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## REMARKS ON THE REGULARITY OF BOUNDARY POINTS IN A RESOLUTIVE COMPACTIFICATION

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**Introduction.** Let  $X$  be a strong harmonic space in the sense of Bauer [2] and suppose that constant functions are harmonic. In the previous paper [5], the author studied the regularity of boundary points in a resolute compactification of  $X$  and discussed characterization of regularity, existence of regular points, strong regularity and pseudo-strong regularity, characterization of harmonic boundary and consideration in the case of open subsets. In this paper we shall use the same notations and definitions as in [5], and we shall give some supplementary remarks.

In §1, we recall the notations and terminologies used in [5]. We reform characterization of the regularity in Theorem 1 of §2. Theorem 2 in §3 is the extremal characterization of pseudo-strong regularity in the case where  $X$  is a Brelot space. The trace filters of neighborhoods of boundary points in the Wiener compactification  $X^w$  of  $X$  is of some interest. Using this filters we can construct in §4 a family of completely regular filters in a metrizable and resolute compactification  $X^*$  of  $X$ . A regular boundary point  $x$  is said to have a local property if  $x$  is regular for every  $\overline{U(x) \cap X}$ , where  $U(x)$  is a neighborhood of  $x$ . The main results of this paper are in §5. It is shown that a regular point  $x$  does not possess a local property in general and  $x$  has a local property if and only if  $x$  is pseudo-strongly regular. Further the related problems are investigated. In the final section, we consider a relatively compact open set  $G$  of a Brelot space and obtain the result, if  $G$  is minimally bounded, then the set of all regular points is dense in the boundary  $\partial G$  of  $G$ , which is a generalization of a result of Bauer [1].

### 1. Preliminaries

Let  $X$  be a *strong* harmonic space in the sense of Bauer [2] on which constant functions are harmonic, and  $X^*$  be a resolute compactification of  $X$ . On the boundary  $\Delta = X^* \setminus X$  we define the harmonic boundary  $\Gamma = \{x \in \Delta; \lim_{a \rightarrow x} p(a) = 0 \text{ for every strictly positive potential } p \text{ on } X\}$ . For  $f \in C(\Delta)$ , i.e., a continuous

real valued function on  $\Delta$ , the Dirichlet solution of  $f$  is denoted by  $H_f$ . A point  $x \in \Delta$  is termed to be *regular* if  $\lim_{a \rightarrow x} H_f(a) = f(x)$  for every  $f \in C(\Delta)$ . A point  $x \in \Delta$  is called *pseudo-strongly regular* if  $\lim_{a \rightarrow x} p(a) = 0$  for every bounded potential  $p$  harmonic in a neighborhood of  $x$ . Every pseudo-strongly regular point is regular but the converse does not hold in general. We set

$$S^+ = \{v; \text{superharmonic functions non-negative on } X\}$$

and

$$\mathcal{M}_x = \{\mu; \text{probability measures on } \Delta \text{ satisfying}$$

$$\int \underline{v} d\mu \leq \overline{u}_v(x) + \underline{p}_v(x) \text{ for every } v \in S^+\},$$

where  $\underline{f}$  (resp.  $\overline{f}$ ) is the lower (resp. upper) semicontinuous extension of  $f$  on  $\Delta$  and  $\overline{u}_v$  is the greatest harmonic minorant of  $v$  and  $\underline{p}_v$  is the potential part of  $v$ .

The main results of our previous paper [5] are the following: a point  $x \in \Gamma$  is regular if and only if  $\mathcal{M}_x = \{\varepsilon_x\}$ , where  $\varepsilon_x$  is the Dirac measure at  $x$ . As a collorary we obtain: if

$$\lim_{\substack{U(x) \\ U(x)}} [\lim_{a \rightarrow x} R_1^{X \setminus U(x)}(a)] < 1,$$

then  $x$  is regular, where  $\mathcal{U}(x)$  is a fundamental system of neighborhoods  $U(x)$  of  $x$ . The harmonic boundary is the  $S^+$ -Šilov boundary. For an open subset  $G$  of  $X$ , every regular point is pseudo-strongly regular, thus a regular point has a local property in this case.

