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Author(s)	Ikegami, Teruo
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REMARKS ON THE REGULARITY OF BOUNDARY POINTS IN A RESOLUTIVE COMPACTIFICATION

TERUO IKEGAMI

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Introduction. Let X be a strong harmonic space in the sense of Bauer [2] and suppose that constant functions are harmonic. In the previous paper [5], the author studied the regularity of boundary points in a resolute compactification of X and discussed characterization of regularity, existence of regular points, strong regularity and pseudo-strong regularity, characterization of harmonic boundary and consideration in the case of open subsets. In this paper we shall use the same notations and definitions as in [5], and we shall give some supplementary remarks.

In §1, we recall the notations and terminologies used in [5]. We reform characterization of the regularity in Theorem 1 of §2. Theorem 2 in §3 is the extremal characterization of pseudo-strong regularity in the case where X is a Brelot space. The trace filters of neighborhoods of boundary points in the Wiener compactification X^W of X is of some interest. Using this filters we can construct in §4 a family of completely regular filters in a metrizable and resolute compactification X^* of X . A regular boundary point x is said to have a local property if x is regular for every $\overline{U(x) \cap X}$, where $U(x)$ is a neighborhood of x . The main results of this paper are in §5. It is shown that a regular point x does not possess a local property in general and x has a local property if and only if x is pseudo-strongly regular. Further the related problems are investigated. In the final section, we consider a relatively compact open set G of a Brelot space and obtain the result, if G is minimally bounded, then the set of all regular points is dense in the boundary ∂G of G , which is a generalization of a result of Bauer [1].

1. Preliminaries

Let X be a *strong* harmonic space in the sense of Bauer [2] on which constant functions are harmonic, and X^* be a resolute compactification of X . On the boundary $\Delta = X^* \setminus X$ we define the harmonic boundary $\Gamma = \{x \in \Delta; \lim_{a \rightarrow x} p(a) = 0$ for every strictly positive potential p on $X\}$. For $f \in C(\Delta)$, i.e., a continuous

real valued function on Δ , the Dirichlet solution of f is denoted by H_f . A point $x \in \Delta$ is termed to be *regular* if $\lim_{a \rightarrow x} H_f(a) = f(x)$ for every $f \in \mathcal{C}(\Delta)$. A point $x \in \Delta$ is called *pseudo-strongly regular* if $\lim_{a \rightarrow x} p(a) = 0$ for every bounded potential p harmonic in a neighborhood of x . Every pseudo-strongly regular point is regular but the converse does not hold in general. We set

$$\mathcal{S}^+ = \{v; \text{superharmonic functions non-negative on } X\}$$

and

$$\mathcal{M}_x = \{\mu; \text{probability measures on } \Delta \text{ satisfying}$$

$$\int \underline{v} \, d\mu \leq \bar{u}_v(x) + \underline{p}_v(x) \text{ for every } v \in \mathcal{S}^+ \},$$

where \underline{f} (resp. \bar{f}) is the lower (resp. upper) semicontinuous extension of f on Δ and \bar{u}_v is the greatest harmonic minorant of v and \underline{p}_v is the potential part of v .

The main results of our previous paper [5] are the following: a point $x \in \Gamma$ is regular if and only if $\mathcal{M}_x = \{\varepsilon_x\}$, where ε_x is the Dirac measure at x . As a corollary we obtain: if

$$\lim_{\mathcal{U}(x)} \overline{[\lim_{a \rightarrow x} R_1^{X \setminus U(x)}(a)]} < 1,$$

then x is regular, where $\mathcal{U}(x)$ is a fundamental system of neighborhoods $U(x)$ of x . The harmonic boundary is the \mathcal{S}^+ -Silov boundary. For an open subset G of X , every regular point is pseudo-strongly regular, thus a regular point has a local property in this case.

