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EICHLER CLASSES ATTACHED TO AUTOMORPHIC FORMS OF DIMENSION —I

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1. Introduction

In his work [1], Eichler discussed a relation between the automorphic forms of dimension $-2k$ ($k$ is an integer and $>1$) for a fuchsian group $\Gamma$ and the cohomology groups of $\Gamma$ with certain modules of polynomials as coefficients, where the cocycles appeared as the periods of 'the generalized abelian integrals' attached to the automorphic forms. Gunning [2] gave a more general form of this relation.

The purpose of the present paper is to give an analogous relation in the case of automorphic forms of dimension $-1$ for a fuchsian group and give an application to Selberg's eigenspace [3].

I express my gratitude to Dr. H. Shimizu for suggesting this problem to me and for his many valuable critical comments during the preparation of this paper.

2. The eigenspace $\mathfrak{M}\left(1, \frac{-3}{2}\right)$

Let

$$ S = \{z = x+iy; x, y \text{ real and } y > 0\} $$

denote the complex upper half-plane and let $G=SL(2, \mathbb{R})$ be the real special linear group of the second degree. Consider direct products

$$ \tilde{S} = S \times \mathbb{R}/(2\pi), $$

$$ \tilde{G} = G \times \mathbb{R}/(2\pi), $$

where $\mathbb{R}/(2\pi)$ denotes the real torus, and let an element $(\sigma, \theta)$ of $\tilde{G}$ operate on $\tilde{S}$ as follows:

$$ \tilde{S}\ni(z, \phi) \rightarrow (z, \phi)(\sigma, \theta) = \left(\frac{az+b}{cz+d}, \phi + \arg(cz+d)+\theta\right) \in \tilde{S}, $$

where $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. The operation of $\tilde{G}$ on $\tilde{S}$ is transitive. $\tilde{S}$ is a
weakly symmetric Riemannian space with the $\tilde{G}$-invariant metric

$$\frac{dx^2 + dy^2}{y^2} + \left( d\phi - \frac{dx}{2y} \right)^2,$$

and with the isometry $\mu$ defined by

$$\mu(z, \phi) = (-\bar{z}, -\phi).$$

The $\tilde{G}$-invariant measure $d(z, \phi)$ associated to the $\tilde{G}$-invariant metric is given by

$$d(z, \phi) = d(x, y, \phi) = \frac{dx \wedge dy \wedge d\phi}{y^2}.$$

The ring $\mathcal{R}(S)$ of $\tilde{G}$-invariant differential operators on $\tilde{S}$ is generated by

$$\frac{\partial}{\partial \phi}$$

and

$$\Delta = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{5}{4} \frac{\partial^2}{\partial \phi^2} + y \frac{\partial}{\partial \phi} \frac{\partial}{\partial x},$$

where $\Delta$ is the Laplace-Beltrami operator of $\tilde{S}$.

For an element $(\sigma, \theta) \in \tilde{G}$, we define a map $T_{(\sigma, \theta)}$ of $C^\infty(\tilde{S})$ into itself by

$$(T_{(\sigma, \theta)} f)(z, \phi) = f((z, \phi)(\sigma, \theta)) = f \left( \frac{az + b}{cz + d}, \phi + \arg (cz + d) + \theta \right),$$

where $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. $(\sigma, \theta) \to T_{(\sigma, \theta)}$ is a representation of $\tilde{G}$.

By the correspondence

$$G \ni \sigma \leftrightarrow (\sigma, 0) \ni \tilde{G} = G \times \mathbb{R}/(2\pi),$$

we identify the group $G = SL(2, \mathbb{R})$ with a subgroup $G \times \{0\}$ of $\tilde{G}$, and for an element $\sigma \in G$ we put $T_{(\sigma, \theta)} = T_\sigma$. Then we have

$$(T_\sigma f)(z, \phi) = f \left( \frac{az + b}{cz + d}, \phi + \arg (cz + d) \right).$$

Let $\Gamma$ be a discrete subgroup of $G$ not containing the element $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, and suppose that $G/\Gamma$ is compact. We also identify the subgroup $\Gamma$ with a subgroup $\Gamma \times \{0\}$ of $\tilde{G}$.

Denote by $C^\infty(\tilde{S}/\Gamma)$ the set of all $C^\infty$ functions on $\tilde{S}$ invariant under $\Gamma$:...
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\[ C^\infty(S/\Gamma) = \{ f(z, \phi) \in C^\infty(S) \mid T_\sigma f = f \text{ for all } \sigma \in \Gamma \} \]

and define \( C^\infty(S/\Gamma) \) similarly.

