Title	Spectral properties of differential operators related to stochastic oscillatory integrals
Author(s)	Sheu, Tian-Wei
Citation	Osaka Journal of Mathematics. 1992, 29(2), p. 203-231
Version Type	VoR
URL	https://doi.org/10.18910/5591
rights	
Note	

Sheu, T.W.
Osaka J. Math.
29 (1992), 203–231

SPECTRAL PROPERTIES OF DIFFERENTIAL OPERATORS RELATED TO STOCHASTIC OSCILLATORY INTEGRALS

TIAN-WEI SHEU

(Received December 21, 1989)
(Revised October 8, 1991)

1. Introduction

Recently, in the case of $\boldsymbol{R}^d$ with the standard flat metric, Schrödinger operator with magnetic fields has been studied by many authors, e.g., B. Simon and A. Iwatsuka. The semi-group generated by this operator can be represented in terms of stochastic oscillatory integrals (see [12] and [20]). Hence probabilistic methods play an important role in the study of this operator and its spectral properties are closely related to results in the stochastic analysis. The purpose of this paper is to study the spectral properties of Schrödinger operator of magnetic fields on $\boldsymbol{R}^2$ with a rotationally invariant Riemannian metric g. It is well known that if g is the standard flat metric, then Schrödinger operator of magnetic fields can have a wide variety of spectral porperties, (see, for example, [3]). In our case, the above variety still remains. In fact, we will show several similar facts to the results obtained by K. Miller-B. Simon, [14], A. Iwatsuka [8],[9],[10], etc.

Let M be a complete Riemannian manifold with a Riemannian metric g, α be a real valued differential 1-form on M and Δ be the Laplace-Beltrami operator on M, We consider a differential operator $L(\alpha)$ on M with the domain $C_0^\infty(M)^C$ defined by

$$L(\alpha)f=-\frac{1}{2}(\Delta f+2\sqrt{-1}\langle df,\alpha\rangle-(\sqrt{-1}\delta\alpha+||\alpha||^2)f)$$

where $\langle\cdot,\cdot\rangle$ and $||\cdot||$ denote the Hermitian inner product and the norm in complexified cotangent bundle $T^*(M)^C$ which are defined from g respectively, and $C_0^\infty(M)^C$ is the space of all complex valued C^∞-functions with compact support in M. In the case of $M=\boldsymbol{R}^d$ with the standard flat metric g, the operator $L(\alpha)$ is usually called the Schrödinger operator with the magnetic field $d\alpha$ (see [3]). Under some mild conditions, $L(\alpha)$ is essentially self-adjoint on $C_0^\infty(M)^C$ (see Section 3). Then $L(\alpha)$ can be uniquely extended to a self-adjoint operator $H(\alpha)$ on $L_C^2(M;dm)$, the Hilbert space of all complex valued functions on M which are square integrable with respect to the Riemannian volume dm. We now consider a diffusion process $(X(t), P_x)$ on M starting at x generated by $\Delta/2$ and $S(t, \alpha)$ be the stochastic line integral of α along the path of the diffusion process $\{X(s); 0 \leqq s \leqq t\}$ i.e.

$$S(t, \alpha) = \int_0^t \alpha(X(s)) \circ dX(s)$$

where $\circ$ stands for the Stratonovich symmetric stochastic integral (see [6]). It is known that the semi-group $\{Q_t\}_{t \geqq 0}$ generated by $H(\alpha)$ can be written in the following form:

$$(Q_t f)(x) = E_x[f(X(t)) e^{\sqrt{-1} S(t, \alpha)}] \qquad \text{for} \quad f \in L^2_C(M; dm) \tag{1.1}$$

where $E_x[\cdot]$ is the expectation with respect to P_x and that $\{Q_t\}_{t \geqq 0}$ is symmetric with respect to dm (see [5], [19]).