2. Characterization of the regularity

We reform characterization of the regularity (Theorem 1 in [5]) in a slightly different form. Let

$$\mathcal{M}'_x = \{\mu; \text{probability measures on } \Delta \text{ satisfying}$$

$$\int \underline{v} \, d\mu \leq \bar{u}_v(x) \text{ for every } v \in \mathcal{S}^+ \}.$$

Clearly we have $\mathcal{M}'_x \subset \mathcal{M}_x$ and $\mathcal{M}'_x = \mathcal{M}_x$ if $x \in \Gamma$. It is noteworthy that \mathcal{M}'_x may be empty whereas $\varepsilon_x \in \mathcal{M}_x$.

Theorem 1. $x \in \Delta$ is regular if and only if $\mathcal{M}'_x = \{\varepsilon_x\}$.

Proof. If x is regular then $x \in \Gamma$, and therefore $\mathcal{M}'_x = \mathcal{M}_x = \{\varepsilon_x\}$ [5]. Next, suppose that \mathcal{M}'_x is not empty and consists of a single measure ε_x , and let $\{a_i\}$ be a net of points converging to x . Let ω_i be a harmonic measure at a_i , i.e.,

$$\int f \, d\omega_i = H_f(a_i) \text{ for every } f \in \mathcal{C}(\Delta).$$

ω_i is a probability measure on Δ . There exists a subnet $\{\omega_{i_\kappa}\}$ of $\{\omega_i\}$ converging to a measure μ vaguely. μ is a probability measure on Δ . Further, $\mu \in \mathcal{M}_x^*$. In fact, let $f \in \mathbf{C}^+(\Delta)$ with $f \leq \underline{\lim} v$, where $v \in \mathcal{S}^+$, then $H_f \leq v$ and $H_f \leq u_v$. Since $\int f d\mu = \lim \int f d\omega_{i_\kappa} = \lim H_f(a_{i_\kappa}) \leq \overline{\lim}_x u_v$ implies $\int (\underline{\lim} v) d\mu \leq \overline{\lim}_x u_v$, we have $\mu = \varepsilon_x$, i.e., ω_i converges to ε_x and x is regular.

3. Extremal characterization of the pseudo-strong regularity in Brelot spaces

In this section, we consider a resolutive compactification of a *Brelot* space X . For $x \in \Delta$, we define

$$\mathcal{S}_x^* = \{H_f + p; f \in \mathbf{C}^+(\Delta), p \text{ is a potential such that } \lim_x p = 0\}$$

and

$$\mathcal{M}_x^* = \{\mu; \text{probability measures on } \Delta \text{ such that}$$

$$\int v d\mu \leq \bar{v}(x) \text{ for every } v \in \mathcal{S}_x^*\}.$$

REMARK 1. $\mu \in \mathcal{M}_x^*$ if and only if $\int v d\mu \leq \overline{\lim}_x H_f$ for every $v \in \mathcal{S}_x^*$, where $v = H_f + p$.

REMARK 2. $\mathcal{M}_x^* = \{\varepsilon_x\}$ implies $\mathcal{M}_x = \{\varepsilon_x\}$; for $\mathcal{M}_x \subset \mathcal{M}_x^*$, i.e., $\mathcal{M}_x^* = \{\varepsilon_x\}$ means that x is regular.

Theorem 2. $x \in \Delta$ is pseudo-strongly regular if and only if $\mathcal{M}_x^* = \{\varepsilon_x\}$.

Proof. Suppose that x is pseudo-strongly regular and that there exists $\mu \in \mathcal{M}_x^*$ such that $\mu \neq \varepsilon_x$. Let $y \in \text{Supp } \mu \setminus \{x\}$ and $f \in \mathbf{C}^+(X^*)$, $f(y) > 0$, $f = 0$ on $U(x)$, where $U(x)$ is a neighborhood of x such that $y \notin \overline{U(x)}$. Put $u = H_f$. There exists a bounded potential p such that $u + p \geq f$ outside a compact subset of X . For, we may find a potential p' such that $u + p' \geq f$ outside a compact subset K of X since $u = h_f$ (for the definition of h_f , see [6]). On $X \setminus K$, $f \leq \min(u + p', \|f\|) \leq \min(u, \|f\|) + \min(p', \|f\|) = u + \min(p', \|f\|) = u + p$. Here $p = \min(p', \|f\|)$ is a bounded potential. Set $p_1 = \hat{R}_p^{X \setminus U(x)}$. By hypothesis, $\lim_x p_1 = 0$. Since $\underline{\lim}(u + p_1) \geq f > 0$ in a neighborhood of y , we have a contradiction that $0 < \int \underline{\lim}(u + p_1) d\mu \leq \overline{\lim}_x u = f(x) = 0$.

