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EICHLER CLASSES ATTACHED TO AUTOMORPHIC FORMS OF DIMENSION −1

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1. Introduction

In his work [1], Eichler discussed a relation between the automorphic forms of dimension $-2k$ (k is an integer and >1) for a fuchsian group Γ and the cohomology groups of Γ with certain modules of polynomials as coefficients, where the cocycles appeared as the periods of 'the generalized abelian integrals' attached to the automorphic forms. Gunning [2] gave a more general form of this relation.

The purpose of the present paper is to give an analogous relation in the case of automorphic forms of dimension -1 for a fuchsian group and give an application to Selberg's eigenspace [3].

I express my gratitude to Dr. H. Shimizu for suggesting this problem to me and for his many valuable critical comments during the preparation of this paper.

2. The eigenspaec $\mathfrak{M}\left(1, -\frac{3}{2}\right)$

Let

$$S = \{z = x+iy;\ x, y \text{ real and } \ y>0\}$$

denote the complex upper half-plane and let $G=SL(2, \boldsymbol{R})$ be the real special linear group of the second degree. Consider direct products

$$\tilde{S} = S\times \boldsymbol{R}/(2\pi)\,,$$
$$\tilde{G} = G\times \boldsymbol{R}/(2\pi)\,,$$

where $\boldsymbol{R}/(2\pi)$ denotes the real torus, and let an element (σ, θ) of $\tilde{G}$ operate on $\tilde{S}$ as follows:

$$\tilde{S}\ni(z, \phi) \rightarrow (z, \phi)(\sigma, \theta) = \left(\frac{az+b}{cz+d}, \phi+\arg(cz+d)+\theta\right)\in\tilde{S}\,,$$

where $\sigma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in G$. The operation of $\tilde{G}$ on $\tilde{S}$ is transitive. $\tilde{S}$ is a weakly symmetric Riemannian space with the $\tilde{G}$-invariant metric

$$\frac{dx^2+dy^2}{y^2}+\left(d\phi-\frac{dx}{2y}\right)^2,$$

and with the isometry μ defined by

$$\mu(z, \phi) = (-\bar{z}, -\phi).$$

The $\tilde{G}$-invariant measure $d(z, \phi)$ associated to the $\tilde{G}$-invariant metric is given by

$$d(z, \phi) = d(x, y, \phi) = \frac{dx\wedge dy\wedge d\phi}{y^2}.$$

The ring $\mathfrak{R}(S)$ of $\tilde{G}$-invariant differential operators on $\tilde{S}$ is generated by

$$\frac{\partial}{\partial\phi}$$

and

$$\tilde{\Delta} \equiv y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\frac{5}{4}\frac{\partial^2}{\partial\phi^2}+y\frac{\partial}{\partial\phi}\frac{\partial}{\partial x},$$

where $\tilde{\Delta}$ is the Laplace-Beltrami operator of $\tilde{S}$.

For an element $(\sigma, \theta)\in\tilde{G}$, we define a map $T_{(\sigma, \theta)}$ of $C^\infty(\tilde{S})$ into itself by

$$(T_{(\sigma, \theta)}f)(z, \phi) = f((z, \phi)(\sigma, \theta)) = f\left(\frac{az+b}{cz+d}, \phi+\arg(cz+d)+\theta\right),$$

where $\sigma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$. $(\sigma, \theta) \to T_{(\sigma, \theta)}$ is a representation of $\tilde{G}$.

By the correspondence

$$G\ni\sigma \leftrightarrow (\sigma, 0)\ni\tilde{G} = G\times \boldsymbol{R}/(2\pi),$$

we identify the group $G=SL(2, \boldsymbol{R})$ with a subgroup $G\times\{0\}$ of $\tilde{G}$, and for an element $\sigma\in G$ we put $T_{(\sigma, \theta)}=T_\sigma$. Then we have

$$(T_\sigma f)(z, \phi) = f\left(\frac{az+b}{cz+d}, \phi+\arg(cz+d)\right).$$

Let Γ be a discrete subgroup of G not containing the element $\begin{pmatrix} -1 & 0 \\ 0 & -1\end{pmatrix}$, and suppose that G/Γ is compact. We also identify the subgroup Γ with a subgroup $\Gamma\times\{0\}$ of $\tilde{G}$.

