Ikegami, T.
Osaka J. Math.
17 (1980), 177-186

REMARKS ON THE REGULARITY OF BOUNDARY POINTS IN A RESOLUTIVE COMPACTIFICATION

TERUO IKEGAMI

(Received January 8, 1979)

Introduction. Let X be a strong harmonic space in the sense of Bauer [2] and suppose that constant functions are harmonic. In the previous paper [5], the author studied the regularity of boundary points in a resolutive compactification of X and discussed characterization of regularity, existence of regular points, strong regularity and pseudo-strong regularity, characterization of harmonic boundary and consideration in the case of open subsets. In this paper we shall use the same notations and definitions as in [5], and we shall give some supplementary remarks.

In §1, we recall the notations and terminologies used in [5]. We reform characterization of the regularity in Theorem 1 of §2. Theorem 2 in §3 is the extremal characterization of pseudo-strong regularity in the case where X is a Brelot space. The trace filters of neighborhoods of boundary points in the Wiener compactification X^W of X is of some interest. Using this filters we can construct in §4 a family of completely regular filters in a metrizable and resolutive compactification X^* of X. A regular boundary point x is said to have a local property if x is regular for every $\overline{U(x)\cap X}$, where $U(x)$ is a neighborhood of x. The main results of this paper are in §5. It is shown that a regular point x does not possess a local property in general and x has a local property if and only if x is pseudo-strongly regular. Further the related problems are investigated. In the final section, we consider a relatively compact open set G of a Brelot space and obtain the result, if G is minimally bounded, then the set of all regular points is dense in the boundary ∂G of G, which is a generalization of a result of Bauer [1].

1. Preliminaries

Let X be a *strong* harmonic space in the sense of Bauer [2] on which constant functions are harmonic, and X^* be a resolutive compactification of X. On the boundary $\Delta=X^*\backslash X$ we define the harmonic boundary $\Gamma=\{x\in\Delta;\ \underline{\lim}_{a\to x}p(a)=0$ for every strictly positive potential p on $X\}$. For $f\in C(\Delta)$, i.e., a continuous real valued function on Δ, the Dirichlet solution of f is denoted by H_f. A point $x\in\Delta$ is termed to be *regular* if $\lim_{a\to x} H_f(a)=f(x)$ for every $f\in \boldsymbol{C}(\Delta)$. A point $x\in\Delta$ is called *pseudo-strongly regular* if $\lim_{a\to x} p(a)=0$ for every bounded potential p harmonic in a neighborhood of x. Every pseudo-strongly regular point is regular but the converse does not hold in general. We set

$$\mathcal{S}^+ = \{v;\ \text{superharmonic functions non-negative on } X\}$$

and

$$\mathcal{M}_x = \{\mu;\ \text{probability measures on } \Delta \text{ satisfying} \int \underline{v}\, d\mu \leq \bar{u}_v(x) + \underline{p}_v(x) \text{ for every } v\in\mathcal{S}^+\},$$

where $\underline{f}$ (resp. $\bar{f}$) is the lower (resp. upper) semicontinuous extension of f on Δ and u_v is the greatest harmonic minorant of v and p_v is the potential part of v.

The main results of our previous paper [5] are the following: a point $x\in\Gamma$ is regular if and only if $\mathcal{M}_x=\{\varepsilon_x\}$, where ε_x is the Dirac measure at x. As a collorary we obtain: if

$$\underline{\lim}_{\mathcal{U}(x)} [\overline{\lim_{a\to x}} R_1^{X\setminus U(x)}(a)] < 1,$$

then x is regular, where $\mathcal{U}(x)$ is a fundamental system of neighborhoods $U(x)$ of x. The harmonic boundary is the $\mathcal{S}^+$-Šilov boundary. For an open subset G of X, every regular point is pseudo-strongly regular, thus a regular point has a local property in this case.

2. Characterization of the regularity

We reform characterization of the regularity (Theorem 1 in [5]) in a slightly different form. Let

$$\mathcal{M}'_x = \{\mu;\ \text{probability measures on } \Delta \text{ satisfying} \int \underline{v}\, d\mu \leq \bar{u}_v(x) \text{ for every } v\in\mathcal{S}^+\}.$$

Clearly we have $\mathcal{M}'_x\subset\mathcal{M}_x$ and $\mathcal{M}'_x=\mathcal{M}_x$ if $x\in\Gamma$. It is noteworthy that $\mathcal{M}'_x$ may be empty whereas $\varepsilon_x\in\mathcal{M}_x$.