## 2. Characterization of the regularity

We reform characterization of the regularity (Theorem 1 in [5]) in a slightly different form. Let

$$\mathcal{M}'_x = \{\mu; \text{probability measures on } \Delta \text{ satisfying}$$

$$\int \underline{v} d\mu \leq \overline{u}_v(x) \text{ for every } v \in S^+\}.$$

Clearly we have  $\mathcal{M}'_x \subset \mathcal{M}_x$  and  $\mathcal{M}'_x = \mathcal{M}_x$  if  $x \in \Gamma$ . It is noteworthy that  $\mathcal{M}'_x$  may be empty whereas  $\varepsilon_x \in \mathcal{M}_x$ .

**Theorem 1.**  $x \in \Delta$  is regular if and only if  $\mathcal{M}'_x = \{\varepsilon_x\}$ .

Proof. If  $x$  is regular then  $x \in \Gamma$ , and therefore  $\mathcal{M}'_x = \mathcal{M}_x = \{\varepsilon_x\}$  [5]. Next, suppose that  $\mathcal{M}'_x$  is not empty and consists of a single measure  $\varepsilon_x$ , and let  $\{a_i\}$  be a net of points converging to  $x$ . Let  $\omega_i$  be a harmonic measure at  $a_i$ , i.e.,

$$\int f d\omega_i = H_f(a_i) \text{ for every } f \in C(\Delta).$$

$\omega_i$  is a probability measure on  $\Delta$ . There exists a subnet  $\{\omega_{i_k}\}$  of  $\{\omega_i\}$  converging to a measure  $\mu$  vaguely.  $\mu$  is a probability measure on  $\Delta$ . Further,  $\mu \in \mathcal{M}'_x$ . In fact, let  $f \in C^+(\Delta)$  with  $f \leq \varliminf v$ , where  $v \in S^+$ , then  $H_f \leq v$  and  $H_f \leq u_v$ . Since  $\int f d\mu = \lim \int f d\omega_{i_k} = \lim H_f(a_{i_k}) \leq \overline{\lim}_x u_v$ , implies  $\int (\varliminf v) d\mu \leq \overline{\lim}_x u_v$ , we have  $\mu = \varepsilon_x$ , i.e.,  $\omega_i$  converges to  $\varepsilon_x$  and  $x$  is regular.

### 3. Extremal characterization of the pseudo-strong regularity in Brelot spaces

In this section, we consider a resolutive compactification of a Brelot space  $X$ . For  $x \in \Delta$ , we define

$$S_x^* = \{H_f + p; f \in C^+(\Delta), p \text{ is a potential such that } \lim_x p = 0\}$$

and

$$\mathcal{M}_x^* = \{\mu; \text{probability measures on } \Delta \text{ such that} \\ \int \underline{v} d\mu \leq \overline{v}(x) \text{ for every } v \in S_x^*\}.$$

REMARK 1.  $\mu \in \mathcal{M}_x^*$  if and only if  $\int \underline{v} d\mu \leq \overline{\lim}_x H_f$  for every  $v \in S_x^*$ , where  $v = H_f + p$ .

REMARK 2.  $\mathcal{M}_x^* = \{\varepsilon_x\}$  implies  $\mathcal{M}_x = \{\varepsilon_x\}$ ; for  $\mathcal{M}_x \subset \mathcal{M}_x^*$ , i.e.,  $\mathcal{M}_x^* = \{\varepsilon_x\}$  means that  $x$  is regular.

**Theorem 2.**  $x \in \Delta$  is pseudo-strongly regular if and only if  $\mathcal{M}_x^* = \{\varepsilon_x\}$ .

Proof. Suppose that  $x$  is pseudo-strongly regular and that there exists  $\mu \in \mathcal{M}_x^*$  such that  $\mu \neq \varepsilon_x$ . Let  $y \in \text{Supp } \mu \setminus \{x\}$  and  $f \in C^+(X^*)$ ,  $f(y) > 0$ ,  $f = 0$  on  $U(x)$ , where  $U(x)$  is a neighborhood of  $x$  such that  $y \notin \overline{U(x)}$ . Put  $u = H_f$ . There exists a bounded potential  $p$  such that  $u + p \geq f$  outside a compact subset of  $X$ . For, we may find a potential  $p'$  such that  $u + p' \geq f$  outside a compact subset  $K$  of  $X$  since  $u = h_f$  (for the definition of  $h_f$ , see [6]). On  $X \setminus K$ ,  $f \leq \min(u + p', \|f\|) \leq \min(u, \|f\|) + \min(p', \|f\|) = u + \min(p', \|f\|) = u + p$ . Here  $p = \min(p', \|f\|)$  is a bounded potential. Set  $p_1 = \hat{R}_p^{X \setminus U(x)}$ . By hypothesis,  $\lim_x p_1 = 0$ . Since  $\varliminf(u + p_1) \geq f > 0$  in a neighborhood of  $y$ , we have a contradiction that  $0 < \int \varliminf(u + p_1) d\mu \leq \overline{\lim}_x u = f(x) = 0$ .