Denote by $C^\infty(\tilde{S}/\Gamma)$ the set of all C^∞ functions on $\tilde{S}$ invariant under Γ:

$$C^{\infty}(\tilde{S}/\Gamma) = \{f(z, \phi) \in C^{\infty}(\tilde{S});\ T_{\sigma}f = f \ \text{ for all } \ \sigma \in \Gamma\}$$

and define $C^{\infty}(S/\Gamma)$ similarly.

Now consider the following simultaneous eigenvalue problem in $C^{\infty}(\tilde{S}/\Gamma)$:

$$\text{(A)} \qquad \begin{cases} f \in C^{\infty}(\tilde{S}/\Gamma)\,, \\ \dfrac{\partial}{\partial\phi} f(z, \phi) = -if(z, \phi)\,, \\ \tilde{\Delta} f(z, \phi) = -\dfrac{3}{2} f(z, \phi)\,. \end{cases} \tag{1}$$

We denote by $\mathfrak{M}_{\Gamma}\left(1, -\frac{3}{2}\right) = \mathfrak{M}\left(1, -\frac{3}{2}\right)$ the set of all functions satisfying the above condition (A). The eigenspace $\mathfrak{M}\left(1, -\frac{3}{2}\right)$ is of finite dimension (A. Selberg and T. Tamagawa). We put

$$d_0 = \dim \mathfrak{M}\left(1, -\frac{3}{2}\right) \qquad (<\infty)\,.$$

We shall denote by $\mathfrak{S}_1(\Gamma)$ the linear space of all holomorphic automorphic forms of dimension -1 for the fuchsian group Γ and put

$$d = \dim \mathfrak{S}_1(\Gamma)\,.$$

Then we have the following

Theorem 1. $d \leqq d_0 \leqq 2d$.

Proof. By (1) and the identification of (z, ϕ) with $(z, \phi + 2n\pi)$, the element $f(z, \phi) \in \mathfrak{M}\left(1, -\frac{3}{2}\right)$ is of the form

$$f(z, \phi) = e^{-i\phi} G(z)\,,$$

where $G(z)$ is a function depending only on z. If we put

$$E(z) = y^{-1/2} G(z)\,,$$

then

$$f(z, \phi) = e^{-i\phi} y^{1/2} E(z)\,,$$

and $E(z)$ satisfies the following condition:

$$\text{(B)} \qquad \begin{cases} E(z) \in C^\infty(S)\,, \\ E(\sigma z) = (cz+d)E(z), \quad \text{where} \quad \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\,, \\ 2y \dfrac{\partial}{\partial z}\dfrac{\partial}{\partial \bar{z}} E(z) = i \dfrac{\partial}{\partial \bar{z}} E(z)\,. \end{cases} \tag{2}$$

Conversely, if $E(z)$ is any function satisfying the above condition (B), then the function $e^{-i\phi}y^{1/2}E(z)$ satisfies the condition (A).
Hence

$$\mathfrak{M}\left(1, -\frac{3}{2}\right) = \{e^{-i\phi}y^{1/2}E(z)\,;\ E(z) \ \text{ satisfying the condition (B)}\}\,,$$

and therefore

$$\dim \mathfrak{M}\left(1, -\frac{3}{2}\right) = \dim \mathfrak{E}_\Gamma\,, \tag{3}$$

where $\mathfrak{E}_\Gamma$ denotes the linear space of all functions satisfying the condition (B).