Theorem 1. *$x\in\Delta$ is regular if and only if $\mathcal{M}'_x=\{\varepsilon_x\}$.*

Proof. If x is regular then $x\in\Gamma$, and therefore $\mathcal{M}'_x=\mathcal{M}_x=\{\varepsilon_x\}$ [5]. Next, suppose that $\mathcal{M}'_x$ is not empty and consists of a single measure ε_x, and let $\{a_\iota\}$ be a net of points converging to x. Let ω_ι be a harmonic measure at a_ι, i.e.,

$$\int f\, d\omega_\iota = H_f(a_\iota) \quad \text{for every } f\in\boldsymbol{C}(\Delta).$$

ω_ι is a probability measure on Δ. There exists a subnet $\{\omega_{\iota_\kappa}\}$ of $\{\omega_\iota\}$ converging to a measure μ vaguely. μ is a probability measure on Δ. Further, $\mu\in\mathcal{M}'_x$. In fact, let $f\in C^+(\Delta)$ with $f\leq\underline{\lim}\, v$, where $v\in\mathcal{S}^+$, then $H_f\leq v$ and $H_f\leq u_v$. Since $\int f\,d\mu=\lim\int f\,d\omega_{\iota_\kappa}=\lim H_f(a_{\iota_\kappa})\leq\overline{\lim}_x u_v$ implies $\int(\underline{\lim}\, v)\,d\mu\leq\overline{\lim}_x u_v$, we have $\mu=\varepsilon_x$, i.e., ω_ι converges to ε_x and x is regular.

3. Extremal characterization of the pseudo-strong regularity in Brelot spaces

In this section, we consider a resolutive compactification of a *Brelot* space X. For $x\in\Delta$, we define

$$\mathcal{S}_x^* = \{H_f+p;\, f\in C^+(\Delta),\ p \text{ is a potential such that } \lim_x p = 0\}$$

and

$$\mathcal{M}_x^* = \{\mu;\ \text{probability measures on } \Delta \text{ such that } \int\underline{v}\,d\mu\leq\bar{v}(x) \text{ for every } v\in\mathcal{S}_x^*\}\ .$$

REMARK 1. $\mu\in\mathcal{M}_x^*$ if and only if $\int\underline{v}\,d\mu\leq\overline{\lim}_x H_f$ for every $v\in\mathcal{S}_x^*$, where $v=H_f+p$.

REMARK 2. $\mathcal{M}_x^*=\{\varepsilon_x\}$ implies $\mathcal{M}_x=\{\varepsilon_x\}$; for $\mathcal{M}_x\subset\mathcal{M}_x^*$, i.e., $\mathcal{M}_x^*=\{\varepsilon_x\}$ means that x is regular.

Theorem 2. *$x\in\Delta$ is pseudo-strongly regular if and only if $\mathcal{M}_x^*=\{\varepsilon_x\}$.*

Proof. Suppose that x is pseudo-strongly regular and that there exists $\mu\in\mathcal{M}_x^*$ such that $\mu\neq\varepsilon_x$. Let $y\in Supp\ \mu\setminus\{x\}$ and $f\in C^+(X^*)$, $f(y)>0$, $f=0$ on $U(x)$, where $U(x)$ is a neighborhood of x such that $y\notin\overline{U(x)}$. Put $u=H_f$. There exists a bounded potential p such that $u+p\geq f$ outside a compact subset of X. For, we may find a potential p' such that $u+p'\geq f$ outside a compact subset K of X since $u=h_f$ (for the definition of h_f, see [6]). On $X\setminus K$, $f\leq\min(u+p', ||f||)\leq\min(u, ||f||)+\min(p'\, ||f||)=u+\min(p', ||f||)=u+p$. Here $p=\min(p', ||f||)$ is a bounded potential. Set $p_1=\hat{R}_p^{X\setminus U(x)}$. By hypothesis, $\lim_x p_1=0$. Since $\underline{\lim}(u+p_1)\geq f>0$ in a neighborhood of y, we have a contradiction that $0<\int\underline{\lim}\,(u+p_1)\,d\mu\leq\overline{\lim}_x\, u=f(x)=0$.