Next, we prove the converse. We show first that for every  $y \in \Delta$ ,  $y \neq x$  there exists  $v_y \in S_x^*$  such that  $\varliminf_y v_y > \overline{\lim}_x v_y = 0$ . In fact, there is a function  $v \in S_x^*$  such that  $\varliminf_y v > \overline{\lim}_x v = g(x)$ , where  $v = H_g + p$  (by Remark 2); for otherwise we have  $\varepsilon_y \in \mathcal{M}_x^*$ . Set  $f = \max(g - g(x), 0)$ . Then  $H_f + p \in S_x^*$  and  $\varliminf_y (H_f + p) \geq \varliminf_y (H_g + p) - g(x) > 0 = \overline{\lim}_x H_f = \overline{\lim}_x (H_f + p)$ , i.e., we may take

$v_y = H_f + p$ . Now, let  $U(x)$  be a neighborhood of  $x$ . For every  $y \in \overline{\partial U(x)} \cap \Delta$  we associate with  $v_y$  described above. Then there exists a triple  $(v_y, U(y), \delta_y)$  such that

$$v_y > \delta_y > 0 \text{ on } U(y) \cap X \text{ and } \lim_x v_y = 0$$

A finite number of  $U(y)$ , say  $\{U(y_i)\}$ , covers  $\overline{\partial U(x)} \cap \Delta$ . Set  $\delta = \min \delta_{y_i}$ ,  $v = \sum_i v_{y_i}$  and  $V = \bigcup_i U(y_i)$ . Then  $v > \delta$  on  $V \cap X$  and  $\lim_x v = 0$ . Since  $X$  is a Brelot space we may also find  $\alpha > 0$  such that  $\alpha v > 1$  on  $\overline{\partial U(x)}$ . Then  $\lim_x \hat{R}_1^{\alpha v \cup (x)} = 0$ , i.e.,  $x$  is pseudo-strongly regular.

#### 4. The Wiener compactification

The compactification on which every Wiener function is extended continuously and separates points is called the *Wiener compactification* and is denoted by  $X^W$  [6]. The harmonic boundary of  $X^W$  is denoted by  $\Gamma^W$ .

**Theorem 3.** *Every point of  $\Gamma^W$  is pseudo-strongly regular.*

*Proof.* Let  $U(x)$  be an open neighborhood of  $x \in \Gamma^W$  in  $X^W$ . For a neighborhood  $V(x)$  of  $x$  such that  $\overline{V(x)} \subset U(x)$ ,  $v = \hat{R}_1^{V(x) \cap X}$  is a potential. In fact, since  $\overline{V(x)} \cap X \cap \overline{X \setminus U(x)} \cap \Delta^W = \emptyset$ ,  $q = \min (\hat{R}_1^{X \setminus U(x)}, \hat{R}_1^{V(x) \cap X})$  is a potential ([6], Th. 3.2.23) and  $v \leq q$ .  $v = \hat{R}_1^{X \setminus U(x)}$  on  $V(x) \cap X$  and  $v$  has a limit at  $x$  ([6], Prop. 4.4). Thus  $\lim_x v = \lim_x v = 0$ , i.e.,  $\lim_x \hat{R}_1^{X \setminus U(x)} = 0$ .

Let  $X^*$  be a *metrizable* and resolutive compactification of  $X$ . Then there exists a family of completely regular filters  $\{\mathcal{F}\}$  each of which converges to a point of  $\Delta = X^* \setminus X$  and such that

- A) if a superharmonic function  $v$  on  $X$  is bounded from below and  $\liminf_{\mathcal{F}} v \geq 0$  for every  $\mathcal{F}$ , then  $v \geq 0$ ,
- B) for every  $\mathcal{F}$ , there exists a superharmonic function  $v$  on  $X$  such that  $\lim_{\mathcal{F}} v = 0$  and  $\inf \{v; X \setminus U(x)\} > 0$  for every neighborhood  $U(x)$  of  $x$ , where  $\mathcal{F}$  converges to  $x$ .

Here, a filter  $\mathcal{F}$ , converging to  $x$ , is called to be *completely regular* if  $\lim_{\mathcal{F}} H_f = f(x)$  for every resolutive function  $f$  continuous at  $x$ .