Since each element $F(z) \in \mathfrak{S}_1(\Gamma)$ satisfies the condition (B), we have

$$\mathfrak{M}\left(1, -\frac{3}{2}\right) \supseteq \{e^{-i\phi}y^{1/2}F(z)\,;\ F(z) \in \mathfrak{S}_1(\Gamma)\}\,,$$

and therefore

$$d_0 \geqq d\,. \tag{4}$$

Now we put

$$\frac{\partial}{\partial \bar{z}} E(z) = g(z)\,. \tag{5}$$

By virtue of the condition (B) for $E(z)$, the function $g(z)$ satisfies the following condition:

$$\text{(C)} \qquad \begin{cases} g(z) \in C^\infty(S)\,, \\ g(\sigma z) = (cz+d)(\overline{cz+d})^2 g(z), \quad \text{where} \quad \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\,, \\ 2y \dfrac{\partial}{\partial z} g(z) = ig(z)\,. \end{cases}$$

We shall denote by $\mathfrak{G}_\Gamma$ the linear space of all functions satisfying the above condition (C).

If $g(z)$ is an element of $\mathfrak{G}_\Gamma$, then $g(z) = y^{-1}\overline{F(z)}$, where $F(z) \in \mathfrak{S}_1(\Gamma)$, and the mapping $g \to F$ is a bijection of the linear space $\mathfrak{G}_\Gamma$ onto the linear space $\mathfrak{S}_1(\Gamma)$. Therefore

$$\dim \mathfrak{G}_\Gamma = \dim \mathfrak{S}_1(\Gamma)\,.$$

On the other hand, the differential equation (5) is transformed as follows:

$$\frac{\partial}{\partial \bar{z}} E(z) = y^{-1}\overline{F(z)}, \quad \text{where} \quad F(z) \in \mathfrak{S}_1(\Gamma). \tag{6}$$

Consider the following linear map

$$\frac{\partial}{\partial \bar{z}} : \mathfrak{E}_\Gamma \to \{y^{-1}\overline{F(z)};\ F(z) \in \mathfrak{S}_1(\Gamma)\}.$$

From the above discussion, the kernel of this linear map is $\mathfrak{S}_1(\Gamma)$, and we have

$$\begin{aligned} \dim \mathfrak{E}_\Gamma &= \dim \mathfrak{S}_1(\Gamma) + \dim \frac{\partial}{\partial \bar{z}}(\mathfrak{E}_\Gamma) \\ &\leqq \dim \mathfrak{S}_1(\Gamma) + \dim \{y^{-1}\overline{F(z)};\ F(z) \in \mathfrak{S}_1(\Gamma)\} \\ &= d + d = 2d. \end{aligned}$$

Since $\dim \mathfrak{E}_\Gamma$ is equal to d_0 by (3), we have

$$d_0 \leqq 2d. \tag{7}$$

By (4) and (7), we have $d \leqq d_0 \leqq 2d$. Q. E. D.

Corollary. *The dimension of* $\mathfrak{M}\left(1, -\frac{3}{2}\right)$ *is different from* 0 *if and only if* $\mathfrak{S}_1(\Gamma) \neq \{0\}$.

3. The Eichler periods of integrals

For each function $F(z)$ in $\mathfrak{S}_1(\Gamma)$, consider the differential equation (6) in §2:

$$\frac{\partial}{\partial \bar{z}} E(z) = y^{-1}\overline{F(z)}. \tag{6}$$

The function $\overline{F(z)}$ is a holomorphic function of $\bar{z}$; put $\overline{F(z)} = F^*(\bar{z})$. Then the differential equation (6) has a C^∞ solution of the form

$$E_1(z) = E_1(z, \bar{z}) = \int_{z_0}^{\bar{z}} \frac{-2i}{\xi - z} F^*(\xi)\, d\xi \qquad (\operatorname{Im} \xi < 0),$$

where z_0 is a fixed point in the lower half-plane and the integral is taken along an arbitrary peicewise differentiable arc in the lower half-plane with initial point z_0 and terminal point $\bar{z}$. Therefore the general solutions of class C^∞ of the equation (6) is

$$E(z) = E_1(z) - E_0(z),$$

where $E_0(z)$ denotes an arbitrary holomorphic function on S.