Next, we prove the converse. We show first that for every $y \in \Delta$, $y \neq x$ there exists $v_y \in \mathcal{S}_x^*$ such that $\underline{\lim}_y v_y > \overline{\lim}_x v_y = 0$. In fact, there is a function $v \in \mathcal{S}_x^*$ such that $\underline{\lim}_y v > \overline{\lim}_x v = g(x)$, where $v = H_g + p$ (by Remark 2); for otherwise we have $\varepsilon_y \in \mathcal{M}_x^*$. Set $f = \max(g - g(x), 0)$. Then $H_f + p \in \mathcal{S}_x^*$ and $\underline{\lim}_y (H_f + p) \geq \underline{\lim}_y (H_g + p) - g(x) > 0 = \overline{\lim}_x H_f = \overline{\lim}_x (H_f + p)$, i.e., we may take

$v_y = H_f + p$. Now, let $U(x)$ be a neighborhood of x . For every $y \in \overline{\partial U(x)} \cap \Delta$ we associate with v_y described above. Then there exists a triple $(v_y, U(y), \delta_y)$ such that

$$v_y > \delta_y > 0 \text{ on } U(y) \cap X \text{ and } \lim_x v_y = 0$$

A finite number of $U(y)$, say $\{U(y_i)\}$, covers $\overline{\partial U(x)} \cap \Delta$. Set $\delta = \min \delta_{y_i}$, $v = \sum_i v_{y_i}$ and $V = \bigcup_i U(y_i)$. Then $v > \delta$ on $V \cap X$ and $\lim_x v = 0$. Since X is a Brelot space we may also find $\alpha > 0$ such that $\alpha v > 1$ on $\overline{\partial U(x)}$. Then $\lim_x \hat{R}_1^{X \setminus U(x)} = 0$, i.e., x is pseudo-strongly regular.

4. The Wiener compactification

The compactification on which every Wiener function is extended continuously and separates points is called the *Wiener compactification* and is denoted by X^W [6]. The harmonic boundary of X^W is denoted by Γ^W .

Theorem 3. *Every point of Γ^W is pseudo-strongly regular.*

Proof. Let $U(x)$ be an open neighborhood of $x \in \Gamma^W$ in X^W . For a neighborhood $V(x)$ of x such that $\overline{V(x)} \subset U(x)$, $v = \hat{R}_1^{V(x) \cap X}$ is a potential. In fact, since $\overline{V(x)} \cap X \cap \overline{X \setminus U(x)} \cap \Delta^W = \emptyset$, $q = \min(\hat{R}_1^{X \setminus U(x)}, \hat{R}_1^{V(x) \cap X})$ is a potential ([6], Th. 3.2.23) and $v \leq q$. $v = \hat{R}_1^{X \setminus U(x)}$ on $V(x) \cap X$ and v has a limit at x ([6], Prop. 4.4). Thus $\lim_x v = \lim_x v = 0$, i.e., $\lim_x \hat{R}_1^{X \setminus U(x)} = 0$.

Let X^* be a *metrizable* and resolutive compactification of X . Then there exists a family of completely regular filters $\{\mathcal{F}\}$ each of which converges to a point of $\Delta = X^* \setminus X$ and such that

- A) *if a superharmonic function v on X is bounded from below and $\liminf_{\mathcal{F}} v \geq 0$ for every \mathcal{F} , then $v \geq 0$,*
- B) *for every \mathcal{F} , there exists a superharmonic function v on X such that $\lim_{\mathcal{F}} v = 0$ and $\inf \{v; X \setminus U(x)\} > 0$ for every neighborhood $U(x)$ of x , where \mathcal{F} converges to x .*

Here, a filter \mathcal{F} , converging to x , is called to be *completely regular* if $\lim_{\mathcal{F}} H_f = f(x)$ for every resolutive function f continuous at x .

