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EICHLER CLASSES ATTACHED TO AUTOMORPHIC FORMS OF DIMENSION —I

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1. Introduction

In his work [1], Eichler discussed a relation between the automorphic forms of dimension $-2k$ ($k$ is an integer and $>1$) for a fuchsian group $\Gamma$ and the cohomology groups of $\Gamma$ with certain modules of polynomials as coefficients, where the cocycles appeared as the periods of 'the generalized abelian integrals' attached to the automorphic forms. Gunning [2] gave a more general form of this relation.

The purpose of the present paper is to give an analogous relation in the case of automorphic forms of dimension $-1$ for a fuchsian group and give an application to Selberg's eigenspace [3].

I express my gratitude to Dr. H. Shimizu for suggesting this problem to me and for his many valuable critical comments during the preparation of this paper.

2. The eigenspace $\mathfrak{M}(1, \frac{-3}{2})$

Let

$$S = \{z = x+iy; x, y \text{ real and } y > 0\}$$

denote the complex upper half-plane and let $G=SL(2, \mathbb{R})$ be the real special linear group of the second degree. Consider direct products

$$\tilde{S} = S \times \mathbb{R}/(2\pi),$$

$$\tilde{G} = G \times \mathbb{R}/(2\pi),$$

where $\mathbb{R}/(2\pi)$ denotes the real torus, and let an element $(\sigma, \theta)$ of $\tilde{G}$ operate on $\tilde{S}$ as follows:

$$\tilde{S} \ni (z, \phi) \rightarrow (z, \phi)(\sigma, \theta) = \left(\frac{az+b}{cz+d}, \phi + \arg (cz+d) + \theta\right) \in \tilde{S},$$

where $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. The operation of $\tilde{G}$ on $\tilde{S}$ is transitive. $\tilde{S}$ is a...
weakly symmetric Riemannian space with the $\mathcal{G}$-invariant metric

$$\frac{dx^2 + dy^2}{y^2} + \left( d\phi - \frac{dx}{2y} \right)^2,$$

and with the isometry $\mu$ defined by

$$\mu(z, \phi) = (-\bar{z}, -\phi).$$

The $\mathcal{G}$-invariant measure $d(z, \phi)$ associated to the $\mathcal{G}$-invariant metric is given by

$$d(z, \phi) = d(x, y, \phi) = \frac{dx \wedge dy \wedge d\phi}{y^2}.$$

The ring $\mathfrak{R}(S)$ of $\mathcal{G}$-invariant differential operators on $\bar{S}$ is generated by

$$\frac{\partial}{\partial \phi}$$

and

$$\Delta = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{5}{4} \frac{\partial^2}{\partial \phi^2} + y \frac{\partial}{\partial \phi} \frac{\partial}{\partial x},$$

where $\Delta$ is the Laplace-Beltrami operator of $\bar{S}$.

For an element $(\sigma, \theta) \in \bar{G}$, we define a map $T_{(\sigma, \theta)}$ of $C^\infty(\bar{S})$ into itself by

$$(T_{(\sigma, \theta)} f)(z, \phi) = f((z, \phi)(\sigma, \theta)) = f\left(\frac{az + b}{cz + d}, \phi + \arg(cz + d) + \theta\right),$$

where $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. $(\sigma, \theta) \rightarrow T_{(\sigma, \theta)}$ is a representation of $\bar{G}$.

By the correspondence

$$G \ni \sigma \leftrightarrow (\sigma, 0) \in \bar{G} = G \times \mathbb{R}/(2\pi),$$

we identify the group $G = SL(2, \mathbb{R})$ with a subgroup $G \times \{0\}$ of $\bar{G}$, and for an element $\sigma \in G$ we put $T_{(\sigma, \theta)} = T_{\sigma}$. Then we have

$$(T_{\sigma} f)(z, \phi) = f\left(\frac{az + b}{cz + d}, \phi + \arg(cz + d)\right).$$

Let $\Gamma$ be a discrete subgroup of $G$ not containing the element $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, and suppose that $G/\Gamma$ is compact. We also identify the subgroup $\Gamma$ with a subgroup $\Gamma \times \{0\}$ of $\bar{G}$.

