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Connection formulae of the Ramanujan entire function and a resummation of the basic hypergeometric series ${}_2\varphi_0(0,0;-;q,x)$

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Abstract

We study a connection problem on a q-difference equation satisfied by the Ramanujan entire function. Since one local solution contains a divergent series, we apply the q-Borel-Laplace resummation method to study the connection problem. We show two connection formulae. One is a connection formula between the Ramanujan entire function and the q-Airy function, another is a connection formula for a resummation of a divergent basic hypergeometric series ${}_2\varphi_0(0, 0; -; q, x)$. This thesis is based on author's work [8].

Contents

Chapter 1

Introduction

In this paper, we study a connection problem on a linear q-difference equation satisfied by the Ramanujan entire function. The Ramanujan entire function is originally found by S. Ramanujan in Ramanujan's Lost Notebook [9]. Throughout the paper, we assume that $0 < |q| < 1$. The Ramanujan entire function with the base q is given by

$$\mathrm{A}_q(x) := \sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n}(-x)^n$$

where $(a;q)_n$ is the q-shifted factorial

$$(a;q)_n := \begin{cases} 1, & n = 0, \\ (1-a)(1-aq)\dots(1-aq^{n-1}), & n \geq 1. \end{cases}$$

M. E. H. Ismail has pointed out that the Ramanujan entire function can be considered as a q-analogue of the Airy function [5]. It is known that there exist two different q-analogues of the Airy function. One is the Ramanujan entire function and the other is the q-Airy function found by K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta and Y. Yamada in the study of hypergeometric-type solutions to the q-Painlevé equation of type $(A_1 + A_1')^{(1)}$ [7]. The q-Airy function with the base q is given by

$$\mathrm{Ai}_q(x) := \sum_{n\geq 0} \frac{q^{\frac{n(n-1)}{2}}}{(-q,q;q)_n} x^n,$$

where $(a_1, a_2, \dots, a_m; q)_n := (a_1;q)_n (a_2;q)_n \dots (a_m;q)_n$.

No relations between two different q-analogues of Airy functions have been known, but we show that the Ramanujan entire function and the q-Airy function are related by our connection formula.

1.1 Linear q-difference equations and connection problems

At first, we review the classical results by C. R. Adams [1] in a second order case. Adams studied a local behavior of solutions of linear q-difference equations. We consider the q-difference equations of the form

$$a_2(x)u(q^2x) + a_1(x)u(qx) + a_0(x)u(x) = 0 \tag{1.1}$$

where $a_0(x)$, $a_1(x)$ and $a_2(x)$ are holomorphic functions around $x = 0$. He called the following equation

$$a_2(0)\lambda^2 + a_1(0)\lambda + a_0(0) = 0 \tag{1.2}$$

a characteristic equation at $x = 0$ of (1.1). The roots of the characteristic equation are called *characteristic exponents.* He pointed out that there exist two essentially different cases:

1. $a_2(0)a_0(0) \neq 0$,
2. $a_2(0)a_0(0) = 0$.

In the first case, we can find two solutions of the form

$$u_m(x) = \frac{\theta_q(x)}{\theta_q(\lambda_m x)} \sum_{k\geq 0} u_{m,k}x^k, \quad m = 1, 2 \tag{1.3}$$

where λ_1 and λ_2 (provided that $\lambda_1/\lambda_2 \notin q^{\mathbb{Z}}$) are non-zero exponents. Here, $\theta_q(x) := \sum_{n\in\mathbb{Z}} q^{\frac{n(n-1)}{2}} x^n$ is the theta function of Jacobi. Adams gave a proof of convergence of power series contained in these solutions.

In the second case, either the characteristic equation is a linear equation or an exponent is zero. Therefore, some solutions of (1.1) are not in the form (1.3). After changing the variables and the base, Adams gave a formal solution of the form

$$u(x) = \frac{\theta_q(x)^{\mu+1}}{\theta_q(\lambda x)} \sum_{k\geq 0} u_k x^k \tag{1.4}$$

where λ is a suitable non-zero constant and μ is an integer. We remark that formal solutions of the form (1.4) may contain divergent series. The existence of formal solutions of the form (1.4) around $x = \infty$ is similar to the case of $x = 0$.

We consider a connection problem on a second order linear q-difference equation of the form

$$\left\{(a_0+b_0x)\sigma_q^2+(a_1+b_1x)\sigma_q+(a_2+b_2x)\right\}u(x)=0 \tag{1.5}$$

where σ_q is the q-shift operator $\sigma_q f(x)=f(qx)$. Connection problems on linear q-difference equations are studied by G. D. Birkhoff [2]. He studied connection problems when the first case occurs for both $x=0$ and $x=\infty$, namely, in the case $a_0b_0a_2b_2\neq 0$.

Let $u_1(x),u_2(x)$ be independent solutions of (1.1) around $x=0$ and let $v_1(x),v_2(x)$ be those around $x=\infty$. We take suitable analytic continuation of $u_1(x)$ and $u_2(x)$. Then the connection formula in the matrix form is given by

$$\begin{pmatrix}u_1(x)\\u_2(x)\end{pmatrix}=\begin{pmatrix}C_{11}(x)&C_{12}(x)\\C_{21}(x)&C_{22}(x)\end{pmatrix}\begin{pmatrix}v_1(x)\\v_2(x)\end{pmatrix}.$$

Here, $C_{jk}(x)$ are q-constant, namely, $C_{jk}(qx)=C_{jk}(x)$.

In the case $a_0a_2b_0b_2=0$, some power series which appear in formal solutions may be divergent. In order to study connection problems, we should take a suitable resummation of a divergent series. J.-P. Ramis and C. Zhang introduced a discrete resummation method, called the q-Borel-Laplace transformation of the first kind. The q-Borel-Laplace transformation of the first kind is given as follows.

1. We assume that $f(x)=\sum_{n\geq 0}a_nx^n$ is a formal power series. The q-Borel transformation of the first kind $\mathcal{B}_q^+$ is given by

   $$\left(\mathcal{B}_q^+f\right)(\xi):=\sum_{n\geq 0}a_nq^{\frac{n(n-1)}{2}}\xi^n.$$

   We denote $\varphi_f(\xi)=\left(\mathcal{B}_q^+f\right)(\xi)$. If $f(x)$ is a convergent series, then $\varphi_f(\xi)$ is an entire function.

2. We fix $\lambda\in\mathbb{C}^*\setminus q^{\mathbb{Z}}$. For any entire function $\varphi(\xi)$, the q-Laplace transformation of the first kind $\mathcal{L}_{q,\lambda}^+$ [3, 16] is given by

   $$\left(\mathcal{L}_{q,\lambda}^+\varphi\right)(x):=\frac{1}{1-q}\int_0^{\lambda\infty}\frac{\varphi(\xi)}{\theta_q\left(\frac{\xi}{x}\right)}\frac{d_q\xi}{\xi}=\sum_{n\in\mathbb{Z}}\frac{\varphi(\lambda q^n)}{\theta_q\left(\frac{\lambda q^n}{x}\right)}$$

   where

   $$\int_0^{\lambda\infty}f(t)d_qt:=(1-q)\lambda\sum_{n\in\mathbb{Z}}f(\lambda q^n)q^n$$

   is Jackson's q-integral on $(0,\lambda\infty)$ [4].

