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Connection formulae of the Ramanujan entire function and a resummation of the basic hypergeometric series $\phi_0(O, O; -; q, x)$

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Connection formulae of the Ramanujan entire function and a resummation of the basic hypergeometric series $2\varphi_0(0, 0; -; q, x)$

Takeshi MORITA$^1$

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Abstract

We study a connection problem on a $q$-difference equation satisfied by the Ramanujan entire function. Since one local solution contains a divergent series, we apply the $q$-Borel-Laplace resummation method to study the connection problem. We show two connection formulae. One is a connection formula between the Ramanujan entire function and the $q$-Airy function, another is a connection formula for a resummation of a divergent basic hypergeometric series $2\varphi_0(0,0;-;q,x)$. This thesis is based on author’s work [8].
Contents

1 Introduction 3
  1.1 Linear $q$-difference equations and connection problems . . . 4
  1.2 Connection problem on $q$-difference equation satisfied by the
      Ramanujan entire function . . . . . . . . . . . . . . . . . . . . . 7
  1.3 The theta function and the $q$-exponential functions . . . . . . 10

2 Connection formulae of the Ramanujan entire function and
the resummation of the divergent series $2\varphi_0(0, 0; -; q, x)$ 11
  2.1 Relation between the Ramanujan entire function and the $q$-
      Airy function . . . . . . . . . . . . . . . . . . . . . . . . . . . 11
  2.2 Connection formula of the Ramanujan entire function . . . . . 14
  2.3 Resummation of $2\varphi_0(0, 0; -; q, -x/q)$ . . . . . . . . . 18
      2.3.1 Alternative representation of $q$-exponential functions . 18
      2.3.2 Connection formula for the resummation of $2\varphi_0$ . . 19
  2.4 Summary . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 24
Chapter 1

Introduction

In this paper, we study a connection problem on a linear $q$-difference equation satisfied by the Ramanujan entire function. The Ramanujan entire function is originally found by S. Ramanujan in Ramanujan’s Lost Notebook [9]. Throughout the paper, we assume that $0 < |q| < 1$. The Ramanujan entire function with the base $q$ is given by

$$A_q(x) := \sum_{n \geq 0} \frac{q^{n^2}}{(q; q)_n} (-x)^n$$

where $(a; q)_n$ is the $q$-shifted factorial

$$(a; q)_n := \begin{cases} 1, & n = 0, \\ (1 - a)(1 - aq) \ldots (1 - aq^{n-1}), & n \geq 1. \end{cases}$$

M. E. H. Ismail has pointed out that the Ramanujan entire function can be considered as a $q$-analogue of the Airy function [5]. It is known that there exist two different $q$-analogues of the Airy function. One is the Ramanujan entire function and the other is the $q$-Airy function found by K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta and Y. Yamada in the study of hypergeometric-type solutions to the $q$-Painlevé equation of type $(A_1 + A_1')^{(1)}$ [7]. The $q$-Airy function with the base $q$ is given by

$$Ai_q(x) := \sum_{n \geq 0} \frac{q^{n(n-1)}}{(-q, q; q)_n} x^n,$$

where $(a_1, a_2, \ldots, a_m; q)_n := (a_1; q)_n (a_2; q)_n \ldots (a_m; q)_n$.

No relations between two different $q$-analogues of Airy functions have been known, but we show that the Ramanujan entire function and the $q$-Airy function are related by our connection formula.
1.1 Linear $q$-difference equations and connection problems

At first, we review the classical results by C. R. Adams [1] in a second order case. Adams studied a local behavior of solutions of linear $q$-difference equations. We consider the $q$-difference equations of the form

$$a_2(x)u(q^2x) + a_1(x)u(qx) + a_0(x)u(x) = 0$$  \hspace{1cm} (1.1)

where $a_0(x)$, $a_1(x)$ and $a_2(x)$ are holomorphic functions around $x = 0$. He called the following equation

$$a_2(0)\lambda^2 + a_1(0)\lambda + a_0(0) = 0$$  \hspace{1cm} (1.2)

a characteristic equation at $x = 0$ of (1.1). The roots of the characteristic equation are called characteristic exponents. He pointed out that there exist two essentially different cases:

1. $a_2(0)a_0(0) \neq 0$,
2. $a_2(0)a_0(0) = 0$.

In the first case, we can find two solutions of the form

$$u_m(x) = \frac{\theta_q(x)}{\theta_q(\lambda_m x)} \sum_{k \geq 0} u_{m,k} x^k, \quad m = 1, 2$$  \hspace{1cm} (1.3)

where $\lambda_1$ and $\lambda_2$ (provided that $\lambda_1/\lambda_2 \notin q^{\mathbb{Z}}$) are non-zero exponents. Here, $\theta_q(x) := \sum_{n \in \mathbb{Z}} q^{a(n+1)} x^n$ is the theta function of Jacobi. Adams gave a proof of convergence of power series contained in these solutions.

In the second case, either the characteristic equation is a linear equation or an exponent is zero. Therefore, some solutions of (1.1) are not in the form (1.3). After changing the variables and the base, Adams gave a formal solution of the form

$$u(x) = \frac{\theta_q(x)^{\mu+1}}{\theta_q(\lambda x)} \sum_{k \geq 0} u_k x^k$$  \hspace{1cm} (1.4)

where $\lambda$ is a suitable non-zero constant and $\mu$ is an integer. We remark that formal solutions of the form (1.4) may contain divergent series. The existence of formal solutions of the form (1.4) around $x = \infty$ is similar to the case of $x = 0$. 

We consider a connection problem on a second order linear $q$-difference equation of the form
\[
\{(a_0 + b_0 x)\sigma_q^2 + (a_1 + b_1 x)\sigma_q + (a_2 + b_2 x)\} u(x) = 0 \quad (1.5)
\]
where $\sigma_q$ is the $q$-shift operator $\sigma_q f(x) = f(qx)$. Connection problems on linear $q$-difference equations are studied by G. D. Birkhoff [2]. He studied connection problems when the first case occurs for both $x = 0$ and $x = \infty$, namely, in the case $a_0 b_0 a_2 b_2 \neq 0$.

Let $u_1(x), u_2(x)$ be independent solutions of (1.1) around $x = 0$ and let $v_1(x), v_2(x)$ be those around $x = \infty$. We take suitable analytic continuation of $u_1(x)$ and $u_2(x)$. Then the connection formula in the matrix form is given by
\[
\begin{pmatrix}
u_1(x) \\ u_2(x)
\end{pmatrix} =
\begin{pmatrix}
C_{11}(x) & C_{12}(x) \\ C_{21}(x) & C_{22}(x)
\end{pmatrix}
\begin{pmatrix}
u_1(x) \\ v_2(x)
\end{pmatrix}.
\]
Here, $C_{jk}(x)$ are $q$-constant, namely, $C_{jk}(q x) = C_{jk}(x)$.

In the case $a_0 a_2 b_0 b_2 = 0$, some power series which appear in formal solutions may be divergent. In order to study connection problems, we should take a suitable resummation of a divergent series. J.-P. Ramis and C. Zhang introduced a discrete resummation method, called the $q$-Borel-Laplace transformation of the first kind. The $q$-Borel-Laplace transformation of the first kind is given as follows.

1. We assume that $f(x) = \sum_{n \geq 0} a_n x^n$ is a formal power series. The $q$-Borel transformation of the first kind $B_q^+$ is given by
\[
(B_q^+ f) (\xi) := \sum_{n \geq 0} a_n q^{\frac{n(n-1)}{2}} \xi^n.
\]
We denote $\varphi_f (\xi) = (B_q^+ f) (\xi)$. If $f(x)$ is a convergent series, then $\varphi_f (\xi)$ is an entire function.