Next, we prove the converse. We show first that for every $y\in\Delta$, $y\neq x$ there exists $v_y\in\mathcal{S}_x^*$ such that $\underline{\lim}_y v_y>\overline{\lim}_x v_y=0$. In fact, there is a function $v\in\mathcal{S}_x^*$ such that $\underline{\lim}_y\, v>\overline{\lim}_x\, v=g(x)$, where $v=H_g+p$ (by Remark 2); for otherwise we have $\varepsilon_y\in\mathcal{M}_x^*$. Set $f=\max(g-g(x), 0)$. Then $H_f+p\in\mathcal{S}_x^*$ and $\underline{\lim}_y\,(H_f+p)\geq\underline{\lim}_y\,(H_g+p)-g(x)>0=\overline{\lim}_x\, H_f=\overline{\lim}_x\,(H_f+p)$, *i.e.*, we may take

$v_y = H_f + p$. Now, let $U(x)$ be a neighborhood of x. For every $y \in \overline{\partial U(x)} \cap \Delta$ we associate with v_y described above. Then there exists a triple $(v_y, U(y), \delta_y)$ such that

$$v_y > \delta_y > 0 \text{ on } U(y) \cap X \text{ and } \lim_x v_y = 0$$

A finite number of $U(y)$, say $\{U(y_i)\}$, covers $\overline{\partial U(x)} \cap \Delta$. Set $\delta = \min \delta_{y_i}, v = \Sigma_i v_{y_i}$ and $V = \bigcup_i U(y_i)$. Then $v > \delta$ on $V \cap X$ and $\lim_x v = 0$. Since X is a Brelot space we may also find $\alpha > 0$ such that $\alpha v > 1$ on $\overline{\partial U(x)}$. Then $\lim_x \hat{R}_1^{X \backslash U(x)} = 0$, *i.e.*, x is pseudo-strongly regular.

4. The Wiener compactification

The compactification on which every Wiener function is extended continuously and separates points is called the *Wiener compactification* and is denoted by X^W [6]. The harmonic boundary of X^W is denoted by Γ^W.

Theorem 3. *Every point of Γ^W is pseudo-strongly regular.*

Proof. Let $U(x)$ be an open neighborhood of $x \in \Gamma^W$ in X^W. For a neighborhood $V(x)$ of x such that $\overline{V(x)} \subset U(x)$, $v = \hat{R}^{V(x) \cap X}_{\hat{R}_1^{X \backslash U(x)}}$ is a potential. In fact, since $\overline{V(x) \cap X} \cap \overline{X \backslash U(x)} \cap \Delta^W = \phi$, $q = \min(\hat{R}_1^{X \backslash U(x)}, \hat{R}_1^{V(x) \cap X})$ is a potential ([6], Th. 3.2.23) and $v \leq q$. $v = \hat{R}_1^{X \backslash U(x)}$ on $V(x) \cap X$ and v has a limit at x ([6], Prop. 4.4). Thus $\lim_x v = \underline{\lim}_x v = 0$, *i.e.*, $\lim_x \hat{R}_1^{X \backslash U(x)} = 0$.

Let X^* be a *metrizable* and resolutive compactification of X. Then there exists a family of completely regular filters $\{\mathcal{F}\}$ each of which converges to a point of $\Delta = X^* \backslash X$ and such that

A) *if a superharmonic function v on X is bounded from below and* $\lim$ *inf*$_{\mathcal{F}}$ $v \geq 0$ *for every $\mathcal{F}$, then $v \geq 0$,*

B) *for every $\mathcal{F}$, there exists a superharmonic function v on X such that* $\lim_{\mathcal{F}} v = 0$ *and inf* $\{v;\ X \backslash U(x)\} > 0$ *for every neighborhood $U(x)$ of x, where $\mathcal{F}$ converges to x.*

Here, a filter $\mathcal{F}$, converging to x, is called to be *completely regular* if $\lim_{\mathcal{F}} H_f = f(x)$ for every resolutive function f continuous at x.