Now consider the following simultaneous eigenvalue problem in \( C^\infty(S/\Gamma) \):

\[
\begin{align*}
\frac{\partial}{\partial \phi} f(z, \phi) &= -i f(z, \phi), \\
\Delta f(z, \phi) &= -\frac{3}{2} f(z, \phi).
\end{align*}
\]

We denote by \( \mathcal{M}_\Gamma \left( 1, -\frac{3}{2} \right) = \mathcal{M} \left( 1, -\frac{3}{2} \right) \) the set of all functions satisfying the above condition (A). The eigenspace \( \mathcal{M}_\Gamma \left( 1, -\frac{3}{2} \right) \) is of finite dimension (A. Selberg and T. Tamagawa). We put

\[ d_0 = \dim \mathcal{M}_\Gamma \left( 1, -\frac{3}{2} \right) \quad (<\infty). \]

We shall denote by \( \mathcal{E}_\gamma(\Gamma) \) the linear space of all holomorphic automorphic forms of dimension $-1$ for the fuchsian group \( \Gamma \) and put

\[ d = \dim \mathcal{E}_\gamma(\Gamma). \]

Then we have the following

**Theorem 1.** $d \leq d_0 \leq 2d$.

**Proof.** By (1) and the identification of \((z, \phi)\) with \((z, \phi + 2n\pi)\), the element \( f(z, \phi) \in \mathcal{M}_\Gamma \left( 1, -\frac{3}{2} \right) \) is of the form

\[ f(z, \phi) = e^{-i\phi} G(z), \]

where \( G(z) \) is a function depending only on \( z \). If we put

\[ E(z) = y^{-\gamma/2} G(z), \]

then

\[ f(z, \phi) = e^{-i\phi} y^{\gamma/2} E(z), \]

and \( E(z) \) satisfies the following condition:
\[ E(\sigma z) = (cz+d)E(z), \quad \text{where} \quad \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \quad (2) \]

Conversely, if \( E(z) \) is any function satisfying the above condition (B), then the function \( e^{-i\phi y^{1/2}}E(z) \) satisfies the condition (A).

Hence
\[ \mathfrak{M}\left(1, -\frac{3}{2}\right) = \{ e^{-i\phi y^{1/2}}E(z); \ E(z) \text{ satisfying the condition (B)} \}, \]
and therefore
\[ \dim \mathfrak{M}\left(1, -\frac{3}{2}\right) = \dim \mathfrak{G}_\Gamma, \quad (3) \]
where \( \mathfrak{G}_\Gamma \) denotes the linear space of all functions satisfying the condition (B).

Since each element \( F(z) \in \mathfrak{S}(\Gamma) \) satisfies the condition (B), we have
\[ \mathfrak{M}\left(1, -\frac{3}{2}\right) \supseteq \{ e^{-i\phi y^{1/2}}F(z); \ F(z) \in \mathfrak{S}_1(\Gamma) \}, \]
and therefore
\[ d_0 \geq d. \quad (4) \]

Now we put
\[ \frac{\partial}{\partial \overline{z}} E(z) = g(z). \quad (5) \]

By virtue of the condition (B) for \( E(z) \), the function \( g(z) \) satisfies the following condition:
\[ \begin{cases} g(z) \in C^\infty(S), \\ g(\sigma z) = (cz+d)(\overline{cz+d})^* g(z), \quad \text{where} \quad \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \\ 2y \frac{\partial}{\partial \overline{z}} g(z) = ig(z). \end{cases} \quad (C) \]

We shall denote by \( \mathfrak{S}_\Gamma \) the linear space of all functions satisfying the above condition (C).

If \( g(z) \) is an element of \( \mathfrak{S}_\Gamma \), then \( g(z) = y^{-1} \overline{F(z)} \), where \( F(z) \in \mathfrak{S}_1(\Gamma) \), and the mapping \( g \rightarrow F \) is a bijection of the linear space \( \mathfrak{S}_\Gamma \) onto the linear space \( \mathfrak{S}_1(\Gamma) \). Therefore
\[ \dim \mathfrak{S}_\Gamma = \dim \mathfrak{S}_1(\Gamma). \]
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On the other hand, the differential equation (5) is transformed as follows:

$$
\frac{\partial}{\partial \bar{z}} E(z) = y^{-1} \overline{F(z)}, \text{ where } F(z) \in \mathcal{S}_1(\Gamma). \quad (6)
$$

Consider the following linear map

$$
\frac{\partial}{\partial \bar{z}} : \mathcal{E}_r \rightarrow \{ y^{-1} \overline{F(z)} ; F(z) \in \mathcal{S}_1(\Gamma) \}.
$$