2. We fix $\lambda \in \mathbb{C} \setminus q \mathbb{Z}$. For any entire function $\varphi(\xi)$, the $q$-Laplace transformation of the first kind $L_{q,\lambda}^+$ [3, 16] is given by
\[
(L_{q,\lambda}^+ \varphi) (x) := \frac{1}{1 - q} \int_0^{\lambda \infty} \varphi(\xi) \frac{d_q \xi}{\theta_q \left( \frac{\xi}{x} \right)} = \sum_{n \in \mathbb{Z}} \varphi(\lambda q^n) \theta_q \left( \frac{\lambda q^n}{x} \right)
\]
where
\[
\int_0^{\lambda \infty} f(t) d_q t := (1 - q) \lambda \sum_{n \in \mathbb{Z}} f(\lambda q^n) q^n
\]
is Jackson’s $q$-integral on $(0, \lambda \infty)$ [4].
The $q$-Borel transformation of the first kind may be used for a resummation of a divergent series. We set $D_r := \{ x \in \mathbb{C} | 0 < |x| < r \}$, where $r$ is a positive number. Let $f(x)$ be a holomorphic function on $D_r \setminus [-\lambda; q]$, where 

$[-\lambda; q] := -\lambda q^Z$. The set $[\lambda; q]$ is called a $q$-spiral. We assume that $\hat{f}(x) = \sum_{m \geq 0} a_m x_m$ is a divergent series. We call $\hat{f}(x)$ is an asymptotic series of $f(x)$ for $x \to 0$ on $D_r \setminus [\lambda; q]$ when $f(x)$ satisfies the condition as follows:

**Definition 1.** For any $\epsilon > 0$, any $c > 0$ and any $n \in \mathbb{Z}_{\geq 0}$, there exists a constant $r_0 > 0$ such that, if $x \in D_{r_0} \setminus \bigcup_{k \in \mathbb{Z}} \{ x; |x - \lambda q^k| < \epsilon |q|^k \}$, we have

$$
\left| f(x) - \sum_{m=0}^{n} a_m x^m \right| < c|x|^n.
$$

Zhang [13] shows that the $q$-Borel-Laplace transform of a divergent series with a suitable condition is holomorphic on an open dense domain $\mathbb{C}^* \setminus [-\lambda; q]$, and the original divergent series gives its asymptotic series for $x \to 0$.

Zhang [15] also introduced the following $q$-Borel-Laplace transformation of the second kind. We use the $q$-Borel-Laplace transformation of the second kind to obtain an integral representation of a convergent series around $x = 0$.

1. We assume that $f(x) = \sum_{n \geq 0} a_n x^n$ is a power series such that $|a_n|$ decrease rapidly enough. The $q$-Borel transformation of the second kind $B_q^-$ is given by

$$(B_q^- f)(\xi) := \sum_{n \geq 0} a_n q \frac{n(n-1)}{2} \xi^n.$$

2. For any function $g(\xi)$ which is holomorphic around $\xi = 0$, the $q$-Laplace transformation of the second kind $L_q^-$ is given by

$$(L_q^- g)(x) := \frac{1}{2\pi i} \int_{|\xi| = r} g(\xi) \theta_q \left( \frac{x}{\xi} \right) \frac{d\xi}{\xi},$$

where $r$ is a suitable positive number.

Zhang shows that the $q$-Borel transformation $B_q^-$ is a formal inverse of the $q$-Laplace transformation $L_q^-$ [15]. More precisely, we can show the following lemma.

**Lemma 1.** Let $f(x) = \sum_{n \geq 0} a_n x^n$ be a function such that $(B_q^- f)(\xi)$ is convergent. Then $(L_q^- \circ B_q^- f)(x)$ coincides with $f(x)$ near $x = 0$. 

6
We assume that one solution $u_1(x)$ around $x = 0$ is an entire function and the other formal solution $u_2(x)$ around $x = 0$ contains a divergent series. We also assume that both of the solutions $v_1(x)$, $v_2(x)$ around $x = \infty$ extend to meromorphic functions on $\mathbb{C}^*$. Under these assumptions, connection formulae of (1.5) are given in the following matrix form:

$$
\begin{pmatrix}
  u_1(x) \\
  \tilde{u}_2(x, \lambda)
\end{pmatrix}
= 
\begin{pmatrix}
  C_{11}(x) & C_{12}(x) \\
  \tilde{C}_{21}(x, \lambda) & \tilde{C}_{22}(x, \lambda)
\end{pmatrix}
\begin{pmatrix}
  v_1(x) \\
  v_2(x)
\end{pmatrix}.
$$

Here, $\tilde{u}_2(x, \lambda)$ contains the resummation of the divergent series obtained by the $q$-Borel-Laplace transformation. The functions $C_{11}(x), C_{12}(x)$ are $q$-constants and $\tilde{C}_{21}(x, \lambda), \tilde{C}_{22}(x, \lambda)$ are also $q$-constants with poles at $x \in [-\lambda; q]$. The connection coefficients are analytic function of the parameter $\lambda$. The domain $\mathbb{C}^* \setminus [-\lambda; q]$ can be considered as a $q$-analogue of the Stokes region. The dependence on the parameter $\lambda$ of connection coefficients is called the $q$-Stokes phenomenon [10].

1.2 Connection problem on $q$-difference equation satisfied by the Ramanujan entire function

In Chapter 2, we study a connection problem on a $q$-difference equation satisfied by the Ramanujan entire function. The Ramanujan entire function with the base $q$ satisfies the following $q$-difference equation

$$
(qx \sigma_q^2 - \sigma_q + 1) u(x) = 0. 
$$

We introduce a transformation of variables

$$
x \to -q^3 t^2, \quad q \to q^2
$$

and set $f(t) = u(x)$. Then we give a connection formula between the Ramanujan entire function with the base $q^2$ and the $q$-Airy function with the base $q$. By the transformation (1.7), the equation (1.6) reduces to

$$
(-q^5 t^2 \sigma_q^2 - \sigma_q + 1)f(t) = 0.
$$

The aim of Chapter 2 is to solve a connection problem of (1.8). Equation (1.8) has formal solutions

$$
\begin{align*}
  f_1(t) &= A_q^2(-q^3 t^2) = 0 \varphi_1(-; 0; q^2, q^5 t^2), \\
  f_2(t) &= \theta_q(qt) \theta_q(-q^2 t) \varphi_0(0, 0; -; q^2, qt^2)
\end{align*}
$$

7
around \( t = 0 \). Here, the function \( r \varphi_s(a_1, \ldots, a_r; b_1, \ldots, b_s; q, t) \) is the basic hypergeometric series with the base \( q \):

\[
r \varphi_s(a_1, \ldots, a_r; b_1, \ldots, b_s; q, t) := \sum_{n \geq 0} \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s; q)_n(q; q)_n} \left\{ (-1)^n q^\frac{n(n-1)}{2} \right\} 1 + s - rt^n.
\]

The radius of convergence is \( \infty, 1 \) or 0 according to whether \( r - s < 1, r - s = 1 \) or \( r - s > 1 \).

The function \( f_1(t) \) is the Ramanujan entire function with base \( q^2 \) and the solution \( f_2(t) \) has a divergent series \( 2 \varphi_0(0, 0; -; q^2, qt^2) \). Around \( t = \infty \), equation (1.8) has solutions

\[
v_1(t) = \theta_q(q^2t) \varphi_1 \left( 0; -q; q^{\frac{1}{t}} \right) = \theta_q(q^2t) \operatorname{Ai}_q \left( -\frac{1}{t} \right), \quad (1.11)
\]

\[
v_2(t) = \theta_q(-q^2t) \varphi_1 \left( 0; -q; -q^{\frac{1}{t}} \right) = \theta_q(-q^2t) \operatorname{Ai}_q \left( \frac{1}{t} \right).
\]

Both \( v_1(t) \) and \( v_2(t) \) are represented by the \( q \)-Airy function, which is an entire function. In subsection 2.2, we apply the \( q \)-Borel-Laplace transformation of the second kind to the power series (1.9). Then we obtain the following connection formula for \( f_1(t) \) and deduce a relationship between the Ramanujan entire function and the \( q \)-Airy function.