In fact, consider the Wiener compactification X^W of X . X^* is a quotient space of X^W , i.e., there exists a continuous mapping π of X^W onto X^* fixing each point of X . Let $\mathcal{F}_{\tilde{x}}$ be the trace filter of the filter of sections of neighborhoods of $\tilde{x} \in \Gamma^W$, i.e.,

$$\mathcal{F}_{\tilde{x}} = \{U(\tilde{x}) \cap X; U(\tilde{x}) \text{ is a neighborhood of } \tilde{x} \text{ in } X^W\}.$$

$\mathcal{F}_{\tilde{x}}$ converges to $x = \pi(\tilde{x})$. The family of filters $\{\mathcal{F}_{\tilde{x}}; \tilde{x} \in \Gamma^W\}$ is the desired

one.

For, *A*) follows from the property of Γ^W ([6], Th.3.1.6). As for *B*) let $\tilde{x} \in \Gamma^W$, $\pi(\tilde{x})=x$, $\{U_n(x)\}$ be a fundamental system of neighborhoods of x , and let $\mathcal{F}=\mathcal{F}_{\tilde{x}}$. Then $v=\Sigma_n(1/2^n) \hat{R}_1^{X \setminus U_n(x)}$ fulfills the requirement of *B*). For, given $\varepsilon > 0$, there exists an integer N such that $\Sigma_{N+1}^\infty(1/2^n) < \varepsilon/2$. Since \tilde{x} is pseudo-strongly regular, $\lim_{\tilde{x}} \hat{R}_1^{X \setminus U_n(x)} = 0$ in X^W . Hence $\overline{\lim}_{\tilde{x}} v \leq \varepsilon/2$. $\inf \{v; X \setminus U(x)\} > 0$ is trivially seen. All that remains is to prove $\lim_{\mathcal{F}} H_f = f(x)$ for every resolutive function f continuous at x . We may suppose that $f \geq 0$ and $f(x)=0$. Let $\tilde{f}=f \circ \pi$. Since H_f is a Wiener function, H_f is extended continuously onto X^W . We denote this extended function by \tilde{F} . \tilde{f} is resolutive with respect to X^W . For, since $\underline{\lim}_{\tilde{x}} s \geq \underline{\lim}_{\pi(\tilde{x})} s$, if s is non-negative superharmonic and $\underline{\lim} s \geq f$ on Δ , then $\underline{\lim} s \geq \tilde{f}$ on Δ^W , which implies that $H_f \geq \tilde{H}_{\tilde{f}}^W$ and similarly $\underline{H}_{\tilde{f}}^W \geq H_f$, where H_f^W is the Dirichlet solution with respect to X^W . Noting that $H_f = h_{H_f}$, where h is the operator of Constantinescu-Conea ([6], p.26), we have $v \geq \tilde{H}_{\tilde{f}}^W$ for every $v \geq 0$ superharmonic and $v \geq H_f$ outside a compact subset of X . Hence $H_f \geq \tilde{H}_{\tilde{f}}^W$ and similarly $\underline{H}_{\tilde{f}}^W \geq H_f$. Thus, we have $H_{\tilde{f}}^W = H_f = H_{\tilde{f}}^W$. Therefore $\int (\tilde{F} - \tilde{f}) d\omega^W = 0$ and $\int |\tilde{F} - \tilde{f}| d\omega^W = 0$, i.e., $\tilde{F} = \tilde{f}$ $d\omega^W$ -a.e., where ω^W is the harmonic measure in Δ^W . We shall prove that $\tilde{F}(x)=0$. For otherwise, since \tilde{F} and \tilde{f} are continuous at \tilde{x} , $\tilde{F} \neq \tilde{f}$ in a neighborhood of \tilde{x} , but this is impossible since this neighborhood is not of $d\omega^W$ -harmonic measure zero ([6], Th. 3.2.19).

