Gröbner bases of binomial ideals associated with finite graphs and polyominoes

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Abstract

Binomial ideals appear in various areas of pure mathematics as well as of applied mathematics, including algebraic geometry, commutative algebra, combinatorics and algebraic statistics. In the present dissertation, two classes of binomial ideals are studied. One is the class of toric ideals arising from finite simple connected graphs, and the other is the class of ideals generated by 2-minors called polyomino ideals, which are attached to polyominoes. In the former class, a main result is to show the existence of infinite series of finite simple connected graphs G for which the toric ideal I_G of G is generated by quadratic binomials, but possesses no quadratic Gröbner basis. In the latter class, we challenge the outstanding open problem of classifying all polyomino ideals which are prime. First, we prove that the polyomino ideal of a simple polyomino, i.e., a polyomino with no "holes," is prime. Second, we discuss nonsimple polyominoes which are obtained by removing convex polyominoes from their ambient rectangles and show that the attached polyomino ideals are prime.

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Contents

Introduction

The main motivation of this thesis is to study algebraic properties of various binomial ideals associated with particular combinatorial structures. Binomial ideals have been discussed from various viewpoints including algebraic geometry, commutative algebra, combinatorics and algebraic statistics. In commutative algebra, binomial ideals which are defining ideals of toric rings have been well studied. For a given toric ring, properties such as the Krull dimension, normality and Cohen–Macaulayness have been studied. Here, our aim is to study binomial ideals arising from finite connected graphs and polyominoes, and to characterize their algebraic properties.

We now discuss the contents of this thesis in detail. In Chapter 1, we provide background information to support the study of the binomial ideals. In Section 1.1, we recall the definition of Gröbner bases. In Section 1.2, the definition of toric ideals and toric rings are given. Then, in Section 1.3 and Section 1.4, we give some known results on the toric ideals of graphs and the polyomino ideals, respectively.

In Chapter 2, we study the question of how often the toric ideals of finite connected graphs have a quadratic squarefree Gröbner basis. The results given in Chapter 2 are based on [14]. Let K be a field and G be a finite connected simple graph on the vertex set $[n] = \{1, \ldots, n\}$ with the edge set $E(G) = \{e_1, \ldots, e_d\}$. For a finite graph G, we attach the ring $K[G] = K[t_i t_j \mid \{i, j\} \in E(G)] \subset K[t_1, \ldots, t_n]$. This ring is called the *edge ring* of G and has been well studied for example in [23, 24, 25, 26, 37]. Let $K[\mathbf{x}] = K[x_1, \ldots, x_d]$ be the polynomial ring in d variables over K with each $\deg x_i = 1$. If $e = \{i, j\} \in E(G)$, then we write $\mathbf{t}^e$ for $t_i t_j \in K[t_1, \ldots, t_n]$. We define the surjective ring homomorphism $\pi : K[\mathbf{x}] \to K[G]$ by setting $\pi(x_i) = \mathbf{t}^{e_i}$ for $1 \leq i \leq d$. The *toric ideal* I_G of G is the kernel of π.

In general, if an ideal has a quadratic Gröbner basis then the ideal is generated by quadratic binomials. We study graphs G which disprove the converse. Namely, we look for finite graphs with the following property:

($*$) The toric ideal I_G of G is generated by quadratic binomials, but possesses no quadratic Gröbner basis.

Moreover, a graph G satisfying ($*$) is said to be ($*$)-minimal if there does not exist an induced subgraph G' of G which satisfies ($*$). An example of ($*$)-minimal graph is given in [25, Example 2.1]. In Section 2.1, we give an infinite series of $(*)$-minimal graphs. This result is interesting because [25, Example 2.1] has been the only known $(*)$-minimal graph. Another result of this chapter given in Section 2.2 involves a computer search to search for other graphs satisfying $(*)$. In [25], an algorithm to decide if the toric ideal I_G of a given graph G is generated by quadratic binomials was given. We combine several computer programs in order to find quadratic Gröbner bases. By using the algorithm and the software, we find all finite connected graphs with up to 8 vertices satisfying the above condition $(*)$.

In Chapter 3, we study the ideals generated by inner 2-minors of polyominoes. The results given in this chapter are based on [12, 29, 33]. A *polyomino* is a plane figure obtained by joining squares of equal sizes, which are called cells. Polyominoes have been discussed in recreational mathematics and combinatorics. For example, the tiling problem of an $m \times n$ rectangle using a given set of polyominoes has been discussed. In commutative algebra, polyominoes first appeared in 2012 in [28] by assigning each polyomino the set of its inner 2-minors. These ideals of inner 2-minors are called *polyomino ideals*.

Let $\mathcal{P}$ be a polyomino. We denote by $I_{\mathcal{P}}$ the polyomino ideal attached to $\mathcal{P}$ in a suitable polynomial ring over K. The quotient ring defined by $I_{\mathcal{P}}$ is denoted by $K[\mathcal{P}]$. It is natural to investigate the algebraic properties of $I_{\mathcal{P}}$ and $K[\mathcal{P}]$. One of the most fascinating questions is the classification problem of polyominoes whose polyomino ideals are prime ideals. In previous studies, the class of polyominoes whose polyomino ideals are known to be prime has gradually developed (see Figure 1). The class of polyomino ideals can be regarded as a generalization of the class of ideals of all 2-minors of an $m \times n$-matrix studied in [5, 36] and of the one or two sided ladder studied in [3]. In these papers, it was shown that the polyomino ideals for such a class of polyominoes are prime. In [28], it was shown that for a convex polyomino, the quotient ring $K[\mathcal{P}]$ is a normal Cohen–Macaulay domain. It was also shown that polyomino ideals attached to a row or column convex polyomino are prime ideals. Later, in [8], a classification is given of the convex polyomino ideals which are linearly related. Our starting point is the conjecture suggested by Qureshi in [28]. It is conjectured that the polyomino ideal attached to a simple polyomino is prime. Roughly speaking, a simple polyomino is a polyomino with no "hole."

In Section 3.1, we introduce the concept of balanced polyominoes. Then, an important characterization of balanced polyominoes is given: A polyomino is balanced if and only if the polyomino ideal $I_{\mathcal{P}}$ coincides with the lattice ideal attached to $\mathcal{P}$. In Section 3.2, we prove that the polyomino ideal of a balanced polyomino is prime. We further show that if $\mathcal{P}$ is balanced, then $K[\mathcal{P}]$ is a normal Cohen–Macaulay domain of dimension $|V(\mathcal{P})| - |\mathcal{P}|$, where $|V(\mathcal{P})|$ is the number of vertices in $\mathcal{P}$ and $|\mathcal{P}|$ is the number of cells in $\mathcal{P}$.

Qureshi's conjecture was solved independently by Herzog and Saeedi Madani in [11] and Qureshi, Shibuta and the author in [29]. In [11], Qureshi's conjecture is proved by showing that simple polyominoes are balanced. Their result is briefly introduced in Section 3.2. In Section 3.3, we give the proof of Qureshi's conjecture presented in [29]. We prove the conjecture by using a much simpler argument. We prove that $K[\mathcal{P}]$ is isomorphic to the toric ring associated with a weakly chordal bipartite graph.

Another class of polyominoes studied in this thesis is the class of nonsimple polyominoes. In Section 3.4, we investigate the binomial ideals attached to a special class of nonsimple polyominoes, which are, roughly speaking, "rectangle minus convex." We find a toric toric ring generated by monomials of degree two or three which is isomorphic to the quotient ring of the polyomino ideal of "rectangle minus convex." It is proved in [15] that polyomino ideals of "rectangle minus convex" are prime. It is known that every binomial prime ideal is a toric ideal and every toric ideal is a binomial prime ideal (see [7, 34]). This result is interesting because it is generally not easy to find a toric ring which is isomorphic to a given binomial prime ideal.

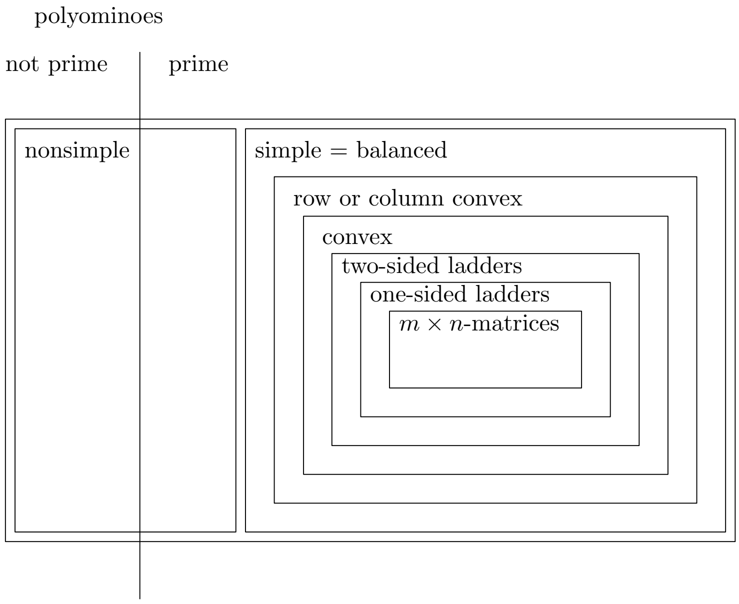


Figure 1: Overview of classes of polyominoes

Chapter 1

Background

In this chapter, we provide background information to support the study of the binomial ideals. We give a definition of Gröbner bases and toric ideals. Then, we introduce known results on the binomial ideals associated with finite graphs and polyominoes.

1.1 Gröbner bases

In this section, we briefly introduce Gröbner bases. Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field K with each $\deg x_i = 1$. Let $\mathrm{Mon}(S)$ be the set of monomials of S. For monomials $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ and $\mathbf{x}^{\mathbf{b}} = x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$ of S, we say that $\mathbf{x}^{\mathbf{b}}$ *divides* $\mathbf{x}^{\mathbf{a}}$ if each $b_i \leq a_i$. We write $\mathbf{x}^{\mathbf{b}} | \mathbf{x}^{\mathbf{a}}$ if $\mathbf{x}^{\mathbf{b}}$ divides $\mathbf{x}^{\mathbf{a}}$. Let $\mathcal{M}$ be a nonempty subset of $\mathrm{Mon}(S)$. A monomial $\mathbf{x}^{\mathbf{a}}$ is said to be a *minimal element* of $\mathcal{M}$ if the following condition is satisfied: If there exists an element $\mathbf{x}^{\mathbf{b}} \in \mathcal{M}$ which divides $\mathbf{x}^{\mathbf{a}}$, then $\mathbf{x}^{\mathbf{a}} = \mathbf{x}^{\mathbf{b}}$. Let $\mathcal{M}^{\min}$ be the set of minimal elements of $\mathcal{M}$. We give some known classical results for Gröbner bases without proving them. The proofs for these results can be found in the literature, for example, in [10, 13].

Proposition 1.1.1 (Dickson's lemma). *Let $\mathcal{M}$ be a nonempty subset of* $\mathrm{Mon}(S)$*. Then $\mathcal{M}^{\min}$ is a finite set.*

Recall that a *partial order* on a set P is a relation $\leq$ on P such that for all $x, y, z \in P$ we have

(a) $x \leq x$ (reflexivity):

(b) $x \leq y$ and $y \leq x \Rightarrow x = y$ (antisymmetry);

(c) $x \leq y$ and $y \leq z \Rightarrow x \leq z$ (transitivity).

A *total order* on a set P is a partial order $\leq$ on P such that, for any $x, y \in P$, one has either $x \leq y$ or $y \leq x$. A *monomial order* on S is a total order on $\mathrm{Mon}(S)$ such that

(a) $1 < u$ for all $1 \neq u \in \mathrm{Mon}(S)$;

(b) if $u, v \in \mathrm{Mon}(S)$ and $u < v$, then $uw < vw$ for all $w \in \mathrm{Mon}(S)$.

We will work with a fixed monomial order $<$ on S. Let $f = \sum_{u \in \mathrm{Mon}(S)} a_u u$. The *initial monomial* of f with respect to $<$ is the biggest monomial with respect to $<$ among the monomials appearing in f. We write $\mathrm{in}_<(f)$ for the initial monomial of f with respect to $<$. The coefficient of $\mathrm{in}_<(f)$ in f is called the *leading coefficient* or the *initial coefficient* of f.

Lemma 1.1.2. *Let u, v be monomials of S and f, g nonzero polynomials of S. Then the following holds.*

(a) *If u divides v, then $u \leq v$;*

(b) $\mathrm{in}_<(uf) = u\,\mathrm{in}_<(f)$*;*

(c) $\mathrm{in}_<(fg) = \mathrm{in}_<(f)\,\mathrm{in}_<(g)$*;*

(d) $\mathrm{in}_<(f+g) \leq \max\{\mathrm{in}_<(f), \mathrm{in}_<(g)\}$ *with equality if* $\mathrm{in}_<(f) \neq \mathrm{in}_<(g)$.