For each $\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma$, the function $E_1(z)$ is transformed as follows:

$$E_1(\sigma z) = \int_{z_0}^{\sigma\bar{z}} \frac{-2i}{\xi-\sigma z} F^*(\xi)\, d\xi = \int_{z_0}^{\sigma\bar{z}} \frac{-2i}{\xi-\sigma z}\frac{F^*(\sigma^{-1}\xi)}{-c\xi+a} d\xi$$

by $F^*(\sigma^{-1}\xi)=(-c\xi+a)F^*(\xi)$.
Putting $\sigma^{-1}\xi=t$ we have

$$\int_{z_0}^{\sigma\bar{z}} \frac{-2i}{\xi-\sigma z}\frac{F^*(\sigma^{-1}\xi)}{-c\xi+a} d\xi = (cz+d)\int_{\sigma^{-1}z_0}^{\bar{z}} \frac{-2i}{t-z} F^*(t)\, dt = (cz+d)$$
$$\times\int_{z_0}^{\sigma\bar{z}} \frac{-2i}{t-z} F^*(t)\, dt + (cz+d)\int_{\sigma^{-1}z_0}^{z_0} \frac{-2i}{t-z} F^*(t)\, dt\,.$$

Hence we have

$$E(\sigma z) = (cz+d)E(z)+(cz+d)E_0(z)+(cz+d)C(\sigma\,;z)-E_0(\sigma z)\,,$$

where $\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma$, and

$$C(\sigma\,;z) = \int_{\sigma^{-1}z_0}^{z_0} \frac{-2i}{\xi-z} F^*(\xi)\, d\xi\,.$$

Hereafter we put

$$[*] = (cz+d)E_0(z)+(cz+d)C(\sigma\,;z)-E_0(\sigma z)\,.$$

Let $\mathfrak{N}$ be the set of all holomorphic functions on S and let Γ operate on $\mathfrak{N}$ as follows:

$$h(z)\cdot\sigma \equiv h(\sigma\,;z) = \frac{1}{cz+d}h(\sigma z)\,,$$

where $\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma$ and $h(z)\in\mathfrak{N}$. Then $\mathfrak{N}$ forms a Γ-right module. A map $\sigma\to h(\sigma\,;z)$ of Γ into $\mathfrak{N}$ is called *a* (*non-homogeneous*) *1-cochain*, and if a cochain satisfies the equation

$$h(z)(\sigma\tau) = h(\sigma\,;z)\cdot\tau+h(\tau\,;z)\,,$$

then it is called *a 1-cocycle*. A 1-cocycle of the form $h(\sigma\,;z)=n(z)\cdot\sigma-n(z)$, where $n(z)$ is a fixed element of $\mathfrak{N}$ is called *a coboundary*. The cocycles form an additive group $Z^1(\Gamma, \mathfrak{N})$, and the coboundaries form a subgroup $B^1(\Gamma, \mathfrak{N})$ of $Z^1(\Gamma, \mathfrak{N})$. The factor group $Z^1(\Gamma, \mathfrak{N})/B^1(\Gamma, \mathfrak{N})$ is denoted by $H^1(\Gamma, \mathfrak{N})$ and called *the first cohomology group of* Γ *in* $\mathfrak{N}$, and its elements are called *cohomology classes*.

For each function $F(z)$ in $\mathfrak{S}_1(\Gamma)$, the map

$$\sigma \to C(\sigma\,;z), \quad \text{where} \quad C(\sigma\,;z) = \int_{\sigma^{-1}z_0}^{z_0} \frac{-2i}{\xi - z} F^*(\xi)\,d\xi \tag{8}$$

is clearly a cocycle. We shall call it *the Eichler period of the integral* $E_1(z)$ *on* σ, where

$$E_1(z) = \int_{z_0}^{\bar{z}} \frac{-2i}{\xi - z} F^*(\xi)\,d\xi\,.$$

This cocycle depends on $F(z)$, and moreover on the constant z_0 ocurring in the definition of $E_1(z)$. But a change of z_0 would add a coboundary to $C(\sigma\,;z)$. Therefore the cohomology class of (8) depends only on $F(z)$. We shall call it *the Eichler class of* $F(z)$.