**Theorem.** For any \( t \in \mathbb{C}^* \), we have

\[
f_1(t) = \left( \frac{1}{(q, q^2; q_\infty)} - v_1(t) \right) + \left( \frac{1}{(q, q; q_\infty)} - v_2(t) \right)
\]

where \( (a; q)_\infty := \lim_{n \to \infty} (a; q)_n \).

In subsection 2.3, we give a connection formula of a resummation of the formal solution \( f_2(t) \). Namely, since the solution \( f_2(t) \) contains a divergent series \( 2 \varphi_0(0, 0; -; q^2, qt^2) \), we construct a true solution by applying the \( q \)-Borel-Laplace transformation of the first kind to the divergent series. Then we obtain a true solution which is defined on \( \mathbb{C}^* \setminus [-\lambda; q] \). We denote

\[
2 f_0 (0, 0; -q, \lambda, -x/q) := \mathcal{L}_{q, \lambda}^+ \circ \mathcal{B}_q^+ 2 \varphi_0(0, 0; -q, -x/q).
\]

We remark that the \( q \)-Borel-Laplace transform (of the first kind) of a divergent series \( 2 \varphi_0(0, 0; -q, -x/q) \) is defined on the set \( \mathbb{C}^* \setminus [-\lambda; q] \) and the original divergent series \( 2 \varphi_0(0, 0; -q, -x/q) \) gives its asymptotic series for
$x \to 0$. We set $\tilde{f}_2(t; \lambda) = \theta_q(qt)\theta_q(-q^2t)2f_0(0, 0; -: q^2, \lambda, qt^2)$, then we have

the following connection formula between $\tilde{f}_2(t; \lambda)$ and $v_1(t), v_2(t)$.

**Theorem.** For any $t \in \mathbb{C}^*$ such that $t^2 \in \mathbb{C}^* \setminus [\lambda/q; q^2]$, we have

$$
\theta_q(qt)\theta_q(-q^2t)2f_0(0, 0; -: q^2, \lambda, qt^2) = \frac{(q^2; q^2)_\infty \theta_q(qt)\theta_q(-q^2t)}{\theta_q^2 \left( -\frac{\lambda}{q^2} \right) \theta_q^2 \left( -\frac{\lambda}{q^{3/2}} \right)} 2\theta_q(q^2t)v_1(t) \\
+ \frac{(q^2; q^2)_\infty \theta_q(qt)\theta_q(-q^2t)}{\theta_q^2 \left( -\frac{\lambda}{q^2} \right) \theta_q^2 \left( -\frac{\lambda}{q^{3/2}} \right)} 2\theta_q(-q^2t)v_2(t).
$$

Zhang and Ismail [6] studied an asymptotic behavior of the Ramanujan entire function and obtained an asymptotic formula as follows:

$$
A_q(x) = \frac{\theta_{q^2}(-\frac{q}{x})}{(q^2, q; q^2)_\infty} \Phi_1 \left( 0; q^2, \frac{q^2}{x} \right) \\
+ \frac{\theta_{q^2}(-\frac{1}{x})}{(q^2, q; q^2)_\infty} \frac{q}{q-1} \Phi_1 \left( 0; q^3, \frac{q^3}{x} \right).
$$

We can derive an asymptotic expansion which is essentially equivalent to (1.13) in Proposition 4 as a corollary to our connection formula.
1.3 The theta function and the $q$-exponential functions

The theta function of Jacobi with the base $q$

$$
\theta_q(x) = \sum_{n \in \mathbb{Z}} q^{\frac{n(n-1)}{2}} x^n, \quad \forall x \in \mathbb{C}^*
$$

is holomorphic on $\mathbb{C}^*$. Our theta function is not a standard one but we follow the notation in [10]. The theta function has the following properties:

1. Jacobi’s triple product identity

$$
\theta_q(x) = (q, -x, -q/x; q)_\infty .
$$

2. The $q$-difference equation

$$
\theta_q(q^k x) = q^{-\frac{k(k-1)}{2}} x^{-k} \theta_q(x), \quad \forall k \in \mathbb{Z}.
$$

3. The inversion formula

$$
\theta_q(1/x) = \theta_q(x)/x.
$$

For any $\lambda \in \mathbb{C}^*$, the function $\theta(-\lambda x)/\theta(\lambda x)$ satisfies a $q$-difference equation $u(qx) = -u(x)$, which is also satisfied by the function $u(x) = e^{\pi i (\log x / \log q)}$.

Two different $q$-exponential functions are given by

$$
e_q(x) := 1 \varphi_0(0; -; q, x) = \sum_{n \geq 0} \frac{x^n}{(q; q)_n} = \frac{1}{(x; q)_\infty},
$$

$$
E_q(x) := 0 \varphi_0(-; -; q, -x) = \sum_{n \geq 0} q^{-\frac{n(n-1)}{2}} x^n = (-x; q)_\infty.
$$

These two $q$-exponential functions are related to each other as follows:

$$
e_q(x)E_q(-x) = 1, \quad e_{q^{-1}}(x) = E_q(-qx).
$$
Chapter 2

Connection formulae of the Ramanujan entire function and the resummation of the divergent series

\[ 2\varphi_0(0, 0; -; q, x) \]

In this chapter, we study a relation between the aforementioned two \( q \)-analogues of the Airy function from the viewpoint of connection problems on \( q \)-difference equations.

2.1 Relation between the Ramanujan entire function and the \( q \)-Airy function

To find a relation between the Ramanujan entire function and the \( q \)-Airy function, we compare \( q \)-difference equations satisfied by each of the functions. The Ramanujan entire function satisfies the following \( q \)-difference equation

\[ (qXX\sigma_q^2 - \sigma_q + 1)u(x) = 0. \]  \hspace{1cm} (2.1)

The \( q \)-Airy function satisfies the second order linear \( q \)-difference equation

\[ (\sigma_q^2 + x\sigma_q - 1)u(x) = 0. \] \hspace{1cm} (2.2)
We can transform the $q$-difference equation satisfied by the Ramanujan entire function with base $q^2$
\[
\left(q^2x\sigma_q^2 - \sigma_x^2 + 1\right) u(x) = 0 \tag{2.3}
\]
into a $q$-difference equation
\[
(-q^5t^2\sigma_q^2 - \sigma_q + 1) f(t) = 0 \tag{2.4}
\]
by changing the independent variable $x$ to $t$ by $-q^3t^2 = x$. We can also transform (2.4) into (2.2) by changing the independent variable $t \to 1/t$ and a suitable change of a depending variable as we will show in the proof of the next proposition. In this way, we will find a relation between the Ramanujan entire function and the $q$-Airy function.

The first task is to give local formal solutions of (2.4) around the origin and the infinity.

**Proposition 1.** The equation (2.4) has solutions
\[
\begin{align*}
    f_1(t) &= \theta_1(qt)\theta_1(-q^2t) = A_{q^2}(-q^3t^2), \tag{2.5} \\
    f_2(t) &= \theta_1(qt)\theta_1(-q^2t)\varphi_0(0, 0; -q^2, qt^2). \tag{2.6}
\end{align*}
\]
around the origin. The first solution $f_1(t)$ is a convergent series and represented by the Ramanujan entire function with the base $q^2$. The second solution $f_2(t)$ contains a divergent series.

