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EICHLER CLASSES ATTACHED TO AUTOMORPHIC FORMS OF DIMENSION —I

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1. Introduction

In his work [1], Eichler discussed a relation between the automorphic forms of dimension $-2k$ (k is an integer and >1) for a fuchsian group Γ and the cohomology groups of Γ with certain modules of polynomials as coefficients, where the cocycles appeared as the periods of 'the generalized abelian integrals' attached to the automorphic forms. Gunning [2] gave a more general form of this relation.

The purpose of the present paper is to give an analogous relation in the case of automorphic forms of dimension -1 for a fuchsian group and give an application to Selberg's eigenspace [3].

I express my gratitude to Dr. H. Shimizu for suggesting this problem to me and for his many valuable critical comments during the preparation of this paper.

2. The eigenspace $\mathfrak{M}\left(1, -\frac{3}{2}\right)$

Let

$$S = \{z = x + iy; x, y \text{ real and } y > 0\}$$

denote the complex upper half-plane and let $G = SL(2, \mathbf{R})$ be the real special linear group of the second degree. Consider direct products

$$\tilde{S} = S \times \mathbf{R}/(2\pi),$$

$$\tilde{G} = G \times \mathbf{R}/(2\pi),$$

where $\mathbf{R}/(2\pi)$ denotes the real torus, and let an element (σ, θ) of \tilde{G} operate on \tilde{S} as follows:

$$\tilde{S} \ni (z, \phi) \rightarrow (z, \phi)(\sigma, \theta) = \left(\frac{az+b}{cz+d}, \phi + \arg(cz+d) + \theta \right) \in \tilde{S},$$

where $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. The operation of \tilde{G} on \tilde{S} is transitive. \tilde{S} is a

weakly symmetric Riemannian space with the \tilde{G} -invariant metric

$$\frac{dx^2 + dy^2}{y^2} + \left(d\phi - \frac{dx}{2y} \right)^2,$$

and with the isometry μ defined by

$$\mu(z, \phi) = (-\bar{z}, -\phi).$$

The \tilde{G} -invariant measure $d(z, \phi)$ associated to the \tilde{G} -invariant metric is given by

$$d(z, \phi) = d(x, y, \phi) = \frac{dx \wedge dy \wedge d\phi}{y^2}.$$

The ring $\mathfrak{R}(S)$ of \tilde{G} -invariant differential operators on \tilde{S} is generated by

$$\frac{\partial}{\partial \phi}$$

and

$$\tilde{\Delta} \equiv y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{5}{4} \frac{\partial^2}{\partial \phi^2} + y \frac{\partial}{\partial \phi} \frac{\partial}{\partial x},$$

where $\tilde{\Delta}$ is the Laplace-Beltrami operator of \tilde{S} .

For an element $(\sigma, \theta) \in \tilde{G}$, we define a map $T_{(\sigma, \theta)}$ of $C^\infty(\tilde{S})$ into itself by

$$(T_{(\sigma, \theta)} f)(z, \phi) = f((z, \phi)(\sigma, \theta)) = f\left(\frac{az+b}{cz+d}, \phi + \arg(cz+d) + \theta\right),$$

where $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. $(\sigma, \theta) \rightarrow T_{(\sigma, \theta)}$ is a representation of \tilde{G} .

By the correspondence

$$G \ni \sigma \leftrightarrow (\sigma, 0) \ni \tilde{G} = G \times \mathbf{R}/(2\pi),$$

we identify the group $G = SL(2, \mathbf{R})$ with a subgroup $G \times \{0\}$ of \tilde{G} , and for an element $\sigma \in G$ we put $T_{(\sigma, \theta)} = T_\sigma$. Then we have

$$(T_\sigma f)(z, \phi) = f\left(\frac{az+b}{cz+d}, \phi + \arg(cz+d)\right).$$

Let Γ be a discrete subgroup of G not containing the element $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, and suppose that G/Γ is compact. We also identify the subgroup Γ with a subgroup $\Gamma \times \{0\}$ of \tilde{G} .

Denote by $C^\infty(\tilde{S}/\Gamma)$ the set of all C^∞ functions on \tilde{S} invariant under Γ :

$$C^\infty(\tilde{S}/\Gamma) = \{f(z, \phi) \in C^\infty(\tilde{S}); T_\sigma f = f \text{ for all } \sigma \in \Gamma\}$$

and define $C^\infty(S/\Gamma)$ similarly.