Throughout this paper, we always assume that $M = \boldsymbol{R}^2$ and the Riemannian metric g is given by

$$g = (g_{ij}(r, 0)) = \begin{pmatrix} 1 & 0 \\ 0 & g(r)^2 \end{pmatrix} \tag{1.2}$$

in polar coordinates (r, θ) where $g(r)$ is a positive C^2-function on $(0, \infty)$ satisfying the condition (2.1) stated in Section 2. We also assume that the differential 1-form can be written in the form

$$\alpha(r, \theta) = k(r) d\theta$$

where $k(r)$ is a positive smooth real valued function on $(0, \infty)$ with $\lim_{r \to 0} k(r) = 0$. It is easily seen that $||\alpha||(r, \theta) = k(r)/g(r)$ and $||d\alpha||(r, \theta) = |k'(r)|/g(r)$ for $(r, \theta) \in (0, \infty) \times S^1$. Then we can show that $L(\alpha)$ is essentially self-adjoint on $C_0^\infty(\boldsymbol{R}^2)^C$. In this paper, we will concern with the spectral properties of $H(\alpha)$ in this case. We will first show that if $\lim_{r \to \infty} ||d\alpha||(r, \theta) = \lim_{r \to \infty} |k'(r)|/g(r) = 0$, $H(\alpha)$ has spectrum $[0, \infty)$. Furthermore, we can describe a fine structure of this spectrum in terms of the limit $\lim_{r \to \infty} ||\alpha||(r, \theta) = \lim_{r \to \infty} k(r)/g(r)$. These are analogous to the results by K. Miller and B. Simon [14] (see also A. Iwatsuka [9]), (for details, see Theorem 2.1 in Section 2). We will next show by combining results in the functional analysis and probabilistic methods that if $\lim_{r \to \infty} ||d\alpha||(r, \theta) = \lim_{r \to \infty} |k'(r)|/g(r) = \infty$, the spectrum of $H(\alpha)$ is discrete (see Theorem 2.2). We will also give a remark in the case of $\lim_{r \to \infty} ||d\alpha||(r, \theta) = \lim_{r \to \infty} |k'(r)|/g(r) = \lambda_0 > 0$. In this case, $H(\alpha)$ has point spectrum. Furthermore, in the case of $k(r) = \lambda_0 \int_0^r g(\xi) d\xi$, $\lambda_0 > 0$, it holds that α is harmonic and the multiplicity of the least eigenvalue of $H(\alpha)$ is infinity, (For details, see Remark 2.1). Finally we assume that there exist constant $r_0 > 0$ and $p > \alpha_2 + 1$ such that

$$k(r) = r^p \qquad \text{for} \quad r \geqq r_0$$

where α_2 is the constant given by (2.1) below. Then it is clear that $\lim ||d\alpha||(r, \theta) = \infty$. In this case, we will give a result on the asymptotic distribution of eigenvalues of $H(\alpha)$. In the case of $M = \boldsymbol{R}^2$ with the standard flat metric g, this problem was discussed by Colin de Verdiere [2] (also see Tamura [21]), and our result is easily obtained from their results.

The organization of this paper is as follows: In Section 2, we will summarize our main results. In Section 3, we will prove that the operator $L(\alpha)$ is essentially self-adjoint on $C_0^\infty(\boldsymbol{R}^2)^{\boldsymbol{C}}$. Section 4 will be devoted to the proof of Theorem 2.1. In Section 5, we will give the proof of Theorem 2.2. The proof of Theorem 2.3 will be given in Section 6.

2. The problems and the results

As we stated in the introduction, throughout this paper we always assume that the Riemannian metric g on $\boldsymbol{R}^2$ is given by

$$g = (g_{ij}(r, \theta)) = \begin{pmatrix} 1 & 0 \\ 0 & g(r)^2 \end{pmatrix}$$

where $g(r)$ is a positive continuous function on $(0, \infty)$ such that there exist $K > 0$, $\alpha_1, \alpha_2 \geqq 1$ and $0 < r_1 < r_2$ satisfying

$$(2.1) \qquad g(r) = r^{\alpha_1} \quad \text{for} \quad 0 < r \leqq r_1, \qquad g(r) = K r^{\alpha_2} \quad \text{for} \quad r \geqq r_2 .$$

and g has derivatives of every order on $(0, \infty)$. Then the Laplace-Beltrami operator Δ with respect to the Riemannian metric g can be written in the form

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{g'(r)}{g(r)} \frac{\partial}{\partial r} + \frac{1}{g(r)^2} \frac{\partial^2}{\partial \theta^2} \qquad \text{for} \quad (r, \theta) \in (0, \infty) \times S^1 .$$