In fact, consider the Wiener compactification X^W of X. X^* is a quotient space of X^W, *i.e.*, there exists a continuous mapping π of X^W onto X^* fixing each point of X. Let $\mathcal{F}_{\tilde{x}}$ be the trace filter of the filter of sections of neighborhoods of $\tilde{x} \in \Gamma^W$, *i.e.*,

$$\mathcal{F}_{\tilde{x}} = \{U(\tilde{x}) \cap X;\ U(\tilde{x}) \text{ is a neighborhood of } \tilde{x} \textit{ in } X^W\}\ .$$

$\mathcal{F}_{\tilde{x}}$ converges to $x = \pi(\tilde{x})$. The family of filters $\{\mathcal{F}_{\tilde{x}};\ \tilde{x} \in \Gamma^W\}$ is the desired one.

For, A) follows from the property of Γ^W ([6], Th.3.1.6). As for B) let $\tilde{x}\in\Gamma^W$, $\pi(\tilde{x})=x$, $\{U_n(x)\}$ be a fundamental system of neighborhoods of x, and let $\mathcal{F}=\mathcal{F}_{\tilde{x}}$. Then $v=\Sigma_n(1/2^n)\,\hat{R}_1^{X\setminus U_n(x)}$ fulfills the requirement of B). For, given $\varepsilon>0$, there exists an integer N such that $\Sigma_{N+1}^{\infty}(1/2^n)<\varepsilon/2$. Since $\tilde{x}$ is pseudo-strongly regular, $\lim_{\tilde{x}}\hat{R}_1^{X\setminus U_n(x)}=0$ in X^W. Hence $\overline{\lim}_{\tilde{x}}\,v\leq\varepsilon/2$. $\inf\{v;\,X\setminus U(x)\}>0$ is trivially seen. All that remains is to prove $\lim_{\mathcal{F}}H_f=f(x)$ for every resolutive function f continuous at x. We may suppose that $f\geq 0$ and $f(x)=0$. Let $\tilde{f}=f\circ\pi$. Since H_f is a Wiener function, H_f is extended continuously onto X^W. We denote this extended function by $\tilde{F}$. $\tilde{f}$ is resolutive with respect to X^W. For, since $\underline{\lim}_{\tilde{x}}\,s\geq\underline{\lim}_{\pi(\tilde{x})}\,s$, if s is non-negative superharmonic and $\underline{\lim}\,s\geq f$ on Δ, then $\underline{\lim}\,s\geq\tilde{f}$ on Δ^W, which implies that $H_f\geq\bar{H}_{\tilde{f}}^W$ and similarly $\underline{H}_{\tilde{f}}^W\geq H_f$, where $H_{\tilde{f}}^W$ is the Dirichlet solution with respect to X^W. Noting that $H_f=h_{H_f}$, where h is the operator of Constantinescu-Conea ([6], p. 26), we have $v\geq\bar{H}_{\tilde{F}}^W$ for every $v\geq 0$ superharmonic and $v\geq H_f$ outside a compact subset of X. Hence $H_f\geq\bar{H}_{\tilde{F}}^W$ and similarly $\underline{H}_{\tilde{F}}^W\geq H_f$. Thus, we have $H_{\tilde{F}}^W=H_f=H_{\tilde{f}}^W$. Therefore $\int(\tilde{F}-\tilde{f})d\omega^W=0$ and $\int|\tilde{F}-\tilde{f}|\,d\omega^W=0$, *i.e.*, $\tilde{F}=\tilde{f}\;d\omega^W-a.e.$, where ω^W is the harmonic measure in Δ^W. We shall prove that $\tilde{F}(x)=0$. For otherwise, since $\tilde{F}$ and $\tilde{f}$ are continuous at $\tilde{x}$, $\tilde{F}\neq\tilde{f}$ in a neighborhood of $\tilde{x}$, but this is impossible since this neighborhood is not of $d\omega^W$-harmonic measure zero ([6], Th. 3.2.19).

5. The local property of regular points

Let X^* be a resolutive compactification of X. We consider $G=X\cap U(x)$, where $U(x)$ is an open neighborhood of $x\in\Delta$. The closure $\bar{G}$ in X^* is a compactification. The boundary of $\bar{G}$ is denoted by $\Delta(G)$. $\Delta(G)=\partial G\cup\delta$, where $\partial G=\Delta(G)\cap X$ and $\delta=\Delta(G)\cap\Delta$. Obviously we have $x\in\delta$.