Recall that an ideal is said to be a *monomial ideal* if it is generated by monomials. Let $I \subset S$ be a monomial ideal. It follows that I is generated by a subset $\mathcal{N} \subset \mathrm{Mon}(S) \cap I$ if and only if $(I \cap \mathcal{N})^{\min} \subset \mathcal{N}$. Hence, $(I \cap \mathcal{N})^{\min}$ is the unique minimal system of monomial generators of I. Dickson's lemma guarantees that $(I \cap \mathcal{N})^{\min}$ is a finite set. Therefore, every monomial ideal I of S is finitely generated. Let I be a nonzero ideal of S. The *initial ideal* of I with respect to $<$ is the monomial ideal of S which is generated by $\{\mathrm{in}_<(f) \mid 0 \neq f \in I\}$. We denote the initial ideal of I by $\mathrm{in}_<(I)$. The set $(\mathrm{in}_<(I) \cap \mathrm{Mon}(S))^{\min}$ is the minimal system of generators of $\mathrm{in}_<(I)$. Since $\mathrm{in}_<(I) \cap \mathrm{Mon}(S) = \{\mathrm{in}_<(f) \mid 0 \neq f \in I\}$, there exists a finite number of nonzero polynomials $g_1, \ldots, g_s \in I$ such that $\mathrm{in}_<(I)$ is generated by their initial monomials given by $\mathrm{in}_<(g_1), \mathrm{in}_<(g_2), \ldots, \mathrm{in}_<(g_s)$.

Definition 1.1.3. Let I be a nonzero ideal of S. A finite set of nonzero polynomials $\{g_1, g_2, \ldots, g_s\}$ with each $g_i \in I$ is said to be a *Gröbner basis* of I with respect to $<$ if $\mathrm{in}_<(I)$ is generated by the monomials $\mathrm{in}_<(g_1), \mathrm{in}_<(g_2), \ldots, \mathrm{in}_<(g_s)$.

It is known that a Gröbner basis of I with respect to $<$ exists. If $\mathcal{G}$ is a Gröbner basis of I with respect to $<$, then every finite set $\mathcal{G}'$ with $\mathcal{G} \subset \mathcal{G}' \subset I$ is also a Gröbner basis of I with respect to $<$ and if $f_1, \ldots, f_s$ are nonzero polynomials belonging to I with each $\mathrm{in}_<(f_i) = \mathrm{in}_<(g_i)$ then $\{f_1, \ldots, f_s\}$ is a Gröbner basis of I with respect to $<$.

Now we note some known facts about Gröbner bases. An important property of Gröbner bases is the following:

Proposition 1.1.4. *Let I be an ideal and $\mathcal{G}$ be a Gröbner basis of I with respect to some monomial order. Then, $\mathcal{G}$ is a generating set of I.*

A Gröbner basis $\{g_1, \ldots, g_s\}$ is called *reduced* if the following conditions are satisfied:

(a) The coefficient of $\text{in}_<(g_i)$ is 1 for $1 \leq i \leq s$;

(b) If $i \neq j$, then none of the monomials appearing in g_j are divisible by $\text{in}_<(g_i)$.

Proposition 1.1.5. *For a given monomial order $<$, a reduced Gröbner basis exists and is uniquely determined.*

Proposition 1.1.6. *Let I and J be nonzero ideals of S. Then, $I = J$ if and only if the reduced Gröbner basis of I with respect to a given monomial order $<$ coincides with that of J.*

If the degree of all monomials appearing in a polynomial are equal to q, then the polynomial is called *homogeneous* polynomial of degree q. An ideal is called a *homogeneous* ideal if the ideal is generated by homogeneous polynomials. A *binomial ideal* is an ideal which is generated by binomials. The following proposition is elementary but it is important for our study of binomial ideals.

Proposition 1.1.7. *Let $I \subset S$ be an ideal and $<$ a monomial order on S.*

(a) *If I is homogeneous, then the reduced Gröbner basis of I with respect to $<$ consists of homogeneous polynomials.*

(b) *If I is a binomial ideal, then the reduced Gröbner basis of I with respect to $<$ consists of binomials.*

The ideals studied in this thesis are homogeneous binomial ideals. In particular, the reduced Gröbner bases of our ideals consist of homogeneous binomials. We say a monomial $x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$ is *squarefree* if each $a_i \in \{0, 1\}$, $1 \leq i \leq n$. A Gröbner basis $\mathcal{G}$ of a binomial ideal I consisting of binomials is called a *squarefree* Gröbner basis if every (initial and non-initial) monomial appearing in $\mathcal{G}$ is squarefree.

Let $I \subset S$ be an ideal. A finite set $\mathcal{G}_u \subset I$ is called a *universal Gröbner basis* if it is a Gröbner basis with respect to any monomial order. By the following proposition, a universal Gröbner basis always exists.

Proposition 1.1.8. *Let $I \subset S$ be an ideal. Then, there exists only finitely many initial ideals for I.*

A homogeneous binomial $f = f^+ - f^- \in I$ is called a *primitive binomial* in I if there does not exist any homogeneous binomial $h = h^+ - h^- \in I$ with $\deg h < \deg f$ such that $h^+|f^+$ and $h^-|f^-$. The following proposition is important for finding a universal Gröbner basis for a homogeneous binomial ideal.

Proposition 1.1.9. *Let $I \subset S$ be a homogeneous binomial ideal. The set of primitive binomials in I is a universal Gröbner basis of I.*

Note that a universal Gröbner basis consisting of primitive binomials is not necessarily reduced (see e.g., [34, Chapter 4]).

Next, we state the elimination theorem. Let $S = K[x_1, \dots, x_n]$ be a polynomial ring. We write $B_{i_1,\dots,i_m}$ for the subset of S consisting of those polynomials $f \in S$ such that each monomial appearing in f is a monomial in the variables $x_{i_1}, x_{i_2}, \dots, x_{i_m}$, where $1 \le i_1 < i_2 < \cdots < i_m \le n$. If $f, g \in B_{i_1,\dots,i_m}$, then the sum and the product of f and g also belong to $B_{i_1,\dots,i_m}$. Thus, $B_{i_1,\dots,i_m}$ itself is a polynomial ring. A monomial order $<$ on S can naturally induce the monomial order $<'$ on $B_{i_1,\dots,i_m}$ defined as follows: If monomials u and v belong to $B_{i_1,\dots,i_m}$, we have $u <' v$ if and only if $u < v$ in S. For the sake of simplicity, we denote the naturally induced monomial order $<'$ on $B_{i_1,\dots,i_m}$ by $<$.

Proposition 1.1.10 (The elimination theorem)**.** *Let $<$ be a monomial order on S and $\mathcal{G}$ be a Gröbner basis of an ideal $I \subset S$ with respect to $<$. Suppose that*

- *For each $g \in \mathcal{G}$, one has $g \in B_{i_1,\dots,i_m}$ if $\mathrm{in}_<(g) \in B_{i_1,\dots,i_m}$.*

Then $\mathcal{G} \cap B_{i_1,\dots,i_m}$ is a Gröbner basis of $I \cap B_{i_1,\dots,i_m}$ with respect to $<$ on $B_{i_1,\dots,i_m}$.

1.2 Toric ideals and toric rings

Let $A = (a_{ij})_{\substack{1\le i\le n\\ 1\le j\le d}}$ be an $n \times d$ matrix and

$$\mathbf{a}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix}, \quad 1 \le j \le d \tag{1.1}$$

be the column vectors of A.

A matrix $A \in \mathbb{Z}^{n\times d}$ is called a *configuration matrix* if there exists $\mathbf{c} \in \mathbb{R}^n$ such that

$$\mathbf{a}_j \cdot \mathbf{c} = 1, \quad 1 \le j \le d.$$

Let $t_1, \dots, t_n$ be variables. Let $A \in \mathbb{Z}^{n\times d}$ be a configuration matrix. With each column vector $\mathbf{a}_j$, we associate the monomial $\mathbf{t}^{\mathbf{a}_j} = t_1^{a_{1j}} t_2^{a_{2j}} \cdots t_n^{a_{nj}}$, allowing negative powers. We define the *toric ring* $K[A]$ associated with a configuration matrix A by $K[A] = K[\mathbf{t}^{\mathbf{a}_j} \mid 1 \le j \le d]$.

Note that, in this thesis, we sometimes define a toric ring for a set of monomials of the same degree without defining its configuration matrix.

Let $\pi : S \to K[A]$ be a surjective ring homomorphism with the setting $\pi(x_j) = \mathbf{t}^{\mathbf{a}_j}$. Then the *toric ideal* of A denoted by I_A is the kernel of π.

Equivalently, we can define the toric ideal as follows. For a given configuration matrix $A \in \mathbb{Z}^{n\times d}$, we define $\ker_{\mathbb{Z}} A = \{\mathbf{b} \in \mathbb{Z}^d \mid A\mathbf{b} = \mathbf{0}\}$.

Lemma 1.2.1. *If a column vector*

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_d \end{bmatrix} \in \mathbb{Z}^d$$

belongs to $\ker_{\mathbb{Z}} A$*, then*

$$b_1 + \cdots + b_d = 0.$$

Now for a given column vector $\mathbf{b} = [b_1, \ldots, b_d]^{\mathrm{T}} \in \ker_{\mathbb{Z}} A$, we define the binomial $f_{\mathbf{b}} \in K[x_1, \ldots, x_d]$ as

$$f_{\mathbf{b}} = \prod_{b_i > 0} x_i^{b_i} - \prod_{b_j < 0} x_j^{-b_j}.$$

It is clear that the binomial ideal

$$(\{f_{\mathbf{b}} \mid \mathbf{b} \in \ker_{\mathbb{Z}} A\})$$

coincides with I_A. From Lemma 1.2.1, we see that the toric ideals are generated by homogeneous binomials.

Therefore, it follows from Proposition 1.1.7 that every reduced Gröbner basis of I_A for a given configuration matrix A consists of homogeneous binomials.

1.3 Toric ideals and toric rings associated with finite simple graphs

Let G be a finite simple connected graph on the vertex set $[n] = \{1, 2, \ldots, n\}$ and the edge set $E(G) = \{e_1, \ldots, e_d\}$. Recall that a finite graph G is called *simple* if it has no loops and no multiple edges. Given a subset W of $[n]$, we define the *induced subgraph* of G on W to be the subgraph G_W on W consisting of those edges $\{i, j\} \in E(G)$ with $\{i, j\} \subset W$. A *complete graph* on $[n]$ is a finite simple graph consisting of all possible edges on $[n]$. The complementary graph $\overline{G}$ of a finite graph G is the finite graph on $[n]$ whose edge set $E(\overline{G})$ consists of $\{i, j\} \subset [n]$ with $\{i, j\} \notin E(G)$.

A *cycle* in G of length q is a subgraph C of G such that

$$E(C) = \{\{i_1, i_2\}, \{i_2, i_3\}, \ldots, \{i_{q-1}, i_q\}, \{i_q, i_1\}\}$$

where $i_1, \ldots i_q$ are vertices of G and $i_j \neq i_k$ if $j \neq k$. A cycle is called an *even (odd)* cycle if its length q is even (odd).

A graph is called *bipartite* if there is a decomposition $[n] = V_1 \cup V_2$ such that every edge of G is of the form $\{i, j\}$ with $i \in V_1$ and $j \in V_2$. Bipartite graphs can be characterized as follows:

Proposition 1.3.1. *A finite graph G is a bipartite graph if and only if there exist no cycles of odd length in G.*

We now introduce the toric ideals arising from finite graphs. Let K be a field and $K[\mathbf{t}] = K[t_1, \ldots, t_n]$ the polynomial ring in n variables over K. If $e = \{i, j\} \in E(G)$, then we write $\mathbf{t}^e$ for $t_i t_j \in K[\mathbf{t}]$. The edge ring of G denoted by $K[G]$ is a subring of $K[\mathbf{t}]$ with $K[\mathbf{t}^{e_1}, \ldots, \mathbf{t}^{e_d}]$. Let $K[\mathbf{x}] = K[x_1, \ldots, x_d]$ be the polynomial ring in d variables over K with each $\deg x_i = 1$. We define the surjective ring homomorphism $\pi : K[\mathbf{x}] \to K[G]$ by setting $\pi(x_i) = \mathbf{t}^{e_i}$ for $1 \leq i \leq d$. The *toric ideal* I_G of G is the kernel of π.

A walk of length q of G connecting $v_1 \in V(G)$ and $v_{q+1} \in V(G)$ is a finite sequence of the form

$$\Gamma = (\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_q, v_{q+1}\}) \tag{1.2}$$

with each $\{v_k, v_{k+1}\} \in E(G)$. An *even* (*odd*) *walk* is a walk of even (odd) length. A walk Γ of the form (1.2) is called *closed* if $v_{q+1} = v_1$. A cycle in G can be naturally regarded as a closed walk

$$C = (\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_q, v_1\}) \tag{1.3}$$

with $q \geq 3$ and $v_i \neq v_j$ for all $1 \leq i < j \leq q$. A *chord* of a cycle (1.3) is an edge $e \in E(G)$ of the form $e = \{v_i, v_j\}$ for some $1 \leq i < j \leq q$ with $e \notin E(C)$. If a cycle (1.3) is even, an *even-chord* (*odd-chord*) of (1.3) is a chord $e = \{v_i, v_j\}$ with $1 \leq i < j \leq q$ such that $j - i$ is odd (even). If $e = \{v_i, v_j\}$ and $e' = \{v_{i'}, v_{j'}\}$ are chords of a cycle (1.3) with $1 \leq i < j \leq q$ and $1 \leq i' < j' \leq q$, then we say that e and e' *cross* in C if the following conditions are satisfied:

(a) either $i < i' < j < j'$ or $i' < i < j' < j$;

(b) either $\{\{v_i, v_{i'}\}, \{v_j, v_{j'}\}\} \subset E(C)$ or $\{\{v_i, v_{j'}\}, \{v_j, v_{i'}\}\} \subset E(C)$.