In fact, consider the Wiener compactification  $X^W$  of  $X$ .  $X^*$  is a quotient space of  $X^W$ , i.e., there exists a continuous mapping  $\pi$  of  $X^W$  onto  $X^*$  fixing each point of  $X$ . Let  $\mathcal{F}_{\tilde{x}}$  be the trace filter of the filter of sections of neighborhoods of  $\tilde{x} \in \Gamma^W$ , i.e.,

$$\mathcal{F}_{\tilde{x}} = \{U(\tilde{x}) \cap X; U(\tilde{x}) \text{ is a neighborhood of } \tilde{x} \text{ in } X^W\}.$$

$\mathcal{F}_{\tilde{x}}$  converges to  $x = \pi(\tilde{x})$ . The family of filters  $\{\mathcal{F}_{\tilde{x}}; \tilde{x} \in \Gamma^W\}$  is the desired

one.

For,  $A$ ) follows from the property of  $\Gamma^W$  ([6], Th.3.1.6). As for  $B$ ) let  $\tilde{x} \in \Gamma^W$ ,  $\pi(\tilde{x}) = x$ ,  $\{U_n(x)\}$  be a fundamental system of neighborhoods of  $x$ , and let  $\mathcal{F} = \mathcal{F}_{\tilde{x}}$ . Then  $v = \sum_n (1/2^n) \hat{R}_1^{X \setminus U_n(x)}$  fulfills the requirement of  $B$ ). For, given  $\varepsilon > 0$ , there exists an integer  $N$  such that  $\sum_{N+1}^{\infty} (1/2^n) < \varepsilon/2$ . Since  $\tilde{x}$  is pseudo-strongly regular,  $\lim_{\tilde{x}} \hat{R}_1^{X \setminus U_n(x)} = 0$  in  $X^W$ . Hence  $\lim_{\tilde{x}} v \leq \varepsilon/2$ .  $\inf \{v; X \setminus U(x)\} > 0$  is trivially seen. All that remains is to prove  $\lim_{\mathcal{F}} H_f = f(x)$  for every resolutive function  $f$  continuous at  $x$ . We may suppose that  $f \geq 0$  and  $f(x) = 0$ . Let  $\tilde{f} = f \circ \pi$ . Since  $H_f$  is a Wiener function,  $H_f$  is extended continuously onto  $X^W$ . We denote this extended function by  $\tilde{F}$ .  $\tilde{f}$  is resolutive with respect to  $X^W$ . For, since  $\lim_{\tilde{x}} s \geq \lim_{\pi(\tilde{x})} s$ , if  $s$  is non-negative superharmonic and  $\lim s \geq f$  on  $\Delta$ , then  $\lim s \geq \tilde{f}$  on  $\Delta^W$ , which implies that  $H_f \geq H_{\tilde{f}}^W$  and similarly  $H_{\tilde{f}}^W \geq H_f$ , where  $H_{\tilde{f}}^W$  is the Dirichlet solution with respect to  $X^W$ . Noting that  $H_f = h_{H_f}$ , where  $h$  is the operator of Constantinescu-Conea ([6], p.26), we have  $v \geq H_{\tilde{f}}^W$  for every  $v \geq 0$  superharmonic and  $v \geq H_f$  outside a compact subset of  $X$ . Hence  $H_f \geq H_{\tilde{f}}^W$  and similarly  $H_{\tilde{f}}^W \geq H_f$ . Thus, we have  $H_{\tilde{f}}^W = H_f = H_{\tilde{f}}^W$ . Therefore  $\int (\tilde{F} - \tilde{f}) d\omega^W = 0$  and  $\int |\tilde{F} - \tilde{f}| d\omega^W = 0$ , i.e.,  $\tilde{F} = \tilde{f} d\omega^W - a.e.$ , where  $\omega^W$  is the harmonic measure in  $\Delta^W$ . We shall prove that  $\tilde{F}(x) = 0$ . For otherwise, since  $\tilde{F}$  and  $\tilde{f}$  are continuous at  $\tilde{x}$ ,  $\tilde{F} \neq \tilde{f}$  in a neighborhood of  $\tilde{x}$ , but this is impossible since this neighborhood is not of  $d\omega^W$ -harmonic measure zero ([6], Th. 3.2.19).