The equation (2.4) has solutions
\[
\begin{align*}
    v_1(t) &= \theta_1(q^2t)\varphi_1(0; -q; q, -1/t) = \theta_1(q^2t)\Lambda_{q^2}\left(-\frac{1}{t}\right) \tag{2.7} \\
    v_2(t) &= \theta_1(-q^2t)\varphi_1(0; -q; q, -1/t) = \theta_1(-q^2t)\Lambda_{q^2}\left(\frac{1}{t}\right) \tag{2.8}
\end{align*}
\]
avoid around the infinity. Both solutions contain convergent series and represented by the $q$-Airy function with the base $q$.

**Proof.** Equation (2.4) has a unique holomorphic solution at $t = 0$:
\[
f_1(t) = A_{q^2}(-q^3t^2).
\]
We define $\tilde{f}(t)$ by $f(t) = \theta_1(qt)\theta_1(-q^2t)\tilde{f}(t)$. Since the theta function satisfies the relation
\[
\sigma_q\left\{\theta_1(qt)\theta_1(-q^2t)\right\} = -\frac{1}{q^3t^2}\theta_1(qt)\theta_1(-q^2t),
\]
we can further transform (2.4) to
\[
(\sigma_q - \beta(t)) \tilde{f}(t) = 0
\]
where $\beta(t) = q^3t^2\theta_1(qt)\theta_1(-q^2t)$. The function $\theta_1$ is defined by
\[
\theta_1(z) = \sum_{n=0}^{\infty} \frac{z^n}{q^n},
\]
and we have
\[
\theta_1(q^2t)\theta_1(-q^2t) = q^3t^2\theta_1(qt)\theta_1(-q^2t).
\]
In this way, we will find a relation between the Ramanujan entire function and the $q$-Airy function.
the function \( \tilde{f}(t) \) satisfies the equation
\[
\left( \sigma_q^2 - \sigma_q - q^3t^2 \right) \tilde{f}(t) = 0.
\] (2.9)

The equation (2.9) has a unique power series solution of the form \( \tilde{f}(t) = \sum_{n \geq 0} b_n t^n \) where \( b_0 = 1 \), which is
\[
\tilde{f}(t) = 2\varphi_0(0, 0; -; q^2, qt^2) = \sum_{n \geq 0} \frac{1}{(q^2; q^2)_n} \left\{ (q^2)^{n+1} \right\}^{-1} (qt^2)^n.
\]

Therefore, the equation (2.4) has another (formal) solution as follows:
\[
f_2(t) = \theta_q(qt)\theta_q(-q^2t)\varphi_0(0, 0; -; q^2, qt^2).
\]

We consider local solutions of (2.4) around \( t = \infty \). We set
\[
z(t) = \frac{1}{\theta_q(-q^2t)} f(t).
\]

Then \( z(t) \) satisfies
\[
\left( -\sigma_q^2 + \frac{1}{q^2t}\sigma_q + 1 \right) z(t) = 0.
\]

We set \( x = 1/t \) and \( u(x) = z(1/t) \). Then \( u(x) \) satisfies the \( q \)-Airy equation
\[
\left( \sigma_q^2 + x\sigma_q - 1 \right) u(x) = 0.
\]

We take local solutions of the \( q \)-Airy equation around \( x = 0 \) as follows:
\[
u_1(x) = -\frac{\theta_q(x/q^2)}{\theta_q(-x/q^2)} \text{Ai}_q(-x), \quad u_2(x) = \text{Ai}_q(x).
\]

Therefore (2.4) has solutions around \( x = \infty \) as follows:
\[
v_1(t) = \theta_q(q^2t) \text{Ai}_q \left( -\frac{1}{t} \right) = \theta_q(q^2t)\varphi_1 \left( 0; -q; q, \frac{1}{t} \right), \quad \text{(2.10)}
\]
\[
v_2(t) = \theta_q(-q^2t) \text{Ai}_q \left( \frac{1}{t} \right) = \theta_q(-q^2t)\varphi_1 \left( 0; -q; q, -\frac{1}{t} \right). \quad \text{(2.11)}
\]

The Ramanujan entire function is a convergent solution of the equation (2.4) around \( t = 0 \). The solution \( f_1(t) \) is a unique holomorphic solution at \( t = 0 \) with the condition \( f_1(0) = 1 \).

The \( q \)-Airy functions are solutions of (2.4) around \( t = \infty \). We show a connection formula between the Ramanujan entire function (2.5) and the \( q \)-Airy functions (2.7), (2.8) in the next subsection 2.2.

\[ \square \]
2.2 Connection formula of the Ramanujan entire function

In the following, we give a formula which relates the Ramanujan entire function (with the base $q^2$) with the $q$-Airy function as a corollary to our connection formula between $f_1(t)$ and $v_1(t)$, $v_2(t)$.

We set $g(\tau) = (B_q f_1)(\tau)$. We see that $g(\tau) = \sum_{n \geq 0} g_n \tau^n$, $g_0 = 1$ is a convergent series in our case. To begin with, we note that the $q$-Borel transformation has the following operational relation:

**Lemma 2.** For any $l, m \in \mathbb{Z}_{\geq 0}$, we have

$$B_q^{-1}( \tau^m \sigma_q^l ) = q^{-\frac{m(m-1)}{2}} \tau^m \sigma_q^{l-m} B_q^{-1}.$$ 

The operational relation for the $q$-Borel transformation is given by Zhang [15]. We apply the $q$-Borel transformation $B_q^{-1}$ to the function $f_1(t)$.

1. The $q$-difference equation satisfied by $(B_q^{-1} f_1)(\tau)$.

By the operational relation in Lemma 2, the function $g(\tau)$ satisfies a first order $q$-difference equation

$$g(q\tau) = (1 + q^2 \tau)(1 - q^2 \tau)g(\tau). \tag{2.12}$$

The $q$-difference equation (2.12) has a solution

$$h(\tau) = \frac{1}{(-q^2 \tau; q)_\infty (q^2 \tau; q)_\infty}.$$ 

We consider the Maclaurin series of the function $h(\tau)$, which is a formal power series solution of (2.12). Since there exists a unique power series solution $g(\tau)$ with a condition $g(0) = 1$, the Maclaurin series of $h(\tau)$ coincides with the solution $g(\tau)$. Then we obtain the following infinite product representation of $g(\tau)$:

$$g(\tau) = \frac{1}{(-q^2 \tau; q)_\infty (q^2 \tau; q)_\infty}.$$ 

The function $g(\tau)$ has simple poles at

$$\left\{ \tau; \tau = \pm q^{-2k}, \forall k \in \mathbb{Z}_{\geq 0} \right\}.$$ 

In particular, the radius of convergence of $g(\tau)$ is $|q|^{-2}$. 

14
2. Residue calculus for $(L_q^- \circ B_q^- f_1)(t)$.

We fix a number $r > 0$ such that $|q|^{-1} < r < |q|^{-2}$ and define a set $C_l = \{ \tau \in \mathbb{C} ||\tau| = r|q|^{-l} \}$, for $l \in \mathbb{Z}_{\geq 0}$. Applying the $q$-Laplace transformation $L_q^-$ to $g(\tau)$, we obtain an integral representation of $f_1(t)$ by Lemma 1.

\[
f_1(t) = \frac{1}{2\pi i} \int_{C_0} g(\tau) \theta_q \left( \frac{t}{\tau} \right) \frac{d\tau}{\tau} = \frac{1}{2\pi i} \int_{|\tau|=r|q|^{-l}} g(\tau) \theta_q \left( \frac{t}{\tau} \right) \frac{d\tau}{\tau} \]

\[= - \sum_{k \geq 0} \text{Res} \left\{ g(\tau) \theta_q \left( \frac{t}{\tau} \right) \frac{1}{\tau}; \tau = -q^{-2-k} \right\} - \sum_{k \geq 0} \text{Res} \left\{ g(\tau) \theta_q \left( \frac{t}{\tau} \right) \frac{1}{\tau}; \tau = q^{-2-k} \right\}.
\]  

(2.13)

At first, we show the integral on $|\tau| = r|q|^{-l}$ goes to zero when $l \to \infty$.