Denote by $C^\infty(\bar{S}/\Gamma)$ the set of all $C^\infty$ functions on $\bar{S}$ invariant under $\Gamma$:...
Eichler Classes Attached to Automorphic Forms of Dimension $-1$

$$C^\infty(S/\Gamma) = \{ f(z, \phi) \in C^\infty(S) ; \ T_\sigma f = f \ \text{for all} \ \sigma \in \Gamma \}$$

and define $C^\infty(S/\Gamma)$ similarly.

Now consider the following simultaneous eigenvalue problem in $C^\infty(S/\Gamma)$:

$$f \in C^\infty(S/\Gamma), \ \begin{cases} \frac{\partial}{\partial \phi} f(z, \phi) = -if(z, \phi), \\ \Delta f(z, \phi) = -\frac{3}{2} f(z, \phi). \end{cases} \quad (A) \quad (1)$$

We denote by $\mathcal{M}_f(1, -\frac{3}{2}) = \mathcal{M}(1, -\frac{3}{2})$ the set of all functions satisfying the above condition (A). The eigenspace $\mathcal{M}(1, -\frac{3}{2})$ is of finite dimension (A. Selberg and T. Tamagawa). We put

$$d_0 = \dim \mathcal{M}(1, -\frac{3}{2}) \quad (<\infty).$$

We shall denote by $\mathcal{O}_1(\Gamma)$ the linear space of all holomorphic automorphic forms of dimension $-1$ for the fuchsian group $\Gamma$ and put

$$d = \dim \mathcal{O}_1(\Gamma).$$

Then we have the following

**Theorem 1.** $d \leq d_0 \leq 2d.$

**Proof.** By (1) and the identification of $(z, \phi)$ with $(z, \phi + 2n\pi)$, the element $f(z, \phi) \in \mathcal{M}(1, -\frac{3}{2})$ is of the form

$$f(z, \phi) = e^{-i\phi} G(z),$$

where $G(z)$ is a function depending only on $z$. If we put

$$E(z) = y^{1/2} G(z),$$

then

$$f(z, \phi) = e^{-i\phi} y^{1/2} E(z),$$

and $E(z)$ satisfies the following condition:
Conversely, if $E(z)$ is any function satisfying the above condition (B), then the function $e^{-i\phi}y^{1/2}E(z)$ satisfies the condition (A). Hence

$$\mathcal{M}\left(1, -\frac{3}{2}\right) = \{e^{-i\phi}y^{1/2}E(z); E(z) \text{ satisfying the condition (B)}\},$$

and therefore

$$\dim \mathcal{M}\left(1, -\frac{3}{2}\right) = \dim \mathcal{G}_\Gamma,$$

(3)

where $\mathcal{G}_\Gamma$ denotes the linear space of all functions satisfying the condition (B).

Since each element $F(z) \in \mathcal{G}_1(\Gamma)$ satisfies the condition (B), we have

$$\mathcal{M}\left(1, -\frac{3}{2}\right) \supset \{e^{-i\phi}y^{1/2}F(z); F(z) \in \mathcal{G}_1(\Gamma)\},$$

and therefore

$$d_0 \geq d.$$  

(4)

Now we put

$$\frac{\partial}{\partial \bar{z}}E(z) = g(z).$$  

(5)

By virtue of the condition (B) for $E(z)$, the function $g(z)$ satisfies the following condition:

$$(C) \quad \left\{ \begin{array}{l}
g(z) \in C^\infty(S), \\
g(\sigma z) = (cz+d)(\bar{c}z+d)^xg(z), \quad \text{where} \quad \sigma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma, \\
2y \frac{\partial}{\partial z}g(z) = ig(z).
\end{array} \right.$$  

We shall denote by $\mathcal{G}_\Gamma$ the linear space of all functions satisfying the above condition (C).

