. Hence $a_n R_i = R_i$ for almost all $i \in \boldsymbol{I}$, that is, $a_n \in C(P_i)$ and so $f(x) \in C(P_i[x])$. This implies that $I^* R_i[x]_{P_i'[x]} = R_i[x]_{P_i'[x]}$ for almost all $i \in \boldsymbol{I}$. Since $I^* Q[x]$ is an essential right ideal of $Q[x]$, we have $T_l = I^* Q[x] T_l = I^* T_l$ for almost all $l \in \boldsymbol{L}$. Similarly $S_{jk}^* = I S_{jk}^*$ for almost all $jk \in \boldsymbol{J}_j$. Therefore $R[x]$ is a Krull ring.

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