**Proposition 2.** For any $t \in \mathbb{C}^*$, we have

\[
\lim_{l \to \infty} \left| \frac{1}{2\pi i} \int_{|\tau|=r|q|^{-l}} g(\tau) \theta_q \left( \frac{t}{\tau} \right) \frac{d\tau}{\tau} \right| = 0.
\]

For the proof, we prepare two lemmas on the $q$-shifted factorials.

**Lemma 3.** For any $l \in \mathbb{Z}_{> 0}$, we have

\[
(\tau q^{2l}; q)_l = \tau^l (-1)^l q^{\frac{(l-1)}{2}} \left( \frac{1}{\tau q}; q \right)_l.
\]

The lemma above can be proved directly.

**Lemma 4.** For any $\tau \in C_0$, we have

\[
\left| \left( \frac{1}{\tau q}; q \right)_l \right| \leq \left( \frac{1}{r|q|}; |q| \right)_l < \left( \frac{1}{r|q|}; |q| \right)_\infty.
\]
Proof. Since \(|q|^{-1} < r < |q|^{-2}\), we have \(|\tau q^j| > 1\) \((j = 1, 0, -1, \cdots)\), where \(\tau \in C_0\). We remark that

\[
\left| 1 - \frac{1}{\tau q^j} \right| \geq 1 - \frac{1}{r|q|^j}, \quad j = 1, 0, -1, \cdots.
\]

Therefore we have an estimation

\[
\left| \left( \frac{1}{\tau q^j} ; q \right)_l \right| \geq \left( \frac{1}{r|q|} ; |q| \right)_l > \left( \frac{1}{r|q|} ; |q| \right)_\infty.
\]

Thus we obtain

\[
\frac{1}{\left( \frac{1}{\tau q^j} ; q \right)_l} \leq \frac{1}{\left( \frac{1}{r|q|} ; |q| \right)_l} < \frac{1}{\left( \frac{1}{r|q|} ; |q| \right)_\infty}.
\]

We are ready to prove Proposition 2.

Proof. We put \(M := 1/\left( \frac{1}{r|q|} ; |q| \right)_\infty\) and \(N := \max_{\tau \in C_0} |g(\tau)\theta_q(t/\tau)|\).

\[
\frac{1}{2\pi i} \int_{|\tau|=r|q|^{-l}} g(\tau)\theta_q \left( \frac{t}{\tau} \right) \frac{d\tau}{\tau} = \frac{1}{2\pi i} \int_{C_0} g(\tau q^{-l})\theta_q \left( \frac{tq^l}{\tau} \right) \frac{d\tau}{\tau} = \frac{1}{2\pi i} \int_{C_0} g(\tau) \frac{q^{l(l-1)/2} \theta_q \left( \frac{t}{\tau} \right)}{(r\tau)^{2l-2} - |q^{2l-2}q^{-l-1} ; q|} \frac{d\tau}{\tau} = \frac{1}{2\pi i} \int_{C_0} g(\tau) \theta_q \left( \frac{t}{\tau} \right) \frac{d\tau}{\tau}.
\]

Therefore we have

\[
\left| \frac{1}{2\pi i} \int_{|\tau|=r|q|^{-l}} g(\tau)\theta_q \left( \frac{t}{\tau} \right) \frac{d\tau}{\tau} \right| \leq \frac{1}{2\pi} |q|^{(l-5)/2} |t|^{l-1} \int_{C_0} \frac{1}{\left( \frac{1}{r|q|} ; |q| \right)_\infty} \frac{d\tau}{\tau} = M^2 N |q|^{(l-5)/2} |t|^{l-1} r^{-l} \to 0 \quad (l \to \infty)
\]

as desired. \(\square\)
Hence \( f_1(t) \) has the expression

\[
f_1(t) = - \sum_{k \geq 0} \operatorname{Res} \left\{ \frac{g(\tau)}{(q/\tau)^{k+1}} \frac{1}{\tau} : \tau = -q^{-2-k} \right\} + \sum_{k \geq 0} \operatorname{Res} \left\{ \frac{g(\tau)}{(q/\tau)^{k+1}} \frac{1}{\tau} : \tau = q^{-2-k} \right\}.
\]

We calculate the residues by the following lemma.

**Lemma 5.** For any \( k \in \mathbb{Z}_{>0}, \lambda \in \mathbb{C}^* \), we have:

(a) \[ \operatorname{Res} \left\{ \frac{1}{(\tau/\lambda; q)_\infty} \frac{1}{\tau} : \tau = \lambda q^{-k} \right\} = (-1)^{k+1} q^{\frac{k(k+1)}{2}} (q; q)_k (q; q)_\infty, \]

(b) \[ \frac{1}{(\lambda q^{-k}; q)_\infty} = \frac{(\lambda^{-1} - k^{(k+1)})}{(\lambda; q)_\infty (q/\lambda; q)_k}, \quad \lambda \neq q^\mathbb{Z}. \]

By Lemma 5, we have

\[
\operatorname{Res}_{\tau=q^{-2-k}} g(\tau) \theta_q \left( \frac{t}{\tau} \right) \frac{1}{\tau} = \frac{\theta_q(q^{-2+k} t) (-1)^{k+1} q^{\frac{k(k+1)}{2}}}{(-q^{-k}; q)_\infty (q; q)_k (q; q)_\infty}.
\]

\[
= \frac{q^{\frac{k(k+1)}{2}}}{(-1; q)_\infty (-q; q)_k (q^2 t)^{-k} q^{-\frac{k(k-1)}{2}}} \theta_q(q^2 t) \frac{(-1)^{k+1} q^{\frac{k(k+1)}{2}}}{(q; q)_k (q; q)_\infty} \theta_q(-q^2 t) \frac{1}{(q, -1; q)_\infty (-q; q)_k (q^{-2+k} t) \frac{1}{\tau}}.
\]

In the same way, we calculate other residues.

By taking the sum of all the residues, we obtain

\[
f_1(t) = \frac{\theta_q(q^2 t)}{(q, -1; q)_\infty} \varphi_1 \left( 0, -q; q, \frac{1}{t} \right) + \frac{\theta_q(-q^2 t)}{(q, -1; q)_\infty} \varphi_1 \left( 0, -q; q, -\frac{1}{t} \right).
\]

Therefore, we have reached the following connection formula.

**Theorem 2.** For any \( x \in \mathbb{C}^* \), we have a connection formula

\[
f_1(t) = \frac{1}{(q, -1; q)_\infty} v_1(t) + \frac{1}{(q, -1; q)_\infty} v_2(t).
\]
This connection formula derives a relation between the Ramanujan entire function with base $q^2$ and the $q$-Airy functions with base $q$, namely,

$$
\Lambda_{q^2}(-q^3t^2) = \frac{1}{\vartheta_q(q^2 t)} \theta_q(q^2 t) \text{Ai}_q \left( -\frac{1}{t} \right)
$$

$$
+ \frac{1}{\vartheta_q(-q^2 t)} \theta_q(-q^2 t) \text{Ai}_q \left( \frac{1}{t} \right).
$$

### 2.3 Resummation of $2\varphi_0(0, 0; -; q, -x/q)$

We recall that the formal solution $f_2(t)$ contains a divergent basic hypergeometric series. We construct a solution $\tilde{f}_2(t, \lambda)$ by the $q$-Borel-Laplace transformation of the first kind, which is holomorphic on $\mathbb{C}^* \setminus [-\lambda; q]$. Then we can consider a connection formula between the solution $\tilde{f}_2(t, \lambda)$ and $v_1(t)$, $v_2(t)$, which contains a continuous parameter $\lambda$. In subsection 2.3.1, we show a relation between the $q$-exponential function $e_q(x)$ and convergent series $0\varphi_1$ with the base $q^2$. We also show a connection formula for a resummation of the divergent series $2\varphi_0(0, 0; -; q, -x/q)$ by applying Proposition 3 in subsection 2.3.2. Then we obtain a connection formula between the resummation $\tilde{f}_2(t, \lambda)$ and $v_1(t)$, $v_2(t)$ in Theorem 4.