Now consider the following simultaneous eigenvalue problem in $C^\infty(\tilde{S}/\Gamma)$:

$$(A) \quad \begin{cases} f \in C^\infty(\tilde{S}/\Gamma), \\ \frac{\partial}{\partial \phi} f(z, \phi) = -if(z, \phi), \\ \tilde{\Delta} f(z, \phi) = -\frac{3}{2}f(z, \phi). \end{cases} \quad (1)$$

We denote by $\mathfrak{M}_\Gamma\left(1, -\frac{3}{2}\right) = \mathfrak{M}\left(1, -\frac{3}{2}\right)$ the set of all functions satisfying the above condition (A). The eigenspace $\mathfrak{M}\left(1, -\frac{3}{2}\right)$ is of finite dimension (A. Selberg and T. Tamagawa). We put

$$d_0 = \dim \mathfrak{M}\left(1, -\frac{3}{2}\right) \quad (< \infty).$$

We shall denote by $\mathfrak{S}_1(\Gamma)$ the linear space of all holomorphic automorphic forms of dimension -1 for the fuchsian group Γ and put

$$d = \dim \mathfrak{S}_1(\Gamma).$$

Then we have the following

Theorem 1. $d \leq d_0 \leq 2d$.

Proof. By (1) and the identification of (z, ϕ) with $(z, \phi + 2m\pi)$, the element $f(z, \phi) \in \mathfrak{M}\left(1, -\frac{3}{2}\right)$ is of the form

$$f(z, \phi) = e^{-i\phi} G(z),$$

where $G(z)$ is a function depending only on z . If we put

$$E(z) = y^{-1/2} G(z),$$

then

$$f(z, \phi) = e^{-i\phi} y^{1/2} E(z),$$

and $E(z)$ satisfies the following condition:

$$(B) \quad \begin{cases} E(z) \in C^\infty(S), \\ E(\sigma z) = (cz+d)E(z), \quad \text{where } \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \\ 2y \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} E(z) = i \frac{\partial}{\partial \bar{z}} E(z). \end{cases} \quad (2)$$

Conversely, if $E(z)$ is any function satisfying the above condition (B), then the function $e^{-i\phi} y^{1/2} E(z)$ satisfies the condition (A).

Hence

$$\mathfrak{M}\left(1, -\frac{3}{2}\right) = \{e^{-i\phi} y^{1/2} E(z); E(z) \text{ satisfying the condition (B)}\},$$

and therefore

$$\dim \mathfrak{M}\left(1, -\frac{3}{2}\right) = \dim \mathfrak{E}_\Gamma, \quad (3)$$

where \mathfrak{E}_Γ denotes the linear space of all functions satisfying the condition (B).

Since each element $F(z) \in \mathfrak{S}_1(\Gamma)$ satisfies the condition (B), we have

$$\mathfrak{M}\left(1, -\frac{3}{2}\right) \supseteq \{e^{-i\phi} y^{1/2} F(z); F(z) \in \mathfrak{S}_1(\Gamma)\},$$

and therefore

$$d_0 \geq d. \quad (4)$$

Now we put

$$\frac{\partial}{\partial \bar{z}} E(z) = g(z). \quad (5)$$

By virtue of the condition (B) for $E(z)$, the function $g(z)$ satisfies the following condition:

$$(C) \quad \begin{cases} g(z) \in C^\infty(S), \\ g(\sigma z) = (cz+d) \overline{(cz+d)} g(z), \quad \text{where } \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \\ 2y \frac{\partial}{\partial z} g(z) = i g(z). \end{cases}$$

We shall denote by \mathfrak{G}_Γ the linear space of all functions satisfying the above condition (C).

If $g(z)$ is an element of \mathfrak{G}_Γ , then $g(z) = y^{-1} \overline{F(z)}$, where $F(z) \in \mathfrak{S}_1(\Gamma)$, and the mapping $g \rightarrow F$ is a bijection of the linear space \mathfrak{G}_Γ onto the linear space $\mathfrak{S}_1(\Gamma)$. Therefore

$$\dim \mathfrak{G}_\Gamma = \dim \mathfrak{S}_1(\Gamma).$$

On the other hand, the differential equation (5) is transformed as follows :

$$\frac{\partial}{\partial \bar{z}} E(z) = y^{-1} \overline{F(z)}, \quad \text{where } F(z) \in \mathfrak{S}_1(\Gamma). \quad (6)$$

Consider the following linear map

$$\frac{\partial}{\partial \bar{z}} : \mathfrak{G}_\Gamma \rightarrow \{y^{-1} \overline{F(z)}; F(z) \in \mathfrak{S}_1(\Gamma)\}.$$