The q-Borel transformation of the first kind may be used for a resummation of a divergent series. We set $D_r := \{x \in \mathbb{C} | 0 < |x| < r\}$, where r is a positive number. Let $f(x)$ be a holomorphic function on $D_r \setminus [\lambda; q]$, where $[-\lambda; q] := -\lambda q^{\mathbb{Z}}$. The set $[\lambda; q]$ is called a q-spiral. We assume that $\hat{f}(x) = \sum_{m \geq 0} a_m x_m$ is a divergent series. We call $\hat{f}(x)$ is an asymptotic series of $f(x)$ for $x \to 0$ on $D_r \setminus [\lambda; q]$ when $f(x)$ satisfies the condition as follows:

Definition 1. *For any $\epsilon > 0$, any $c > 0$ and any $n \in \mathbb{Z}_{\geq 0}$, there exists a constant $r_0 > 0$ such that, if $x \in D_{r_0} \setminus \cup_{k \in \mathbb{Z}} \{x; |x - \lambda q^k| < \epsilon |q|^k\}$, we have*

$$\left| f(x) - \sum_{m=0}^{n} a_m x^m \right| < c|x|^n.$$

Zhang [13] shows that the q-Borel-Laplace transform of a divergent series with a suitable condition is holomorphic on an open dense domain $\mathbb{C}^* \setminus [-\lambda; q]$, and the original divergent series gives its asymptotic series for $x \to 0$.

Zhang [15] also introduced the following q-Borel-Laplace transformation of the second kind. We use the q-Borel-Laplace transformation of the second kind to obtain an integral representation of a convergent series around $x = 0$.

1. We assume that $f(x) = \sum_{n \geq 0} a_n x^n$ is a power series such that $|a_n|$ decrease rapidly enough. The q-Borel transformation of the second kind $\mathcal{B}_q^-$ is given by

$$(\mathcal{B}_q^- f)(\xi) := \sum_{n \geq 0} a_n q^{-\frac{n(n-1)}{2}} \xi^n.$$

2. For any function $g(\xi)$ which is holomorphic around $\xi = 0$, the q-Laplace transformation of the second kind $\mathcal{L}_q^-$ is given by

$$\left(\mathcal{L}_q^- g\right)(x) := \frac{1}{2\pi i} \int_{|\xi| = r} g(\xi) \theta_q \left(\frac{x}{\xi}\right) \frac{d\xi}{\xi},$$

   where r is a suitable positive number.

Zhang shows that the q-Borel transformation $\mathcal{B}_q^-$ is a formal inverse of the q-Laplace transformation $\mathcal{L}_q^-$ [15]. More precisely, we can show the following lemma.

Lemma 1. *Let $f(x) = \sum_{n \geq 0} a_n x^n$ be a function such that $(\mathcal{B}_q^- f)(\xi)$ is convergent. Then $\left(\mathcal{L}_q^- \circ \mathcal{B}_q^- f\right)(x)$ coincides with $f(x)$ near $x = 0$.*

We assume that one solution $u_1(x)$ around $x = 0$ is an entire function and the other formal solution $u_2(x)$ around $x = 0$ contains a divergent series. We also assume that both of the solutions $v_1(x)$, $v_2(x)$ around $x = \infty$ extend to meromorphic functions on $\mathbb{C}^*$. Under these assumptions, connection formulae of (1.5) are given in the following matrix form:

$$\begin{pmatrix} u_1(x) \\ \tilde{u}_2(x,\lambda) \end{pmatrix} = \begin{pmatrix} C_{11}(x) & C_{12}(x) \\ \tilde{C}_{21}(x,\lambda) & \tilde{C}_{22}(x,\lambda) \end{pmatrix} \begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix}.$$

Here, $\tilde{u}_2(x, \lambda)$ contains the resummation of the divergent series obtained by the q-Borel-Laplace transformation. The functions $C_{11}(x), C_{12}(x)$ are q-constants and $\tilde{C}_{21}(x, \lambda), \tilde{C}_{22}(x, \lambda)$ are also q-constants with poles at $x \in [-\lambda; q]$. The connection coefficients are analytic function of the parameter λ. The domain $\mathbb{C}^* \setminus [-\lambda; q]$ can be considered as a q-analogue of the Stokes region. The dependance on the parameter λ of connection coefficients is called the q-Stokes phenomenon [10].

1.2 Connection problem on q-difference equation satisfied by the Ramanujan entire function

In Chapter 2, we study a connection problem on a q-difference equation satisfied by the Ramanujan entire function. The Ramanujan entire function with the base q satisfies the following q-difference equation

$$\left(qx\sigma_q^2 - \sigma_q + 1\right) u(x) = 0. \tag{1.6}$$

We introduce a transformation of variables

$$x \to -q^3t^2, \quad q \to q^2 \tag{1.7}$$

and set $f(t) = u(x)$. Then we give a connection formula between the Ramanujan entire function with the base q^2 and the q-Airy function with the base q. By the transformation (1.7), the equation (1.6) reduces to

$$(-q^5t^2\sigma_q^2 - \sigma_q + 1)f(t) = 0. \tag{1.8}$$

The aim of Chapter 2 is to solve a connection problem of (1.8). Equation (1.8) has formal solutions

$$f_1(t) = \mathrm{A}_{q^2}(-q^3t^2) = {}_0\varphi_1(-; 0; q^2, q^5t^2), \tag{1.9}$$

$$f_2(t) = \theta_q(qt)\theta_q(-q^2t)_2\varphi_0(0, 0; -; q^2, qt^2) \tag{1.10}$$

around $t=0$. Here, the function ${}_r\varphi_s(a_1,\dots,a_r;b_1,\dots,b_s;q,t)$ is the basic hypergeometric series with the base q:

$$
\begin{aligned}
&{}_r\varphi_s(a_1,\dots,a_r;b_1,\dots,b_s;q,t)\\
&\quad:=\sum_{n\geq 0}\frac{(a_1,\dots,a_r;q)_n}{(b_1,\dots,b_s;q)_n(q;q)_n}\left\{(-1)^n q^{\frac{n(n-1)}{2}}\right\}^{1+s-r} t^n.
\end{aligned}
$$

The radius of convergence is $\infty, 1$ or 0 according to whether $r-s<1, r-s=1$ or $r-s>1$.

The function $f_1(t)$ is the Ramanujan entire function with base q^2 and the solution $f_2(t)$ has a divergent series ${}_2\varphi_0(0,0;-;q^2,qt^2)$. Around $t=\infty$, equation (1.8) has solutions

$$v_1(t)=\theta_q(q^2t)\,{}_1\varphi_1\left(0;-q;q,\frac{1}{t}\right)=\theta_q(q^2t)\operatorname{Ai}_q\left(-\frac{1}{t}\right), \tag{1.11}$$

$$v_2(t)=\theta_q(-q^2t)\,{}_1\varphi_1\left(0;-q;q,-\frac{1}{t}\right)=\theta_q(-q^2t)\operatorname{Ai}_q\left(\frac{1}{t}\right). \tag{1.12}$$

Both $v_1(t)$ and $v_2(t)$ are represented by the q-Airy function, which is an entire function. In subsection 2.2, we apply the q-Borel-Laplace transformation of the second kind to the power series (1.9). Then we obtain the following connection formula for $f_1(t)$ and deduce a relationship between the Ramanujan entire function and the q-Airy function.

Theorem. *For any $t\in\mathbb{C}^*$, we have*

$$f_1(t)=\frac{1}{(q,-1;q)_\infty}v_1(t)+\frac{1}{(q,-1;q)_\infty}v_2(t)$$

where $(a;q)_\infty:=\lim_{n\to\infty}(a;q)_n$.

In subsection 2.3, we give a connection formula of a resummation of the formal solution $f_2(t)$. Namely, since the solution $f_2(t)$ contains a divergent series ${}_2\varphi_0(0,0;-;q^2,qt^2)$, we construct a true solution by applying the q-Borel-Laplace transformation of the first kind to the divergent series. Then we obtain a true solution which is defined on $\mathbb{C}^*\setminus[-\lambda;q]$. We denote

$${}_2f_0\left(0,0;-;q,\lambda,-x/q\right):=\mathcal{L}^+_{q,\lambda}\circ\mathcal{B}^+_q\,{}_2\varphi_0(0,0;-;q,-x/q).$$

We remark that the q-Borel-Laplace transform (of the first kind) of a divergent series ${}_2\varphi_0(0,0;-;q,-x/q)$ is defined on the set $\mathbb{C}^*\setminus[-\lambda;q]$ and the original divergent series ${}_2\varphi_0(0,0;-;q,-x/q)$ gives its asymptotic series for

$x \to 0$. We set $\tilde{f}_2(t;\lambda) = \theta_q(qt)\theta_q(-q^2t)_2f_0(0,0;-;q^2,\lambda,qt^2)$, then we have the following connection formula between $\tilde{f}_2(t;\lambda)$ and $v_1(t)$, $v_2(t)$.