5. The local property of regular points

Let X^* be a resolutive compactification of X . We consider $G=X \cap U(x)$, where $U(x)$ is an open neighborhood of $x \in \Delta$. The closure \bar{G} in X^* is a compactification. The boundary of \bar{G} is denoted by $\Delta(G)$. $\Delta(G)=\partial G \cup \delta$, where $\partial G=\Delta(G) \cap X$ and $\delta=\Delta(G) \cap \Delta$. Obviously we have $x \in \delta$.

Proposition 1. \bar{G} is a resolutive compactification.

Proof. Let $f \in C^+(\Delta(G))$ and f_1 be a finite continuous extension of $f|_\delta$ onto Δ , where $f|_\delta$ is the restriction of f onto δ . Denoting by s_1 (resp. s_2) a hyperharmonic function on G , bounded from below, $\underline{\lim} s_1 \geq f - H_{f_1}$ on ∂G , $s_1 \geq 0$ outside a compact subset of X (resp. a hyperharmonic function on X , bounded from below, $\underline{\lim} s_2 \geq f_1$ on Δ), we have

$$\underline{\lim} (s_1 + s_2) \geq \begin{cases} f - H_{f_1} + H_{f_1} = f & \text{on } \partial G \\ f_1 = f & \text{on } \delta \end{cases}$$

Hence, $\tilde{H}_f^G \leq \tilde{H}_{f-H_{f_1}}^G + H_{f_1}$, and similarly $\underline{H}_f^G \geq \underline{H}_{f-H_{f_1}}^G + H_{f_1}$, where H_f^G is the Dirichlet solution with respect to \bar{G} and for the definition of $H_f^{G,x}$ we refer to

[6]. Thus we have $\bar{H}_f^G = \underline{H}_f^G = H_{f-H_{f_1}}^{G,X} + H_{f_1}$, since $\bar{H}_{f-H_{f_1}}^{G,X} = \underline{H}_{f-H_{f_1}}^{G,X}$ ([6], Th. 1.2.7).

Proposition 2. *If x is irregular for X^* then x is irregular for \bar{G} .*

Proof. Suppose that x is regular for \bar{G} . For a function $f \in \mathcal{C}(\Delta)$, let

$$\varphi = \begin{cases} f & \text{on } \delta \\ H_f & \text{on } \partial G \end{cases}$$

It is easily seen that φ is resolutive and $H_\varphi^G = H_f$ on G . From this we derive

$$\lim_x H_f = \lim_x H_\varphi^G = \varphi(x) = f(x)$$

which implies that x is regular for X^* .

The following example shows that the converse does not hold in general.

EXAMPLE. Let $X = \{|z| < 1\} \setminus \{-1/2, 1/2\}$. We identify the two points $-1/2$ and $1/2$, and denote it by e . The Green function of $\{|z| < 1\}$ with pole at $1/2$ is denoted by u_0 . We consider the compactification of X such that $\Delta = \{|z| = 1\} \cup \{e\}$, and the harmonic structure given by u_0 -harmonic functions, i.e., the quotient of usual harmonic functions by u_0 . The compactification X^* is resolutive and $H_f = f(e)$ (the constant function). Let $G = X \setminus K$, where $K = \{iy; y \text{ is real and } |y| \leq 1/2\}$. e is regular for X^* but it is irregular for \bar{G} .

A strictly positive superharmonic function v_0 on X satisfying $\lim_x v_0 = 0$ is called a *weak barrier* of x .

In a resolutive compactification of a Brelot space, if Γ contains at least two points every regular point has a weak barrier. In the above example e has no weak barrier. We know an example of an irregular point with weak barrier ([7], p. 253). If X is a Brelot space, the existence of a (strong) barrier v_0 at x , i.e., v_0 is a positive superharmonic function satisfying $\lim_x v_0 = 0$ and $\inf \{v_0; X \setminus U(x)\} > 0$ for every open neighborhood $U(x)$ of x , is equivalent to $\lim_x R_1^{X \setminus K} = 0$ for every compact set K .