If $g(z)$ is an element of $\mathcal{G}_\Gamma$, then $g(z) = y^{-1} \overline{F(\overline{z})}$, where $F(z) \in \mathcal{G}_1(\Gamma)$, and the mapping $g \rightarrow F$ is a bijection of the linear space $\mathcal{G}_\Gamma$ onto the linear space $\mathcal{G}_1(\Gamma)$. Therefore

$$\dim \mathcal{G}_\Gamma = \dim \mathcal{G}_1(\Gamma).$$
On the other hand, the differential equation (5) is transformed as follows:

\[ \frac{\partial}{\partial \bar{z}} E(z) = y^{-1} \overline{F(z)}, \text{ where } F(z) \in \mathfrak{S}_1(\Gamma). \]  

(6)

Consider the following linear map

\[ \frac{\partial}{\partial \bar{z}} : \mathfrak{C}_r \to \{ y^{-1} \overline{F(z)} ; F(z) \in \mathfrak{S}_1(\Gamma) \}. \]

From the above discussion, the kernel of this linear map is \( \mathfrak{S}_1(\Gamma) \), and we have

\[ \text{dim } \mathfrak{C}_r = \text{dim } \mathfrak{S}_1(\Gamma) + \text{dim } \frac{\partial}{\partial \bar{z}} (\mathfrak{C}_r) \]

\[ \leq \text{dim } \mathfrak{S}_1(\Gamma) + \text{dim } \{ y^{-1} \overline{F(z)} ; F(z) \in \mathfrak{S}_1(\Gamma) \} \]

\[ = d + d = 2d. \]

Since \( \text{dim } \mathfrak{C}_r \) is equal to \( d_0 \) by (3), we have

\[ d_0 \leq 2d. \]  

(7)

By (4) and (7), we have \( d \leq d_0 \leq 2d \). Q. E. D.

**Corollary.** The dimension of \( \mathfrak{M}(1, -\frac{3}{2}) \) is different from 0 if and only if \( \mathfrak{S}_1(\Gamma) \neq \{0\} \).

**3. The Eichler periods of integrals**

For each function \( F(z) \) in \( \mathfrak{S}_1(\Gamma) \), consider the differential equation (6) in §2:

\[ \frac{\partial}{\partial \bar{z}} E(z) = y^{-1} \overline{F(z)}. \]  

(6)

The function \( \overline{F(z)} \) is a holomorphic function of \( \bar{z} \); put \( \overline{F(z)} = F^*(\bar{z}) \). Then the differential equation (6) has a \( C^\infty \) solution of the form

\[ E_0(z) = E_0(z, \bar{z}) = \int_{z_0}^{\bar{z}} \frac{-2i}{\xi - z} F^*(\xi) d\xi \quad (\text{Im } \xi < 0), \]

where \( z_0 \) is a fixed point in the lower half-plane and the integral is taken along an arbitrary piecewise differentiable arc in the lower half-plane with initial point \( z_0 \) and terminal point \( \bar{z} \). Therefore the general solutions of class \( C^\infty \) of the equation (6) is

\[ E(z) = E_0(z) - E_0(z), \]
where $E_0(z)$ denotes an arbitrary holomorphic function on $S$.

For each $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, the function $E_0(z)$ is transformed as follows:

$$E_0(\sigma z) = \int_{x_0, \xi}^{E_0(\xi)} \frac{-2i}{z - \sigma z} F^*(\xi) d\xi = \int_{x_0, \xi}^{E_0(\xi)} \frac{-2i}{z - \sigma z - c\xi + a} F^*(\xi) d\xi$$

by $F^*_{\sigma^{-1}}(\xi) = (-c\xi + a) F^*(\xi)$.