It should be noted that a cocycle $C(\sigma\,;z)$ satisfying the condition

$$[*] = 0\,,$$

is a coboundary.

4. A theorem of monomorphism and its application

Consider the following map from $\mathfrak{S}_1(\Gamma)$ into $H^1(\Gamma, \mathfrak{N})$

$$\varphi : F(z) \to C(\sigma\,;z)\,, \tag{9}$$

where $$C(\sigma\,;z) = \int_{\sigma^{-1}z_0}^{z_0} \frac{-2i}{\xi - z} F^*(\xi)\,d\xi\,,\ F^*(\xi) = \overline{F(\bar{\xi})}\,.$$

The map is obviously a homomorphism of $\mathfrak{S}_1(\Gamma)$ into $H^1(\Gamma, \mathfrak{N})$. We shall prove that $\varphi : \mathfrak{S}_1(\Gamma) \to H^1(\Gamma, \mathfrak{N})$ is one-one.

For the cocycle $C(\sigma\,;z)$ such that $\varphi(F_c(z)) = C(\sigma\,;z)$ with $F_c(z) \in \mathfrak{S}_1(\Gamma)$, we put

$$E_1(z) = \int_{z_0}^{\bar{z}} \frac{-2i}{\xi - z} F_c^*(\xi)\,d\xi \qquad (F_c^*(\xi) = \overline{F_c(\bar{\xi})})\,,$$

then $E_1(z)$ is a C^∞ function satisfying the conditions

$$C(\sigma\,;z) = E_1(z)\cdot\sigma - E_1(z) \tag{10}$$

and

$$\frac{\partial}{\partial \bar{z}} E_1(z) = y^{-1}\overline{F_c(z)}\,. \tag{11}$$

The differential form $\phi(z) = \bar{\partial} E_1(z)$ is a C^∞ differential form on D[1], and

1) D denotes a fundamental domain of Γ in S. As mentioned in §2 G/Γ is compact, so D is compact.

by (11) it satisfies the relation $\phi(\sigma z)=(cz+d)\phi(z)$ for all $\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma$. For an arbitrary element $F(z)$ of $\mathfrak{S}_1(\Gamma)$, $\theta(z)=F(z)dz$ is a holomorphic differential form on D and satisfies the relation $\theta(\sigma z)=\frac{1}{cz+d}\theta(z)$ for all $\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma$. Therefore the differential form $\phi(z)\wedge\theta(z)$ is invariant under the group Γ.

We consider the following integral over D,

$$\int_D \phi(z)\wedge\theta(z)\,. \tag{12}$$

From the above discussion, the integral converges and is independent of the choice of fundamental domain of Γ in S. If $g_1(z)$, $g_2(z)$ are two C^∞ functions associated with the same cocycle $C(\sigma\,;z)$ by the relation (10), then we have

$$g_0(\sigma z) = (cz+d)g_0(z)\,,$$

where $g_0(z)=g_1(z)-g_2(z)$. Then the differential form $g_0(z)F(z)dz$ is invariant under the group Γ. Therefore

$$\begin{aligned}\int_D \bar\partial g_1(z)\wedge F(z)dz-\int_D \bar\partial g_2(z)\wedge F(z)dz &= \int_D \bar\partial g_0(z)\wedge F(z)dz \\ &= \int_D \frac{\partial}{\partial\bar z}(g_0(z)F(z))d\bar z\wedge dz \\ &= \int_{\partial D} g_0(z)F(z)dz \\ &= 0\,,\end{aligned}$$

where ∂D denotes the boundary of D. Consequently, the integral (12) depends only on $C(\sigma\,;z)$. We shall denote it by

$$\langle C(\sigma\,;z),\, F(z)\rangle = \int_D \phi(z)\wedge\theta(z)\,.$$

If the cocycle $C(\sigma\,;z)$ is an element of $B^1(\Gamma, \mathfrak{N})$, then $\phi(z)\equiv 0$; and hence

$$\langle C(\sigma\,;z),\, F(z)\rangle = 0$$

for all $F(z)\in\mathfrak{S}_1(\Gamma)$. Therefore, if there exists a function $F(z)$ in $\mathfrak{S}_1(\Gamma)$ such that $\langle C(\sigma\,;z), F(z)\rangle\neq 0$, then $C(\sigma\,;z)\notin B^1(\Gamma, \mathfrak{N})$.