Theorem. *For any $t \in \mathbb{C}^*$ such that $t^2 \in \mathbb{C}^* \setminus [\lambda/q;q^2]$, we have*

$$\begin{aligned}\theta_q(qt)\theta_q(-q^2t)_2f_0(0,0;-;q^2,\lambda,qt^2) =& \frac{(q^2;q^2)_\infty \theta_q(qt)\theta_q(-q^2t)}{\theta_{q^2}\left(-\frac{\lambda}{q^2}\right)\theta_{q^2}\left(\frac{-\lambda}{q^3t^2}\right)} \frac{\theta_q\left(\frac{\lambda}{q^3t}\right)}{2\theta_q(q^2t)} v_1(t) \\ &+ \frac{(q^2;q^2)_\infty \theta_q(qt)\theta_q(-q^2t)}{\theta_{q^2}\left(-\frac{\lambda}{q^2}\right)\theta_{q^2}\left(\frac{-\lambda}{q^3t^2}\right)} \frac{\theta_q\left(-\frac{\lambda}{q^3t}\right)}{2\theta_q(-q^2t)} v_2(t).\end{aligned}$$

Zhang and Ismail [6] studied an asymptotic behavior of the Ramanujan entire function and obtained an asymptotic formula as follows:

$$\begin{aligned}\mathrm{A}_q(x) =& \frac{\theta_{q^2}\left(-\frac{q}{x}\right)}{(q^2,q;q^2)_\infty} {}_1\varphi_1\left(0;q;q^2,\frac{q^2}{x}\right) \\ &+ \frac{\theta_{q^2}\left(-\frac{1}{x}\right)}{(q^2,q;q^2)_\infty}\frac{q}{q-1} {}_1\varphi_1\left(0;q^3;q^2,\frac{q^3}{x}\right).\end{aligned} \tag{1.13}$$

We can derive an asymptotic expansion which is essentially equivalent to (1.13) in Proposition 4 as a corollary to our connection formula.

1.3 The theta function and the q-exponential functions

The theta function of Jacobi with the base q

$$\theta_q(x) = \sum_{n\in\mathbb{Z}} q^{\frac{n(n-1)}{2}} x^n, \qquad \forall x \in \mathbb{C}^*$$

is holomorphic on $\mathbb{C}^*$. Our theta function is not a standard one but we follow the notation in [10]. The theta function has the following properties:

1. Jacobi's triple product identity

$$\theta_q(x) = (q, -x, -q/x; q)_\infty \,. \tag{1.14}$$

2. The q-difference equation

$$\theta_q(q^k x) = q^{-\frac{k(k-1)}{2}} x^{-k} \theta_q(x), \quad \forall k \in \mathbb{Z}.$$

3. The inversion formula

$$\theta_q\left(1/x\right) = \theta_q(x)/x.$$

For any $\lambda \in \mathbb{C}^*$, the function $\theta(-\lambda x)/\theta(\lambda x)$ satisfies a q-difference equation $u(qx) = -u(x)$, which is also satisfied by the function $u(x) = e^{\pi i\left(\frac{\log x}{\log q}\right)}$.

Two different q-exponential functions are given by

$$e_q(x) := {}_1\varphi_0(0; -; q, x) = \sum_{n\geq 0} \frac{x^n}{(q;q)_n} = \frac{1}{(x;q)_\infty},$$

$$E_q(x) := {}_0\varphi_0(-; -; q, -x) = \sum_{n\geq 0} \frac{q^{\frac{n(n-1)}{2}}}{(q;q)_n} x^n = (-x;q)_\infty.$$

These two q-exponential functions are related to each other as follows:

$$e_q(x) E_q(-x) = 1, \quad e_{q^{-1}}(x) = E_q(-qx).$$

Chapter 2

Connection formulae of the Ramanujan entire function and the resummation of the divergent series ${}_2\varphi_0(0,0;-;q,x)$

In this chapter, we study a relation between the aforementioned two q-analogues of the Airy function from the viewpoint of connection problems on q-difference equations.

2.1 Relation between the Ramanujan entire function and the q-Airy function

To find a relation between the Ramanujan entire function and the q-Airy function, we compare q-difference equations satisfied by each of the functions. The Ramanujan entire function satisfies the following q-difference equation

$$\left(qx\sigma_q^2 - \sigma_q + 1\right) u(x) = 0. \tag{2.1}$$

The q-Airy function satisfies the second order linear q-difference equation

$$\left(\sigma_q^2 + x\sigma_q - 1\right) u(x) = 0. \tag{2.2}$$

We can transform the q-difference equation satisfied by the Ramanujan entire function with base q^2

$$\left(q^2 x\sigma_{q^2}^2 - \sigma_{q^2} + 1\right) u(x) = 0 \tag{2.3}$$

into a q-difference equation

$$\left(-q^5 t^2 \sigma_q^2 - \sigma_q + 1\right) f(t) = 0 \tag{2.4}$$

by changing the independent variable x to t by $-q^3t^2 = x$. We can also transform (2.4) into (2.2) by changing the independent variable $t \to 1/t$ and a suitable change of a depending variable as we will show in the proof of the next proposition. In this way, we will find a relation between the Ramanujan entire function and the q-Airy function.

The first task is to give local formal solutions of (2.4) around the origin and the infinity.

Proposition 1. *The equation* (2.4) *has solutions*

$$f_1(t) = {}_0\varphi_1(-;0;q^2,q^5t^2) = \mathrm{A}_{q^2}(-q^3t^2), \tag{2.5}$$
$$f_2(t) = \theta_q(qt)\theta_q(-q^2t)\,{}_2\varphi_0(0,0;-;q^2,qt^2). \tag{2.6}$$

around the origin. The first solution $f_1(t)$ is a convergent series and represented by the Ramanujan entire function with the base q^2. The second solution $f_2(t)$ contains a divergent series.

The equation (2.4) *has solutions*

$$v_1(t) = \theta_q(q^2t)\,{}_1\varphi_1\left(0;-q;q,\frac{1}{t}\right) = \theta_q(q^2t)\,\mathrm{Ai}_q\left(-\frac{1}{t}\right) \tag{2.7}$$
$$v_2(t) = \theta_q(-q^2t)\,{}_1\varphi_1\left(0;-q;q,-\frac{1}{t}\right) = \theta_q(-q^2t)\,\mathrm{Ai}_q\left(\frac{1}{t}\right) \tag{2.8}$$

around the infinity. Both solutions contain convergent series and represented by the q-Airy function with the base q.