In addition, we consider a differential 1-form α on $\boldsymbol{R}^2$ given by

$$\alpha(r, \theta) = k(r) d\theta$$

where $k(r)$ is a positive smooth real valued function on $(0, \infty)$ with $\lim_{r \to 0} k(r) = 0$. Then $L(\alpha)$ can be written in the form

$$L(\alpha) = -\frac{1}{2}\left(\frac{\partial^2}{\partial r^2} + \frac{g'(r)}{g(r)} \frac{\partial}{\partial r} + \frac{1}{g(r)^2} \frac{\partial^2}{\partial \theta^2}\right) - \sqrt{-1} \frac{k(r)}{g(r)^2} \frac{\partial}{\partial \theta} + \frac{1}{2} \frac{k(r)^2}{g(r)^2}$$

in polar coordinates (r, θ). We obtain the following lemma.

Lemma 2.1. (1) *$L(\alpha)$ is essentially self-adjoint on $C_0^\infty(\boldsymbol{R}^2)^{\boldsymbol{C}}$.*
(2) *$L(\alpha)$ can be uniquely extended to a self-adjoint operator $H(\alpha)$ defined on $L_{\boldsymbol{C}}^2(\boldsymbol{R}^2; dm)$.*

The proof of Lemma 2.1 will be given in Section 3.

We also consider a Sturm-Liouville operator A on $(0, \infty)$ given by

$$(2.2)\qquad A = \frac{1}{2}\left(\frac{\partial^2}{\partial r^2} + \frac{g'(r)}{g(r)}\frac{\partial}{\partial r}\right) \qquad r \in (0, \infty).$$

Then the boundaries 0 and ∞ are entrance and natural with respect to the operator A in the sense of Feller [13] respectively. Hence there exists a unique diffusion process $(X(t), P_x)$ on $\boldsymbol{R}^2$ generated by $\Delta/2$ (cf. [7]).

We now consider the stochastic line intergral $S(t, \alpha)$ of α along the path of the diffusion process $\{X(s); 0 \leqq s \leqq t\}$. Then there exists a symmetric continuous semigroup $\{Q_t\}_{t \geq 0}$ on $L^2_C(\boldsymbol{R}^2; dm)$ such that

$$(2.3)\qquad (Q_t f)(x) = E_x[f(X(t))e^{\sqrt{-1}S(t,\alpha)}] \qquad \text{for } f \in L^2_C(\boldsymbol{R}^2; dm).$$

Furthermore, the generator of the semi-group $\{Q_t\}_{t \geq 0}$ on $L^2_C(\boldsymbol{R}^2; dm)$ coincides with $-H(\alpha)$. We now state our results.

Theorem 2.1. *Suppose that* $\lim_{r\to\infty} k'(r)/g(r) = 0$. *Then* $H(\alpha)$ *has spectrum* $[0, \infty)$. *Furthermore,*

(1) *If* $\lim_{r\to\infty} k(r)/g(r) = \infty$, *then the spectrum of* $H(\alpha)$ *is a dense point spectrum in* $[0, \infty)$ *and there is a complete set of eigenfunctions.*

(2) *If* $\lim_{r\to\infty} k(r)/g(r) = c \neq 0$, *then the spectrum of* $H(\alpha)$ *is a dense point spectrum in* $[0, \frac{1}{2}c^2]$ *and absolutely continuous in* $[\frac{1}{2}c^2, \infty)$.

(3) *If* $\lim_{r\to\infty} k(r)/g(r) = 0$, *then the spectrum of* $H(\alpha)$ *is absolutely continuous in* $[0, \infty)$.

Theorem 2.2. *If* $\lim_{r\to\infty} k'(r)/g(r) = \infty$, *then the spectrum of* $H(\alpha)$ *is discrete, and the least eigenvalue of* $H(\alpha)$ *is positive.*