A *minimal* cycle of G is a cycle that has no chords. If C_1 and C_2 are cycles of G that have no common vertices, then a *bridge* between C_1 and C_2 is an edge $\{i, j\}$ of G with $i \in V(C_1)$ and $j \in V(C_2)$.

The toric ideal I_G is generated by the binomials associated with even closed walks. Given an even closed walk $\Gamma = (e_{i_1}, e_{i_2}, \ldots, e_{i_{2q}})$ of G, we write f_Γ for the binomial

$$f_\Gamma = \prod_{k=1}^{q} x_{i_{2k-1}} - \prod_{k=1}^{q} x_{i_{2k}} \in I_G.$$

The following is known ([37, Proposition 3.1], [25, Lemma 1.1] and [34, Chapter 9]).

Proposition 1.3.2. *Let G be a connected graph. Then, I_G is generated by all the binomials f_Γ, where Γ is an even closed walk of G. In particular, $I_G = (0)$ if and only if G has at most one cycle and the cycle is odd.*

Note that, for a binomial $f \in I_G$, $\deg(f) = 2$ if and only if there exists an even cycle C of G of length 4 such that $f = f_C$. On the other hand, a criterion for the existence of a set of quadratic binomial generators of I_G is given in [25, Theorem 1.2].

Proposition 1.3.3. *Let G be a finite connected graph. Then, I_G is generated by quadratic binomials if and only if the following conditions are satisfied:*

(a) *If C is an even cycle of G of length ≥ 6, then either C has an even-chord or C has three odd-chords e, e' and e'' such that e and e' cross in C;*

(b) *If C_1 and C_2 are minimal odd cycles with exactly one common vertex, then there exists an edge $\{i, j\} \notin E(C_1) \cup E(C_2)$ with $i \in V(C_1)$ and $j \in V(C_2)$;*

(c) *If C_1 and C_2 are minimal odd cycles with no common vertices, then there exist at least two bridges between C_1 and C_2.*

Even if $I_G = (0)$, we say that "I_G is generated by quadratic binomials" and "I_G possesses a quadratic Gröbner basis."

If G is bipartite, then the following is shown in [24]:

Proposition 1.3.4. *Let G be a bipartite graph. Then the following conditions are equivalent:*

(a) *Every cycle of length ≥ 6 has a chord in G;*

(b) *I_G possesses a quadratic Gröbner basis;*

(c) *I_G is generated by quadratic binomials.*

If G is not bipartite, then the conditions (b) and (c) are not equivalent. The following example is known from [25, Example 2.1].

Example 1.3.5. Let G be the graph in Figure 1.1. Then, I_G is generated by quadratic binomials. However, I_G has no quadratic Gröbner bases.

1.4 Polyominoes and polyomino ideals

In order to define polyominoes, first we introduce some notation. We consider the natural partial order on $\mathbb{N}^2$ as follows: $(i, j) \leq (k, l)$ if and only if $i \leq k$ and $j \leq l$. Let $a, b \in \mathbb{N}^2$ with $a \leq b$ and $a \neq b$, then the set $[a, b] = \{c \in \mathbb{N}^2 \mid a \leq c \leq b\}$ is called an *interval* of $\mathbb{N}^2$. If $a = (i, j)$ and $b = (k, l)$, then the interval $[a, b]$ is called a *proper interval* if $i < k$ and $j < l$. Further, an interval is called a *horizontal (*or *vertical) edge interval* if $j = l$ $(i = k)$.

For a proper interval $[a, b]$, the elements a, b together with $c = (k, j)$ and $d = (i, l)$ are called the corners of $[a, b]$. The elements a, b are the *diagonal corners* and c, d are the *anti-diagonal corners* of $[a, b]$.

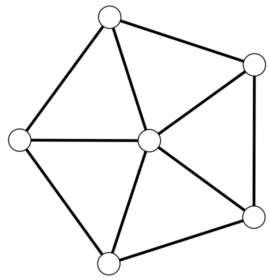


Figure 1.1: Wheel with 6 vertices.

The interval $C = [a, a + (1, 1)]$ is called a *cell* in $\mathbb{N}^2$ with lower left corner a. The *edges* of a cell $C = [a, a + (1, 1)]$ are the sets $\{a, a + (1, 0)\}$, $\{a, a + (0, 1)\}$, $\{a + (1, 0), a + (1, 1)\}$ and $\{a + (0, 1), a + (1, 1)\}$. We denote the set of edges of C by $E(C)$. The corners of a cell C are called the *vertices* of C and the set of vertices of C is denoted by $V(C)$.

Let $\mathcal{P}$ be a finite collection of cells. Two cells $C, D \in \mathcal{P}$ are said to be *connected* in $\mathcal{P}$, if there exists a finite sequence of cells $\mathcal{C} : C = C_1, C_2, \dots, C_m = D$ in $\mathcal{P}$ such that each of C_i and C_{i+1}, $i = 1, \dots, m-1$ have an edge in common. A finite collection of cells $\mathcal{P}$ is called a *polyomino* if for every pair of cells in $C, D \in \mathcal{P}$, C and D are connected in $\mathcal{P}$. For example, the collection of cells shown in Figure 1.4 is a polyomino.

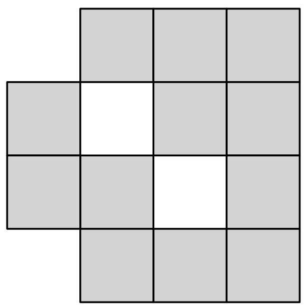


Figure 1.2: A polyomino

Let $\mathcal{P}$ be a polyomino. We set $V(\mathcal{P}) = \bigcup_{C\in\mathcal{P}} V(C)$ and $E(\mathcal{P}) = \bigcup_{C\in\mathcal{P}} E(C)$ and call them the *vertex set* of $\mathcal{P}$ and the *edge set* of $\mathcal{P}$, respectively.

Let $\mathcal{P}$ be a polyomino and K be a field. Let $S = K[x_{ij} \mid (i, j) \in V(\mathcal{P})]$ be the polynomial ring over K. A 2–minor $x_{ij}x_{kl} - x_{il}x_{kj}$ is called an *inner minor* of $\mathcal{P}$ if all the cells $[(r, s), (r+1, s+1)]$ with $i \leq r \leq k-1$ and $j \leq s \leq l-1$ belong to $\mathcal{P}$. In such a case, the interval $[(i, j), (k, l)]$ is called an *inner interval* of $\mathcal{P}$. The ideal $I_{\mathcal{P}}$ generated by all inner minors of $\mathcal{P}$ is called the *polyomino ideal* of $\mathcal{P}$. A *horizontal* *cell interval* of $\mathcal{P}$ is an inner interval $[(i,j),(k,l)]$ of $\mathcal{P}$ with $l = j+1$ and $k > i$. A *vertical cell interval* in $\mathcal{P}$ is defined similarly. A polyomino $\mathcal{P}$ is called *row convex* if for any two vertices (i,j), (k,l) in $V(\mathcal{P})$ with $i < k$ and $l = j+1$, $[(i,j),(k,j+1)]$ is an inner interval of $\mathcal{P}$. Similarly a polyomino $\mathcal{P}$ is called *column convex* if for any two vertices (i,j), (k,l) with $j < l$ and $k = i+1$, $[(i,j),(i+1,l)]$ is an inner minor of $\mathcal{P}$. Finally, a polyomino is called *convex* if it is row convex and column convex. We now introduce a known result from [28].

Proposition 1.4.1. *Let $\mathcal{P}$ be a row or column convex polyomino. Then $I_{\mathcal{P}}$ is a prime ideal.*

Let $\mathcal{P}$ be a polyomino and let $[a,b]$ be an interval with the property $\mathcal{P} \subset [a,b]$. According to [28], a polyomino $\mathcal{P}$ is called *simple*, if for any cell C not belonging to $\mathcal{P}$, there exists a path $C = C_1, C_2, \ldots, C_m = D$ with $C_i \notin \mathcal{P}$ for $i = 1, \ldots, m$ and such that D is not a cell of $[a,b]$. Roughly speaking, a simple polyomino is a polyomino with no "holes." For example the polyomino displayed in Figure 1.3 is not simple.

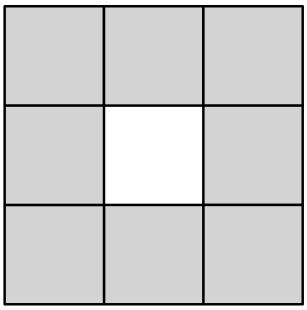


Figure 1.3: A polyomino that is not simple but is prime

In [28], the following conjecture is suggested.

Conjecture 1.4.2 (Qureshi's conjecture [28])**.** Let $\mathcal{P}$ be a simple polyomino. Then $I_{\mathcal{P}}$ is a prime ideal.

The converse of Qureshi's conjecture is not true. In fact, the polyomino in Figure 1.3 is not simple but its polyomino ideal is prime. The motivation of our study of polyomino ideal is to solve this conjecture. In Chapter 3 of this thesis, we solve this conjecture and discuss the primeness of polyomino ideals of wider class of polyominoes.

Chapter 2

Toric ideals generated by quadratic binomials with no quadratic Gröbner bases

Let G be a finite connected simple graph on the vertex set $[n] = \{1, 2, \ldots, n\}$ with $E(G) = \{e_1, \ldots, e_d\}$ its edge set. Let $K[G]$ be the toric ring and I_G the toric ideal as defined in Section 1.3

It is clear that if a toric ideal I_G has a quadratic Gröbner basis, then the ideal is generated by quadratic binomials. However, the converse is not true [25].

We study finite connected simple graphs G satisfying the following condition:

($*$) I_G is generated by quadratic binomials and I_G possesses no quadratic Gröbner basis.

Historically, examples of homogeneous binomial ideals generated by quadratic binomials that have no quadratic Gröbner basis have been discussed. Such examples are presented by Ohsugi and Hibi in [25], Roos and Sturmfels in [30] and Schenck and Stillman in [31].

We say that a finite connected simple graph G is ($*$)-*minimal* if G satisfies the condition ($*$) and if no induced subgraph H ($\neq G$) satisfies the condition ($*$). A ($*$)-minimal graph is given in [25, Example 2.1].

A nontrivial infinite series of ($*$)-minimal finite graphs is given in Section 2.1. In Section 2.2, we implement a combinatorial characterization for I_G to be generated by quadratic binomials (Proposition 1.3.3), and, by using computer search, we classify the finite graphs G with up to 8 vertices satisfying the condition ($*$).

2.1 Toric ideals of the suspension of graphs

In this section, we study the existence of quadratic Gröbner bases of toric ideals of the suspension of graphs.

Let G be a graph with the vertex set $V(G) = [n] = \{1, 2, \dots, n\}$ and edge set $E(G)$. Recall that if a graph G' on the vertex set $V(G') \subset V(G)$ satisfies $E(G') = \{\{i, j\} \in E(G) \mid i, j \in V(G')\}$, then G' is called an *induced subgraph* of G. The following proposition is a fundamental and important fact regarding the toric ideals of graphs.

Proposition 2.1.1 ([23])**.** *Let G' be an induced subgraph of a graph G. Then, $K[G']$ is a combinatorial pure subring of $K[G]$. In particular,*

(a) *If I_G possesses a quadratic Gröbner basis, then so does $I_{G'}$.*

(b) *If I_G is generated by quadratic binomials, then so is $I_{G'}$.*

The *suspension* of the graph G is a graph $\widehat{G}$ whose vertex set is $[n + 1] = V(G) \cup \{n + 1\}$ and edge set is $E(G) \cup \{\{i, n + 1\} \mid i \in V(G)\}$. Note that, any graph G is an induced subgraph of its suspension $\widehat{G}$. The *edge ideal* of G is the monomial ideal $I(G)$ of $K[\mathbf{t}]$ which is generated by $\{t_i t_j \mid \{i, j\} \in E(G)\}$. See, e.g., [10, Chapter 9].

It is easy to see that the edge ring $K[\widehat{G}]$ of the suspension $\widehat{G}$ of G is isomorphic to the *Rees algebra*

$$\mathcal{R}(I(G)) = \bigoplus_{j=0}^{\infty} I(G)^j s^j = K[t_1, \dots, t_n, \{t_i t_j s\}_{\{i,j\} \in E(G)}]$$

of the edge ideal $I(G)$ of G.