#### 2.3.1 Alternative representation of $q$-exponential functions

We show an alternative representation of $q$-exponential functions, which plays an important role later on.

By the triple product identity, we have the following relation between two $q$-exponential functions

$$
e_q(x) = \frac{(q; q)_\infty}{\vartheta_q(-x)} E_q \left( \frac{-q}{x} \right)
$$

on $\mathbb{C}^* \setminus [1; q]$. The aim of this subsection is to show a behavior of $e_q(x/q)$ around $x = \infty$ given in the following proposition.

**Proposition 3.** For any $x \in \mathbb{C}^* \setminus [1; q]$, the function $e_q(x/q)$ has the following representation

$$
e_q \left( \frac{x}{q} \right) = \frac{(q; q)_\infty}{\vartheta_q \left( \frac{-x}{q} \right)} 0\varphi_1 \left( -: q; q^2, q^5 \frac{x}{x^2} \right) - \frac{(q; q)_\infty}{\vartheta_q \left( \frac{-x}{q} \right)} \frac{q^2}{(1-q)x} 0\varphi_1 \left( -: q^3, q^2, q^7 \frac{x}{x^2} \right).
$$
We obtain Proposition 3 from the following lemma.

Lemma 6. We have

\[
E_q(-x) = \varphi_1(-q; q^2, qx^2) - \frac{q^2}{(1-q)x} \varphi_1(-q^3; q^2, q^3x^2).
\] (2.15)

Proof. We separate the function \(0\varphi_0(-; -q, x)\) by even terms and odd terms.

\[
0\varphi_0(-; -q, x) = \sum_{k \geq 0} \frac{1}{(q; q)_k} (-1)^k \frac{q^{k-1}}{2} x^k
\]

\[
= \sum_{k \geq 0} \frac{(-1)^{2k} q^{2k(2k-1)/2}}{(q; q)_{2k}} x^{2k} + \sum_{k \geq 0} \frac{(-1)^{2k+1} q^{(2k+1)/2}}{(q; q)_{2k+1}} x^{2k+1}
\]

\[
= \sum_{k \geq 0} \frac{(-1)^{2k} q^{2k(2k-1)/2}}{(q; q)_{2k}} x^{2k} - \frac{x}{1-q} \sum_{k \geq 0} \frac{(-1)^{2k} q^{(2k+1)/2}}{(q^2; q)_{2k}} x^{2k}
\]

\[
= \sum_{k \geq 0} \frac{(-1)^k (q^2)^{k(k-1)/2}}{(q, q^2; q^2)_k} x^{2k} - \frac{x}{1-q} \sum_{k \geq 0} \frac{(-1)^k (q^2)^{k(k-1)/2}}{(q^2, q^2; q^2)_k} x^{2k}
\]

\[
= 0\varphi_1(-; q^2, qx^2) - \frac{q^2}{(1-q)x} 0\varphi_1(-; q^3, q^2, q^3 x^2).
\]

Since

\[
E_q(x) = 0\varphi_0(-; -q, x), \quad |x| < 1,
\]

we have the result. \(\square\)

Then Lemma 6 implies

\[
E_q\left(-\frac{q^2}{x}\right) = 0\varphi_1\left(-; q^2, \frac{q^5}{x^2}\right) - \frac{q^2}{(1-q)x} 0\varphi_1\left(-; q^3, q^2, \frac{q^7}{x^2}\right).
\]

Thus we have shown Proposition 3.

2.3.2 Connection formula for the resummation of \(2\varphi_0\)

Since \(f_2(t)\) contains a divergent series \(2\varphi_0(0, 0; -q, -x/q)\), we need to give a resummation of the divergent series by the \(q\)-Borel-Laplace transformation of the first kind. We set

\[
2f_0(0, 0; -q, \lambda, -x/q) := \mathcal{L}^{+}_{q, \lambda} \circ \mathcal{B}^{+}_{q} 2\varphi_0(0, 0; -q, -x/q).
\]
Theorem 3. For any \( x \in \mathbb{C}^* \setminus [-\lambda; q] \), we have

\[
2f_0 \left( 0, 0; -; q, \lambda, -\frac{x}{q} \right) = (q; q)_{\infty} \frac{\theta_{q^2} \left( -\frac{\lambda^2}{q^2} \right)}{\theta_{q} \left( -\frac{\lambda}{q} \right) \theta_{q} \left( \frac{\lambda}{q} \right)} \varphi_1 \left( 0; q^2; \frac{q^2}{x} \right)
+ \frac{(q; q)_{\infty}}{1-q} \frac{\theta_{q^2} \left( -\frac{\lambda^2}{x} \right)}{\theta_{q} \left( -\frac{\lambda}{x} \right) \theta_{q} \left( \frac{\lambda}{x} \right)} \frac{\lambda}{x} \varphi_1 \left( 0; q^2; \frac{q^2}{x} \right).
\]

Proof. We apply the \( q \)-Borel transformation \( \mathcal{B}_q^+ \) to the divergent series \( v(x) = 2\varphi_0(0, 0; -; q, -x/q) \). Let \( \varphi(\xi) = e_q(\xi/q) \). Then \((\mathcal{B}_q^+ v)(\xi)\) is the Maclaurin expansion of \( \varphi(\xi) \). The function \( \varphi(\xi) \) is holomorphic around \( \xi = 0 \). By Proposition 3,

\[
\varphi(\xi) = \frac{(q; q)_{\infty}}{\theta_{q} \left( -\frac{\xi}{q} \right)} \varphi_1 \left( -; q^2; \frac{q^5}{\xi^2} \right) - \frac{(q; q)_{\infty}}{\theta_{q} \left( -\frac{\xi}{q} \right) (1-q)\xi} \varphi_1 \left( -; q^2; \frac{q^5}{\xi^2} \right).
\]

We set

\[
\varphi_1(\xi) = \frac{1}{\theta_{q} \left( -\frac{\xi}{q} \right)} \varphi_1 \left( -; q^2; \frac{q^5}{\xi^2} \right),
\]

\[
\varphi_2(\xi) = \frac{1}{\theta_{q} \left( -\frac{\xi}{q} \right)} \varphi_1 \left( -; q^2; \frac{q^5}{\xi^2} \right).
\]

Both \( \varphi_1(\xi) \) and \( \varphi_2(\xi) \) are meromorphic functions on \( \mathbb{C}^* \), which leads to the convergence of \( \mathcal{L}_{q,\lambda}^+ \varphi \) as we shall show now.