Putting $\sigma^{-1} \xi = t$ we have

$$\int_{x_0, \xi}^{E_0(\xi)} \frac{-2i}{z - \sigma z - c\xi + a} F^*(\xi) d\xi = (cz + d) \int_{\sigma^{-1} x_0}^{t - z} \frac{-2i}{t - z} F^*(t) dt = (cz + d) \int_{\sigma^{-1} x_0}^{t - z} \frac{-2i}{t - z} F^*(t) dt.$$

Hence we have

$$E(\sigma z) = (cz + d) E_0(z) + (cz + d) E_0(z) C(\sigma ; z) - E_0(\sigma z),$$

where $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, and

$$C(\sigma ; z) = \int_{x_0, \xi}^{E_0(\xi)} \frac{-2i}{z - \sigma z - c\xi + a} F^*(\xi) d\xi.$$

Hereafter we put

$$[\ast] = (cz + d) E_0(z) + (cz + d) C(\sigma ; z) - E_0(\sigma z).$$

Let $\mathcal{H}$ be the set of all holomorphic functions on $S$ and let $\Gamma$ operate on $\mathcal{H}$ as follows:

$$h(z) \cdot \sigma = h(\sigma ; z) = \frac{1}{cz + d} h(\sigma z),$$

where $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $h(z) \in \mathcal{H}$. Then $\mathcal{H}$ forms a $\Gamma$-right module. A map $\sigma \rightarrow h(\sigma ; z)$ of $\Gamma$ into $\mathcal{H}$ is called a (non-homogeneous) 1-cochain, and if a cochain satisfies the equation

$$h(z)(\sigma \tau) = h(\sigma ; z) \cdot \tau + h(\tau ; z),$$

then it is called a 1-cocycle. A 1-cocycle of the form $h(\sigma ; z) = n(z) \cdot \sigma - n(z)$, where $n(z)$ is a fixed element of $\mathcal{H}$ is called a coboundary. The cocycles form an additive group $Z^1(\Gamma, \mathcal{H})$, and the coboundaries form a subgroup $B^1(\Gamma, \mathcal{H})$ of $Z^1(\Gamma, \mathcal{H})$. The factor group $Z^1(\Gamma, \mathcal{H})/B^1(\Gamma, \mathcal{H})$ is denoted by $H^1(\Gamma, \mathcal{H})$ and called the first cohomology group of $\Gamma$ in $\mathcal{H}$, and its elements are called cohomology classes.
For each function $F(z)$ in $\Theta(\Gamma)$, the map

$$\sigma \to C(\sigma ; z), \text{ where } C(\sigma ; z) = \int_{\sigma^{-1}z_0}^{z_0} \frac{-2iF^*(\xi)d\xi}{\xi - z} \quad (8)$$

is clearly a cocycle. We shall call it the **Eichler period of the integral** $E_\lambda(z)$ on $\sigma$, where

$$E_\lambda(z) = \int_{z_0}^{\bar{z}} \frac{-2iF^*(\xi)d\xi}{\xi - z}.$$

This cocycle depends on $F(z)$, and moreover on the constant $z_0$ occurring in the definition of $E_\lambda(z)$. But a change of $z_0$ would add a coboundary to $C(\sigma ; z)$. Therefore the cohomology class of (8) depends only on $F(z)$. We shall call it the **Eichler class of $F(z)$**.

It should be noted that a cocycle $C(\sigma ; z)$ satisfying the condition

$$[\ast] = 0,$$

is a coboundary.

4. A theorem of monomorphism and its application

Consider the following map from $\Theta(\Gamma)$ into $H^1(\Gamma, \mathbb{R})$

$$\phi: F(z) \rightarrow C(\sigma ; z), \quad (9)$$

where

$$C(\sigma ; z) = \int_{\sigma^{-1}z_0}^{z_0} \frac{-2iF^*(\xi)d\xi}{\xi - z}, \quad F^*(\xi) = \overline{F(\xi)}.$$

The map is obviously a homomorphism of $\Theta(\Gamma)$ into $H^1(\Gamma, \mathbb{R})$. We shall prove that $\phi: \Theta(\Gamma) \rightarrow H^1(\Gamma, \mathbb{R})$ is one-one.