From the above discussion, the kernel of this linear map is $\mathcal{S}_1(\Gamma)$, and we have

$$
\dim \mathcal{E}_r = \dim \mathcal{S}_1(\Gamma) + \dim \frac{\partial}{\partial \bar{z}} (\mathcal{E}_r)
$$

$$
\leq \dim \mathcal{S}_1(\Gamma) + \dim \{ y^{-1} \overline{F(z)} ; F(z) \in \mathcal{S}_1(\Gamma) \}
$$

$$
= d + d = 2d.
$$

Since $\dim \mathcal{E}_r$ is equal to $d_o$ by (3), we have

$$
d_o \leq 2d. \quad (7)
$$

By (4) and (7), we have $d \leq d_o \leq 2d$. Q.E.D.

**Corollary.** The dimension of $\mathcal{M}(1, -\frac{3}{2})$ is different from 0 if and only if $\mathcal{S}_1(\Gamma) \neq \{0\}$.

3. The Eichler periods of integrals

For each function $F(z)$ in $\mathcal{S}_1(\Gamma)$, consider the differential equation (6) in §2:

$$
\frac{\partial}{\partial \bar{z}} E(z) = y^{-1} \overline{F(z)}. \quad (6)
$$

The function $\overline{F(z)}$ is a holomorphic function of $\bar{z}$; put $\overline{F(z)} = F^*(z)$. Then the differential equation (6) has a $C^\infty$ solution of the form

$$
E_c(z) = E_1(z, \bar{z}) = \int_{z_0}^{\bar{z}} \frac{-2i}{\xi-z} F^*(\xi) d\xi \quad (\text{Im } \xi < 0),
$$

where $z_0$ is a fixed point in the lower half-plane and the integral is taken along an arbitrary piecewise differentiable arc in the lower half-plane with initial point $z_0$ and terminal point $\bar{z}$. Therefore the general solutions of class $C^\infty$ of the equation (6) is

$$
E(z) = E_1(z) - E_2(z),
$$
where \( E_0(z) \) denotes an arbitrary holomorphic function on \( S \).

For each \( \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \), the function \( E_0(z) \) is transformed as follows:

\[
E_\sigma(z) = \int_{z_0}^{z} \frac{-2i}{t - z} F^*(\xi) \, d\xi = \int_{z_0}^{z} \frac{-2i}{x_0 \xi - \sigma z} \, d\xi
\]

by \( F^*(\sigma^{-1} \xi) = (-c \xi + a) F^*(\xi) \).

Putting \( \sigma^{-1} \xi = t \) we have

\[
\int_{z_0}^{z} \frac{-2i}{t - z} F^*(\sigma^{-1} \xi) \, d\xi = (cz + d) \int_{z_0}^{z} \frac{-2i}{t - z} F^*(t) \, dt = (cz + d)
\]

\[
\times \int_{z_0}^{z} \frac{-2i}{t - z} F^*(t) \, dt + (cz + d) \int_{z_0}^{z} \frac{-2i}{t - z} F^*(t) \, dt.
\]

Hence we have

\[
E(\sigma z) = (cz + d) E(z) + (cz + d) E_0(z) + (cz + d) C(\sigma z) - E_0(\sigma z),
\]

where \( \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \), and

\[
C(\sigma z) = \int_{z_0}^{z} \frac{-2i}{x_0 \xi - \sigma z} F^*(\xi) \, d\xi.
\]

Hereafter we put

\[
[*] = (cz + d) E_0(z) + (cz + d) C(\sigma z) - E_0(\sigma z).
\]

Let \( R \) be the set of all holomorphic functions on \( S \) and let \( \Gamma \) operate on \( R \) as follows:

\[
h(z) \circ \sigma = h(\sigma z) = \frac{1}{cz + d} h(\sigma z),
\]

where \( \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \) and \( h(z) \in R \). Then \( R \) forms a \( \Gamma \)-right module. A map \( \sigma \rightarrow h(\sigma z) \) of \( \Gamma \) into \( R \) is called a (non-homogeneous) 1-cochain, and if a cochain satisfies the equation

\[
h(z)(\sigma \tau) = h(\sigma; z) \cdot \tau + h(\tau; z),
\]