From the above discussion, the kernel of this linear map is $\mathfrak{S}_1(\Gamma)$, and we have

$$\begin{aligned} \dim \mathfrak{G}_\Gamma &= \dim \mathfrak{S}_1(\Gamma) + \dim \frac{\partial}{\partial \bar{z}} (\mathfrak{G}_\Gamma) \\ &\leq \dim \mathfrak{S}_1(\Gamma) + \dim \{y^{-1} \overline{F(z)}; F(z) \in \mathfrak{S}_1(\Gamma)\} \\ &= d + d = 2d. \end{aligned}$$

Since $\dim \mathfrak{G}_\Gamma$ is equal to d_0 by (3), we have

$$d_0 \leq 2d. \quad (7)$$

By (4) and (7), we have $d \leq d_0 \leq 2d$. Q. E. D.

Corollary. *The dimension of $\mathfrak{M}\left(1, -\frac{3}{2}\right)$ is different from 0 if and only if $\mathfrak{S}_1(\Gamma) \neq \{0\}$.*

3. The Eichler periods of integrals

For each function $F(z)$ in $\mathfrak{S}_1(\Gamma)$, consider the differential equation (6) in §2 :

$$\frac{\partial}{\partial \bar{z}} E(z) = y^{-1} \overline{F(z)}. \quad (6)$$

The function $\overline{F(z)}$ is a holomorphic function of \bar{z} ; put $\overline{F(z)} = F^*(\bar{z})$. Then the differential equation (6) has a C^∞ solution of the form

$$E_1(z) = E_1(z, \bar{z}) = \int_{z_0}^{\bar{z}} \frac{-2i}{\xi - z} F^*(\xi) d\xi \quad (\text{Im } \xi < 0),$$

where z_0 is a fixed point in the lower half-plane and the integral is taken along an arbitrary peicewise differentiable arc in the lower half-plane with initial point z_0 and terminal point \bar{z} . Therefore the general solutions of class C^∞ of the equation (6) is

$$E(z) = E_1(z) - E_0(z),$$

where $E_0(z)$ denotes an arbitrary holomorphic function on S .

For each $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, the function $E_1(z)$ is transformed as follows :

$$E_1(\sigma z) = \int_{z_0}^{\sigma \bar{z}} \frac{-2i}{\xi - \sigma z} F^*(\xi) d\xi = \int_{z_0}^{\sigma \bar{z}} \frac{-2i}{\xi - \sigma z} \frac{F^*(\sigma^{-1}\xi)}{-c\xi + a} d\xi$$

by $F^*(\sigma^{-1}\xi) = (-c\xi + a)F^*(\xi)$.

Putting $\sigma^{-1}\xi = t$ we have

$$\begin{aligned} \int_{z_0}^{\sigma \bar{z}} \frac{-2i}{\xi - \sigma z} \frac{F^*(\sigma^{-1}\xi)}{-c\xi + a} d\xi &= (cz + d) \int_{\sigma^{-1}z_0}^z \frac{-2i}{t - z} F^*(t) dt = (cz + d) \\ &\times \int_{z_0}^{\sigma \bar{z}} \frac{-2i}{t - z} F^*(t) dt + (cz + d) \int_{\sigma^{-1}z_0}^{z_0} \frac{-2i}{t - z} F^*(t) dt. \end{aligned}$$

Hence we have

$$E(\sigma z) = (cz + d)E(z) + (cz + d)E_0(z) + (cz + d)C(\sigma; z) - E_0(\sigma z),$$

where $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, and

$$C(\sigma; z) = \int_{\sigma^{-1}z_0}^{z_0} \frac{-2i}{\xi - z} F^*(\xi) d\xi.$$

Hereafter we put

$$[*] = (cz + d)E_0(z) + (cz + d)C(\sigma; z) - E_0(\sigma z).$$

Let \mathfrak{N} be the set of all holomorphic functions on S and let Γ operate on \mathfrak{N} as follows :

$$h(z) \cdot \sigma \equiv h(\sigma; z) = \frac{1}{cz + d} h(\sigma z),$$

where $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $h(z) \in \mathfrak{N}$. Then \mathfrak{N} forms a Γ -right module. A map $\sigma \rightarrow h(\sigma; z)$ of Γ into \mathfrak{N} is called a (non-homogeneous) 1-cochain, and if a cochain satisfies the equation

$$h(z)(\sigma\tau) = h(\sigma; z) \cdot \tau + h(\tau; z),$$

then it is called a 1-cocycle. A 1-cocycle of the form $h(\sigma; z) = n(z) \cdot \sigma - n(z)$, where $n(z)$ is a fixed element of \mathfrak{N} is called a coboundary. The cocycles form an additive group $Z^1(\Gamma, \mathfrak{N})$, and the coboundaries form a subgroup $B^1(\Gamma, \mathfrak{N})$ of $Z^1(\Gamma, \mathfrak{N})$. The factor group $Z^1(\Gamma, \mathfrak{N})/B^1(\Gamma, \mathfrak{N})$ is denoted by $H^1(\Gamma, \mathfrak{N})$ and called the first cohomology group of Γ in \mathfrak{N} , and its elements are called cohomology classes.