Proof. Equation (2.4) has a unique holomorphic solution at $t = 0$:

$$f_1(t) = \mathrm{A}_{q^2}(-q^3t^2).$$

We define $\tilde{f}(t)$ by $f(t) = \theta_q(qt)\theta_q(-q^2t)\tilde{f}(t)$. Since the theta function satisfies the relation

$$\sigma_q\left\{\theta_q(qt)\theta_q(-q^2t)\right\} = -\frac{1}{q^3t^2}\theta_q(qt)\theta_q(-q^2t),$$

the function $\tilde{f}(t)$ satisfies the equation

$$\left(\sigma_q^2 - \sigma_q - q^3 t^2\right) \tilde{f}(t) = 0. \tag{2.9}$$

The equation (2.9) has a unique power series solution of the form $\tilde{f}(t) = \sum_{n\geq 0} b_n t^n$ where $b_0 = 1$, which is

$$\tilde{f}(t) = {}_2\varphi_0(0, 0; -; q^2, qt^2) = \sum_{n\geq 0} \frac{1}{(q^2; q^2)_n} \left\{(q^2)^{\frac{n(n-1)}{2}}\right\}^{-1} (qt^2)^n.$$

Therefore, the equation (2.4) has another (formal) solution as follows:

$$f_2(t) = \theta_q(qt)\theta_q(-q^2 t)_2\varphi_0(0, 0; -; q^2, qt^2).$$

We consider local solutions of (2.4) around $t = \infty$. We set

$$z(t) = \frac{1}{\theta_q(-q^2 t)} f(t).$$

Then $z(t)$ satisfies

$$\left(-\sigma_q^2 + \frac{1}{q^2 t}\sigma_q + 1\right) z(t) = 0.$$

We set $x = 1/t$ and $u(x) = z(1/t)$. Then $u(x)$ satisfies the q-Airy equation

$$\left(\sigma_q^2 + x\sigma_q - 1\right) u(x) = 0.$$

We take local solutions of the q-Airy equation around $x = 0$ as follows:

$$u_1(x) = -\frac{\theta_q(x/q^2)}{\theta_q(-x/q^2)} \operatorname{Ai}_q(-x), \quad u_2(x) = \operatorname{Ai}_q(x).$$

Therefore (2.4) has solutions around $x = \infty$ as follows:

$$v_1(t) = \theta_q(q^2 t) \operatorname{Ai}_q\left(-\frac{1}{t}\right) = \theta_q(q^2 t)_1\varphi_1\left(0; -q; q, \frac{1}{t}\right), \tag{2.10}$$

$$v_2(t) = \theta_q(-q^2 t) \operatorname{Ai}_q\left(\frac{1}{t}\right) = \theta_q(-q^2 t)_1\varphi_1\left(0; -q; q, -\frac{1}{t}\right). \tag{2.11}$$

□

The Ramanujan entire function is a convergent solution of the equation (2.4) around $t = 0$. The solution $f_1(t)$ is a unique holomorphic solution at $t = 0$ with the condition $f_1(0) = 1$.

The q-Airy functions are solutions of (2.4) around $t = \infty$. We show a connection formula between the Ramanujan entire function (2.5) and the q-Airy functions (2.7), (2.8) in the next subsection 2.2.

2.2 Connection formula of the Ramanujan entire function

In the following, we give a formula which relates the Ramanujan entire function (with the base q^2) with the q-Airy function as a corollary to our connection formula between $f_1(t)$ and $v_1(t)$, $v_2(t)$.

We set $g(\tau) = (\mathcal{B}_q^- f_1)(\tau)$. We see that $g(\tau) = \sum_{n\geq 0} g_n\tau^n$, $g_0 = 1$ is a convergent series in our case. To begin with, we note that the q-Borel transformation has the following operational relation:

Lemma 2. *For any $l, m \in \mathbb{Z}_{\geq 0}$, we have*

$$\mathcal{B}_q^-(t^m\sigma_q^l) = q^{-\frac{m(m-1)}{2}}\tau^m\sigma_q^{l-m}\mathcal{B}_q^-.$$

The operational relation for the q-Borel transformation is given by Zhang [15]. We apply the q-Borel transformation $\mathcal{B}_q^-$ to the function $f_1(t)$.

1. The q-difference equation satisfied by $(\mathcal{B}_q^- f_1)(\tau)$.

   By the operational relation in Lemma 2, the function $g(\tau)$ satisfies a first order q-difference equation

   $$g(q\tau) = (1 + q^2\tau)(1 - q^2\tau)g(\tau). \tag{2.12}$$

   The q-difference equation (2.12) has a solution

   $$h(\tau) = \frac{1}{(-q^2\tau; q)_\infty(q^2\tau; q)_\infty}.$$

   We consider the Maclaurin series of the function $h(\tau)$, which is a formal power series solution of (2.12). Since there exists a unique power series solution $g(\tau)$ with a condition $g(0) = 1$, the Maclaurin series of $h(\tau)$ coincides with the solution $g(\tau)$. Then we obtain the following infinite product representation of $g(\tau)$:

   $$g(\tau) = \frac{1}{(-q^2\tau; q)_\infty(q^2\tau; q)_\infty}.$$

   The function $g(\tau)$ has simple poles at

   $$\left\{\tau; \tau = \pm q^{-2-k}, \quad \forall k \in \mathbb{Z}_{\geq 0}\right\}.$$

   In particular, the radius of convergence of $g(\tau)$ is $|q|^{-2}$.

2. Residue calculus for $(\mathcal{L}_q^- \circ \mathcal{B}_q^- f_1)(t)$.

We fix a number $r > 0$ such that $|q|^{-1} < r < |q|^{-2}$ and define a set $C_l = \{\tau \in \mathbb{C} \,|\, |\tau| = r|q|^{-l}\}$, for $l \in \mathbb{Z}_{\geq 0}$. Applying the q-Laplace transformation $\mathcal{L}_q^-$ to $g(\tau)$, we obtain an integral representation of $f_1(t)$ by Lemma 1.

$$\begin{aligned}
f_1(t) =& \frac{1}{2\pi i} \oint_{C_0} g(\tau)\theta_q\left(\frac{t}{\tau}\right)\frac{d\tau}{\tau} \\
=& \frac{1}{2\pi i} \int_{|\tau|=r|q|^{-l}} g(\tau)\theta_q\left(\frac{t}{\tau}\right)\frac{d\tau}{\tau} \\
&- \sum_{k\geq 0}^{l-1} \mathrm{Res}\left\{g(\tau)\theta_q\left(\frac{t}{\tau}\right)\frac{1}{\tau}; \tau = -q^{-2-k}\right\} \\
&- \sum_{k\geq 0}^{l-1} \mathrm{Res}\left\{g(\tau)\theta_q\left(\frac{t}{\tau}\right)\frac{1}{\tau}; \tau = q^{-2-k}\right\}.
\end{aligned} \tag{2.13}$$

At first, we show the integral on $|\tau| = r|q|^{-l}$ goes to zero when $l \to \infty$.

Proposition 2. *For any $t \in \mathbb{C}^*$, we have*

$$\lim_{l\to\infty} \left| \frac{1}{2\pi i} \int_{|\tau|=r|q|^{-l}} g(\tau)\theta_q\left(\frac{t}{\tau}\right)\frac{d\tau}{\tau} \right| = 0.$$

For the proof, we prepare two lemmas on the q-shifted factorials.

Lemma 3. *For any $l \in \mathbb{Z}_{>0}$, we have*

$$(\tau q^{2-l}; q)_l = \tau^l (-1)^l q^{\frac{l(l-3)}{2}} \left(\frac{1}{\tau q}; q\right)_l.$$

The lemma above can be proved directly.

Lemma 4. *For any $\tau \in C_0$, we have*

$$\frac{1}{\left|\left(\frac{1}{\tau q}; q\right)_l\right|} \leq \frac{1}{\left(\frac{1}{r|q|}; |q|\right)_l} < \frac{1}{\left(\frac{1}{r|q|}; |q|\right)_\infty}.$$

Proof. Since $|q|^{-1} < r < |q|^{-2}$, we have $|\tau q^j| > 1 \quad (j = 1, 0, -1, \cdots)$, where $\tau \in C_0$. We remark that

$$\left|1 - \frac{1}{\tau q^j}\right| \geq 1 - \frac{1}{r|q|^j}, \quad j = 1, 0, -1, \cdots .$$

Therefore we have an estimation

$$\left|\left(\frac{1}{\tau q}; q\right)_l\right| \geq \left(\frac{1}{r|q|}; |q|\right)_l > \left(\frac{1}{r|q|}; |q|\right)_\infty .$$

Thus we obtain

$$\frac{1}{\left|\left(\frac{1}{\tau q}; q\right)_l\right|} \leq \frac{1}{\left(\frac{1}{r|q|}; |q|\right)_l} < \frac{1}{\left(\frac{1}{r|q|}; |q|\right)_\infty}.$$

□

We are ready to prove Proposition 2.