Proposition 1. *$\bar{G}$ is a resolutive compactification.*

Proof. Let $f\in \boldsymbol{C}^+(\Delta(G))$ and f_1 be a finite continuous extension of $f|\delta$ onto Δ, where $f|\delta$ is the restriction of f onto δ. Denoting by s_1 (resp. s_2) a hyperharmonic function on G, bounded from below, $\underline{\lim}\,s_1\geq f-H_{f_1}$ on ∂G, $s_1\geq 0$ outside a compact subset of X (resp. a hyperharmonic function on X, bounded from below, $\underline{\lim}\,s_2\geq f_1$ on Δ), we have

$$\underline{\lim}\,(s_1+s_2)\geq\begin{cases} f-H_{f_1}+H_{f_1}=f & \text{on } \partial G\\ f_1=f & \text{on } \delta\end{cases}$$

Hence, $\bar{H}_f^G\leq\bar{H}_{f-H_{f_1}}^{G,X}+H_{f_1}$, and similarly $\underline{H}_f^G\geq\underline{H}_{f-H_{f_1}}^{G,X}+H_{f_1}$, where H_f^G is the Dirichlet solution with respect to $\bar{G}$ and for the definition of $H_f^{G,X}$ we refer to

[6]. Thus we have $\bar{H}_f^G=\underline{H}_f^G=H_{f-H_{f_1}}^{G,X}+H_{f_1}$, since $\bar{H}_{f-H_{f_1}}^{G,X}=\underline{H}_{f-H_{f_1}}^{G,X}$ ([6], Th. 1.2.7).

Proposition 2. *If x is irregular for X^* then x is irregular for $\bar{G}$.*

Proof. Suppose that x is regular for $\bar{G}$. For a function $f\in \boldsymbol{C}(\Delta)$, let

$$\varphi=\begin{cases} f & \text{on } \delta \\ H_f & \text{on } \partial G \end{cases}$$

It is easily seen that φ is resolutive and $H_\varphi^G=H_f$ on G. From this we derive

$$\lim_x H_f=\lim_x H_\varphi^G=\varphi(x)=f(x)$$

which implies that x is regular for X^*.

The following example shows that the converse does not hold in general.

EXAMPLE. Let $X=\{|z|<1\}\setminus\{-1/2, 1/2\}$. We identify the two points $-1/2$ and $1/2$, and denote it by e. The Green function of $\{|z|<1\}$ with pole at $1/2$ is denoted by u_0. We consider the compactification of X such that $\Delta=\{|z|=1\}\cup\{e\}$, and the harmonic structure given by u_0-harmonic functions, i.e., the quotient of usual harmonic functions by u_0. The compactification X^* is resolutive and $H_f=f(e)$ (the constant function). Let $G=X\setminus K$, where $K=\{iy;\ y \textit{ is real and } |y|\leq 1/2\}$. e is regular for X^* but it is irregular for $\bar{G}$.

A strictly positive superharmonic function v_0 on X satisfying $\lim_x v_0=0$ is called a ***weak barrier*** of x.

In a resolutive compactification of a Brelot space, if Γ contains at least two points every regular point has a weak barrier. In the above example e has no weak barrier. We know an example of an irregular point with weak barrier ([7], p. 253) If X is a Brelot space, the existence of a (strong) barrier v_0 at x, *i.e.*, v_0 is a positive superharmonic function satisfying $\lim_x v_0=0$ and $\inf\{v_0;\ X\setminus U(x)\}>0$ for every open neighborhood $U(x)$ of x, is equivalent to $\lim_x R_1^{X\setminus K}=0$ for every compact set K.

Theorem 4. *Suppose that x has a weak barrier. Then x is regular for X^* if and only if x is regular for $X\setminus K$ for every compact subset K of X.*