For each function $F(z)$ in $\mathfrak{S}_1(\Gamma)$, the map

$$\sigma \rightarrow C(\sigma; z), \quad \text{where } C(\sigma; z) = \int_{\sigma^{-1}z_0}^{z_0} \frac{-2i}{\xi - z} F^*(\xi) d\xi \quad (8)$$

is clearly a cocycle. We shall call it *the Eichler period of the integral $E_1(z)$ on σ* , where

$$E_1(z) = \int_{z_0}^{\bar{z}} \frac{-2i}{\xi - z} F^*(\xi) d\xi.$$

This cocycle depends on $F(z)$, and moreover on the constant z_0 occurring in the definition of $E_1(z)$. But a change of z_0 would add a coboundary to $C(\sigma; z)$. Therefore the cohomology class of (8) depends only on $F(z)$. We shall call it *the Eichler class of $F(z)$* .

It should be noted that a cocycle $C(\sigma; z)$ satisfying the condition

$$[*] = 0,$$

is a coboundary.

4. A theorem of monomorphism and its application

Consider the following map from $\mathfrak{S}_1(\Gamma)$ into $H^1(\Gamma, \mathfrak{R})$

$$\varphi : F(z) \rightarrow C(\sigma; z), \quad (9)$$

where
$$C(\sigma; z) = \int_{\sigma^{-1}z_0}^{z_0} \frac{-2i}{\xi - z} F^*(\xi) d\xi, \quad F^*(\xi) = \overline{F(\bar{\xi})}.$$

The map is obviously a homomorphism of $\mathfrak{S}_1(\Gamma)$ into $H^1(\Gamma, \mathfrak{R})$. We shall prove that $\varphi : \mathfrak{S}_1(\Gamma) \rightarrow H^1(\Gamma, \mathfrak{R})$ is one-one.

For the cocycle $C(\sigma; z)$ such that $\varphi(F_c(z)) = C(\sigma; z)$ with $F_c(z) \in \mathfrak{S}_1(\Gamma)$, we put

$$E_1(z) = \int_{z_0}^{\bar{z}} \frac{-2i}{\xi - z} F_c^*(\xi) d\xi \quad (F_c^*(\xi) = \overline{F_c(\bar{\xi})}),$$

then $E_1(z)$ is a C^∞ function satisfying the conditions

$$C(\sigma; z) = E_1(z) \cdot \sigma - E_1(z) \quad (10)$$

and

$$\frac{\partial}{\partial \bar{z}} E_1(z) = y^{-1} \overline{F_c(z)}. \quad (11)$$

The differential form $\phi(z) = \bar{\partial} E_1(z)$ is a C^∞ differential form on D^1 , and

1) D denotes a fundamental domain of Γ in S . As mentioned in §2 G/Γ is compact, so D is compact.

by (11) it satisfies the relation $\phi(\sigma z) = (cz + d)\phi(z)$ for all $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. For an arbitrary element $F(z)$ of $\mathfrak{S}_1(\Gamma)$, $\theta(z) = F(z)dz$ is a holomorphic differential form on D and satisfies the relation $\theta(\sigma z) = \frac{1}{cz + d}\theta(z)$ for all $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Therefore the differential form $\phi(z) \wedge \theta(z)$ is invariant under the group Γ .

We consider the following integral over D ,

$$\int_D \phi(z) \wedge \theta(z). \quad (12)$$

From the above discussion, the integral converges and is independent of the choice of fundamental domain of Γ in S . If $g_1(z)$, $g_2(z)$ are two C^∞ functions associated with the same cocycle $C(\sigma; z)$ by the relation (10), then we have

$$g_0(\sigma z) = (cz + d)g_0(z),$$

where $g_0(z) = g_1(z) - g_2(z)$. Then the differential form $g_0(z)F(z)dz$ is invariant under the group Γ . Therefore