Theorem 4. *Suppose that x has a weak barrier. Then x is regular for X^* if and only if x is regular for $X \setminus K$ for every compact subset K of X .*

Proof. By Proposition 2, it is enough to prove the “only if” part. Suppose for a moment that x is irregular for \bar{G} , where $G = X \setminus K$. Then $x \in \Gamma$. We shall see that there exists $f_0 \in \mathcal{C} = \{f \in \mathcal{C}^+(\Delta \cup \partial K); f = 0 \text{ on } \partial K\}$ such that $\underline{\lim}_x H_{f_0}^G < \overline{\lim}_x H_{f_0}^G$. In fact, if we have $\lim_x H_f^G = f(x)$ for every $f \in \mathcal{C}$, then $\lim_x H_g^G = g(x)$ for every $g \geq 0$ continuous on $\Delta \cup \partial K$. For, letting

$$g_1 = \begin{cases} g & \text{on } \Delta \\ 0 & \text{on } \partial K \end{cases}$$

and

$$g_2 = \begin{cases} 0 & \text{on } \Delta \\ g & \text{on } \partial K \end{cases}$$

we have $\lim_x H_{g_1}^G = g_1(x) = g(x)$ and $0 \leq H_{g_2}^G \leq \|g_2\| H_\psi^G$, where ψ is the characteristic function of ∂K . From $1 - \psi \in \mathcal{C}$, it is derived that $\lim_x H_\psi^G = 0$ and $\lim_x H_{g_2}^G = 0$. Select a number γ such that

$$\underline{\lim}_x H_{f_0}^G < \gamma < \overline{\lim}_x H_{f_0}^G \text{ and } f_0(x) \neq \gamma.$$

By the theorem of Hahn-Banach, there exists a probability measure on $\Delta \cup \partial K$ such that

$$\gamma = \int f_0 d\mu \text{ and } \int \underline{\lim} v d\mu \leq \overline{\lim}_x u, \quad \text{for every } v \in \mathcal{S}^+(G).$$

Obviously $\mu \neq \varepsilon_x$. Since $\int \underline{\lim} v_0 d\mu \leq \overline{\lim}_x v_0 = 0$, where v_0 is a weak barrier of x , we have $\text{Supp } \mu \subset \Delta$. Take a point $y \in \text{Supp } \mu \setminus \{x\}$ and $g \in \mathcal{C}^+(\Delta)$ such that $g(x) = 0$ and $g > 0$ in a neighborhood of y . We have

$$H_{g_1}^G = H_g \quad \text{on } G,$$

where

$$g_1 = \begin{cases} g & \text{on } \Delta \\ H_g & \text{on } \partial K \end{cases}$$

We may find a potential on G with $\underline{\lim} (H_{g_1}^G + p) \geq g_1$ on $\Delta \cup \partial K$. Hence

$$\int \underline{\lim} (H_{g_1}^G + p) d\mu \geq \int g_1 d\mu = \int_{\Delta} g_1 d\mu > 0.$$

On the other hand,

$$\int \underline{\lim} (H_{g_1}^G + p) d\mu \leq \overline{\lim}_x H_{g_1}^G = \overline{\lim}_x H_g = g(x) = 0,$$

which is a contradiction.

Let $x \in \Delta$ be regular for X^* . If x is regular for every $\overline{X \cap U(x)}$, then x is said to have the *local property*.

Theorem 5. x has the local property if and only if x is pseudo-strongly regular.

Proof. We need to prove the “only if” part. We shall prove $\lim_x R_1^{X \setminus U(x)} = 0$ for every $U(x)$. Let $G = X \cap U(x)$ and $f \in \mathcal{C}^+(\Delta(G))$ such that $f(x) = 0$ and $f = 1$ on $\partial \overline{G}$. Consider a non-negative superharmonic function s with $\underline{\lim} s \geq f$ on $\Delta(G)$. We define

$$s_1 = \begin{cases} 1 & \text{on } X \setminus U(x) \\ \min(1, s) & \text{on } U(x). \end{cases}$$

s_1 is superharmonic on X and $R_1^{X \setminus U(x)} \leq s_1$. Therefore $R_1^{X \setminus U(x)} \leq H_f^G$ in G , and $\lim_x H_f^G = f(x) = 0$ implies $\lim_x R_1^{X \setminus U(x)} = 0$.