Proof. By Proposition 2, it is enough to prove the "only if" part. Suppose for a moment that x is irregular for $\bar{G}$, where $G=X\setminus K$. Then $x\in\Gamma$. We shall see that there exists $f_0\in\mathcal{C}=\{f\in \boldsymbol{C}^+(\Delta\cup\partial K);\ f=0 \text{ on } \partial K\}$ such that $\underline{\lim}_x H_{f_0}^G<\overline{\lim}_x H_{f_0}^G$. In fact, if we have $\lim_x H_f^G=f(x)$ for every $f\in\mathcal{C}$, then $\lim_x H_g^G=g(x)$ for every $g\geq 0$ continuous on $\Delta\cup\partial K$. For, letting

$$g_1=\begin{cases} g & \text{on } \Delta \\ 0 & \text{on } \partial K \end{cases}$$

and

$$g_2=\begin{cases} 0 & \text{on } \Delta \\ g & \text{on } \partial K \end{cases}$$

we have $\lim_x H^G_{g_1}=g_1(x)=g(x)$ and $0\leq H^G_{g_2}\leq ||g_2|| H^G_\psi$, where ψ is the characteristic function of ∂K. From $1-\psi\in C$, it is derived that $\lim_x H^G_\psi=0$ and $\lim_x H^G_{g_2}=0$. Select a number γ such that

$$\underline{\lim}_x H^G_{f_0}<\gamma<\overline{\lim}_x H^G_{f_0} \text{ and } f_0(x)\neq\gamma .$$

By the theorem of Hahn-Banach, there exists a probability measure on $\Delta\cup\partial K$ such that

$$\gamma=\int f_0\, d\mu \text{ and } \int \underline{\lim}\, v\, d\mu\leq \overline{\lim}_x u_v \quad \text{for every } v\in \mathcal{S}^+(G).$$

Obviously $\mu\neq\varepsilon_x$. Since $\int\underline{\lim}\, v_0\, d\mu\leq\overline{\lim}_x v_0=0$, where v_0 is a weak barrier of x, we have $Supp\ \mu\subset\Delta$. Take a point $y\in Supp\ \mu\setminus\{x\}$ and $g\in \boldsymbol{C}^+(\Delta)$ such that $g(x)=0$ and $g>0$ in a neighborhood of y. We have

$$H^G_{g_1}=H_g \qquad \text{on } G,$$

where

$$g_1=\begin{cases} g & \text{on } \Delta \\ H_g & \text{on } \partial K \end{cases}$$

We may find a potential on G with $\underline{\lim}\,(H^G_{g_1}+p)\geq g_1$ on $\Delta\cup\partial K$. Hence

$$\int\underline{\lim}\,(H^G_{g_1}+p)\, d\mu\geq\int g_1\, d\mu=\int_\Delta g_1\, d\mu>0 .$$

On the other hand,

$$\int\underline{\lim}\,(H^G_{g_1}+p)\, d\mu\leq\overline{\lim}_x H^G_{g_1}=\overline{\lim}_x H_g=g(x)=0 ,$$

which is a contradiction.

Let $x\in\Delta$ be regular for X^*. If x is regular for every $\overline{X\cap U(x)}$, then x is said to have the *local property*.

Theorem 5. *x has the local property if and only if x is pseudo-strongly regular.*

Proof. We need to prove the "only if" part. We shall prove $\lim_x R_1^{X\setminus U(x)}=0$ for every $U(x)$. Let $G=X\cap U(x)$ and $f\in \boldsymbol{C}^+(\Delta(G))$ such that $f(x)=0$ and $f=1$ on ∂G. Consider a non-negative superharmonic function s with $\underline{\lim}\, s\geq f$ on $\Delta(G)$. We define

$$s_1=\begin{cases}1 & \text{on } X\backslash U(x)\\ \min(1,s) & \text{on } U(x)\,.\end{cases}$$

s_1 is superharmonic on X and $R_1^{X\backslash U(x)}\le s_1$. Therefore $R_1^{X\backslash U(x)}\le H_f^G$ in G, and $\lim_x H_f^G=f(x)=0$ implies $\lim_x R_1^{X\backslash U(x)}=0$.

Lemma. *Suppose that x is regular for X^* and $\lim_x R_1^{X\backslash U(x)}=0$ for a neighborhood $U(x)$ of x. Let $U_1(x)$ be a neighborhood of x with $\overline{U_1(x)}\subset U(x)$, and let $\delta=U_1(x)\cap\Delta$, $G=U(x)\cap X$. If $f,g\in \boldsymbol{C}(\Delta(G))$ and $f=g$ on δ, then $\overline{\lim}_x H_f^G=\overline{\lim}_x H_g^G$.*