## 5. The local property of regular points

Let  $X^*$  be a resolutive compactification of  $X$ . We consider  $G = X \cap U(x)$ , where  $U(x)$  is an open neighborhood of  $x \in \Delta$ . The closure  $\bar{G}$  in  $X^*$  is a compactification. The boundary of  $\bar{G}$  is denoted by  $\Delta(G)$ .  $\Delta(G) = \partial G \cup \delta$ , where  $\partial G = \Delta(G) \cap X$  and  $\delta = \Delta(G) \cap \Delta$ . Obviously we have  $x \in \delta$ .

**Proposition 1.**  $\bar{G}$  is a resolutive compactification.

Proof. Let  $f \in C^+(\Delta(G))$  and  $f_1$  be a finite continuous extension of  $f|_{\delta}$  onto  $\Delta$ , where  $f|_{\delta}$  is the restriction of  $f$  onto  $\delta$ . Denoting by  $s_1$  (resp.  $s_2$ ) a hyperharmonic function on  $G$ , bounded from below,  $\lim s_1 \geq f - H_{f_1}$  on  $\partial G$ ,  $s_1 \geq 0$  outside a compact subset of  $X$  (resp. a hyperharmonic function on  $X$ , bounded from below,  $\lim s_2 \geq f_1$  on  $\Delta$ ), we have

$$\lim (s_1 + s_2) \geq \begin{cases} f - H_{f_1} + H_{f_1} = f & \text{on } \partial G \\ f_1 = f & \text{on } \delta \end{cases}$$

Hence,  $H_f^G \leq H_{f-H_{f_1}}^G + H_{f_1}$ , and similarly  $H_f^G \geq H_{f-H_{f_1}}^G + H_{f_1}$ , where  $H_f^G$  is the Dirichlet solution with respect to  $\bar{G}$  and for the definition of  $H_f^{G,X}$  we refer to

[6]. Thus we have  $\bar{H}_f^G = \underline{H}_f^G = H_{f-\bar{H}_{f_1}}^G + H_{f_1}$ , since  $\bar{H}_{f-\bar{H}_{f_1}}^G = \underline{H}_{f-\bar{H}_{f_1}}^G$  ([6], Th. 1.2.7).

**Proposition 2.** *If  $x$  is irregular for  $X^*$  then  $x$  is irregular for  $\bar{G}$ .*

Proof. Suppose that  $x$  is regular for  $\bar{G}$ . For a function  $f \in C(\Delta)$ , let

$$\varphi = \begin{cases} f & \text{on } \delta \\ H_f & \text{on } \partial G \end{cases}$$

It is easily seen that  $\varphi$  is resolutive and  $H_\varphi^G = H_f$  on  $G$ . From this we derive

$$\lim_x H_f = \lim_x H_\varphi^G = \varphi(x) = f(x)$$

which implies that  $x$  is regular for  $X^*$ .

The following example shows that the converse does not hold in general.

EXAMPLE. Let  $X = \{|z| < 1\} \setminus \{-1/2, 1/2\}$ . We identify the two points  $-1/2$  and  $1/2$ , and denote it by  $e$ . The Green function of  $\{|z| < 1\}$  with pole at  $1/2$  is denoted by  $u_0$ . We consider the compactification of  $X$  such that  $\Delta = \{|z| = 1\} \cup \{e\}$ , and the harmonic structure given by  $u_0$ -harmonic functions, i.e., the quotient of usual harmonic functions by  $u_0$ . The compactification  $X^*$  is resolutive and  $H_f = f(e)$  (the constant function). Let  $G = X \setminus K$ , where  $K = \{iy; y \text{ is real and } |y| \leq 1/2\}$ .  $e$  is regular for  $X^*$  but it is irregular for  $\bar{G}$ .

A strictly positive superharmonic function  $v_0$  on  $X$  satisfying  $\lim_x v_0 = 0$  is called a *weak barrier* of  $x$ .

In a resolutive compactification of a Brelot space, if  $\Gamma$  contains at least two points every regular point has a weak barrier. In the above example  $e$  has no weak barrier. We know an example of an irregular point with weak barrier ([7], p. 253). If  $X$  is a Brelot space, the existence of a (strong) barrier  $v_0$  at  $x$ , i.e.,  $v_0$  is a positive superharmonic function satisfying  $\lim_x v_0 = 0$  and  $\inf \{v_0; X \setminus U(x)\} > 0$  for every open neighborhood  $U(x)$  of  $x$ , is equivalent to  $\lim_x R_1^{X \setminus K} = 0$  for every compact set  $K$ .