Proof. We put $M := 1/\left(\frac{1}{r|q|}; |q|\right)_\infty$ and $N := \max_{\tau \in C_0} |g(\tau)\theta_q(t/\tau)|$.

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{|\tau|=r|q|^{-l}} g(\tau)\theta_q\left(\frac{t}{\tau}\right)\frac{d\tau}{\tau} \\
&= \frac{1}{2\pi i}\oint_{C_0} g(\tau q^{-l})\theta_q\left(\frac{tq^l}{\tau}\right)\frac{d\tau}{\tau} \\
&= \frac{1}{2\pi i}\oint_{C_0} \frac{g(\tau)}{(\tau q^{2-l}, -\tau q^{2-l}; q)_l} q^{-\frac{l(l-1)}{2}}\left(\frac{t}{\tau}\right)^{-l}\theta_q\left(\frac{t}{\tau}\right)\frac{d\tau}{\tau} \\
&= \frac{1}{2\pi i}\oint_{C_0} \frac{\tau^{-2l}q^{l(l-3)}g(\tau)}{(1/\tau q, -1/\tau q; q)_l} q^{-\frac{l(l-1)}{2}}\left(\frac{t}{\tau}\right)^{-l}\theta_q\left(\frac{t}{\tau}\right)\frac{d\tau}{\tau} \\
&= \frac{1}{2\pi i} q^{\frac{l(l-5)}{2}} t^{-l}\oint_{C_0} \frac{g(\tau)\theta_q\left(\frac{t}{\tau}\right)\tau^{-l}}{(1/\tau q, -1/\tau q; q)_l}\frac{d\tau}{\tau}.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\left|\frac{1}{2\pi i}\int_{|\tau|=r|q|^{-l}} g(\tau)\theta_q\left(\frac{t}{\tau}\right)\frac{d\tau}{\tau}\right| &\leq \frac{1}{2\pi}|q|^{\frac{l(l-5)}{2}}|t|^{-l}\oint_{C_0}\frac{1}{\left(\frac{1}{r|q|}; |q|\right)_\infty^2} r^{-l}N\left|\frac{d\tau}{\tau}\right| \\
&= M^2 N|q|^{\frac{l(l-5)}{2}}|t|^{-l}r^{-l} \to 0 \quad (l \to \infty)
\end{aligned}$$

as desired. □

Hence $f_1(t)$ has the expression

$$\begin{aligned} f_1(t) = &- \sum_{k\geq 0}^{\infty} \mathrm{Res}\left\{ g(\tau)\theta_q\left(\frac{t}{\tau}\right)\frac{1}{\tau}; \tau = -q^{-2-k} \right\} \\ &- \sum_{k\geq 0}^{\infty} \mathrm{Res}\left\{ g(\tau)\theta_q\left(\frac{t}{\tau}\right)\frac{1}{\tau}; \tau = q^{-2-k} \right\}. \end{aligned}$$

We calculate the residues by the following lemma.

Lemma 5. *For any $k \in \mathbb{Z}_{>0}$, $\lambda \in \mathbb{C}^*$, we have:*

(a) $\mathrm{Res}\left\{ \frac{1}{(\tau/\lambda; q)_\infty}\frac{1}{\tau} : \tau = \lambda q^{-k} \right\} = \frac{(-1)^{k+1}q^{\frac{k(k+1)}{2}}}{(q;q)_k(q;q)_\infty}$,

(b) $\frac{1}{(\lambda q^{-k}; q)_\infty} = \frac{(-\lambda)^{-k} q^{\frac{k(k+1)}{2}}}{(\lambda; q)_\infty \, (q/\lambda; q)_k}, \quad \lambda \notin q^{\mathbb{Z}}.$

By Lemma 5, we have

$$\begin{aligned} &\mathrm{Res}_{\tau=q^{-2-k}}\, g(\tau)\theta_q\left(\frac{t}{\tau}\right)\frac{1}{\tau} = \frac{\theta_q(q^{2+k}t)}{(-q^{-k};q)_\infty}\frac{(-1)^{k+1}q^{\frac{k(k+1)}{2}}}{(q;q)_k(q;q)_\infty} \\ &= \frac{q^{\frac{k(k+1)}{2}}}{(-1;q)_\infty(-q;q)_k}(q^2t)^{-k}q^{-\frac{k(k-1)}{2}}\theta_q(q^2t)\frac{(-1)^{k+1}q^{\frac{k(k+1)}{2}}}{(q;q)_k(q;q)_\infty} \\ &= -\frac{\theta_q(q^2t)}{(q,-1;q)_\infty}\frac{1}{(-q;q)_k(q;q)_k}(-1)^k q^{\frac{k(k-1)}{2}}\left(\frac{1}{t}\right)^k. \end{aligned}$$

In the same way, we calculate other residues.

By taking the sum of all the residues, we obtain

$$f_1(t) = \frac{\theta_q(q^2t)}{(q,-1;q)_\infty}{}_1\varphi_1\left(0,-q;q,\frac{1}{t}\right) + \frac{\theta_q(-q^2t)}{(q,-1;q)_\infty}{}_1\varphi_1\left(0,-q;q,-\frac{1}{t}\right).$$

Therefore, we have reached the following connection formula.

Theorem 2. *For any $x \in \mathbb{C}^*$, we have a connection formula*

$$f_1(t) = \frac{1}{(q,-1;q)_\infty}v_1(t) + \frac{1}{(q,-1;q)_\infty}v_2(t).$$

This connection formula derives a relation between the Ramanujan entire function with base q^2 and the q-Airy functions with base q, namely,

$$\begin{aligned}\mathrm{A}_{q^2}(-q^3t^2) =& \frac{1}{(q,-1;q)_\infty}\theta_q(q^2t)\,\mathrm{Ai}_q\left(-\frac{1}{t}\right)\\ &+\frac{1}{(q,-1;q)_\infty}\theta_q(-q^2t)\,\mathrm{Ai}_q\left(\frac{1}{t}\right).\end{aligned}$$

2.3 Resummation of ${}_2\varphi_0(0,0;-;q,-x/q)$

We recall that the formal solution $f_2(t)$ contains a divergent basic hypergeometric series. We construct a solution $\tilde{f}_2(t,\lambda)$ by the q-Borel-Laplace transformation of the first kind, which is holomorphic on $\mathbb{C}^*\setminus[-\lambda;q]$. Then we can consider a connection formula between the solution $\tilde{f}_2(t,\lambda)$ and $v_1(t)$, $v_2(t)$, which contains a continuous parameter λ. In subsection 2.3.1, we show a relation between the q-exponential function $e_q(x)$ and convergent series ${}_0\varphi_1$ with the base q^2. We also show a connection formula for a resummation of the divergent series ${}_2\varphi_0(0,0;-;q,-x/q)$ by applying Proposition 3 in subsection 2.3.2. Then we obtain a connection formula between the resummation $\tilde{f}_2(t,\lambda)$ and $v_1(t)$, $v_2(t)$ in Theorem 4.

2.3.1 Alternative representation of q-exponential functions

We show an alternative representation of q-exponential functions, which plays an important role later on.

By the triple product identity, we have the following relation between two q-exponential functions

$$e_q(x)=\frac{(q;q)_\infty}{\theta_q(-x)}E_q\left(-\frac{q}{x}\right) \tag{2.14}$$

on $\mathbb{C}^*\setminus[1;q]$. The aim of this subsection is to show a behavior of $e_q(x/q)$ around $x=\infty$ given in the following proposition.