We now characterize graphs G such that $I_{\widehat{G}}$ is generated by quadratic binomials. Recall that the *complementary* graph $\overline{G}$ of G is the graph whose vertex set is $[n]$ and whose edges are the non-edges of G. A *chordal graph* is a finite graph such that each cycle of length greater than 3 has a chord. Note that every induced subgraph of a chordal graph is also chordal. Moreover, a graph G is said to be *co-chordal* if $\overline{G}$ is chordal. A graph G is called a *$2K_2$-free graph* if it is connected and does not contain two independent edges as an induced subgraph. For a connected graph G, the following hold in general:

- G is $2K_2$-free $\Leftrightarrow$ any cycle of $\overline{G}$ of length 4 has a chord in $\overline{G}$;
- G is co-chordal $\Rightarrow$ G is $2K_2$-free,

Moreover, it is known (e.g., [2]) that

Lemma 2.1.2. *Let G be a connected graph. Then,*

(a) *If G is co-chordal, then any cycle of G of length ≥ 5 has a chord;*

(b) *If G is $2K_2$-free, then any cycle of G of length ≥ 6 has a chord.*

The toric ideals I_G of $2K_2$-free graphs G have been studied in [4, 26]. The edge ideals $I(G)$ of $2K_2$-free graphs G are studied by many researchers, such as in [20] and [21]. (Note that these papers call $2K_2$-free graphs a different name.)

Now, we characterize the toric ideals $I_{\widehat{G}}$ of $\widehat{G}$ that are generated by quadratic binomials in terms of $2K_2$-free graphs.

Theorem 2.1.3. *Let G be a finite connected graph. Then, the following conditions are equivalent:*

(a) *$I_{\widehat{G}}$ is generated by quadratic binomials;*

(b) *G is $2K_2$-free and I_G is generated by quadratic binomials;*

(c) *G is $2K_2$-free and satisfies condition* (a) *in Proposition 1.3.3.*

Proof. ((a) $\Rightarrow$ (b)) Suppose $I_{\widehat{G}}$ is generated by quadratic binomials. Using Proposition 2.1.1, we know I_G is also generated by quadratic binomials. We have to show that G is $2K_2$-free. For a proof by contradiction, let us assume that G is not $2K_2$-free. Then we have a pair of disjoint edges, say $\{i,j\},\{k,l\} \in E(G)$ such that none of $\{i,k\},\{i,l\},\{j,k\},\{j,l\}$ is an edge in G. By the definition of suspension of graphs, $\widehat{G}$ has the graph in Figure 2.1 as an induced subgraph. This contradicts the fact that $I_{\widehat{G}}$ is generated by quadratic binomials. Hence G is $2K_2$-free.

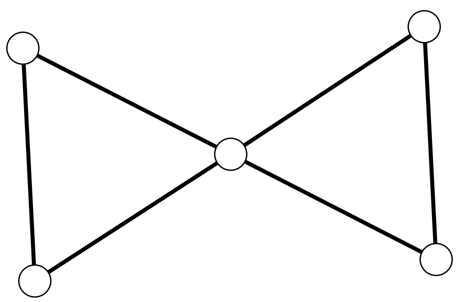


Figure 2.1: Two triangles with one common vertex.

((b) $\Rightarrow$ (c)) is obvious by Proposition 1.3.3.

((c) $\Rightarrow$ (b)) Suppose G is $2K_2$-free and satisfies the condition (a) in Proposition 1.3.3. Let C_1, C_2 be a pair of minimal odd cycles of G. Suppose C_1 and C_2 have a vertex v in common. Then, there exist an edge $\{i,j\}$ in C_1 and an edge $\{k,l\}$ in C_2 such that $i,j,k,l \neq v$. Since G is $2K_2$-free, at least one of $\{i,k\},\{i,l\},\{j,k\},\{j,l\}$ is an edge in G. Therefore, the condition (b) in Proposition 1.3.3 is satisfied. Next, we show that condition (c) in Proposition 1.3.3 is also satisfied. Let C_1, C_2 be the vertex disjoint pair of minimal odd cycles of G. For each pair of edges $\{i,j\}$ in C_1

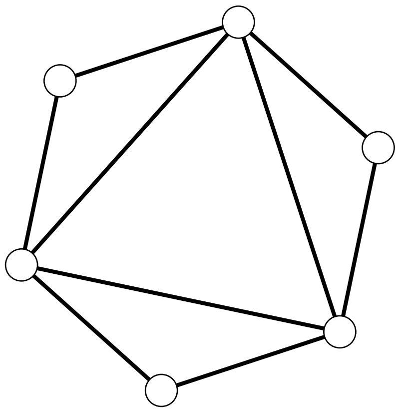


Figure 2.2: An even cycle with three odd chords.

and $\{k, l\}$ in C_2, we have at least one of $\{i, k\}, \{i, l\}, \{j, k\}, \{j, l\}$ in $E(G)$. Since C_1 has at least 3 vertices, we have at least 2 bridges between C_1 and C_2.

((b) $\Rightarrow$ (a)) Suppose I_G is generated by quadratic binomials and G is $2K_2$-free. We show that $I_{\widehat{G}}$ is generated by quadratic binomials. We now check conditions (a), (b) and (c) in Proposition 1.3.3. Note that if there exist cycles in $\widehat{G}$ which do not satisfy the condition (a), (b) or (c) in Proposition 1.3.3, then the vertex $n+1$ must be contained in these cycles. Otherwise, the cycles which do not contain $n+1$ also appear in G, contradicting the fact that I_G is generated by quadratic binomials. Let C be an even cycle of length ≥ 6 in $\widehat{G}$. If the cycle C contains the vertex $n+1$, then since $n+1$ is incident with any other vertex, we have an even chord. Hence the condition (a) in Proposition 1.3.3 is satisfied.

Now let C_1 and C_2 be minimal odd cycles in $\widehat{G}$ having exactly one common vertex v. If $v = n+1$, then minimality of the cycles requires that C_1 and C_2 are cycles of length 3. Then, we have edges $\{i, j\}$ in C_1 and $\{k, l\}$ in C_2. Since G is $2K_2$-free, we have at least one of $\{i, k\}, \{i, l\}, \{j, k\}, \{j, l\}$ in $E(\widehat{G})$. Suppose $v \neq n+1$. We may assume the vertex $n+1$ is contained in C_1. Then, since the vertex $n+1$ is incident with any other vertex, we have the condition (b) in Proposition 1.3.3.

Finally, assume that C_1 and C_2 are minimal cycles in $\widehat{G}$ having no common vertex. Also, assume that $n+1$ is contained in C_1. Then, it is clear that we have at least three bridges since $n+1$ is incident with any other vertex in $\widehat{G}$. Thus, we have the condition (c) in Proposition 1.3.3. Hence, the three conditions are equivalent, as desired. $\square$

Example 2.1.4. There is no implication between the conditions (1) I_G is generated by quadratic binomials and (2) G is $2K_2$-free.

(a) Let G be the graph in Figure 2.2. Then, I_G is not generated by quadratic binomials. On the other hand, G is $2K_2$-free.

(b) If G is a bipartite graph consisting of a cycle C of length 6 and a chord of C, then I_G is generated by two quadratic binomials. On the other hand, G is not $2K_2$-free.

These examples show that both (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1) are not true.

By using the theory of the Rees algebra of edge ideals, we now give a necessary condition for $I_{\widehat{G}}$ to have a quadratic Gröbner basis as follows:

Proposition 2.1.5. *Let G be a connected graph. If $I_{\widehat{G}}$ possesses a quadratic Gröbner basis, then G is co-chordal.*

Proof. Suppose $I_{\widehat{G}}$ has a quadratic Gröbner basis. It is known from [10, Corollary 10.1.8] that each power of the edge ideal $I(G)$ of G has a linear resolution. Hence, $I(G)$ itself has a linear resolution. By Fröberg's theorem [10, Theorem 9.2.3], G is co-chordal, as required. $\square$

The converse of Proposition 2.1.5 is not true. See Example 2.1.10 later in this section. However, if G is bipartite, then we have equivalent conditions as below.

Theorem 2.1.6. *Let G be a bipartite graph. Then, the following conditions are equivalent:*

(a) *$I_{\widehat{G}}$ is generated by quadratic binomials;*

(b) *$I_{\widehat{G}}$ possesses a quadratic Gröbner basis;*

(c) *G is $2K_2$-free;*

(d) *G is co-chordal.*

Proof. First, (d) $\Rightarrow$ (c) is trivial. By Proposition 2.1.5, we have (b) $\Rightarrow$ (d).

((c) $\Rightarrow$ (a)) Suppose G is $2K_2$-free. Since G is bipartite, Proposition 1.3.1 shows that we have no odd cycles in G. By Lemma 2.1.2, every even cycle of G has an even chord. Therefore, Proposition 1.3.4 guarantees that I_G is generated by quadratic binomials. Hence, (a) follows from Theorem 2.1.3.

((a) $\Leftrightarrow$ (b)) Since G is bipartite, any odd cycle of $\widehat{G}$ has the vertex $n+1$. Then by [27, Proposition 5.5], there exists a bipartite graph G' such that $I_{\widehat{G}} = I_{G'}$. By Proposition 1.3.4, $I_{G'}$ is generated by quadratic binomials if and only if $I_{G'}$ possesses a quadratic Gröbner basis. Thus, the four conditions are equivalent, as desired. $\square$

If G is not bipartite, then the conditions (a) and (b) in Theorem 2.1.6 are not equivalent. In fact,

Example 2.1.7. Let G be a cycle of length 5. Then, $\overline{G}$ is also a cycle of length 5. Hence G is not co-chordal, but is $2K_2$-free. By Theorem 2.1.3 and Proposition 2.1.5, $I_{\widehat{G}}$ is generated by quadratic binomials and has no quadratic Gröbner bases.

We say a graph G is *sort-closed* if there exists a labeling of vertices of G which satisfies the following condition:

- Each vertex of G is identified with a vertex in a regular n-gon in a plane labeled clockwise from 1 to n. For each pair of vertex disjoint edges $\{i, j\}, \{k, l\} \in E(G)$, if they do not intersect in a plane, then the pair of intersecting edges determined by $\{i, j, k, l\}$ also belong to $E(G)$.

The following proposition immediately follows from [34, Chapter 9].

Proposition 2.1.8. *Let G be a connected sort-closed graph. Then, I_G possesses a quadratic Gröbner basis.*

Proof. It is known from [34, Theorem 9.1] that the toric ideal of the complete graph K_n has a Gröbner basis $\mathcal{G}_n$ consisting of quadratic binomials $x_i x_j - x_k x_l$ attached to a cycle of length 4, where the initial monomial $x_i x_j$ comes from a non-intersecting pair of edges and the non-initial monomial $x_k x_l$ comes from a pair of intersecting edges. We claim that a Gröbner basis of I_G can be obtained from this Gröbner basis by eliminating the variables that do not appear in G. Since G is sort-closed, it is easy to see that the condition in the elimination theorem (Proposition 1.1.10) is satisfied. Thus, I_G has a quadratic Gröbner basis. $\square$

Recall that a finite connected simple graph G is called $(*)$-*minimal* if G satisfies the condition

$(*)$ I_G is generated by quadratic binomials and I_G possesses no quadratic Gröbner bases

and if no induced subgraph H $(\neq G)$ of G satisfies the condition $(*)$. The suspension graph $\widehat{G}$ given in Example 2.1.7 is a $(*)$-minimal graph. We now generalize this example and give a nontrivial infinite series of $(*)$-minimal graphs.

Theorem 2.1.9. *Let G be the graph on the vertex set $[n]$ whose complement is a cycle of length n. If $n \geq 5$, then $\widehat{G}$ is $(*)$-minimal, i.e., $\widehat{G}$ satisfies the following:*

(a) *$I_{\widehat{G}}$ is generated by quadratic binomials;*

(b) *$I_{\widehat{G}}$ has no quadratic Gröbner bases;*

(c) *For any induced subgraph H $(\neq \widehat{G})$ of $\widehat{G}$, the toric ideal I_H of H possesses a quadratic Gröbner basis.*

Proof. Since a cycle of length $n \geq 5$ is not chordal, (b) follows from Proposition 2.1.5. We will show (c) using Proposition 2.1.8. Since we have Proposition 2.1.1, it suffices to show that I_G and $I_{G'}$ have quadratic Gröbner bases, where G' is the graph with n vertices whose complementary graph $\overline{G'}$ is a path of length $n-1$. First, we show that G is sort-closed. Let the edges $\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}, \{1, n\}$ form a cycle in $\overline{G}$. Then, all the possible intersecting pairs of edges in G belong to $E(G)$. Thus, G is sort-closed. Next, we show that G' is sort-closed. Let $\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}$ be the path of length $n-1$ contained in $\overline{G'}$. Then, all the possible intersecting pairs of edges again belong to $E(G')$. Thus, G' is sort-closed. Hence we have (c).

Next, we will prove (a). By the condition (b), I_G is generated by quadratic binomials. Since $\overline{G}$ is a cycle of length ≥ 5, $\overline{G}$ has no cycle of length 4 and so G is $2K_2$-free. By applying Theorem 2.1.3, we have (a).