$$\begin{aligned} \int_D \bar{\partial}g_1(z) \wedge F(z)dz - \int_D \bar{\partial}g_2(z) \wedge F(z)dz &= \int_D \bar{\partial}g_0(z) \wedge F(z)dz \\ &= \int_D \frac{\partial}{\partial \bar{z}}(g_0(z)F(z))d\bar{z} \wedge dz \\ &= \int_{\partial D} g_0(z)F(z)dz \\ &= 0, \end{aligned}$$

where ∂D denotes the boundary of D . Consequently, the integral (12) depends only on $C(\sigma; z)$. We shall denote it by

$$\langle C(\sigma; z), F(z) \rangle = \int_D \phi(z) \wedge \theta(z).$$

If the cocycle $C(\sigma; z)$ is an element of $B^1(\Gamma, \mathfrak{R})$, then $\phi(z) \equiv 0$; and hence

$$\langle C(\sigma; z), F(z) \rangle = 0$$

for all $F(z) \in \mathfrak{S}_1(\Gamma)$. Therefore, if there exists a function $F(z)$ in $\mathfrak{S}_1(\Gamma)$ such that $\langle C(\sigma; z), F(z) \rangle \neq 0$, then $C(\sigma; z) \notin B^1(\Gamma, \mathfrak{R})$.

Let $F(z)$ be a nontrivial function in $\mathfrak{S}_1(\Gamma)$ and $C(\sigma; z)$ the image of $F(z)$ by φ . If we put

$$\psi(z) = y^{-1} \overline{F(z)} d\bar{z}$$

and

$$\theta(z) = F(z) dz,$$

then the integral

$$\int_D \psi(z) \wedge \theta(z) = \int_D y^{-1} \overline{F(z)} F(z) dz \wedge d\bar{z}$$

is not equal to zero. On the other hand, let

$$E_1(z) = \int_{z_0}^z \frac{-2i}{\xi - z} F^*(\xi) d\xi.$$

Then the function $E_1(z)$ satisfies the following conditions:

$$C(\sigma; z) = E_1(z) \cdot \sigma - E_1(z)$$

and

$$\frac{\partial}{\partial \bar{z}} E_1(z) = y^{-1} \overline{F(z)}.$$

Hence

$$\bar{\partial} E_1(z) = \frac{\partial}{\partial \bar{z}} E_1(z) d\bar{z} = y^{-1} \overline{F(z)} d\bar{z} = \psi(z),$$

and therefore

$$\langle C(\sigma; z), F(z) \rangle \neq 0.$$

Consequently, if $F(z) \neq 0$, then for the corresponding cocycle $C(\sigma; z)$, $\langle C(\sigma; z), F(z) \rangle \neq 0$.

By the two results obtained above, we see that if $F(z) \neq 0$, then the corresponding cycles $C(\sigma; z)$ does not belong to $B^1(\Gamma, \mathfrak{R})$. Therefore $F(z) \neq 0$ is equivalent to $C(\sigma; z) \notin B^1(\Gamma, \mathfrak{R})$; and hence the kernel of φ is equal to zero.

We have the following

Theorem 2.²⁾ *The map $F(z) \rightarrow C(\sigma; z)$ defined by (9) is a monomorphism of $\mathfrak{S}_1(\Gamma)$ into $H^1(\Gamma, \mathfrak{R})$.*

Corollary. *With the same notations as in §2, we have*

$$d = d_0.$$

Proof. In §2, we have considered the following linear map

2) The possibility of this theorem was indicated by Gunning ([2], p. 56).

$$\frac{\partial}{\partial \bar{z}} : \mathfrak{E}_\Gamma \rightarrow \{y^{-1}\overline{F(z)}; F(z) \in \mathfrak{E}_1(\Gamma)\},$$

and proved that the map is into. Next we prove $\frac{\partial}{\partial \bar{z}}(\mathfrak{E}_\Gamma) = \{0\}$. For every nontrivial function $F(z)$ of $\mathfrak{E}_1(\Gamma)$, there exists an inverse image of $y^{-1}\overline{F(z)}$ if and only if the cocycle $C(\sigma; z)$ corresponding to $F(z)$ by φ belongs to $B^1(\Gamma, \mathfrak{N})$ (cf. §3 in this paper). On the other hand, by Theorem 2 the cocycle $C(\sigma; z)$ corresponding to the nontrivial $F(z)$ does not belong to $B^1(\Gamma, \mathfrak{N})$. Therefore

$$\frac{\partial}{\partial \bar{z}}(\mathfrak{E}_\Gamma) = \{0\}^{3)},$$

and hence, we have

$$d = d_0. \quad \text{Q. E. D.}$$

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3) Therefore all the functions $E(z)$ in \mathfrak{E}_Γ are holomorphic.