Proposition 3. *For any $x\in\mathbb{C}^*\setminus[1;q]$, the function $e_q(x/q)$ has the following representation*

$$e_q\left(\frac{x}{q}\right)=\frac{(q;q)_\infty}{\theta_q\left(-\frac{x}{q}\right)}{}_0\varphi_1\left(-;q;q^2,\frac{q^5}{x^2}\right)-\frac{(q;q)_\infty}{\theta_q\left(-\frac{x}{q}\right)}\frac{q^2}{(1-q)x}{}_0\varphi_1\left(-;q^3;q^2,\frac{q^7}{x^2}\right).$$

We obtain Proposition 3 from the following lemma.

Lemma 6. *We have*

$$E_q(-x) = {}_0\varphi_1\left(-;q;q^2,qx^2\right) - \frac{q^2}{(1-q)x}{}_0\varphi_1\left(-;q^3;q^2,q^3x^2\right). \qquad (2.15)$$

Proof. We separate the function ${}_0\varphi_0\left(-;-;q,x\right)$ by even terms and odd terms.

$$\begin{aligned}
&{}_0\varphi_0\left(-;-;q,x\right) = \sum_{k\geq 0}\frac{1}{(q;q)_k}(-1)^k q^{\frac{k(k-1)}{2}}x^k \\
&= \sum_{k\geq 0}\frac{(-1)^{2k}q^{\frac{2k(2k-1)}{2}}}{(q;q)_{2k}}x^{2k} + \sum_{k\geq 0}\frac{(-1)^{2k+1}q^{\frac{(2k+1)2k}{2}}}{(q;q)_{2k+1}}x^{2k+1} \\
&= \sum_{k\geq 0}\frac{(-1)^{2k}q^{\frac{2k(2k-1)}{2}}}{(q;q)_{2k}}x^{2k} - \frac{x}{(1-q)}\sum_{k\geq 0}\frac{(-1)^{2k}q^{\frac{(2k+1)2k}{2}}}{(q^2;q)_{2k}}x^{2k} \\
&= \sum_{k\geq 0}\frac{\left\{(-1)^k(q^2)^{\frac{k(k-1)}{2}}\right\}^2 q^k}{(q,q^2;q^2)_k}x^{2k} - \frac{x}{(1-q)}\sum_{k\geq 0}\frac{\left\{(-1)^k(q^2)^{\frac{k(k-1)}{2}}\right\}^2 q^{3k}}{(q^2,q^3;q^2)_k}x^{2k} \\
&= {}_0\varphi_1\left(-;q;q^2,qx^2\right) - \frac{q^2}{(1-q)x}{}_0\varphi_1\left(-;q^3;q^2,q^3x^2\right).
\end{aligned}$$

Since

$$E_q\left(x\right) = {}_0\varphi_0\left(-;-;q,-x\right), \quad |x|<1,$$

we have the result. ☐

Then Lemma 6 implies

$$E_q\left(-\frac{q^2}{x}\right) = {}_0\varphi_1\left(-;q;q^2,\frac{q^5}{x^2}\right) - \frac{q^2}{(1-q)x}{}_0\varphi_1\left(-;q^3;q^2,\frac{q^7}{x^2}\right).$$

Thus we have shown Proposition 3.

2.3.2 Connection formula for the resummation of ${}_2\varphi_0$

Since $f_2(t)$ contains a divergent series ${}_2\varphi_0(0,0;-;q,-x/q)$, we need to give a resummation of the divergent series by the q-Borel-Laplace transformation of the first kind. We set

$${}_2f_0\left(0,0;-;q,\lambda,-x/q\right) := \mathcal{L}^+_{q,\lambda}\circ\mathcal{B}^+_q\,{}_2\varphi_0(0,0;-;q,-x/q).$$

Theorem 3. *For any $x \in \mathbb{C}^* \setminus [-\lambda; q]$, we have*

$$
\begin{aligned}
{}_2f_0\left(0,0;-;q,\lambda,-\frac{x}{q}\right) =& (q;q)_\infty \frac{\theta_{q^2}\left(-\frac{\lambda^2}{qx}\right)}{\theta_q\left(-\frac{\lambda}{q}\right)\theta_q\left(\frac{\lambda}{x}\right)} {}_1\varphi_1\left(0;q;q^2,\frac{q^2}{x}\right) \\
&+ \frac{(q;q)_\infty}{1-q} \frac{\theta_{q^2}\left(-\frac{\lambda^2}{x}\right)}{\theta_q\left(-\frac{\lambda}{q}\right)\theta_q\left(\frac{\lambda}{x}\right)} \frac{\lambda}{x} {}_1\varphi_1\left(0;q^3;q^2,\frac{q^3}{x}\right).
\end{aligned}
$$

Proof. We apply the q-Borel transformation $\mathcal{B}_q^+$ to the divergent series $v(x) = {}_2\varphi_0(0,0;-;q,-x/q)$. Let $\varphi(\xi) = e_q(\xi/q)$. Then $(\mathcal{B}_q^+ v)(\xi)$ is the Maclaurin expansion of $\varphi(\xi)$. The function $\varphi(\xi)$ is holomorphic around $\xi = 0$. By Proposition 3,

$$
\varphi(\xi) = \frac{(q;q)_\infty}{\theta_q\left(-\frac{\xi}{q}\right)} {}_0\varphi_1\left(-;q;q^2,\frac{q^5}{\xi^2}\right) - \frac{(q;q)_\infty}{\theta_q\left(-\frac{\xi}{q}\right)} \frac{q^2}{(1-q)\xi} {}_0\varphi_1\left(-;q^3;q^2,\frac{q^7}{\xi^2}\right).
$$

We set

$$
\begin{aligned}
\varphi_1(\xi) &= \frac{1}{\theta_q\left(-\frac{\xi}{q}\right)} {}_0\varphi_1\left(-;q;q^2,\frac{q^5}{\xi^2}\right), \\
\varphi_2(\xi) &= \frac{1}{\theta_q\left(-\frac{\xi}{q}\right)} \frac{1}{\xi} {}_0\varphi_1\left(-;q^3;q^2,\frac{q^7}{\xi^2}\right).
\end{aligned}
$$

Both $\varphi_1(\xi)$ and $\varphi_2(\xi)$ are meromorphic functions on $\mathbb{C}^*$, which leads to the convergence of $\mathcal{L}_{q,\lambda}^+\varphi$ as we shall show now.

We apply the q-Laplace transformation $\mathcal{L}_{q,\lambda}^+$ to the functions $\varphi_1(\xi)$ and $\varphi_2(\xi)$.

$$
\begin{aligned}
\left(\mathcal{L}^+_{q,\lambda}\varphi_1\right)(x) &= \sum_{n\in\mathbb{Z}} \frac{\varphi_1(\lambda q^n)}{\theta_q\left(\frac{\lambda q^n}{x}\right)} = \sum_{n\in\mathbb{Z}} \frac{1}{\theta_q\left(\frac{\lambda q^n}{x}\right)} \frac{1}{\theta_q\left(-\frac{\lambda q^n}{q}\right)} {}_0\varphi_1\left(-;q;q^2,\frac{q^5}{\lambda^2 q^{2n}}\right) \\
&= \frac{1}{\theta_q\left(-\frac{\lambda}{q}\right)\theta_q\left(\frac{\lambda}{x}\right)} \sum_{n\in\mathbb{Z}} q^{n(n-1)} \left(-\frac{\lambda^2}{qx}\right)^n \sum_{m\geq 0} \frac{(q^2)^{m(m-1)}}{(q;q^2;q^2)_m} \left(\frac{q^{5-2n}}{\lambda^2}\right)^m \\
&= \frac{1}{\theta_q\left(-\frac{\lambda}{q}\right)\theta_q\left(\frac{\lambda}{x}\right)} \sum_{n-m\in\mathbb{Z}} (q^2)^{\frac{(n-m)(n-m-1)}{2}} \left(-\frac{\lambda^2}{qx}\right)^{n-m} \\
&\qquad\qquad \times \sum_{m\geq 0} \frac{(-1)^m (q^2)^{\frac{m(m-1)}{2}}}{(q;q^2;q^2)_m} \left(\frac{q^2}{x}\right)^m \\
&= \frac{\theta_{q^2}\left(-\frac{\lambda^2}{qx}\right)}{\theta_q\left(-\frac{\lambda}{q}\right)\theta_q\left(\frac{\lambda}{x}\right)} {}_1\varphi_1\left(0;q;q^2,\frac{q^2}{x}\right).
\end{aligned}
$$