Moreover, since $\overline{G}$ is the cycle of length $n \geq 5$, Thus, we have (a) by Theorem 2.1.3 as desired. □

Even if G is co-chordal, $\widehat{G}$ may be $(*)$-minimal:

Example 2.1.10. Let G be the graph whose complement is the chordal graph in Figure 2.2. Then, $I_{\widehat{G}}$ is generated by quadratic binomials since G is co-chordal (and hence $2K_2$-free) and $I_G = (0)$. On the other hand, computational experiments in Section 2.2 will show that $\widehat{G}$ is $(*)$-minimal.

2.2 Computational experiments

In this section, we use various computer programs to enumerate all finite connected simple graphs G with up to 8 vertices that satisfy the condition $(*)$. Proposition 1.3.3 is key to our enumeration method.

Proposition 1.3.3 gives an algorithm to determine if a toric ideal I_G is generated by quadratic binomials. Since the criteria in Proposition 1.3.3 are characterized by cycles of G, we need to find all even cycles and minimal odd cycles of G to implement the algorithm. CyPath [35] enabled us to find all cycles. CyPath is an enumeration program of paths and cycles implemented by T. Uno. The algorithm from Proposition 1.3.3 is used at step (2) of the following procedure to search for graphs satisfying $(*)$.

(1) Use nauty [17] to generate all connected simple graphs with n vertices up to graph isomorphism.

(2) By using Proposition 1.3.3, classify graphs G whose toric ideals I_G are generated by quadratic binomials.
If I_G is generated by quadratic binomials, then G is added to the list of candidates of graphs satisfying $(*)$.

(3) For each graph G in the list of candidates, repeat (a), (b) and (c) 10000 times or a quadratic Gröbner basis is found.

   (a) Rondomly choose a weight vector w on each iteration.

   (b) Compute a Gröbner basis of the toric ideal I_G with respect to the chosen weight vector w with Risa/Asir [22].

(c) If the Gröbner basis is quadratic, exclude the graph G from the list of candidates.

(4) Check the Koszulness of the edge ring $K[G]$ of each candidate graph using Macaulay2 [9]. It is known (see, for example, [25]) that if $K[G]$ is not Koszul then I_G has no quadratic Gröbner bases. If $K[G]$ is not Koszul, then we exclude the graph G from the candidates. If Koszulness cannot be determined, compute all Gröbner bases using TiGERS [16].

In our experimentation, we computed step (2) 10000 times in the case of 8 vertices. We thus obtained 214 graphs as remaining candidates, 213 of which were not Koszul using Macaulay2. The last is indeterminable by computational methods in our environment. However, Theorem 2.1.9 tells us that it has no quadratic Gröbner basis, because it is the suspension of the complement graph of a cycle whose length is 7. Thus, we completed the classification of finite graphs with 8 vertices. Table 2.1 shows the numbers of (1) the connected simple graphs, (2) the graphs whose toric ideals I_G are generated by quadratic binomials (including the number of zero ideals), and (4) the graphs satisfying $(*)$ (including the number of the graphs which have degree 1 vertices).

vertices	(1)	(2)		(4)	
3	2	2	(2)	0	
4	6	6	(3)	0	
5	21	20	(7)	0	
6	112	95	(14)	1	(0)
7	853	568	(34)	14	(2)
8	11117	4578	(78)	214	(51)

Table 2.1:

The graph with 6 vertices satisfying $(*)$ is the graph given in Example 2.1.7. The list of graphs with 7 vertices satisfying $(*)$ are given on the next page. Figures 2.5, 2.8, 2.14, 2.15 and 2.16 are $(*)$-minimal. Figure 2.15 belongs to the infinite series presented in Theorem 2.1.9 and Figure 2.5 is the $(*)$-minimal graph introduced in Example 2.1.10. The list of graphs with 8 vertices satisfying $(*)$ is available at

URL: `http://cr.math.sci.osaka-u.ac.jp/~a-shikama/website/minimalgraphs/minimalgraphs.html`

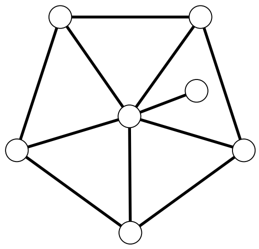


Figure 2.3:

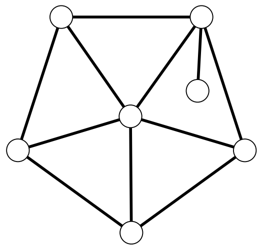


Figure 2.4:

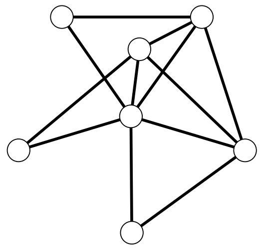


Figure 2.5:

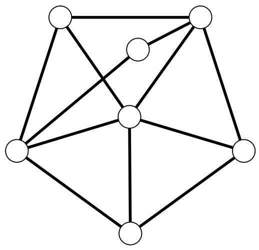
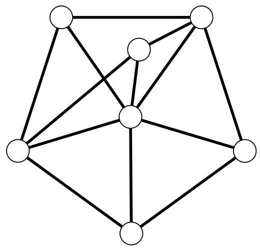


Figure 2.6:



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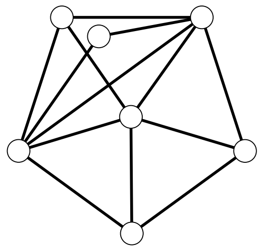

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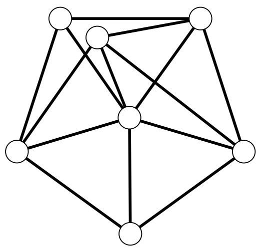

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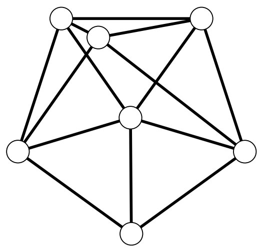


Figure 2.10:

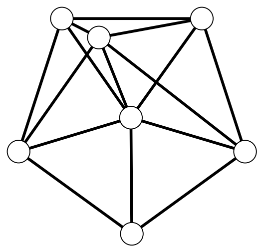
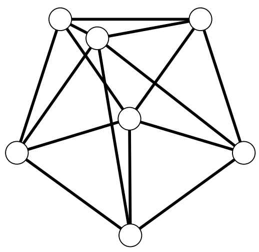


Figure 2.11:



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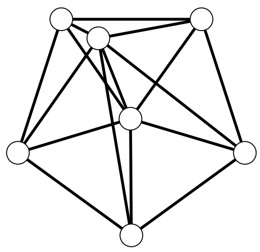

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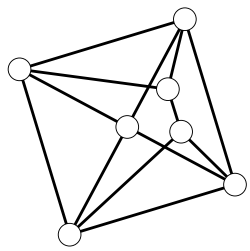

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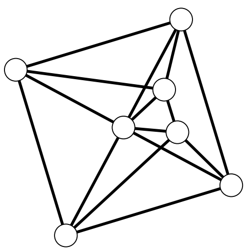

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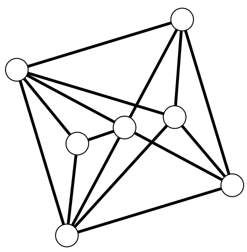


Figure 2.16:

Chapter 3

Polyominoes and polyomino ideals

In this chapter, we classify some classes of polyominoes whose polyomino ideals are prime. In Section 3.1, we introduce the concept of balanced polyominoes and show that the polyomino ideals of balanced polyominoes are prime. In Section 3.2, we show that initial ideals of polyomino ideals of balanced polyominoes are generated by squarefree monomials with respect to any monomial order. In Section 3.3, we show that the polyomino ideals of simple polyominoes are prime. This result completely answers Conjecture 1.4.2 presented by Qureshi in [28]. Finally, in Section 3.4 we discuss the polyomino ideals of the polyominoes obtained by removing a convex polyomino from its ambient rectangle. We find a toric ring generated by monomials of degree two or three which is isomorphic to the ring $K[\mathcal{P}]$ associated with such polyominoes, which immediately implies that $I_{\mathcal{P}}$ is prime.

3.1 The ideal of inner minors of a polyomino

Let $\mathcal{P}$ be a polyomino and let K be a field. Let $S = K[x_{ij} \mid (i,j) \in V(\mathcal{P})]$ and let $I_{\mathcal{P}} \subset S$ be the polyomino ideal as defined in Section 1.4. We set $K[\mathcal{P}] = S/I_{\mathcal{P}}$.

According to [28], an integer value function $\alpha : V(\mathcal{P}) \to \mathbb{Z}$ is called *admissible* if, for all maximal horizontal and vertical edge intervals $\mathcal{I}$ of $\mathcal{P}$, it holds that

$$\sum_{a \in \mathcal{I}} \alpha(a) = 0.$$

Figure 3.1 shows an example of an admissible labeling of a polyomino. For a given admissible labeling α, we define the binomial

$$f_\alpha = \prod_{\substack{a \in V(\mathcal{P}) \\ \alpha(a) > 0}} x_a^{\alpha(a)} - \prod_{\substack{a \in V(\mathcal{P}) \\ \alpha(a) < 0}} x_a^{-\alpha(a)},$$

Let $J_{\mathcal{P}}$ be the ideal generated by the binomials f_α where α is an admissible labeling of $\mathcal{P}$. It is easy to see that $I_{\mathcal{P}} \subset J_{\mathcal{P}}$. We say a polyomino $\mathcal{P}$ is *balanced*, if

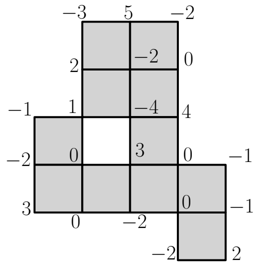


Figure 3.1: An admissible labeling

$I_{\mathcal{P}} = J_{\mathcal{P}}$ holds. This is if and only if for any admissible labeling α, the binomial f_α belongs to $I_{\mathcal{P}}$.

Consider the free abelian group $G = \bigoplus_{(i,j)\in V(\mathcal{P})} \mathbb{Z}_{e_{ij}}$ with basis elements e_{ij}. To any cell $C = [(i,j),(i+1,j+1)]$ of $\mathcal{P}$ we attach the element $b_C = e_{ij} + e_{i+1,j+1} - e_{i+1,j} - e_{i,j+1}$ in G and let $\Lambda \in G$ be the lattice spanned by these elements.

Lemma 3.1.1. *The elements b_C form a $\mathbb{Z}$-basis of Λ and hence* $\operatorname{rank}_{\mathbb{Z}} \Lambda = |\mathcal{P}|$. *Moreover Λ is saturated. In other words, G/Λ is torsionfree.*

Proof. We order the basis elements e_{ij} lexicographically. Then the leading term of b_C is e_{ij}. This shows that the elements b_C are linearly independent and hence form a $\mathbb{Z}$-basis of Λ. We may complete this basis of Λ by the elements e_{ij} for which (i,j) is not a lower left corner of a cell of $\mathcal{P}$ to get a basis of G. This shows that G/Λ is free and hence torsionfree. $\square$

The lattice ideal attached to Λ is generated by all binomials

$$f_v = \prod_{\substack{a\in V(\mathcal{P})\\ v_a>0}} x_a^{v_a} - \prod_{\substack{a\in V(\mathcal{P})\\ v_a<0}} x_a^{-v_a}$$

with $v = (v_a)_{a\in V(\mathcal{P})} \in \Lambda$.

Proposition 3.1.2. *Let $\mathcal{P}$ be a balanced polyomino. Then $I_{\mathcal{P}} = J_{\mathcal{P}} = I_\Lambda$.*

Proof. First, since each inner 2-minor is just a sum of suitable elements b_C, it is clear that $I_{\mathcal{P}} \subset I_\Lambda$. It is enough to show that for any $v \in \Lambda$, there exists an admissible labeling α of $\mathcal{P}$ such that $v_\alpha = \alpha(a)$ for all $a \in V(\mathcal{P})$. Since the elements b_C form a $\mathbb{Z}$-basis of Λ, there exist integers $z_C \in \mathbb{Z}(C \in P)$ such that $v = \sum_{C\in P} z_C b_C$. We set $\alpha = \sum_{C\in\mathcal{P}} z_C\alpha_C$ where for $C = [(i,j),(i+1,j+1)]$,

$$\alpha_C((k,l)) = \begin{cases} 1, & \text{if } (k,l) = (i,j) \text{ or } (k,l) = (i+1,j+1), \\ -1, & \text{if } (k,l) = (i+1,j) \text{ or } (k,l) = (i,j+1), \\ 0, & \text{otherwise.} \end{cases}$$

Then, $\alpha(a) = v_a$ for all $a \in V(\mathcal{P})$. Since each α_C is an admissible labeling and since any linear combination of admissible labelings is again admissible, this concludes the proof. $\square$

Corollary 3.1.3. *If $\mathcal{P}$ is a balanced polyomino, then $I_{\mathcal{P}}$ is a prime ideal of height $|\mathcal{P}|$, where $|\mathcal{P}|$ is the number of cells in $\mathcal{P}$.*

Proof. By Proposition 3.1.2, $I_{\mathcal{P}} = I_\Lambda$ and by Lemma 3.1.1, Λ is saturated. It follows that $I_{\mathcal{P}}$ is a prime ideal, see [18, Theorem 7.4]. Next, it follows from [6, Corollary 2.2] (or [18, Proposition 7.5]) that height $I_{\mathcal{P}}$ = height $I_\Lambda = \operatorname{rank}_{\mathbb{Z}} \Lambda$, Since we have that $\operatorname{rank}_{\mathbb{Z}} \Lambda = |\mathcal{P}|$ from Lemma 3.1.1, this completes the proof. □

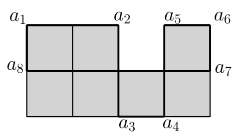
3.2 Primitive binomials of balanced polyominoes

The purpose of this section is to identify the primitive binomials of polyomino ideals of balanced polyominoes. This will allow us to show that the initial ideal of $I_{\mathcal{P}}$ is a squarefree monomial ideal for any monomial order.