then it is called a 1-cocycle. A 1-cocycle of the form \( h(\sigma z) = n(z) \cdot \sigma - n(z) \) where \( n(z) \) is a fixed element of \( R \) is called a coboundary. The cocycles form an additive group \( Z^1(\Gamma, R) \), and the coboundaries form a subgroup \( B^1(\Gamma, R) \) of \( Z^1(\Gamma, R) \). The factor group \( Z^1(\Gamma, R) / B^1(\Gamma, R) \) is denoted by \( H^1(\Gamma, R) \) and called the first cohomology group of \( \Gamma \) in \( R \), and its elements are called cohomology classes.
For each function $F(z)$ in $\mathcal{E}_\lambda(\Gamma)$, the map

$$\sigma \to C(\sigma ; z), \quad \text{where} \quad C(\sigma ; z) = \int_{\sigma^{-1}z_0}^{z_0} \frac{-2i}{\xi - z} F^*(\xi) d\xi$$  \hspace{1cm} (8)

is clearly a cocycle. We shall call it the Eichler period of the integral $E_\lambda(z)$ on $\sigma$, where

$$E_\lambda(z) = \int_{z_0}^z \frac{-2i}{\xi - z} F^*(\xi) d\xi.$$

This cocycle depends on $F(z)$, and moreover on the constant $z_0$ occurring in the definition of $E_\lambda(z)$. But a change of $z_0$ would add a coboundary to $C(\sigma ; z)$. Therefore the cohomology class of (8) depends only on $F(z)$. We shall call it the Eichler class of $F(z)$.

It should be noted that a cocycle $C(\sigma ; z)$ satisfying the condition

$$[\ast] = 0,$$

is a coboundary.

4. A theorem of monomorphism and its application

Consider the following map from $\mathcal{E}_\lambda(\Gamma)$ into $H^1(\Gamma, \mathbb{R})$

$$\varphi: F(z) \to C(\sigma ; z),$$

where

$$C(\sigma ; z) = \int_{\sigma^{-1}z_0}^{z_0} \frac{-2i}{\xi - z} F^*(\xi) d\xi, \quad F^*(\xi) = \overline{F(\xi)}.$$

The map is obviously a homomorphism of $\mathcal{E}_\lambda(\Gamma)$ into $H^1(\Gamma, \mathbb{R})$. We shall prove that $\varphi: \mathcal{E}_\lambda(\Gamma) \to H^1(\Gamma, \mathbb{R})$ is one-one.

For the cocycle $C(\sigma ; z)$ such that $\varphi(F_\epsilon(z)) = C(\sigma ; z)$ with $F_\epsilon(z) \in \mathcal{E}_\lambda(\Gamma)$, we put

$$E_\epsilon(z) = \int_{z_0}^{z} \frac{-2i}{\xi - z} F_\epsilon^*(\xi) d\xi \quad (F_\epsilon^*(\xi) = \overline{F^*_\epsilon(\xi)}),$$

then $E_\epsilon(z)$ is a $C^\infty$ function satisfying the conditions

$$C(\sigma ; z) = E_\epsilon(z) \cdot \sigma - E_\epsilon(z)$$ \hspace{1cm} (10)

and

$$\frac{\partial}{\partial z} E_\epsilon(z) = y^{-1} \overline{F^*_\epsilon(z)}. \hspace{1cm} (11)$$

The differential form $\phi(z) = \partial E_\epsilon(z)$ is a $C^\infty$ differential form on $D^\lambda$, and

1) $D$ denotes a fundamental domain of $\Gamma$ in $S$. As mentioned in §2 $G/\Gamma$ is compact, so $D$ is compact.
by (11) it satisfies the relation $\phi(\sigma z) = (cz + d)\phi(z)$ for all $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

For an arbitrary element $F(z)$ of $\mathfrak{S}_1(\Gamma)$, $\theta(z) = F(z)dz$ is a holomorphic differential form on $D$ and satisfies the relation $\theta(\sigma z) = \frac{1}{cz + d} \theta(z)$ for all $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Therefore the differential form $\phi(z) \wedge \theta(z)$ is invariant under the group $\Gamma$.

We consider the following integral over $D$,

$$\int_D \phi(z) \wedge \theta(z). \quad (12)$$

From the above discussion, the integral converges and is independent of the choice of fundamental domain of $\Gamma$ in $S$. If $g_1(z), g_2(z)$ are two $C^\infty$ functions associated with the same cocycle $C(\sigma ; z)$ by the relation (10), then we have

$$g_1(\sigma z) = (cz + d)g_2(z),$$

where $g_1(z) = g_1(z) - g_2(z)$. Then the differential form $g_1(z)F(z)dz$ is invariant under the group $\Gamma$. Therefore