Lemma. *Suppose that x is regular for X^* and $\lim_x R_1^{X \setminus U(x)} = 0$ for a neighborhood $U(x)$ of x . Let $U_1(x)$ be a neighborhood of x with $\overline{U_1(x)} \subset U(x)$, and let $\delta = U_1(x) \cap \Delta$, $G = U(x) \cap X$. If $f, g \in \mathbf{C}(\Delta(G))$ and $f = g$ on δ , then $\overline{\lim}_x H_f^G = \overline{\lim}_x H_g^G$.*

Proof. Since $H_f^G - H_g^G = H_{f-g}^G$ it is sufficient to show that $f \in \mathbf{C}(\Delta(G))$ and $f = 0$ on δ implies $\overline{\lim}_x H_f^G = 0$. Let $U_2(x)$ be a neighborhood of x such that $\overline{U_2(x)} \subset U_1(x)$, and $\delta' = U_2(x) \cap \Delta$. For a function $\varphi \in \mathbf{C}^+(\Delta)$ with $\varphi \leq \|f\|$ and $\varphi = \|f\|$ on $\Delta \setminus \delta$ and $\varphi(x) = 0$, there exist a potential p and $s \in \mathcal{S}^+(G)$ such that

$$\begin{aligned} \underline{\lim} (H_\varphi + \varepsilon p) &\geq \varphi && \text{on } \Delta \\ \underline{\lim} (R_{\|f\|}^{X \setminus U(x)} + \varepsilon s) &\geq \|f\| && \text{on } \partial U(x) \end{aligned}$$

for every $\varepsilon > 0$. Setting $v = H_\varphi + \|f\| R_1^{X \setminus U(x)} + \varepsilon(p + s)$ we can readily seen that $v \geq H_f^G$ and $H_\varphi + \|f\| R_1^{X \setminus U(x)} \geq H_f^G$. Hence $\overline{\lim}_x H_f^G \leq \lim_x H_\varphi + \|f\| \lim_x R_1^{X \setminus U(x)} = 0$.

Theorem 6. *If x is regular for X^* and $\lim_x R_1^{X \setminus U(x)} = 0$ then x is regular for $\overline{X \cap U(x)}$.*

Proof. Let $G = X \cap U(x)$. Suppose that x is irregular for \overline{G} . Then there exists $f \in \mathbf{C}^+(\Delta(G))$ such that $\text{Supp } f \subset \delta = U_1(x) \cap \Delta$, where $\overline{U_1(x)} \subset U(x)$ and $\overline{\lim}_x H_f^G \neq \lim_x H_f^G$. We may construct a probability measure μ on $\Delta(G)$ such that $\mu \neq \delta_x$ and

$$\int \underline{\lim}_x v \, d\mu \leq \overline{\lim}_x u_v \quad \text{for every } v \in \mathcal{S}^+(G).$$

We assert that $\text{Supp } \mu \subset \overline{\delta}$, for if $g \in \mathbf{C}^+(\Delta(G))$ and $g = 0$ on δ then $0 \leq \int g \, d\mu \leq \overline{\lim}_x H_g^G = 0$ by the above Lemma. There exists $y \in \text{Supp } \mu \setminus \{x\}$. Since $y \in \overline{\delta}$ and $\overline{\delta} \cap (X^* \setminus U(x)) = \emptyset$ we have $y \notin \overline{\partial G}$. Hence we can find $U(y)$ such that $\overline{U(y)} \subset U(x)$. Let $F \in \mathbf{C}^+(X^*)$ with $F(y) > 0$ and $F(x) = 0$, and let

$$F_1 = \begin{cases} F & \text{on } U(y) \\ h_F & \text{on } G \setminus U(y) \end{cases}$$

There exists a potential q on X such that for every $\varepsilon > 0$ we may find a compact subset K_ε of X so that

$$h_F + \varepsilon q \geq F \text{ and } h_F - \varepsilon q \leq F \text{ on } X \setminus K_\varepsilon.$$