REMARK 2.1. We now consider the case of $\lim_{r\to\infty} k'(r)/g(r) = \lambda_0 > 0$. Let H_m be the operator defined by (4.4). Since $\lim_{r\to\infty} k(r)/g(r) = \infty$, and so $\lim_{r\to\infty} V(r) = \infty$, where $V(r)$ is given by (3.2) below. Hence H_m has only discrete spectrum $\{\lambda^n_m\}$, $n = 1, 2, \cdots$. Let $\varphi^n_m(r)$ be the normalized eigenfunction corresponding to λ^n_m. Then, as in Section 4, $\varphi^n_m(r)e^{-\sqrt{-1}m\theta}$ is an eigenfunction of $H(\alpha)$ corresponding to λ^n_m. We consider the special case, $k(r) = \lambda_0 \int_0^r g(\xi)d\xi$, $\lambda_0 > 0$. Then

$$\delta\alpha = 0, \quad d\alpha(r, \theta) = \lambda_0 g(r) dr \wedge d\theta,$$

and so

$$\Delta\alpha = 0,$$

where d is the exterior differential operator, δ is the dual operator of d and

$\Delta=-(d\delta+\delta d)$. Hence α is harmonic.

Let us set

$$v_m(r)=\exp\left\{\int_1^r \frac{m-k(\xi)}{g(\xi)}d\xi\right\}$$

and

$$u_m(r,\theta)=v_m(r)\,e^{-\sqrt{-1}m\theta},\ m=0,\pm1,\pm2,\cdots.$$

Then $u_m\in L^2_C(\boldsymbol{R}^2;dm)$, and it is easy to see

$$H(\alpha)\,u_m(r,\theta)=\frac{\lambda_0}{2}u_m(\mathrm{r},\theta).$$

Hence $u_m(r,\theta)$ is an eigenfunction of $H(\alpha)$ with respect to eigenvalue $\lambda_0/2$. Moreover, we set, for $\varphi\in C_0^\infty(\boldsymbol{R}^2)^C$.

$$\varphi(r,\theta)=\sum_{m=-\infty}^{\infty}\phi(m,r)\,e^{-\sqrt{-1}m\theta}$$

where $\phi(m,r)$ is a Fourier coefficient of $\theta\mapsto\varphi(r,\theta)$, i.e.

$$\phi(m,r)=\frac{1}{2\pi}\int_0^{2\pi}\varphi(r,\theta)\,e^{\sqrt{-1}m\theta}\,d\theta.$$

Then we have

$$((H(\alpha)-\frac{\lambda_0}{2})\,\varphi,\varphi)=\frac{1}{2}\sum_{m=-\infty}^{\infty}\int_{\boldsymbol{R}^2}|\frac{\partial}{\partial r}(v_m^{-1}(r)\,\phi(m,r))|^2|v_m(r)|^2g(r)\,dr\,d\theta\geqq0.$$

Hence the least eigenvalue of $H(\alpha)$ is $\lambda_0/2$ and the eigenfunctions corresponding to $\lambda_0/2$ are:

$$u_m(r,\theta),\ m=0,\pm1,\pm2,\cdots.$$

Furthermore, the multiplicity of $\lambda_0/2$ is ∞. In this case, if g is the standard flat metric, then

$$\alpha(x,y)=ydx-xdy\quad(x,y)\in\boldsymbol{R}^2.$$

Therefore α is the vector potential of a constant magnetic filed on $\boldsymbol{R}^2$. It is a classical result that the multiplicity of all eigenvalues is infinity in this case, (see [3], [19]). For the spectral property of $H(\alpha)$ with perturbed constant magnetic fields, (see [1], [8]).

In the case of $\lim_{r\to\infty}||d\alpha||(r,\theta)=\lim_{r\to\infty}k'(r)/g(r)=\infty$, let $0<\lambda_1\leqq\lambda_2\leqq\cdots$ be the eigenvalues of $H(\alpha)$. Then we set, for $\lambda>0$

$$N(\lambda)=\sum_{\lambda_n<\lambda}1. \tag{2.4}$$

Now we are in a position to state a final result.

Theorem 2.3. *Suppose that there exist $r_0>0$ and $p>\alpha_2+1$ such that*

$$k(r)=r^p \quad \text{for every} \quad r\geqq r_0\,. \tag{2.5}$$

Then, as $\lambda\to\infty$

$$N(\lambda)\sim\int_{\boldsymbol{R}^2}\nu_{b(x)}(\lambda)\,m(dx) \tag{2.6}$$

with

$$\begin{aligned}\nu_{b(x)}(\lambda)&=\frac{b(x)}{2\pi}\sum_{n=0}^{\infty}\Big((\lambda-(2n+1)\frac{b(x)}{2})_+\Big)^0,\ 0\leqq\lambda<\infty\,,\\ b(x)&=\|d\alpha\|\,(r,\theta)\qquad \text{for}\quad x=(r,\theta)\in(0,\infty)\times S^1\end{aligned} \tag{2.7}$$

where $0^0=0$ and f_+ denotes the positive part of a function f.