$$\int_D \partial g_1(z) \wedge F(z)dz - \int_D \partial g_2(z) \wedge F(z)dz = \int_D \partial \left( g_1(z)F(z) \right)dz \wedge dz$$

$$= \int_{\partial D} g_1(z)F(z)dz$$

$$= 0,$$

where $\partial D$ denotes the boundary of $D$. Consequently, the integral (12) depends only on $C(\sigma ; z)$. We shall denote it by

$$\langle C(\sigma ; z), F(z) \rangle = \int_D \phi(z) \wedge \theta(z).$$

If the cocycle $C(\sigma ; z)$ is an element of $B^l(\Gamma, \mathfrak{N})$, then $\phi(z) \equiv 0$; and hence

$$\langle C(\sigma ; z), F(z) \rangle = 0$$

for all $F(z) \in \mathfrak{S}_1(\Gamma)$. Therefore, if there exists a function $F(z)$ in $\mathfrak{S}_1(\Gamma)$ such that $\langle C(\sigma ; z), F(z) \rangle \neq 0$, then $C(\sigma ; z) \in B^l(\Gamma, \mathfrak{N})$.

Let $F(z)$ be a nontrivial function in $\mathfrak{S}_1(\Gamma)$ and $C(\sigma ; z)$ the image of $F(z)$ by $\varphi$. If we put
\[ \psi(z) = y^{-1}F(z)\,dz \]

and

\[ \theta(z) = F(z)\,dz , \]

then the integral

\[ \int_D \psi(z) \wedge \theta(z) = \int_D y^{-1}F(z)F(z)\,dz \wedge d\bar{z} \]

is not equal to zero. On the other hand, let

\[ E_1(z) = \int_{z_0}^{\xi} \frac{-2i}{\xi - z} F^*(\xi)\,d\xi . \]

Then the function \( E_1(z) \) satisfies the following conditions:

\[ C(\sigma ; z) = E_1(z) \cdot \sigma - E_1(z) \]

and

\[ \frac{\partial}{\partial \bar{z}} E_1(z) = y^{-1}F(z) . \]

Hence

\[ \bar{\delta}E_1(z) = \frac{\partial}{\partial \bar{z}} E_1(z)\,d\bar{z} = y^{-1}F(z)\,d\bar{z} = \psi(z) , \]

and therefore

\[ \langle C(\sigma ; z), F(z) \rangle \neq 0 . \]

Consequently, if \( F(z) \neq 0 \), then for the corresponding cocycle \( C(\sigma ; z) \),

\[ \langle C(\sigma ; z), F(z) \rangle \neq 0 . \]

By the two results obtained above, we see that if \( F(z) \neq 0 \), then the corresponding cycles \( C(\sigma ; z) \) does not belong to \( B^i(\Gamma, \mathcal{R}) \). Therefore \( F(z) \neq 0 \) is equivalent to \( C(\sigma ; z) \notin B^i(\Gamma, \mathcal{R}) \); and hence the kernel of \( \varphi \) is equal to zero.

We have the following

**Theorem 2.** The map \( F(z) \mapsto C(\sigma ; z) \) defined by (9) is a monomorphism of \( \mathcal{E}_1(\Gamma) \) into \( H^1(\Gamma, \mathcal{R}) \).

**Corollary.** With the same notations as in §2, we have

\[ d = d_{\alpha} . \]

Proof. In §2, we have considered the following linear map

---

2) The possibility of this theorem was indicated by Gunning ([2], p. 56).
\[ \frac{\partial}{\partial z} : \mathcal{E}_\Gamma \to \{ y^{-1} \overline{F(z)} ; F(z) \in \mathcal{F}(\Gamma) \}, \]

and proved that the map is into. Next we prove \( \frac{\partial}{\partial z}(\mathcal{E}_\Gamma) = \{0\} \). For every nontrivial function \( F(z) \) of \( \mathcal{F}(\Gamma) \), there exists an inverse image of \( y^{-1} \overline{F(z)} \) if and only if the cocycle \( C(\sigma ; z) \) corresponding to \( F(z) \) by \( \varphi \) belongs to \( B'(\Gamma, \mathcal{R}) \) (cf. § 3 in this paper). On the other hand, by Theorem 2 the cocycle \( C(\sigma ; z) \) corresponding to the nontrivial \( F(z) \) does not belong to \( B'(\Gamma, \mathcal{R}) \). Therefore

\[ \frac{\partial}{\partial z}(\mathcal{E}_\Gamma) = \{0\} \]

and hence, we have

\[ d = d_0. \]

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References


3) Therefore all the functions \( E(z) \) in \( \mathcal{E}_\Gamma \) are holomorphic.