For the cocycle $C(\sigma ; z)$ such that $\phi(F_\lambda(z)) = C(\sigma ; z)$ with $F_\lambda(z) \in \Theta(\Gamma)$, we put

$$E_\lambda(z) = \int_{z_0}^{\bar{z}} \frac{-2iF^*_\lambda(\xi)d\xi}{\xi - z} \quad (F^*_\lambda(\xi) = \overline{F_\lambda(\xi)}),$$

then $E_\lambda(z)$ is a $C^\infty$ function satisfying the conditions

$$C(\sigma ; z) = E_\lambda(z) \cdot \sigma - E_\lambda(z) \quad (10)$$

and

$$\frac{\partial}{\partial \bar{z}} E_\lambda(z) = y^{-1}F_\lambda(z). \quad (11)$$

The differential form $\phi(z) = \partial E_\lambda(z)$ is a $C^\infty$ differential form on $D^\infty$, and

---

1) $D$ denotes a fundamental domain of $\Gamma$ in $S$. As mentioned in §2 $G/\Gamma$ is compact, so $D$ is compact.
by (11) it satisfies the relation \( \phi(\sigma z) = (cz+d)\phi(z) \) for all \( \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \).

For an arbitrary element \( F(z) \) of \( \mathfrak{S}_1(\Gamma) \), \( \theta(z) = F(z)dz \) is a holomorphic differential form on \( D \) and satisfies the relation \( \theta(\sigma z) = \frac{1}{cz+d} \theta(z) \) for all \( \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \). Therefore the differential form \( \phi(z) \wedge \theta(z) \) is invariant under the group \( \Gamma \).

We consider the following integral over \( D \),

\[
\int_D \phi(z) \wedge \theta(z) .
\]

From the above discussion, the integral converges and is independent of the choice of fundamental domain of \( \Gamma \) in \( S \). If \( g_1(z), g_2(z) \) are two \( C^\infty \) functions associated with the same cocycle \( C(\sigma ; z) \) by the relation (10), then we have

\[
g_1(\sigma z) = (cz+d)g_1(z),
\]

where \( g_1(\sigma z) = g_1(z) - g_2(z) \). Then the differential form \( g_1(\sigma z)F(z)dz \) is invariant under the group \( \Gamma \). Therefore

\[
\int_D \delta g_1(\sigma z) \wedge F(z)dz - \int_D \delta g_2(\sigma z) \wedge F(z)dz = \int_D \delta g_1(z) \wedge F(z)dz
\]

\[
= \int_D \frac{\partial}{\partial \bar{z}} (g_1(z)F(z))d\bar{z} \wedge dz
\]

\[
= \int_{\partial D} g_1(z)F(z)dz
\]

\[
= 0 ,
\]

where \( \partial D \) denotes the boundary of \( D \). Consequently, the integral (12) depends only on \( C(\sigma ; z) \). We shall denote it by

\[
\langle C(\sigma ; z), F(z) \rangle = \int_D \phi(z) \wedge \theta(z) .
\]

If the cocycle \( C(\sigma ; z) \) is an element of \( B'(\Gamma, \mathcal{H}) \), then \( \phi(z) \equiv 0 \); and hence

\[
\langle C(\sigma ; z), F(z) \rangle = 0
\]

for all \( F(z) \in \mathfrak{S}_1(\Gamma) \). Therefore, if there exists a function \( F(z) \) in \( \mathfrak{S}_1(\Gamma) \) such that \( \langle C(\sigma ; z), F(z) \rangle \neq 0 \), then \( C(\sigma ; z) \in B'(\Gamma, \mathcal{H}) \).