The primitive binomials in $I_{\mathcal{P}}$ are determined by cycles. A sequence of vertices $\mathcal{C} = a_1, \ldots, a_m$ in $V(\mathcal{P})$ with $a_m = a_1$ and $a_i \neq a_j$ for $1 \leq i < j \leq m-1$ is called a *cycle* in $\mathcal{P}$ if the following conditions hold:

(a) $[a_i, a_{i+1}]$ is a horizontal or vertical edge interval of $\mathcal{P}$ for all $i = 1, \ldots, m-1$;

(b) for $i = 1, \ldots, m-1$, if $[a_i, a_{i+1}]$ is a horizontal edge interval of $\mathcal{P}$, then $[a_{i+1}, a_{i+2}]$ is a vertical edge interval of $\mathcal{P}$, and vice versa. Here, $a_{m+1} = a_2$.



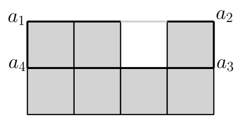


Figure 3.2: A cycle (left) and a non-cycle (right) in $\mathcal{P}$

Note that by definition of the cycle, $m-1$ is even. Given a cycle $\mathcal{C}$, we attach to $\mathcal{C}$ the binomial

$$f_{\mathcal{C}} = \prod_{i=1}^{(m-1)/2} x_{a_{2i-1}} - \prod_{i=1}^{(m-1)/2} x_{a_{2i}}$$

A cycle $\mathcal{C}$ in $\mathcal{P}$ is said to be primitive if each maximal interval of $\mathcal{P}$ contains at most two vertices of $\mathcal{C}$.

Theorem 3.2.1. *Let $\mathcal{P}$ be a balanced polyomino.*

(a) *Let $\mathcal{C}$ be a primitive cycle in $\mathcal{P}$. Then $f_{\mathcal{C}} \in I_{\mathcal{P}}$.*

(b) *Let $f \in I_{\mathcal{P}}$ be a primitive binomial. Then there exists a primitive cycle $\mathcal{C}$ in $\mathcal{P}$ such that $f = \pm f_{\mathcal{C}}$.*

Proof. (a) Let $\mathcal{C} = a_1, \ldots, a_m$ be a primitive cycle in $\mathcal{P}$. We set the labeling α of $\mathcal{P}$ by setting $\alpha(a) = 0$ if $a \notin \mathcal{C}$ and $\alpha(a_i) = (-1)^{i+1}$ for $i = 1, \ldots, m$. We claim that α is an admissible labeling of $\mathcal{P}$. For each maximal horizontal or vertical edge interval $\mathcal{I}$, If $\mathcal{I} \cap \mathcal{C} = \emptyset$ then we have $\alpha(a) = 0$ for all $a \in \mathcal{I}$. If $\mathcal{I} \cap \mathcal{C} \neq \emptyset$, Then since the cycle is primitive we have exactly two vertices that are consecutive in the cycle, say $a_i, a_{i+1} \in \mathcal{I} \cap \mathcal{C}$ such that $\alpha(a_i) \neq 0, \alpha(a_{i+1}) \neq 0$. By the definition of the labeling, each two consecutive vertices in $\mathcal{C}$ have opposite signs. Therefore we see that α is admissible and so $f = f_\alpha \in J_{\mathcal{P}}$. Hence, since $\mathcal{P}$ is balanced, we have $f \in I_{\mathcal{P}}$.

(b) Let $f \in I_{\mathcal{P}}$ be a primitive binomial. Since $\mathcal{P}$ is balanced and f is irreducible, [28, Theorem 3.8(a)] implies that there exists an admissible labeling α such that $f = f_\alpha$. We choose $a_1 \in V(\mathcal{P})$ such that $\alpha(a_1) > 0$. Let $\mathcal{I}_1$ be the maximal horizontal edge interval to which a_1 belongs. Since α is admissible, there exists at least one vertex, say $a_2 \in \mathcal{I}_1$, such that $\alpha(a_2) < 0$. Similarly, let $\mathcal{I}_2$ be the vertical edge interval to which a_2 belongs. Then, we find a vertex $a_3 \in \mathcal{I}_2$ such that $\alpha(a_3) > 0$. In the next step, we find a vertex from a maximal horizontal edge interval to which a_3 belongs. By continuing this operation, we obtain a sequence of vertices of $\mathcal{P}$ $a_1, a_2, \ldots$ such that $\alpha(a_i) < 0$ for even i and $\alpha(a_i) > 0$ for odd i. Since $V(\mathcal{P})$ is a finite set, there exists a number m such that $a_i \neq a_j$ for all $1 \leq i < j < m$ and $a_m = a_i$ for some $i < m$. Let m be the minimum number which satisfies this condition. Then, the sequence $\mathcal{C} : a_i, a_{i+1}, \ldots, a_m$ is a cycle in $\mathcal{P}$. If i is odd, then $f_{\mathcal{C}}^+$ divides f_α^+ and $f_{\mathcal{C}}^-$ divides f_α^-. If i is even, then $f_{\mathcal{C}}^+$ divides f_α^- and $f_{\mathcal{C}}^-$ divides f_α^+. Hence, since f is primitive, $f = \pm f_{\mathcal{C}}$

We now show that $\mathcal{C}$ is a primitive cycle. Suppose that there exists a maximal vertical or horizontal edge interval $\mathcal{I}$ such that at least 3 vertices of $\mathcal{C}$ belong to $\mathcal{I}$. We may assume that $\mathcal{I}$ is a horizontal edge interval. By the definition of cycle in a polyomino, we observe that $a_s, a_{s+1}, a_t \in \mathcal{C}$ with $s + 2 < t$. Also, we see from the definition of cycle that a_{t-1} or a_{t+1} , say a_{t+1}, belongs to $\mathcal{I}$. Since $\mathcal{I}$ is a horizontal edge interval, both s and t are odd. Then, we can construct a subsequence $\mathcal{C}'$ of $\mathcal{C}$ as follows: $\mathcal{C}' : a_i, \ldots a_{s-1} a_s, a_t, a_{t+1}, a_m$. Thus, we see that $\mathcal{C}'$ is a cycle of $\mathcal{P}$ such that $f_{\mathcal{C}'}^+$ divides $f_{\mathcal{C}}^+$ and $f_{\mathcal{C}'}^-$ divides $f_{\mathcal{C}}^-$. This contradicts that $f_{\mathcal{C}} = \pm f$ is a primitive binomial. Thus , $\mathcal{C}$ is a primitive cycle.

By using (a), we obtain $f_{\mathcal{C}} \in I_{\mathcal{P}}$, as desired. □

The above theorem gives the following corollaries. Recall that we say a Gröbner basis $\mathcal{G}$ of a binomial ideal $I \subset S$ is *squarefree* if every monomial appearing in $\mathcal{G}$ is squarefree.

Corollary 3.2.2. *Let $\mathcal{P}$ be a balanced polyomino. Then $I_{\mathcal{P}}$ has a squarefree Gröbner basis with respect to any monomial order.*

Proof. By Corollary 3.1.3, $I_{\mathcal{P}}$ is a prime ideal. It is known that the binomial prime ideals are toric ideals [7, Theorem 5.5]. Also, it is known that the set of primitive binomials of a toric ideal is a universal Gröbner basis (Proposition 1.1.9). Since by Theorem 3.2.1, the primitive binomials of $I_{\mathcal{P}}$ have squarefree initial monomials with respect to any monomial order, we have the conclusion. □

Corollary 3.2.3. *Let $\mathcal{P}$ be a balanced polyomino. Then $K[\mathcal{P}]$ is a normal Cohen–Macaulay domain of dimension $|V(\mathcal{P})| - |\mathcal{P}|$.*

Proof. A toric ring whose toric ideal has a squarefree initial ideal is normal [34, Chapter 8]. By a theorem of Hochster, ([1, Theorem 6.3.5]) a normal toric ring is Cohen–Macaulay. By Proposition 3.1.2, we see that height $I_{\mathcal{P}} = |\mathcal{P}|$. Since $K[\mathcal{P}]$ is Cohen–Macaulay, it holds that the Krull dimension of $K[\mathcal{P}]$ is equal to $|V(\mathcal{P})| - |\mathcal{P}|$, as desired. □

Later Herzog and Saeedi Madani proved the following:

Theorem 3.2.4 ([11])**.** *Let $\mathcal{P}$ be a simple polyomino. Then $\mathcal{P}$ is balanced.*

This theorem and Corollary 3.3.4 given in the next section completely answer Qureshi's conjecture (Conjecture 1.4.2).

3.3 Simple polyominoes are prime

In this section, independent of Theorem 3.2.4, we prove Qureshi's conjecture by finding a toric ring which is isomorphic to the quotient ring $K[\mathcal{P}]$ of a simple polyomino. Our argument is much simpler than that of Theorem 3.2.4 given in [11]. We also give another interesting result: If $\mathcal{P}$ is simple, then $I_{\mathcal{P}}$ has a squarefree quadratic Gröbner basis.

Let $\{V_1, \ldots, V_m\}$ be the set of maximal vertical edge intervals and $\{H_1, \ldots, H_n\}$ be the set of maximal horizontal edge intervals of $\mathcal{P}$. We denote by $G(\mathcal{P})$, the associated bipartite graph of $\mathcal{P}$ with vertex set $\{v_1, \ldots, v_m\} \bigsqcup \{h_1, \ldots, h_n\}$ and edge set

$$E(G(\mathcal{P})) = \{\{v_i, h_j\} \mid V_i \cap H_j \in V(\mathcal{P})\}.$$

Example 3.3.1. Figure 3.3 shows a polyomino $\mathcal{P}$ with maximal vertical and maximal horizontal edge intervals labeled as $\{V_1, \ldots, V_5\}$ and $\{H_1, \ldots, H_4\}$ respectively, and Figure 3.4 shows the associated bipartite graph $G(\mathcal{P})$ of $\mathcal{P}$.

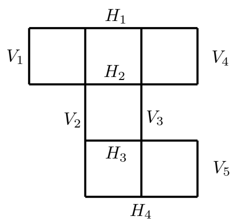


Figure 3.3: Maximal intervals of $\mathcal{P}$

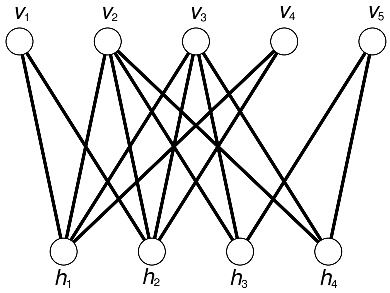


Figure 3.4: Associated bipartite graph of $\mathcal{P}$

Let S be the polynomial ring over a field K with variables x_{ij} with $(i,j) \in V(\mathcal{P})$. Note that $|V_p \cap H_q| \leq 1$. If $V_p \cap H_q = \{(i,j)\}$, then we can write $x_{ij} = x_{V_p \cap H_q}$, when required. For each cycle $\mathcal{C} : v_{i_1}, h_{j_1}, v_{i_2}, h_{j_2}, \ldots, v_{i_r}, h_{j_r}$ in $G(\mathcal{P})$, we associate a binomial in S given by $f_{\mathcal{C}} = x_{V_{i_1} \cap H_{j_1}} x_{V_{i_2} \cap H_{j_2}} \cdots x_{V_{i_r} \cap H_{j_r}} - x_{V_{i_2} \cap H_{j_1}} x_{V_{i_3} \cap H_{j_2}} \cdots x_{V_{i_1} \cap H_{j_r}}$.

Note that if $\mathcal{C} : v_{i_1}, h_{j_1}, v_{i_2}, h_{j_2}, \ldots, v_{i_r}, h_{j_r}$ defines a cycle in $G(\mathcal{P})$, then the sequence of vertices $\mathcal{C}_{\mathcal{P}} : V_{i_1} \cap H_{j_1}, V_{i_2} \cap H_{j_1}, V_{i_2} \cap H_{j_2}, \ldots, V_{i_r} \cap H_{j_r}, V_{i_1} \cap H_{j_r}$ is a primitive cycle in $\mathcal{P}$ and vice versa. Also, $f_{\mathcal{C}} = f_{\mathcal{C}_{\mathcal{P}}}$.