Since $h_F + \varepsilon q \geq F_1$ and $h_F - \varepsilon q \leq F_1$ on $G \setminus (K_\varepsilon \cap U(y))$, we have $h_F \geq \bar{h}_{F_1}^G \geq \underline{h}_{F_1}^G \geq h_F$, i.e., $h_F = \underline{h}_{F_1}^G$. Thus we have a potential p on G such that $\underline{h}_{F_1}^G + p \geq F_1$ outside a compact subset of G and, in particular, in a neighborhood of y . Hence we are led to a contradiction

$$0 < \int \underline{\lim} (h_{F_1}^G + p) d\mu \leq \overline{\lim}_x h_{F_1}^G = \overline{\lim}_x h_F = 0.$$

Let G^α be the closure of G in $X^{\mathcal{A}}$ (the one-point compactification of X). Then G^α is a resolutive compactification [5]. The boundary of G^α is $\partial G \cup \{\mathcal{A}\}$. We denote the Dirichlet solution on G^α by H_f^α . If the boundary function f on $\Delta(G)$ is resolutive for \bar{G} and is constant α on $\delta = \bar{G} \cap \Delta$ then

$$f' = \begin{cases} f & \text{on } \partial G \\ \alpha & \text{at } \mathcal{A} \end{cases}$$

is resolutive for G^α , and conversely if f' is resolutive for G^α then

$$f = \begin{cases} f' & \text{on } \partial G \\ f'(\mathcal{A}) & \text{on } \delta \end{cases}$$

is resolutive for \bar{G} . In both cases $H_f^\alpha = H_f^G$. $x \in \partial G$ is regular for \bar{G} if and only if it is regular for G^α . Hence regular point $x \in \partial G$ for \bar{G} is strongly regular [5].

6. Relatively compact open sets

In this section, we shall assume that X is a *Brelot* space.

Let G be a relatively compact open subset of X . The *outer boundary* of G is defined to be the boundary of \bar{G} and is denoted by $B(G)$. The harmonic boundary of G and the set of regular points for \bar{G} is denoted by $\Gamma(G)$ and $R(G)$ respectively. G termed to be *minimally bounded* if the interior of \bar{G} coincides with G . G is minimally bounded if and only if $\partial G = B(G)$.

Theorem 7. $B(G) \subset \overline{R(\bar{G})} \subset \Gamma(G)$ ([1], Satz 17)

Proof. It is sufficient to prove that for every $x \in B(G)$ and for every regular region D containing x there exists $y \in R(G) \cap D$. Since $x \in B(G)$ we may find $z \in D \setminus \bar{G}$. Consider a regular region V containing z and $\bar{V} \subset D \setminus \bar{G}$. The reduced function $v = (\hat{R}_1^{X \setminus D})_{X \setminus \bar{V}}$ (the reduced function considered in the harmonic space $X \setminus \bar{V}$) is continuous on G and $\alpha = \inf \{v; \partial G\} < \inf \{v; \partial G \setminus D\} = 1$. $v - \alpha$ is a weak barrier at any point of $E = \{y \in \partial G; v(y) = \alpha\} \neq \emptyset$ and all points of E are regular.

Corollary ([1], *Korollar to Satz 17*). *If G is minimally bounded, then $\partial G = \overline{R(G)} = \Gamma(G)$.*

REMARK. We know that in a Bauer space $\Gamma(G)$ is the $\mathcal{S}^+(G)$ -Šilov boundary [5], while if G is *weakly dermining*, $\overline{R(G)}$ is the $(\mathcal{C}(\bar{G}) \cap \mathcal{S}(G))$ -Šilov boundary [3]. It is also known that under the axiom of polarity $\partial G \setminus R(G)$ is polar [4], therefore $\overline{R(G)} = \Gamma(G)$. However it is still an open question whether it is true or not for an *arbitrary* relatively compact open subset G of a Brelot space.

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Department of Mathematics
Osaka City University
Sugimoto-cho, Sumiyoshi-ku
Osaka 558, Japan