Proof. Since $H_f^G-H_g^G=H_{f-g}^G$ it is sufficient to show that $f\in\boldsymbol{C}(\Delta(G))$ and $f=0$ on δ implies $\overline{\lim}_x H_f^G=0$. Let $U_2(x)$ be a neighborhood of x such that $\overline{U_2(x)}\subset U_1(x)$, and $\delta'=U_2(x)\cap\Delta$. For a function $\varphi\in\boldsymbol{C}^+(\Delta)$ with $\varphi\le||f||$ and $\varphi=||f||$ on $\Delta\backslash\delta$ and $\varphi(x)=0$, there exist a potential p and $s\in\mathcal{S}^+(G)$ such that

$$\begin{aligned}&\underline{\lim}\,(H_\varphi+\varepsilon p)\ge\varphi && \text{on } \Delta\\ &\underline{\lim}\,(R_{||f||}^{X\backslash U(x)}+\varepsilon s)\ge||f|| && \text{on } \partial U(x)\end{aligned}$$

for every $\varepsilon>0$. Setting $v=H_\varphi+||f||R_1^{X\backslash U(x)}+\varepsilon(p+s)$ we can readily seen that $v\ge H_f^G$ and $H_\varphi+||f||R_1^{X\backslash U(x)}\ge H_f^G$. Hence $\overline{\lim}_x H_f^G\le\lim_x H_\varphi+||f||\lim_x R_1^{X\backslash U(x)}=0$.

Theorem 6. *If x is regular for X^* and $\lim_x R_1^{X\backslash U(x)}=0$ then x is regular for $\overline{X\cap U(x)}$.*

Proof. Let $G=X\cap U(x)$. Suppose that x is irregular for $\bar{G}$. Then there exists $f\in\boldsymbol{C}^+(\Delta(G))$ such that *Supp* $f\subset\delta=U_1(x)\cap\Delta$, *where* $\overline{U_1(x)}\subset U(x)$ and $\underline{\lim}_x H_f^G\ne\overline{\lim}_x H_f^G$. We may construct a probability measure μ on $\Delta(G)$ such that $\mu\ne\varepsilon_x$ and

$$\int\underline{\lim}\,v\,d\mu\le\overline{\lim}_x u_v\qquad\text{for every } v\in\mathcal{S}^+(G)\,.$$

We assert that *Supp* $\mu\subset\bar{\delta}$, for if $g\in\boldsymbol{C}^+(\Delta(G))$ and $g=0$ on δ then $0\le\int g\,d\mu\le\overline{\lim}_x H_g^G=0$ by the above Lemma. There exists $y\in$ *Supp* $\mu\backslash\{x\}$. Since $y\in\bar{\delta}$ and $\bar{\delta}\cap(X^*\backslash U(x))=\phi$ we have $y\notin\overline{\partial G}$. Hence we can find $U(y)$ such that $\overline{U(y)}\subset U(x)$. Let $F\in\boldsymbol{C}^+(X^*)$ with $F(y)>0$ and $F(x)=0$, and let

$$F_1=\begin{cases}F & \text{on } U(y)\\ h_F & \text{on } G\backslash U(y)\end{cases}$$

There exists a potential q on X such that for every $\varepsilon>0$ we may find a compact subset K_ε of X so that

$$h_F+\varepsilon q\geq F \text{ and } h_F-\varepsilon q\leq F \text{ on } X\backslash K_\varepsilon .$$

Since $h_F+\varepsilon q\geq F_1$ and $h_F-\varepsilon q\leq F_1$ on $G\backslash(K_\varepsilon\cap U(y))$, we have $h_F\geq \bar{h}^G_{F_1}\geq \underline{h}^G_{F_1}\geq h_F$, *i.e.*, $h_F=h^G_{F_1}$. Thus we have a potential p on G such that $h^G_{F_1}+p\geq F_1$ outside a compact subset of G and, in particular, in a neighborhood of y. Hence we are led to a contradiction

$$0<\int \underline{\lim}\,(h^G_{F_1}+p)d\mu\leq \overline{\lim}_x\, h^G_{F_1}=\overline{\lim}_x\, h_F=0\,.$$