We set the edge ring of $G(\mathcal{P})$ by $K[G(\mathcal{P})] = K[v_p h_q \mid \{p,q\} \in E(G(\mathcal{P}))] \subset T = K[v_1, \ldots, v_m, h_1, \ldots, h_n]$. Let $\pi : S \to T$ be the surjective K-algebra homomorphism defined by $\varphi(x_{ij}) = v_p h_q$, where $\{(i,j)\} = V_p \cap H_q$. We denote by $L_{\mathcal{P}}$, the toric ideal of $K[G(\mathcal{P})]$. It is known from Proposition 1.3.3 that $L_{\mathcal{P}}$ is generated by the binomials associated with cycles in $G(\mathcal{P})$.

Let $\mathcal{P}$ be a polyomino and let $[a,b]$ an interval with the property $\mathcal{P} \subset [a,b]$. We recall from graph theory that a graph is called *weakly chordal* if every cycle of length greater than 4 has a chord. To prove the following lemma, we introduce some notation. We say that a cycle $\mathcal{C}_{\mathcal{P}} : a_1, a_2, \ldots, a_m$ in $\mathcal{P}$ with $a_m = a_1$ has a *self-crossing* if there exist indices i and j such that $a_i, a_{i+1} \in V_k$ and $a_j, a_{j+1} \in H_l$, and $a_i, a_{i+1}, a_j, a_{j+1}$ are all distinct, and $V_k \cap H_l \neq \emptyset$. For example, the left side

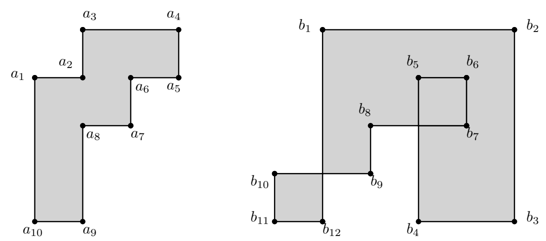


Figure 3.5: Cycles with no self-crossing (left) and two self-crossings (right)

of Figure 3.5 is a cycle with no self-crossing and the right is a cycle with two self-crossings. When there is a self-crossing, if $\mathcal{C}$ is the associated cycle in $G(\mathcal{P})$, then it holds that $\{v_k, h_l\} \in E(G(\mathcal{P}))$, which gives a chord in $\mathcal{C}$.

Let $\mathcal{C}_{\mathcal{P}} : a_1, a_2, \ldots, a_r$ be a cycle in $\mathcal{P}$ that does not have any self-crossing. Then, we call the area bounded by edge intervals $[a_i, a_{i+1}]$ for $i \in \{1, \ldots, r-1\}$ the *interior* of $\mathcal{C}_{\mathcal{P}}$. Moreover, we call a cell C an *interior* cell of $\mathcal{C}_{\mathcal{P}}$ if C belongs to the interior of $\mathcal{C}_{\mathcal{P}}$.

Lemma 3.3.2. *Let $\mathcal{P}$ be a simple polyomino. Then, the graph $G(\mathcal{P})$ is weakly chordal.*

Proof. Let $\mathcal{C}$ be a cycle of $G(\mathcal{P})$ of length $2r$ with $n \geq 3$ and $\mathcal{C}_{\mathcal{P}}$ be the associated primitive cycle in $\mathcal{P}$. We have no self-crossings in $\mathcal{C}_{\mathcal{P}}$ since otherwise, by the definition of self-crossing, we know that $\mathcal{C}$ would have a chord.

Let $\mathcal{C} : v_{i_1}, h_{j_1}, v_{i_2}, h_{j_2}, \dots, v_{i_r}, h_{j_r}$ and $\mathcal{C}_{\mathcal{P}} : V_{i_1} \cap H_{j_1}, V_{i_2} \cap H_{j_1}, V_{i_2} \cap H_{j_2}, \dots, V_{i_r} \cap H_{j_r}, V_{i_1} \cap H_{j_r}$. We may write $a_1 = V_{i_1} \cap H_{j_1}, a_2 = V_{i_2} \cap H_{j_1}, a_3 = V_{i_2} \cap H_{j_2}, \dots, a_{2r-1} = V_{i_r} \cap H_{j_r}, a_{2r} = V_{i_1} \cap H_{j_r}$. Further, we may assume that a_1 and a_2 belong to the same maximal horizontal edge interval. Then, a_{2r} and a_1 belongs to the same maximal vertical edge interval.

First, we show that every interior cell of $\mathcal{C}_{\mathcal{P}}$ belongs to $\mathcal{P}$. Suppose that we have an interior cell C of $\mathcal{C}_{\mathcal{P}}$ which does not belong to $\mathcal{P}$. Let $\mathcal{J}$ be any interval such that $\mathcal{P} \subset \mathcal{J}$. Then, using the definition of a simple polyomino, we obtain a path of cells $C = C_1, C_2, \dots, C_t$ with $C_i \notin P$, $i = 1, \dots, t$ and C_t is a boundary cell in $\mathcal{J}$. This shows that $V(C_1) \cup V(C_2) \cup \dots \cup V(C_t)$ intersects at least one of $[a_i, a_{i+1}]$ for $i \in \{1, \dots, 2r-1\}$ or $[a_r, a_1]$, which is not possible because $\mathcal{C}_{\mathcal{P}}$ is a cycle in $\mathcal{P}$. Hence $C \in \mathcal{P}$. Thus, an interval in the interior of $\mathcal{C}_{\mathcal{P}}$ is an inner interval of $\mathcal{P}$.

Let $\mathcal{I}$ be the maximal inner interval of $\mathcal{C}_{\mathcal{P}}$ to which a_1 and a_2 belong and let b, c be corner vertices of $\mathcal{I}$. We may assume that a_1 and c are the diagonal corners and a_2 and b are the anti-diagonal corners of $\mathcal{I}$. If $b, c \in V(\mathcal{C}_{\mathcal{P}})$, then the primitivity of $\mathcal{C}$ implies that $\mathcal{C}$ is a cycle of length 4. We may assume that $b \notin V(\mathcal{C}_{\mathcal{P}})$. Let H' be the maximal horizontal edge interval which contains b and c. The maximality of $\mathcal{I}$ implies that $H' \cap V(\mathcal{C}_{\mathcal{P}}) \neq \emptyset$. For example, see Figure 3.6. Therefore, $\{v_{i_1}, h'\}$ is a chord in $\mathcal{C}$, as desired.

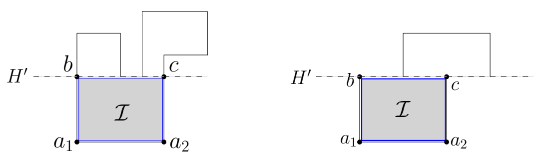


Figure 3.6: Examples of maximal inner intervals of polyominoes

□

Theorem 3.3.3. *Let $\mathcal{P}$ be a simple polyomino. Then $I_{\mathcal{P}} = L_{\mathcal{P}}$.*

Proof. First, we show that $I_{\mathcal{P}} \subset L_{\mathcal{P}}$. Let $f = x_{ij}x_{kl} - x_{il}x_{kj} \in I_{\mathcal{P}}$. Then, there exist maximal vertical edge intervals V_p and V_q and maximal horizontal edge intervals

H_m and H_n of $\mathcal{P}$ such that $(i,j),(i,l) \in V_p$, $(k,j),(k,l) \in V_q$ and $(i,j),(k,j) \in H_m$, $(i,l),(k,l) \in H_n$, leading to $\pi(x_{ij}x_{kl}) = v_pv_qh_mh_n = \pi(x_{il}x_{kj})$. Thus, $f \in L_P$.

Next, we show that $L_{\mathcal{P}} \subset I_{\mathcal{P}}$. Using Proposition 1.3.4, we know that the toric ideal of weakly chordal bipartite graph is minimally generated by quadratic binomials associated with cycles of length 4. It suffices to show that $f_{\mathcal{C}} \in I_P$ where $\mathcal{C}$ is a cycle of length 4 in $G(\mathcal{P})$.

Let $\mathcal{I}$ be an interval such that $\mathcal{P} \subset \mathcal{I}$. Let $\mathcal{C} : h_1, v_1, h_2, v_2$. Then, $\mathcal{C}_{\mathcal{P}} : a_{11} = H_1 \cap V_1, a_{21} = H_2 \cap V_1$, $a_{22} = H_2 \cap V_2$ and $a_{12} = H_1 \cap V_2$ is the associated cycle in $\mathcal{P}$ which also determines an interval in $\mathcal{I}$. Let a_{11} and a_{22} be the diagonal corners of this interval. We need to show that $[a_{11}, a_{22}]$ is an inner interval in $\mathcal{P}$. Assume that $[a_{11}, a_{22}]$ is not an inner interval of $\mathcal{P}$, that is, there exists a cell $C \in [a_{11}, a_{22}]$ which does not belong to $\mathcal{P}$. Using the fact that $\mathcal{P}$ is a simple polyomino, we obtain a path of cells $C = C_1, C_2, \ldots, C_r$ with $C_i \notin \mathcal{P}$, $i = 1, \ldots, r$ and C_r is a boundary cell in $\mathcal{I}$. Then, $V(C_1 \cup \ldots \cup C_r)$ intersects at least one of the maximal intervals H_1, H_2, V_1, V_2, say H_1, which contradicts the fact that H_1 is an interval in $\mathcal{P}$. Thus, every cell C_1 in the interval $[a_{11}, a_{22}]$ belongs to $\mathcal{P}$. Hence, $[a_{11}, a_{22}]$ is an inner interval of $\mathcal{P}$ and $f_{\mathcal{C}} \in I_{\mathcal{P}}$.

□

Corollary 3.3.4. *Let $\mathcal{P}$ be a simple polyomino. Then $K[\mathcal{P}]$ is a normal Cohen–Macaulay domain.*

Proof. By using Proposition 1.3.4, we know that $L_{\mathcal{P}} = I_{\mathcal{P}}$ has a squarefree quadratic Gröbner basis with respect to a suitable monomial order. By the theorem of Sturmfels [34], it holds that $K[\mathcal{P}]$ is normal and then following a theorem of Hochster [1, Theorem 6.3.5], it holds that $K[\mathcal{P}]$ is Cohen–Macaulay. □

3.4 Toric rings of nonsimple polyominoes

In this section, we generalize the result given in [32]. In general, it is interesting, but not so easy, to find a toric ring which is isomorphic to the quotient ring of a given binomial prime ideal. In [32], the author find a toric ring which is isomorphic to $K[\mathcal{P}]$, where $\mathcal{P}$ is a polyomino obtained by removing a rectangle from its ambient rectangle.

We work with a polyomino which is obtained by removing a convex polyomino from its ambient rectangle and give its toric representation. It is proved in [15] that the polyomino ideals of this class is prime.

Recall from [19] that a simple polyomino is called a *one-sided ladder* if it is of the type shown in Figure 3.7. The sequence of vertices $a_1, \ldots, a_s$ of the corners of one-sided ladder $\mathcal{P}$ excluding the opposite corner of the ladder is called the *defining sequence* of $\mathcal{P}$ if each a_i and a_{i+1} are in the horizontal or vertical position. For example, the sequence $a_1, a_2, \ldots, a_{11}$ in Figure 3.7 is the defining sequence of this one-sided ladder.

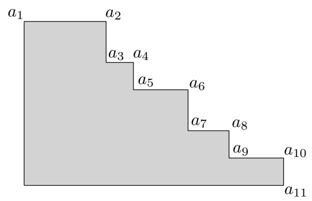


Figure 3.7: A one-sided ladder

To prove the main theorem, we give some properties of convex polyominoes.

Lemma 3.4.1. *Let $\mathcal{P}$ be a convex polyomino and let $\mathcal{I}$ be the unique minimal interval such that $\mathcal{P} \subset \mathcal{I}$. Then,*

(a) *$\mathcal{I} \setminus \mathcal{P}$ consists of at most 4 connected components;*

(b) *each connected component of $\mathcal{I} \setminus \mathcal{P}$ contains exactly one corner vertex of $\mathcal{I}$;*

(c) *each connected component of $\mathcal{I} \setminus \mathcal{P}$ is a one-sided ladder.*

Let $\mathcal{P}$ be a convex polyomino. A vertex of $\mathcal{P}$ is called an *outside corner* if it belongs to exactly one cell of $\mathcal{P}$. On the other hand, a vertex of $\mathcal{P}$ is called an *inside corner* if it belongs to three cells of $\mathcal{P}$. A vertex is called an *interior vertex* if it belongs to four cells of $\mathcal{P}$. The *boundary vertices* of $\mathcal{P}$ are the vertices of $\mathcal{P}$ which are not interior vertices. A cell of $\mathcal{P}$ is called an *interior cell* if all of its 4 vertices are interior vertices. A cell of $\mathcal{P}$ is called a *boundary cell* if it is not an interior cell. We denote by $\partial\mathcal{P}$ the set of boundary vertices of $\mathcal{P}$.