In the same way, we obtain

$$
\left(\mathcal{L}^+_{q,\lambda}\varphi_2\right)(x) = -\frac{\theta_{q^2}\left(-\frac{\lambda^2}{x}\right)}{\theta_q\left(-\frac{\lambda}{q}\right)\theta_q\left(\frac{\lambda}{x}\right)} \frac{\lambda}{xq^2} {}_1\varphi_1\left(0;q^3;q^2,\frac{q^3}{x}\right).
$$

Therefore we obtain

$$
\begin{aligned}
{}_2f_0\left(0,0;-;q,\lambda,-\frac{x}{q}\right) &= \mathcal{L}^+_{q,\lambda}\circ\mathcal{B}^+_q {}_2\varphi_0\left(0,0;-;q,-\frac{x}{q}\right) \\
&= (q;q)_\infty \frac{\theta_{q^2}\left(-\frac{\lambda^2}{qx}\right)}{\theta_q\left(-\frac{\lambda}{q}\right)\theta_q\left(\frac{\lambda}{x}\right)} {}_1\varphi_1\left(0;q;q^2,\frac{q^2}{x}\right) \\
&\qquad + \frac{(q;q)_\infty}{1-q} \frac{\theta_{q^2}\left(-\frac{\lambda^2}{x}\right)}{\theta_q\left(-\frac{\lambda}{q}\right)\theta_q\left(\frac{\lambda}{x}\right)} \frac{\lambda}{x} {}_1\varphi_1\left(0;q^3;q^2,\frac{q^3}{x}\right).
\end{aligned}
$$

□

A resummation of the formal solution $f_2(t)$ in (2.6) is given by $\tilde{f}_2(t,\lambda) = \theta_q(qt)\theta_q(-q^2t){}_2f_0(0,0;-;q^2,\lambda,qt^2)$. We obtain the connection formula between $\tilde{f}_2(t,\lambda)$ and $v_1(t)$, $v_2(t)$ from Theorem 3.

Theorem 4. *For any $t \in \mathbb{C}^*$ such that $t^2 \in \mathbb{C}^* \setminus [\lambda/q; q^2]$, we have*

$$\begin{aligned}\theta_q(qt)\theta_q(-q^2t)_2f_0(0,0;-;q^2,\lambda,qt^2) =& \frac{(q^2;q^2)_\infty \theta_q(qt)\theta_q(-q^2t)}{\theta_{q^2}\left(-\frac{\lambda}{q^2}\right)\theta_{q^2}\left(\frac{-\lambda}{q^3t^2}\right)} \frac{\theta_q\left(\frac{\lambda}{q^3t}\right)}{2\theta_q(q^2t)} v_1(t) \\ &+ \frac{(q^2;q^2)_\infty \theta_q(qt)\theta_q(-q^2t)}{\theta_{q^2}\left(-\frac{\lambda}{q^2}\right)\theta_{q^2}\left(\frac{-\lambda}{q^3t^2}\right)} \frac{\theta_q\left(-\frac{\lambda}{q^3t}\right)}{2\theta_q(-q^2t)} v_2(t)\end{aligned}$$

where $v_1(t)$, $v_2(t)$ are the local solutions at the infinity:

$$\begin{aligned}v_1(t) &= \theta_q(q^2t)_1\varphi_1\left(0;-q;q,\frac{1}{t}\right) = \theta_q(q^2t)\,\mathrm{Ai}_q\left(-\frac{1}{t}\right), \\ v_2(t) &= \theta_q(-q^2t)_1\varphi_1\left(0;-q;q,-\frac{1}{t}\right) = \theta_q(-q^2t)\,\mathrm{Ai}_q\left(\frac{1}{t}\right).\end{aligned}$$

We prepare some relations between some functions with different bases before proving Theorem 4. It is easily checked that

$$(a;q)_{2m} = (a;q^2)_m(aq;q^2)_m, \quad (a;q)_{2m+1} = (1-a)(aq;q^2)_m(aq^2;q^2)_m. \tag{2.16}$$

In (2.16), we set $a \mapsto q^2$ and $q \mapsto q^2$. Then we have

$$(q^2;q^2)_{2m} = (q^2;q^4)_m(q^4;q^4)_m, \tag{2.17}$$

$$(q^2;q^2)_{2m+1} = (1-q^2)(q^4;q^4)_m(q^6;q^4)_m. \tag{2.18}$$

For any $m \in \mathbb{Z}_{\geq 0}$, we have

$$q^{\frac{2m(2m-1)}{2}} = (q^4)^{\frac{m(m-1)}{2}} \times q^m, \quad q^{\frac{(2m+1)2m}{2}} = (q^4)^{\frac{m(m-1)}{2}} \times q^{3m}. \tag{2.19}$$

By the formulae (2.16), (2.17), (2.18) and (2.19), we obtain the following lemma.

Lemma 7. *We have the following relations between functions ${}_1\varphi_1$ with different bases q and q^4:*

$$\begin{aligned}{}_1\varphi_1(0;-q;q,x) &= {}_1\varphi_1(0;q^2;q^4,-qx^2) - \frac{x}{1-q^2}{}_1\varphi_1(0;q^6;q^4;-q^3x^2), \\ {}_1\varphi_1(0;-q;q,-x) &= {}_1\varphi_1(0;q^2;q^4,-qx^2) + \frac{x}{1-q^2}{}_1\varphi_1(0;q^6;q^4;-q^3x^2).\end{aligned}$$

The theta functions with the different bases have the following relations.

Lemma 8. *For any $x \in \mathbb{C}^*$, we have*

$$\theta_q\left(\frac{x}{q}\right) + \theta_q\left(-\frac{x}{q}\right) = 2\theta_{q^4}\left(\frac{x^2}{q}\right),$$
$$\theta_q\left(\frac{x}{q}\right) - \theta_q\left(-\frac{x}{q}\right) = 2\frac{x}{q}\theta_{q^4}\left(qx^2\right).$$

Corollary 1. *If we put $x \mapsto \lambda/(q^2t)$, we have*

$$\theta_q\left(\frac{\lambda}{q^3t}\right) + \theta_q\left(-\frac{\lambda}{q^3t}\right) = 2\theta_{q^4}\left(\frac{\lambda^2}{q^5t^2}\right),$$
$$\theta_q\left(\frac{\lambda}{q^3t}\right) - \theta_q\left(-\frac{\lambda}{q^3t}\right) = 2\frac{\lambda}{q^3t^2}\theta_{q^4}\left(\frac{\lambda^2}{q^3t^2}\right).$$

We give a proof of Theorem 4.