We apply the \( q \)-Laplace transformation \( \mathcal{L}_{q,\lambda}^+ \) to the functions \( \varphi_1(\xi) \) and \( \varphi_2(\xi) \).
\begin{align*}
(L_{q,\lambda}^+ \varphi_1)(x) &= \sum_{n \in \mathbb{Z}} \frac{\varphi_1(\lambda q^n)}{\theta_q \left( \frac{\lambda q^n}{x} \right)} = \sum_{n \in \mathbb{Z}} \frac{1}{\theta_q \left( \frac{\lambda q^n}{x} \right)} \theta_q \left( \frac{-\lambda q^n}{q} \right) \varphi_1 \left( -q; q^2, \frac{q^5}{\lambda^2 q^{2n}} \right) \\
&= \frac{1}{\theta_q \left( -\frac{1}{q} \right)} \theta_q \left( \frac{x}{q} \right) \sum_{n \in \mathbb{Z}} q^n q^{(n-1)} \left( \frac{\lambda^2}{qx} \right)^n \sum_{m \geq 0} (q; q^2; q^2)_m \left( \frac{q^{5-2n}}{\lambda^2} \right)^m \\
&= \frac{1}{\theta_q \left( -\frac{1}{q} \right)} \theta_q \left( \frac{x}{q} \right) \sum_{m \in \mathbb{Z}} q^{(n-m)(n-m-1)} \left( \frac{-\lambda^2}{qx} \right)^{n-m} \\
&\quad \times \sum_{m \geq 0} (-1)^m (q^2)^{m(m-1)} (q; q^2; q^2)_m \left( \frac{q^2}{x} \right)^m \\
&= \frac{\theta_q^2 \left( -\frac{\lambda^2}{qx} \right)}{\theta_q \left( -\frac{1}{q} \right)} \theta_q \left( \frac{x}{q} \right) \varphi_1 \left( -q; q^2, \frac{q^2}{x} \right).
\end{align*}

In the same way, we obtain
\begin{align*}
(L_{q,\lambda}^+ \varphi_2)(x) &= -\frac{\theta_q^2 \left( -\frac{\lambda^2}{x} \right)}{\theta_q \left( -\frac{1}{q} \right)} \theta_q \left( \frac{\lambda q}{x} \right) x q^2 \varphi_1 \left( 0; q^3, q^2, \frac{q^3}{x} \right).
\end{align*}

Therefore we obtain
\begin{align*}
2f_0 \left( 0, 0; -q, q, -\frac{x}{q} \right) &= L_{q,\lambda}^+ \circ B_q^+ 2 \varphi_0 \left( 0, 0; -q, -\frac{x}{q} \right) \\
&= (q; q)_\infty \theta_q \left( -\frac{\lambda^2}{qx} \right) \varphi_1 \left( 0; q^2, \frac{q^2}{x} \right) \\
&\quad + (q; q)_\infty \theta_q \left( -\frac{\lambda^2}{x} \right) \varphi_1 \left( 0; q^3, q^2, \frac{q^3}{x} \right).
\end{align*}

A resummation of the formal solution $f_2(t)$ in (2.6) is given by $	ilde{f}_2(t, \lambda) = \theta_q(qt) \theta_q(-q^2 t) 2f_0(0, 0; -q^2, \lambda, qt^2)$. We obtain the connection formula between $f_2(t, \lambda)$ and $v_1(t), v_2(t)$ from Theorem 3.
Theorem 4. For any $t \in \mathbb{C}^*$ such that $t^2 \in \mathbb{C}^* \setminus [\lambda/q; q^2]$, we have

$$\theta_q(qt)\theta_q(-q^2t)\theta_0(0, 0; -q^2, \lambda, qt^2) = \frac{(q^2; q^2)_\infty \theta_q(qt)\theta_q(-q^2t)}{\theta_q(-\frac{\lambda}{q^2})\theta_q(-\frac{\lambda}{q^2t})} \frac{\theta_q\left(\frac{\lambda}{q^2t}\right)}{2\theta_q(q^2t)} v_1(t)$$

$$+ \frac{(q^2; q^2)_\infty \theta_q(qt)\theta_q(-q^2t)}{\theta_q(-\frac{\lambda}{q^2})\theta_q(-\frac{\lambda}{q^2t})} \frac{\theta_q\left(\frac{\lambda}{q^2t}\right)}{2\theta_q(q^2t)} v_2(t)$$

where $v_1(t)$, $v_2(t)$ are the local solutions at the infinity:

$$v_1(t) = \theta_q(q^2t)\varphi_1\left(0; -q; q_1 \frac{1}{t}\right) = \theta_q(q^2t) Ai_q\left(\frac{1}{t}\right),$$

$$v_2(t) = \theta_q(-q^2t)\varphi_1\left(0; -q; q_1 - \frac{1}{t}\right) = \theta_q(-q^2t) Ai_q\left(\frac{1}{t}\right).$$

We prepare some relations between some functions with different bases before proving Theorem 4. It is easily checked that

$$(a; q)_{2m} = (a; q^2)_m(aq; q^2)_m, \quad (a; q)_{2m+1} = (1 - a)(aq; q^2)_m(aq^2; q^2)_m. \quad (2.16)$$

In (2.16), we set $a \mapsto q^2$ and $q \mapsto q^2$. Then we have

$$(q^2; q^2)_{2m} = (q^2; q^4)_m(q^4; q^4)_m, \quad (2.17)$$

$$(q^2; q^2)_{2m+1} = (1 - q^2)(q^4; q^4)_m(q^6; q^4)_m. \quad (2.18)$$

For any $m \in \mathbb{Z}_{\geq 0}$, we have

$$q^{\frac{2m(2m-1)}{2}} = (q^4)^{\frac{m(m-1)}{2}} \times q^{m}, \quad q^{\frac{(2m+1)2m}{2}} = (q^4)^{\frac{m(m-1)}{2}} \times q^{3m}. \quad (2.19)$$

By the formulae (2.16), (2.17), (2.18) and (2.19), we obtain the following lemma.

Lemma 7. We have the following relations between functions $\varphi_1$ with different bases $q$ and $q^4$:

$$\varphi_1(0; -q; q, x) = \varphi_1(0; q^2; q^4, -qx^2) - \frac{x}{1 - q^2} \varphi_1(0; q^6; q^4; -q^3 x^2),$$

$$\varphi_1(0; -q; q, -x) = \varphi_1(0; q^2; q^4, -qx^2) + \frac{x}{1 - q^2} \varphi_1(0; q^6; q^4; -q^3 x^2).$$
The theta functions with the different bases have the following relations.

**Lemma 8.** For any \(x \in \mathbb{C}^*\), we have
\[
\theta_q \left( \frac{x}{q} \right) + \theta_q \left( -\frac{x}{q} \right) = 2\theta_{q^4} \left( \frac{x^2}{q} \right),
\]
\[
\theta_q \left( \frac{x}{q} \right) - \theta_q \left( -\frac{x}{q} \right) = 2\frac{x}{q}\theta_{q^4} \left( qx^2 \right).
\]

**Corollary 1.** If we put \(x \mapsto \lambda/(q^2 t)\), we have
\[
\theta_q \left( \frac{\lambda}{q^3 t} \right) + \theta_q \left( -\frac{\lambda}{q^3 t} \right) = 2\theta_{q^4} \left( \frac{\lambda^2}{q^5 t^2} \right),
\]
\[
\theta_q \left( \frac{\lambda}{q^3 t} \right) - \theta_q \left( -\frac{\lambda}{q^3 t} \right) = 2\frac{\lambda}{q^4 t^2}\theta_{q^4} \left( \frac{\lambda^2}{q^3 t^2} \right).
\]

We give a proof of Theorem 4.