**Theorem 4.** *Suppose that  $x$  has a weak barrier. Then  $x$  is regular for  $X^*$  if and only if  $x$  is regular for  $X \setminus K$  for every compact subset  $K$  of  $X$ .*

Proof. By Proposition 2, it is enough to prove the “only if” part. Suppose for a moment that  $x$  is irregular for  $\bar{G}$ , where  $G = X \setminus K$ . Then  $x \in \Gamma$ . We shall see that there exists  $f_0 \in \mathcal{C} = \{f \in C^+(\Delta \cup \partial K); f = 0 \text{ on } \partial K\}$  such that  $\lim_x H_{f_0}^G < \overline{\lim}_x H_{f_0}^G$ . In fact, if we have  $\lim_x H_f^G = f(x)$  for every  $f \in \mathcal{C}$ , then  $\lim_x H_g^G = g(x)$  for every  $g \geq 0$  continuous on  $\Delta \cup \partial K$ . For, letting

$$g_1 = \begin{cases} g & \text{on } \Delta \\ 0 & \text{on } \partial K \end{cases}$$

and

$$g_2 = \begin{cases} 0 & \text{on } \Delta \\ g & \text{on } \partial K \end{cases}$$

we have  $\lim_x H_{g_1}^G = g_1(x) = g(x)$  and  $0 \leq H_{g_2}^G \leq \|g_2\| H_\psi^G$ , where  $\psi$  is the characteristic function of  $\partial K$ . From  $1 - \psi \in C$ , it is derived that  $\lim_x H_\psi^G = 0$  and  $\lim_x H_{g_2}^G = 0$ . Select a number  $\gamma$  such that

$$\underline{\lim}_x H_{f_0}^G < \gamma < \overline{\lim}_x H_{f_0}^G \text{ and } f_0(x) \neq \gamma.$$

By the theorem of Hahn-Banach, there exists a probability measure on  $\Delta \cup \partial K$  such that

$$\gamma = \int f_0 d\mu \text{ and } \int \underline{\lim} v d\mu \leq \overline{\lim}_x u, \text{ for every } v \in S^+(G).$$

Obviously  $\mu \neq \varepsilon_x$ . Since  $\int \underline{\lim} v_0 d\mu \leq \overline{\lim}_x v_0 = 0$ , where  $v_0$  is a weak barrier of  $x$ , we have  $\text{Supp } \mu \subset \Delta$ . Take a point  $y \in \text{Supp } \mu \setminus \{x\}$  and  $g \in C^+(\Delta)$  such that  $g(x) = 0$  and  $g > 0$  in a neighborhood of  $y$ . We have

$$H_{g_1}^G = H_g \quad \text{on } G,$$

where

$$g_1 = \begin{cases} g & \text{on } \Delta \\ H_g & \text{on } \partial K \end{cases}$$

We may find a potential on  $G$  with  $\underline{\lim} (H_{g_1}^G + p) \geq g_1$  on  $\Delta \cup \partial K$ . Hence

$$\int \underline{\lim} (H_{g_1}^G + p) d\mu \geq \int g_1 d\mu = \int_\Delta g_1 d\mu > 0.$$

On the other hand,

$$\int \underline{\lim} (H_{g_1}^G + p) d\mu \leq \overline{\lim}_x H_{g_1}^G = \overline{\lim}_x H_g = g(x) = 0,$$

which is a contradiction.

Let  $x \in \Delta$  be regular for  $X^*$ . If  $x$  is regular for every  $\overline{X \cap U(x)}$ , then  $x$  is said to have the *local property*.

**Theorem 5.**  *$x$  has the local property if and only if  $x$  is pseudo-strongly regular.*

**Proof.** We need to prove the “only if” part. We shall prove  $\lim_x R_1^{X \setminus U(x)} = 0$  for every  $U(x)$ . Let  $G = X \cap U(x)$  and  $f \in C^+(\Delta(G))$  such that  $f(x) = 0$  and  $f = 1$  on  $\partial \overline{G}$ . Consider a non-negative superharmonic function  $s$  with  $\underline{\lim} s \geq f$  on  $\Delta(G)$ . We define



$$s_1 = \begin{cases} 1 & \text{on } X \setminus U(x) \\ \min(1, s) & \text{on } U(x). \end{cases}$$

$s_1$  is superharmonic on  $X$  and  $R_1^{X \setminus U(x)} \leq s_1$ . Therefore  $R_1^{X \setminus U(x)} \leq H_f^G$  in  $G$ , and  $\lim_x H_f^G = f(x) = 0$  implies  $\lim_x R_1^{X \setminus U(x)} = 0$ .