Let $F(z)$ be a nontrivial function in $\mathfrak{S}_1(\Gamma)$ and $C(\sigma\,;z)$ the image of $F(z)$ by φ. If we put

$$\psi(z) = y^{-1}\overline{F(z)}\,d\bar{z}$$

and

$$\theta(z) = F(z)\,dz\,,$$

then the integral

$$\int_D \psi(z)\wedge\theta(z) = \int_D y^{-1}\overline{F(z)}F(z)\,dz\wedge d\bar{z}$$

is not equal to zero. On the other hand, let

$$E_1(z) = \int_{z_0}^{\bar{z}} \frac{-2i}{\xi - z} F^*(\xi)\,d\xi\,.$$

Then the function $E_1(z)$ satisfies the following conditions:

$$C(\sigma\,;z) = E_1(z)\cdot\sigma - E_1(z)$$

and

$$\frac{\partial}{\partial\bar{z}}E_1(z) = y^{-1}\overline{F(z)}\,.$$

Hence

$$\bar{\partial}E_1(z) = \frac{\partial}{\partial\bar{z}}E_1(z)\,d\bar{z} = y^{-1}\overline{F(z)}\,d\bar{z} = \psi(z)\,,$$

and therefore

$$\langle C(\sigma\,;z),\,F(z)\rangle \neq 0\,.$$

Consequently, if $F(z)\not\equiv 0$, then for the corresponding cocycle $C(\sigma\,;z)$, $\langle C(\sigma\,;z),\,F(z)\rangle\neq 0$.

By the two results obtained above, we see that if $F(z)\not\equiv 0$, then the corresponding cycles $C(\sigma\,;z)$ does not belong to $B^1(\Gamma,\,\mathfrak{R})$. Therefore $F(z)\not\equiv 0$ is equivalent to $C(\sigma\,;z)\notin B^1(\Gamma,\,\mathfrak{R})$; and hence the kernel of φ is equal to zero.

We have the following

Theorem 2.[2)] *The map $F(z)\rightarrow C(\sigma\,;z)$ defined by* (9) *is a monomorphism of $\mathfrak{S}_1(\Gamma)$ into $H^1(\Gamma,\,\mathfrak{R})$.*

Corollary. *With the same notations as in* § 2, *we have*

$$d = d_0\,.$$

Proof. In § 2, we have considered the following linear map

2) The possibility of this theorem was indicated by Gunning ([2]. p. 56).

$$\frac{\partial}{\partial \bar{z}} : \mathfrak{E}_\Gamma \to \{y^{-1}\overline{F(z)}\,;\, F(z) \in \mathfrak{S}_1(\Gamma)\}\,,$$

and proved that the map is into. Next we prove $\frac{\partial}{\partial \bar{z}}(\mathfrak{E}_\Gamma) = \{0\}$. For every nontrivial function $F(z)$ of $\mathfrak{S}_1(\Gamma)$, there exists an inverse image of $y^{-1}\overline{F(z)}$ if and only if the cocycle $C(\sigma\,;z)$ corresponding to $F(z)$ by φ belongs to $B^1(\Gamma, \mathfrak{R})$ (cf. §3 in this paper). On the other hand, by Theorem 2 the cocycle $C(\sigma\,;z)$ corresponding to the nontrivial $F(z)$ does not belong to $B^1(\Gamma, \mathfrak{R})$. Therefore

$$\frac{\partial}{\partial \bar{z}}(\mathfrak{E}_\Gamma) = \{0\}\,^{3)},$$

and hence, we have

$$d = d_0\,. \qquad \text{Q. E. D.}$$

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3) Therefore all the functions $E(z)$ in $\mathfrak{E}_\Gamma$ are holomorphic.