Let G^Ω be the closure of G in $X^{\mathcal{A}}$ (the one-point compactification of X). Then G^Ω is a resolutive compactification [5]. The boundary of G^Ω is $\partial G \cup \{\mathcal{A}\}$. We denote the Dirichlet solution on G^Ω by H^Ω_f. If the boundary function f on $\Delta(G)$ is resolutive for $\bar{G}$ and is constant α on $\delta=\bar{G}\cap\Delta$ then

$$f'=\begin{cases} f & \text{on } \partial G \\ \alpha & \text{at } \mathcal{A}\end{cases}$$

is resolutive for G^Ω, and conversely if f' is resolutive for G^Ω then

$$f=\begin{cases} f' & \text{on } \partial G \\ f'(\mathcal{A}) & \text{on } \delta\end{cases}$$

is resolutive for $\bar{G}$. In both cases $H^\Omega_{f'}=H^G_f$. $x\in\partial G$ is regular for $\bar{G}$ if and only if it is regular for G^Ω. Hence regular point $x\in\partial G$ for $\bar{G}$ is strongly regular [5].

6. Relatively compact open sets

In this section, we shall assume that X is a *Brelot* space.

Let G be a relatively compact open subset of X. The *outer boundary* of G is defined to be the boundary of $\bar{G}$ and is denoted by $B(G)$. The harmonic boundary of G and the set of regular points for $\bar{G}$ is denoted by $\Gamma(G)$ and $R(G)$ respectively. G termed to be *minimally bounded* if the interior of $\bar{G}$ coincides with G. G is minimally bounded if and only if $\partial G=B(G)$.

Theorem 7. $B(G)\subset\overline{R(G)}\subset\Gamma(G)$ ([1], *Satz* 17)

Proof. It is sufficient to prove that for every $x\in B(G)$ and for every regular region D containing x there exists $y\in R(G)\cap D$. Since $x\in B(G)$ we may find $z\in D\backslash\bar{G}$. Consider a regular region V containing z and $\bar{V}\subset D\backslash\bar{G}$. The reduced function $v=(\hat{R}_1^{X\backslash D})_{X\backslash\bar{V}}$ (the reduced function considered in the harmonic space $X\backslash\bar{V}$) is continuous on G and $\alpha=\inf\{v;\,\partial G\}<\inf\{v;\,\partial G\backslash D\}=1$. $v-\alpha$ is a weak barrier at any point of $E=\{y\in\partial G;\ v(y)=\alpha\}\neq\phi$ and all points of E are regular.

Corollary ([1], *Korollar to Satz* 17). *If G is minimally bounded, then* $\partial G=\overline{R(G)}=\Gamma(G)$.

REMARK. We know that in a Bauer space $\Gamma(G)$ is the $\mathcal{S}^+(G)$-Šilov boundary [5], while if G is *weakly dermining,* $\overline{R(G)}$ is the $(\boldsymbol{C}(\bar{G})\cap\mathcal{S}(G))$-Šilov boundary [3]. It is also known that under the axiom of polarity $\partial G\backslash R(G)$ is polar [4], therefore $\overline{R(G)}=\Gamma(G)$. However it is still an open question whether it is true or not for an *arbitrary* relatively compact open subset G of a Brelot space.

References

[1] H. Bauer: *Šilovscher Rand und Dirichletsches Problem,* Ann. Inst. Fourier **11** (1961), 89–136.

[2] H. Bauer: Harmonische Räume und ihre Potentialtheorie (Lecture Notes in Math. 22) Berlin-Heidelberg-New York, Springer, 1966.

[3] N. Boboc and A. Cornea: *Convex cones of lower semi-continuous functions on compact spaces,* Rev. Roumaine Math. Pures Appl. **12** (1967), 471–525.

[4] C. Constantiescu and A. Cornea: Potential theory on harmonic spaces, (Grundlehren der math. Wissenschaften in Einzeldar. 158) 1972.

[5] T. Ikegami: *On the regularity of boundary points in a resolutive compactification of a harmonic space,* Osaka J. Math. **14** (1977), 271–289.

[6] C. Megea: Compactifications des espaces harmoniques (Lecture Notes in Math. 222) Berlin-Heidelberg-New York, Springer, 1971.

[7] L. Naïm: *Sur le rôle de la frontière de R.S. Martin dans la théories du potentiel,* Ann. Inst. Fourier **7** (1957), 183–281.

Department of Mathematics
Osaka City University
Sugimoto-cho, Sumiyoshi-ku
Osaka 558, Japan