**Lemma.** Suppose that  $x$  is regular for  $X^*$  and  $\lim_x R_1^{X \setminus U(x)} = 0$  for a neighborhood  $U(x)$  of  $x$ . Let  $U_1(x)$  be a neighborhood of  $x$  with  $\overline{U_1(x)} \subset U(x)$ , and let  $\delta = U_1(x) \cap \Delta$ ,  $G = U(x) \cap X$ . If  $f, g \in C(\Delta(G))$  and  $f = g$  on  $\delta$ , then  $\overline{\lim}_x H_f^G = \overline{\lim}_x H_g^G$ .

Proof. Since  $H_f^G - H_g^G = H_{f-g}^G$  it is sufficient to show that  $f \in C(\Delta(G))$  and  $f = 0$  on  $\delta$  implies  $\overline{\lim}_x H_f^G = 0$ . Let  $U_2(x)$  be a neighborhood of  $x$  such that  $\overline{U_2(x)} \subset U_1(x)$ , and  $\delta' = U_2(x) \cap \Delta$ . For a function  $\varphi \in C^+(\Delta)$  with  $\varphi \leq \|f\|$  and  $\varphi = \|f\|$  on  $\Delta \setminus \delta$  and  $\varphi(x) = 0$ , there exist a potential  $p$  and  $s \in S^+(G)$  such that

$$\begin{aligned} \underline{\lim} (H_\varphi + \varepsilon p) &\geq \varphi && \text{on } \Delta \\ \underline{\lim} (R_1^{X \setminus U(x)} + \varepsilon s) &\geq \|f\| && \text{on } \partial U(x) \end{aligned}$$

for every  $\varepsilon > 0$ . Setting  $v = H_\varphi + \|f\| R_1^{X \setminus U(x)} + \varepsilon(p + s)$  we can readily see that  $v \geq H_f^G$  and  $H_\varphi + \|f\| R_1^{X \setminus U(x)} \geq H_f^G$ . Hence  $\overline{\lim}_x H_f^G \leq \lim_x H_\varphi + \|f\| \lim_x R_1^{X \setminus U(x)} = 0$ .

**Theorem 6.** If  $x$  is regular for  $X^*$  and  $\lim_x R_1^{X \setminus U(x)} = 0$  then  $x$  is regular for  $X \cap U(x)$ .

Proof. Let  $G = X \cap U(x)$ . Suppose that  $x$  is irregular for  $\overline{G}$ . Then there exists  $f \in C^+(\Delta(G))$  such that  $\text{Supp } f \subset \delta = U_1(x) \cap \Delta$ , where  $\overline{U_1(x)} \subset U(x)$  and  $\underline{\lim}_x H_f^G \neq \overline{\lim}_x H_f^G$ . We may construct a probability measure  $\mu$  on  $\Delta(G)$  such that  $\mu \neq \varepsilon_x$  and

$$\int \underline{\lim} v \, d\mu \leq \overline{\lim}_x u_v \quad \text{for every } v \in S^+(G).$$

We assert that  $\text{Supp } \mu \subset \delta$ , for if  $g \in C^+(\Delta(G))$  and  $g = 0$  on  $\delta$  then  $0 \leq \int g \, d\mu \leq \overline{\lim}_x H_g^G = 0$  by the above Lemma. There exists  $y \in \text{Supp } \mu \setminus \{x\}$ . Since  $y \in \delta$  and  $\delta \cap (X^* \setminus U(x)) = \emptyset$  we have  $y \notin \partial \overline{G}$ . Hence we can find  $U(y)$  such that  $\overline{U(y)} \subset U(x)$ . Let  $F \in C^+(X^*)$  with  $F(y) > 0$  and  $F(x) = 0$ , and let

$$F_1 = \begin{cases} F & \text{on } U(y) \\ h_F & \text{on } G \setminus U(y) \end{cases}$$

There exists a potential  $q$  on  $X$  such that for every  $\varepsilon > 0$  we may find a compact subset  $K_\varepsilon$  of  $X$  so that

$$h_F + \varepsilon q \geq F \text{ and } h_F - \varepsilon q \leq F \text{ on } X \setminus K_\varepsilon.$$