**Proof.** In Theorem 3, we put \(q \mapsto q^2\) and \(x \mapsto -q^3 t^2\). Then,
\[
2f_0(0, 0; -; q^2, \lambda, q t^2) = \frac{(q^2; q^2)_\infty}{\theta_q^2 \left( -\frac{\lambda}{q^2} \right) \theta_{q^2} \left( -\frac{\lambda}{q^4 t^2} \right)} \left\{ \theta_{q^4} \left( \frac{\lambda^2}{q^5 t^2} \right) \varphi_1 \left( 0; q^2; q^4, -\frac{q}{t^2} \right)
\right.
\]
\[
+ \theta_{q^4} \left( \frac{\lambda^2}{q^3 t^2} \right) \frac{-\lambda}{(1-q^2)q^2 t^2} \varphi_1 \left( 0; q^6; q^4, -\frac{q^3}{t^2} \right) \right\}.
\]

We rewrite the part \{\cdots\} in the right hand side by using Lemma 7 and Corollary 1.
\[\theta_q \left( \frac{\lambda^2}{q^3t^2} \right) \varphi_1 \left( 0; q^2; q^4, -\frac{q^3}{t^2} \right) + \theta_q \left( \frac{\lambda^2}{q^3t^2} \right) \frac{-\lambda}{(1 - q^3t^2q^2)q^3t^2} \varphi_1 \left( 0; q^6; q^4, -\frac{q^3}{t^2} \right) = \]
\[= \left\{ \theta_q \left( \frac{\lambda}{q^2t} \right) + \theta_q \left( -\frac{\lambda}{q^2t} \right) \right\} \varphi_1 \left( 0; q^2; q^4, -\frac{q}{t^2} \right) \]
\[+ \left\{ \theta_q \left( \frac{\lambda}{q^2t} \right) - \theta_q \left( -\frac{\lambda}{q^2t} \right) \right\} \frac{-1}{(1 - q^3t^2)q^3t^2} \varphi_1 \left( 0; q^6; q^4, -\frac{q^3}{t^2} \right) \]
\[= \frac{\theta_q \left( \frac{\lambda}{q^2t} \right)}{2} \left\{ \varphi_1 \left( 0; q^2; q^4, -\frac{q}{t^2} \right) + \frac{-1}{(1 - q^3t^2)q^3t^2} \varphi_1 \left( 0; q^6; q^4, -\frac{q^3}{t^2} \right) \right\} \]
\[+ \frac{\theta_q \left( -\frac{\lambda}{q^2t} \right)}{2} \left\{ \varphi_1 \left( 0; q^2; q^4, -\frac{q}{t^2} \right) + \frac{1}{(1 - q^3t^2)q^3t^2} \varphi_1 \left( 0; q^6; q^4, -\frac{q^3}{t^2} \right) \right\} \]
\[= \frac{\theta_q \left( \frac{\lambda}{q^2t} \right)}{2} \varphi_1 \left( 0; -q; q, \frac{1}{t} \right) + \frac{\theta_q \left( -\frac{\lambda}{q^2t} \right)}{2} \varphi_1 \left( 0; -q; q, \frac{1}{t} \right). \]

Hence we have
\[2f_0(0, 0; -q^2, \lambda, q^2) = \frac{(q^2; q^2)_\infty}{\theta_q \left( \frac{\lambda}{q^2t} \right) \theta_q \left( -\frac{\lambda}{q^2t} \right) 2\theta_q(q^2t) v_1(t) + 2\theta_q(-q^2t)^2 v_2(t)} \]
completing the proof of Theorem 4.

By the theta relations in Lemma 8, we obtain another representation of our connection formula as follows:

**Proposition 4.** For any \( x \in \mathbb{C}^* \), we have
\[A_q \left( -\frac{q^3}{x^2} \right) = \frac{2}{(q, -1; q)_\infty} \theta_q \left( \frac{x^2}{q} \right) \varphi_1(0; q^2; q^4, -qx^2) \]
\[+ \frac{2}{(q, -1; q)_\infty q^2 - 1} \theta_q \left( \frac{x^2}{q^3} \right) \varphi_1(0; q^6; q^4, -q^3 x^2). \]

### 2.4 Summary

We have solved the connection problem for the second order linear \( q \)-difference equation
\[(-q^5t^2q^2 - \sigma_q + 1) f(t) = 0. \quad (2.20) \]
This equation has solutions

\[ f_1(t) = \varphi_1(0; q^2, q^5 t^2) = A_{q^2}(-q^2 t^2), \]
\[ f_2(t) = \theta_q(q t) \theta_q(-q^2 t) \varphi_0(0, 0; -q^2, qt^2) \]

around \( t = 0 \). The solution \( f_1(t) \) is the Ramanujan entire function with the base \( q^2 \). The solution \( f_2(t) \) contains a divergent series \( 2 \varphi_0(0, 0; -q^2, qt^2) \). Thus, we consider \( 2 \varphi_0(0, 0; -q, \lambda, -x/q) := \mathcal{L}_{0,\lambda}^+ \circ \mathcal{B}_q^+ \varphi_0(0, 0; -q, -x/q) \), which is a resummation of a divergent series \( 2 \varphi_0(0, 0; -q, -x/q) \). We remark that the function \( 2 \varphi_0(0, 0; -q, \lambda, -x/q) \) is analytic on \( \mathbb{C}^* \setminus [-\lambda; q] \). We set \( \tilde{f}_2(t, \lambda) := \theta_q(q t) \theta_q(-q^2 t) \varphi_0(0, 0; -q^2, \lambda, qt^2) \). The equation (2.20) also has solutions

\[ v_1(t) = \theta_q(q^2 t) \varphi_1(0; -q; q, \frac{1}{t}) = \theta_q(q^2 t) \text{Ai}_q \left( -\frac{1}{t} \right), \]
\[ v_2(t) = \theta_q(-q^2 t) \varphi_1(0; -q; q, -\frac{1}{t}) = \theta_q(-q^2 t) \text{Ai}_q \left( \frac{1}{t} \right) \]

around \( t = \infty \). Both solutions \( v_1(t) \) and \( v_2(t) \) are represented by the \( q \)-Airy function, so that they are convergent at infinity.

We gave the connection formulae for the equation (2.20) as follows:

\[
\begin{pmatrix}
 f_1(t) \\
 f_2(t, \lambda)
\end{pmatrix} =
\begin{pmatrix}
 C_{11} & C_{12} \\
 C_{21}(t, \lambda) & C_{22}(t, \lambda)
\end{pmatrix}
\begin{pmatrix}
 v_1(t) \\
 v_2(t)
\end{pmatrix}.
\]

The connection formula between \( \tilde{f}_2(t, \lambda) \) and \( v_1(t), v_2(t) \) holds for any \( t \in \mathbb{C}^* \) except poles of \( C_{21}(t, \lambda) \) and \( C_{22}(t, \lambda) \). Connection coefficients are

\[ C_{11} = C_{12} = \frac{1}{(q, -1; q)_\infty}, \]
\[ C_{21}(t, \lambda) = \frac{(q^2; q^2)_\infty \theta_q(q t) \theta_q(-q^2 t) \theta_q\left( \frac{-\lambda}{q^2 t} \right)}{\theta_{q^2}\left( -\frac{\lambda}{q^2} \right) \theta_{q^2}\left( -\frac{\lambda}{q^2 t^2} \right) 2\theta_q(q^2 t)} \]

and

\[ C_{22}(t, \lambda) = \frac{(q^2; q^2)_\infty \theta_q(q t) \theta_q(-q^2 t) \theta_q\left( \frac{-\lambda}{q^2 t} \right)}{\theta_{q^2}\left( -\frac{\lambda}{q^2} \right) \theta_{q^2}\left( -\frac{\lambda}{q^2 t^2} \right) 2\theta_q(-q^2 t)}. \]
We remark that the first connection formula gives a relation between the Ramanujan entire function $A_{q^2}$ and the $q$-Airy function $\text{Ai}_q$:

$$A_{q^2}(-q^3t^2) = \frac{1}{\theta_q(q^2t)} \text{Ai}_q \left( -\frac{1}{t} \right) + \frac{1}{\theta_q(-q^2t)} \text{Ai}_q \left( \frac{1}{t} \right).$$
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