Let \( F(z) \) be a nontrivial function in \( \mathfrak{S}_1(\Gamma) \) and \( C(\sigma ; z) \) the image of \( F(z) \) by \( \varphi \). If we put
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\[ \psi(z) = y^{-1}F(z)\,dz \]

and

\[ \theta(z) = F(z)\,dz , \]

then the integral

\[ \int_{D} \psi(z) \wedge \theta(z) = \int_{D} y^{-1}F(z)\,F(z)\,dz \wedge d\bar{z} \]

is not equal to zero. On the other hand, let

\[ E_{i}(z) = \int_{z_{0}}^{z} \frac{-2i}{\xi - z} F^* (\xi)\,d\xi . \]

Then the function $E_{i}(z)$ satisfies the following conditions:

\[ C(\sigma; z) = E_{i}(z) \cdot \sigma - E_{i}(z) \]

and

\[ \frac{\partial}{\partial \bar{z}} E_{i}(z) = y^{-1}F(z) . \]

Hence

\[ \bar{\partial} E_{i}(z) = \frac{\partial}{\partial \bar{z}} E_{i}(z)\,d\bar{z} = y^{-1}F(z)\,d\bar{z} = \psi(z) , \]

and therefore

\[ \langle C(\sigma; z), F(z) \rangle \neq 0 . \]

Consequently, if $F(z) \neq 0$, then for the corresponding cocycle $C(\sigma; z)$,

\[ \langle C(\sigma; z), F(z) \rangle \neq 0 . \]

By the two results obtained above, we see that if $F(z) \neq 0$, then the corresponding cycles $C(\sigma; z)$ does not belong to $\mathcal{B}^i(\Gamma, \mathcal{R})$. Therefore $F(z) \neq 0$ is equivalent to $C(\sigma; z) \notin \mathcal{B}^i(\Gamma, \mathcal{R})$; and hence the kernel of $\varphi$ is equal to zero.

We have the following

Theorem 2.\(^2\) The map $F(z) \to C(\sigma; z)$ defined by (9) is a monomorphism of $\mathcal{O} \Gamma \Gamma$ into $H^i(\Gamma, \mathcal{R})$.

Corollary. With the same notations as in §2, we have

\[ d = d_{\varphi} . \]

Proof. In §2, we have considered the following linear map

\[ \mathcal{O} \Gamma \Gamma \to \mathcal{R} \Gamma \Gamma \]

2) The possibility of this theorem was indicated by Gunning ([2], p. 56).
\[ \frac{\partial}{\partial \bar{z}} : \mathfrak{C}_{\Gamma} \rightarrow \left\{ y^{-1} \overline{F(z)} ; F(z) \in \mathfrak{C}_{\Gamma}(\Gamma) \right\} , \]

and proved that the map is into. Next we prove \( \frac{\partial}{\partial \bar{z}} (\mathfrak{C}_{\Gamma}) = \{0\} \). For every nontrivial function \( F(z) \) of \( \mathfrak{C}_{\Gamma}(\Gamma) \), there exists an inverse image of \( y^{-1} \overline{F(z)} \) if and only if the cocycle \( C(\sigma ; z) \) corresponding to \( F(z) \) by \( \varphi \) belongs to \( B'(\Gamma, \mathcal{R}) \) (cf. § 3 in this paper). On the other hand, by Theorem 2 the cocycle \( C(\sigma ; z) \) corresponding to the nontrivial \( F(z) \) does not belong to \( B'(\Gamma, \mathcal{R}) \). Therefore

\[ \frac{\partial}{\partial \bar{z}} (\mathfrak{C}_{\Gamma}) = \{0\} \]

and hence, we have

\[ d = d_0. \]

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References


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3) Therefore all the functions \( E(z) \) in \( \mathfrak{C}_{\Gamma} \) are holomorphic.