To each interval $[a, b]$, we attach a polyomino $\mathcal{P}_{[a,b]}$ in the obvious way. Such a polyomino is called a *rectangle*. Hereafter, let $\mathcal{P}$ be a polyomino which is obtained by removing a convex polyomino $\mathcal{Q}$ from its ambient rectangle $\mathcal{P}_{[a,b]}$. We assume $\partial\mathcal{P}_{[a,b]} \cap \partial\mathcal{Q} = \emptyset$; otherwise, $\mathcal{P}$ is a simple polyomino and its toric representation is already well studied in Section 3.3. We further assume that $a = (1, 1)$ and $b = (m, n)$.

We define two types of intervals of $\mathcal{P}$ as follows:

(i) For the lowest corner e among all most left outside corners of $\mathcal{Q}$, let $\mathcal{I}_e = [a, e]$.

(ii) The maximal vertical or horizontal edge intervals $\mathcal{I}$ of $\mathcal{P}$.

For example, for a given polyomino, the intervals of types (i) and (ii) are displayed in Figures 3.8, 3.9.

We denote the set of intervals of types (i) and (ii) by Ω. We define a map $\beta : V(\mathcal{P}) \to K[\{u_\mathcal{I}\}_{\mathcal{I}\in\Omega}]$ by $v \mapsto \prod_{\substack{v\in\mathcal{I}\\ \mathcal{I}\in\Omega}} u_\mathcal{I}$. Now, we define the toric ring and the toric ideal. The toric ring denoted by T is defined as

$$T = K[\beta(v) \mid v \in V(\mathcal{P})] \subset K[\{u_\mathcal{I}\}_{\mathcal{I}\in\Omega}].$$

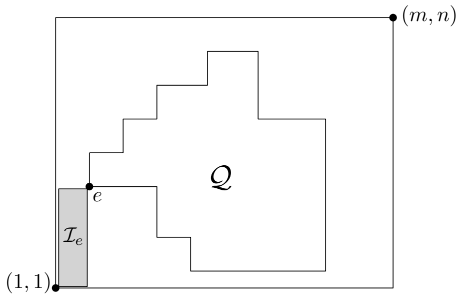


Figure 3.8: (i) Interval $\mathcal{I}_e$ of a polyomino

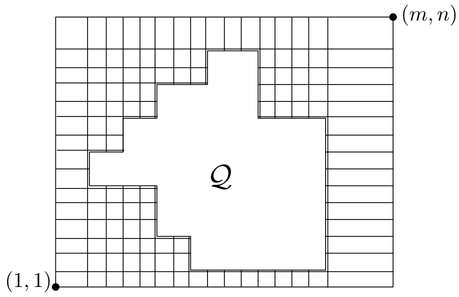


Figure 3.9: (ii) Edge intervals of a polyomino

Let $\varphi : S \to T$ be the surjective ring homomorphism with the setting $\varphi(x_{ij}) = \beta((i,j))$. The toric ideal $L_{\mathcal{P}}$ is the kernel of φ. We claim that $L_{\mathcal{P}} = I_{\mathcal{P}}$. In order to prove this, we will repeatedly use the next lemma.

For any binomial $f = f^+ - f^- \in L_{\mathcal{P}}$, we write V_+ is the set of vertices v such that x_v appear in f^+. V_- is defined similarly. A binomial f in a binomial ideal $I \subset S$ is said to be *redundant* if it can be expressed as a linear combination of binomials in I of lower degree. A binomial is said to be *irredundant* if it is not redundant.

Lemma 3.4.2. *Let $f = f^+ - f^-$ be a binomial of degree ≥ 3 belonging to $L_{\mathcal{P}}$. If there exist three vertices $p, q \in V_+$ and $r \in V_-$ such that p, q are diagonal (anti-diagonal) corners of an inner interval and r is one of the anti-diagonal (diagonal) corners of the inner interval, then f is redundant in $L_{\mathcal{P}}$.*

Proof. Let s be the other corner of the interval determined by p, q and r. Then,

$$\begin{aligned}
f &= f^+ - f^- \\
&= x_p x_q \frac{f^+}{x_p x_q} - f^- \\
&= (x_p x_q - x_r x_s)\frac{f^+}{x_p x_q} + x_r x_s \frac{f^+}{x_p x_q} - x_r \frac{f^-}{x_r} \\
&= (x_p x_q - x_r x_s)\frac{f^+}{x_p x_q} + x_r \left(x_s \frac{f^+}{x_p x_q} - \frac{f^-}{x_r} \right).
\end{aligned}$$

Since $x_p x_q - x_r x_s$ is an inner minor of $\mathcal{P}$ and since $L_{\mathcal{P}}$ is a toric ideal, this completes the proof. □

Theorem 3.4.3. *Let $\mathcal{P} = \mathcal{P}_{[(1,1),(m,n)]} \setminus \mathcal{Q}$ be a polyomino where $\mathcal{Q} \subset \mathcal{P}_{[(1,1),(m,n)]}$ is a convex polyomino. Then $I_{\mathcal{P}} = J_{\mathcal{P}}$.*

Proof. It is easy to see that $I_{\mathcal{P}} \subset L_{\mathcal{P}}$. In order to prove $L_{\mathcal{P}} \subset I_{\mathcal{P}}$, it suffices to show that every binomial of degree 2 in $L_{\mathcal{P}}$ belongs to $I_{\mathcal{P}}$ and that every irredundant binomial in $L_{\mathcal{P}}$ is of degree 2. First, we show that every binomial $f \in L_{\mathcal{P}}$ of degree 2 belongs to $I_{\mathcal{P}}$. Suppose $f = x_p x_q - x_r x_s \in L_{\mathcal{P}}$ is a binomial such that $\{p, q\} \neq \{r, s\}$.

Since $\varphi(x_p x_q) = \varphi(x_r x_s)$, we may assume that $[p, q]$ is an interval which has r and s as its anti-diagonal corners. Assume the pair p and r and the pair s and q belong to the same horizontal edge interval. Then we see that the pair p and s and the pair r and q belong to the same vertical edge interval. If $[p, q]$ is an inner minor of $\mathcal{P}$, then we are done. Suppose that $[p, q]$ is not an inner interval. Then we have either $\mathcal{Q} \subset \mathcal{P}_{[p,q]}$ or $\mathcal{Q} \not\subset \mathcal{P}_{[p,q]}$ and $\mathcal{Q} \cap \mathcal{P}_{[p,q]} \neq \emptyset$. Suppose that $\mathcal{Q} \subset \mathcal{P}_{[p,q]}$. We see that $p \in \mathcal{I}_e$ and $q, r, s \notin \mathcal{I}_e$, where $\mathcal{I}_e$ is the interval given in Figure 3.8. Then, we have $u_{\mathcal{I}_e} | \varphi(x_p)$ and $u_{\mathcal{I}_e} | \varphi(x_r x_s)$, which contradicts $x_p x_q - x_r x_s \in L_{\mathcal{P}}$. Hence, this case is not possible. Suppose $\mathcal{Q} \not\subset \mathcal{P}_{[p,q]}$ and $[p, q]$ is not an inner interval of $\mathcal{P}$. We see that at least one of $[p, r]$, $[p, s]$, $[s, q]$ and $[r, q]$ is not an edge interval in $\mathcal{P}$. Say $[p, r]$ is not an edge interval in $\mathcal{P}$.

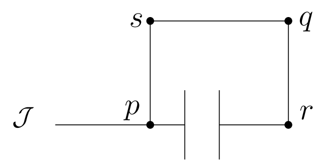


Figure 3.10: A maximal interval

Suppose $\mathcal{J} \in \Omega$ is the maximal horizontal edge interval to which p belongs. Since $x_p x_q - x_r x_s \in L_{\mathcal{P}}$, we see that $u_{\mathcal{J}} | \varphi(x_p)$ and, hence, $u_{\mathcal{J}} | \varphi(x_r x_s)$. This contradicts that neither r nor s belongs to $\mathcal{J}$ (see Figure 3.10). Hence, this case is not possible. Thus, every binomial $f \in L_{\mathcal{P}}$ of degree 2 belongs to $I_{\mathcal{P}}$.

We now show that every binomial $f \in L_{\mathcal{P}}$ with $\deg f \geq 3$ is redundant. Suppose $f = f^+ - f^-$ is an irredundant binomial with $\deg f \geq 3$.

First, we show that there does not exist any vertex $v \in V_+ \cup V_-$ such that $v \in \mathcal{I}_e$, where $\mathcal{I}_e$ is the interval shown in Figure 3.8. To show this, suppose the contrary, that is, that there exists $v_1 \in V_+ \cap \mathcal{I}_e$. Since $\varphi(f^+) = \varphi(f^-)$, we have a vertex $v_1' \in V_-$ such that $v_1' \in \mathcal{I}_e$. We also have a vertex v_2' such that v_1 and v_2' belong to the same maximal vertical edge interval. We see that there exists a vertex $v_2 \in V_+$ such that v_2 and v_1' belong to the same horizontal edge interval of $\mathcal{P}$.

If v_1' and v_1 are in the same horizontal edge interval, then by applying Lemma 3.4.2 to the vertices v_1, v_1', v_2', we obtain that f is redundant, which is a contradiction. Using the same argument, we see that $v_2 \notin \mathcal{I}_e$

Suppose v_1' and v_1 are in the same vertical edge interval. Assume v_1 is lower than v_1'. Using Lemma 3.4.1 (c), we observe that v_1, v_2, v_1' are three corners of an inner interval.Applying Lemma 3.4.2, we see that f is redundant. Similarly, if v_1' is lower than v_1, we obtain that f is redundant. Hence, this case is not possible.

Finally, assume that v_1 and v_1' are not in the same edge intervals. If v_1 and v_2 belong to the same vertical edge interval, then, by applying Lemma 3.4.2 to the vertices v_2, v_1', v_2', we are done. Assume the second coordinate of v_1 is lower than that of v_1'. Let g, h be the other corners of the inner interval defined by v_1' and v_2'. Assume v_1,v_2' and g belong to the same vertical edge interval. Then we have $x_{v_1'}x_{v_2'} - x_g x_h \in L_{\mathcal{P}}$ and

$$\begin{aligned} f &= f^+ - f^- \\ &= f^+ - x_{v_1'}x_{v_2'}\frac{f^-}{x_{v_1'}x_{v_2'}} \\ &= f^+ - (x_{v_1'}x_{v_2'} - x_g x_h)\frac{f^-}{x_{v_1'}x_{v_2'}} - x_g x_h\frac{f^-}{x_{v_1'}x_{v_2'}}. \end{aligned}$$

Let $f' = f'^+ - f'^- = f^+ - x_g x_h \frac{f^-}{x_{v_1'}x_{v_2'}}$ and let V_+' and V_-' be the vertices appearing in f'^+ and f'^-. Note that since f and $x_{v_1'}x_{v_2'} - x_g x_h$ are binomials belonging to $L_{\mathcal{P}}$, it holds that $f' \in L_{\mathcal{P}}$.

Then, by applying Lemma 3.4.2 to the vertices $v_1, v_2 \in V_+'$ and $g \in V_-'$, we obtain that f' is redundant, which implies that f is redundant. Thus, the vertices appearing in f do not belong to $\mathcal{I}_e$. In other words, we have $f \in L_{\mathcal{P}} \cap K[x_{ij} \mid (i,j) \in V(\mathcal{P}) \setminus \mathcal{I}_e]$.

Let $\mathcal{P}'$ be the subpolyomino of $\mathcal{P}$ which consists of all cells of $\mathcal{P}$ having no vertices belonging to $\mathcal{I}_e$. Then, we have $I_{\mathcal{P}'} = I_{\mathcal{P}} \cap K[x_{ij} \mid (i,j) \in V(\mathcal{P}) \setminus \mathcal{I}_e]$. We observe that $\mathcal{P}'$ is a simple polyomino. Notice that $\beta(v)$ for each $v \in \mathcal{P} \setminus \mathcal{I}_e$ is a monomial of degree 2 determined by the maximal horizontal and vertical edge intervals to which v belongs. Then, it is known from Theorem 3.3.3 that $I_{\mathcal{P}'} = I_{\mathcal{P}} \cap K[x_{ij} \mid (i,j) \in V(\mathcal{P}) \setminus \mathcal{I}_e] = L_{\mathcal{P}} \cap K[x_{ij} \mid (i,j) \in V(\mathcal{P}) \setminus \mathcal{I}_e]$. Note that if f is irredundant in $L_{\mathcal{P}}$, then it is also irredundant in $L_{\mathcal{P}} \cap K[x_{ij} \mid (i,j) \in V(\mathcal{P}) \setminus \mathcal{I}_e]$ since we have $L_{\mathcal{P}} \cap K[x_{ij} \mid (i,j) \in V(\mathcal{P}) \setminus \mathcal{I}_e] \subset L_{\mathcal{P}}$. We know that $L_{\mathcal{P}} \cap K[x_{ij} \mid (i,j) \in V(\mathcal{P}) \setminus \mathcal{I}_e]$ is generated by binomials of degree 2 since we have $I_{\mathcal{P}'} = L_{\mathcal{P}} \cap K[x_{ij} \mid (i,j) \in V(\mathcal{P}) \setminus \mathcal{I}_e]$ is generated by binomials of degree 2. This is a contradiction. Hence, the proof is complete. $\square$

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