Since  $h_F + \varepsilon q \geq F_1$  and  $h_F - \varepsilon q \leq F_1$  on  $G \setminus (K_\varepsilon \cap U(y))$ , we have  $h_F \geq \bar{h}_{F_1}^c \geq \underline{h}_{F_1}^c \geq h_F$ , i.e.,  $h_F = h_{F_1}^c$ . Thus we have a potential  $p$  on  $G$  such that  $h_{F_1}^c + p \geq F_1$  outside a compact subset of  $G$  and, in particular, in a neighborhood of  $y$ . Hence we are led to a contradiction

$$0 < \int \underline{\lim} (h_{F_1}^c + p) d\mu \leq \overline{\lim}_x h_{F_1}^c = \overline{\lim}_x h_F = 0.$$

Let  $G^\alpha$  be the closure of  $G$  in  $X^\mathcal{A}$  (the one-point compactification of  $X$ ). Then  $G^\alpha$  is a resolutive compactification [5]. The boundary of  $G^\alpha$  is  $\partial G \cup \{\mathcal{A}\}$ . We denote the Dirichlet solution on  $G^\alpha$  by  $H_f^\alpha$ . If the boundary function  $f$  on  $\Delta(G)$  is resolutive for  $\bar{G}$  and is constant  $\alpha$  on  $\delta = \bar{G} \cap \Delta$  then

$$f' = \begin{cases} f & \text{on } \partial G \\ \alpha & \text{at } \mathcal{A} \end{cases}$$

is resolutive for  $G^\alpha$ , and conversely if  $f'$  is resolutive for  $G^\alpha$  then

$$f = \begin{cases} f' & \text{on } \partial G \\ f'(\mathcal{A}) & \text{on } \delta \end{cases}$$

is resolutive for  $\bar{G}$ . In both cases  $H_{f'}^\alpha = H_f^c$ .  $x \in \partial G$  is regular for  $\bar{G}$  if and only if it is regular for  $G^\alpha$ . Hence regular point  $x \in \partial G$  for  $\bar{G}$  is strongly regular [5].

## 6. Relatively compact open sets

In this section, we shall assume that  $X$  is a *Brelot* space.

Let  $G$  be a relatively compact open subset of  $X$ . The *outer boundary* of  $G$  is defined to be the boundary of  $\bar{G}$  and is denoted by  $B(G)$ . The harmonic boundary of  $G$  and the set of regular points for  $\bar{G}$  is denoted by  $\Gamma(G)$  and  $R(G)$  respectively.  $G$  termed to be *minimally bounded* if the interior of  $\bar{G}$  coincides with  $G$ .  $G$  is *minimally bounded* if and only if  $\partial G = B(G)$ .

**Theorem 7.**  $B(G) \subset \overline{R(G)} \subset \Gamma(G)$  ([1], Satz 17)

**Proof.** It is sufficient to prove that for every  $x \in B(G)$  and for every regular region  $D$  containing  $x$  there exists  $y \in R(G) \cap D$ . Since  $x \in B(G)$  we may find  $z \in D \setminus \bar{G}$ . Consider a regular region  $V$  containing  $z$  and  $\bar{V} \subset D \setminus \bar{G}$ . The reduced function  $v = (\hat{R}_1^{X \setminus D})_{X \setminus \bar{V}}$  (the reduced function considered in the harmonic space  $X \setminus \bar{V}$ ) is continuous on  $G$  and  $\alpha = \inf \{v; \partial G\} < \inf \{v; \partial G \setminus D\} = 1$ .  $v - \alpha$  is a weak barrier at any point of  $E = \{y \in \partial G; v(y) = \alpha\} \neq \emptyset$  and all points of  $E$  are regular.

**Corollary** ([1], *Korollar to Satz 17*). *If  $G$  is minimally bounded, then  $\partial G = \overline{R(G)} = \Gamma(G)$ .*

**REMARK.** We know that in a Bauer space  $\Gamma(G)$  is the  $\mathcal{S}^+(G)$ -Šilov boundary [5], while if  $G$  is *weakly dermining*,  $\overline{R(G)}$  is the  $(\mathcal{C}(\overline{G}) \cap \mathcal{S}(G))$ -Šilov boundary [3]. It is also known that under the axiom of polarity  $\partial G \setminus R(G)$  is polar [4], therefore  $\overline{R(G)} = \Gamma(G)$ . However it is still an open question whether it is true or not for an *arbitrary* relatively compact open subset  $G$  of a Brelot space.

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