Proof. In Theorem 3, we put $q \mapsto q^2$ and $x \mapsto -q^3t^2$. Then,

$$\begin{aligned}{}_2f_0(0,0;-;q^2,\lambda,qt^2) = &\frac{(q^2;q^2)_\infty}{\theta_{q^2}\left(-\frac{\lambda}{q^2}\right)\theta_{q^2}\left(\frac{-\lambda}{q^3t^2}\right)}\left\{\theta_{q^4}\left(\frac{\lambda^2}{q^5t^2}\right){}_1\varphi_1\left(0;q^2;q^4,-\frac{q}{t^2}\right)\right.\\ &\left.+\,\theta_{q^4}\left(\frac{\lambda^2}{q^3t^2}\right)\frac{-\lambda}{(1-q^2)q^3t^2}{}_1\varphi_1\left(0;q^6;q^4,-\frac{q^3}{t^2}\right)\right\}.\end{aligned}$$

We rewrite the part $\{\cdots\}$ in the right hand side by using Lemma 7 and Corollary 1.

$$
\theta_{q^4}\left(\frac{\lambda^2}{q^5t^2}\right){}_1\varphi_1\left(0;q^2;q^4,-\frac{q}{t^2}\right)+\theta_{q^4}\left(\frac{\lambda^2}{q^3t^2}\right)\frac{-\lambda}{(1-q^2)q^3t^2}{}_1\varphi_1\left(0;q^6;q^4,-\frac{q^3}{t^2}\right)
$$

$$
=\left\{\frac{\theta_q\left(\frac{\lambda}{q^3t}\right)+\theta_q\left(-\frac{\lambda}{q^3t}\right)}{2}\right\}{}_1\varphi_1\left(0;q^2;q^4,-\frac{q}{t^2}\right)
$$

$$
+\left\{\frac{\theta_q\left(\frac{\lambda}{q^3t}\right)-\theta_q\left(-\frac{\lambda}{q^3t}\right)}{2}\right\}\frac{-1}{(1-q^2)t}{}_1\varphi_1\left(0;q^6;q^4,-\frac{q^3}{t^2}\right)
$$

$$
=\frac{\theta_q\left(\frac{\lambda}{q^3t}\right)}{2}\left\{{}_1\varphi_1\left(0;q^2;q^4,-\frac{q}{t^2}\right)+\frac{-1}{(1-q^2)t}{}_1\varphi_1\left(0;q^6;q^4,-\frac{q^3}{t^2}\right)\right\}
$$

$$
+\frac{\theta_q\left(-\frac{\lambda}{q^3t}\right)}{2}\left\{{}_1\varphi_1\left(0;q^2;q^4,-\frac{q}{t^2}\right)+\frac{1}{(1-q^2)t}{}_1\varphi_1\left(0;q^6;q^4,-\frac{q^3}{t^2}\right)\right\}
$$

$$
=\frac{\theta_q\left(\frac{\lambda}{q^3t}\right)}{2}{}_1\varphi_1\left(0;-q;q,\frac{1}{t}\right)+\frac{\theta_q\left(-\frac{\lambda}{q^3t}\right)}{2}{}_1\varphi_1\left(0;-q;q,-\frac{1}{t}\right).
$$

Hence we have

$$
{}_2f_0(0,0;-;q^2,\lambda,qt^2)=\frac{(q^2;q^2)_\infty}{\theta_{q^2}\left(-\frac{\lambda}{q^2}\right)\theta_{q^2}\left(\frac{-\lambda}{q^3t^2}\right)}\left\{\frac{\theta_q\left(\frac{\lambda}{q^3t}\right)}{2\theta_q(q^2t)}v_1(t)+\frac{\theta_q\left(-\frac{\lambda}{q^3t}\right)}{2\theta_q(-q^2t)}v_2(t)\right\},
$$

completing the proof of Theorem 4. □

By the theta relations in Lemma 8, we obtain another representation of our connection formula as follows:

Proposition 4. *For any $x\in\mathbb{C}^*$, we have*

$$
\begin{aligned}
\mathrm{A}_{q^2}\left(-\frac{q^3}{x^2}\right)=&\frac{2}{(q,-1;q)_\infty}\theta_{q^4}\left(\frac{x^2}{q}\right){}_1\varphi_1(0;q^2;q^4,-qx^2)\\
&+\frac{2}{(q,-1;q)_\infty}\frac{q^2}{q^2-1}\theta_{q^4}\left(\frac{x^2}{q^3}\right){}_1\varphi_1(0;q^6;q^4,-q^3x^2).
\end{aligned}
$$

2.4 Summary

We have solved the connection problem for the second order linear q-difference equation

$$
\left(-q^5t^2\sigma_q^2-\sigma_q+1\right)f(t)=0. \tag{2.20}
$$

This equation has solutions

$$f_1(t) = {}_0\varphi_1(-;0;q^2,q^5t^2) = \mathrm{A}_{q^2}(-q^3t^2),$$
$$f_2(t) = \theta_q(qt)\theta_q(-q^2t)_2\varphi_0(0,0;-;q^2,qt^2)$$

around $t=0$. The solution $f_1(t)$ is the Ramanujan entire function with the base q^2. The solution $f_2(t)$ contains a divergent series ${}_2\varphi_0(0,0;-;q^2,qt^2)$. Thus, we consider ${}_2f_0(0,0;-;q,\lambda,-x/q) := \mathcal{L}^+_{q,\lambda} \circ \mathcal{B}^+_q {}_2\varphi_0(0,0;-;q,-x/q)$, which is a resummation of a divergent series ${}_2\varphi_0(0,0;-;q,-x/q)$. We remark that the function ${}_2f_0(0,0;-;q,\lambda,-x/q)$ is analytic on $\mathbb{C}^* \setminus [-\lambda;q]$. We set $\tilde{f}_2(t,\lambda) := \theta_q(qt)\theta_q(-q^2t)_2f_0(0,0;-;q^2,\lambda,qt^2)$. The equation (2.20) also has solutions

$$v_1(t) = \theta_q(q^2t)_1\varphi_1\left(0;-q;q,\frac{1}{t}\right) = \theta_q(q^2t)\,\mathrm{Ai}_q\left(-\frac{1}{t}\right),$$
$$v_2(t) = \theta_q(-q^2t)_1\varphi_1\left(0;-q;q,-\frac{1}{t}\right) = \theta_q(-q^2t)\,\mathrm{Ai}_q\left(\frac{1}{t}\right)$$

around $t=\infty$. Both solutions $v_1(t)$ and $v_2(t)$ are represented by the q-Airy function, so that they are convergent at infinity.

We gave the connection formulae for the equation (2.20) as follows:

$$\begin{pmatrix} f_1(t) \\ \tilde{f}_2(t,\lambda) \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21}(t,\lambda) & C_{22}(t,\lambda) \end{pmatrix} \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}.$$

The connection formula between $\tilde{f}_2(t,\lambda)$ and $v_1(t)$, $v_2(t)$ holds for any $t \in \mathbb{C}^*$ except poles of $C_{21}(t,\lambda)$ and $C_{22}(t,\lambda)$. Connection coefficients are

$$C_{11} = C_{12} = \frac{1}{(q,-1;q)_\infty},$$

$$C_{21}(t,\lambda) = \frac{(q^2;q^2)_\infty\theta_q(qt)\theta_q(-q^2t)}{\theta_{q^2}\left(-\frac{\lambda}{q^2}\right)\theta_{q^2}\left(\frac{-\lambda}{q^3t^2}\right)} \frac{\theta_q\left(\frac{\lambda}{q^3t}\right)}{2\theta_q(q^2t)}$$

and

$$C_{22}(t,\lambda) = \frac{(q^2;q^2)_\infty\theta_q(qt)\theta_q(-q^2t)}{\theta_{q^2}\left(-\frac{\lambda}{q^2}\right)\theta_{q^2}\left(\frac{-\lambda}{q^3t^2}\right)} \frac{\theta_q\left(-\frac{\lambda}{q^3t}\right)}{2\theta_q(-q^2t)}.$$

We remark that the first connection formula gives a relation between the Ramanujan entire function A_{q^2} and the q-Airy function Ai_q:

$$\begin{aligned}\mathrm{A}_{q^2}(-q^3t^2) =&\frac{1}{(q,-1;q)_\infty}\theta_q(q^2t)\,\mathrm{Ai}_q\left(-\frac{1}{t}\right)\\&+\frac{1}{(q,-1;q)_\infty}\theta_q(-q^2t)\,\mathrm{Ai}_q\left(\frac{1}{t}\right).\end{aligned}$$

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