REMARK 2.2. If g is the standard flat metric, then Theorems 2.1 and 2.2 mentioned above can be obtained as special examples of general facts. The details can be found in K. Miller-B. Simon [14], A. Iwatsuka [9], [10], H.L. Cycon, G.H. Froese, W. Kirsch and B. Simon [3] and references therein. As we stated in Section 1, if g is the standard flat metric, Theorem 2.3 is a special example of a result of Colin de Verdiere [2]. In $\boldsymbol{R}^3$, Tamura [21] has also discussed the asymptotic distribution of eigenvalues of $H(\alpha)$.

3. Essential self-adjointness

Before proving the main theorem, we will show Lemma 2.1 by using the same idea as the case of the standard flat metric g (see [4], [16]). We first note that the Riemannian volume dm can be written in the form

$$dm=g(r)\,dr\,d\theta\,.$$

For all $u\in L^2_C(\boldsymbol{R}^2;dm)$, the L^2-norm of u is given by

$$\|u\|_2^2=\int_0^\infty\Big(\int_0^{2\pi}|u(r,\theta)|^2\,d\theta\Big)g(r)\,dr\,.$$

We consider a subspace $\tilde{D}$ of $L^2_C(\boldsymbol{R}^2;dm)$ spanned by finite linear combinations of products $v_m(r)\,e^{-\sqrt{-1}m\theta}$ $m=0,\pm1,\pm2,\cdots$ where $v_m(r)\in C_0^\infty((0,\infty))$ if $m\neq0$, and $v_m(r)=\int_0^{2\pi}u(r,\theta)\,d\theta$ for some function $u\in C_0^\infty(\boldsymbol{R}^2)^C$ if $m=0$. Here $C_0^\infty((0,\infty))$ is the space of all C^∞-functions on $(0,\infty)$ with compact supports. Then $\tilde{D}$ is dense in $L^2_C(\boldsymbol{R}^2;dm)$ since

$$C_0^\infty(0,\infty)\otimes L(S^1)\subset\tilde{D}\subset L^2((0,\infty),g(r)\,dr)\otimes L^2_C(S^1,d\theta)=L^2_C(\boldsymbol{R}^2;dm)\,,$$

where $L(S^1)$ is a space generated by finite linear combinations of $S^1=\{e^{-\sqrt{-1}m\theta};$

$m=0, \pm 1, \pm 2, \cdots, 0 \leqq \theta < 2\pi\}$, and $L_C^2(S^1, d\theta)$ is the Hilbert space of complex valued square-integrable functions on S^1 with respect to $d\theta$. For this proof, it is sufficient to show that $L(\alpha)$ is essentially self-adjoint on $\tilde{D}$. We also note that for a fixed integer m, $L(\alpha)$ acts on functions of the form $v(r)\, e^{-\sqrt{-1}m\theta}$ by

$$L(\alpha)\,(v(r)\, e^{-\sqrt{-1}m\theta}) = (L_m\, v(r))\, e^{-\sqrt{-1}m\theta}$$

where

$$L_m = -\frac{1}{2}\left[\frac{\partial^2}{\partial r^2} + \frac{g'(r)}{g(r)}\frac{\partial}{\partial r} - \left(\frac{m-k(r)}{g(r)}\right)^2\right]. \tag{3.1}$$

So we only need to prove that L_m is essentially self-adjoint with domain $\mathrm{Dom}(L_m)$, where $\mathrm{Dom}(L_m)$ is $\{v(r) = \int_0^{2\pi} u(r, \theta)\, d\theta\,;\, u \in C_0^\infty(\boldsymbol{R}^2)^C\}$ if $m=0$, and on $C_0^\infty((0, \infty))$ if $m \neq 0$.

We first consider the case of $m \neq 0$. To do this, let U be a unitary operator from $L^2((0, \infty), g(r)\, dr)$ into $L^2((0, \infty), dr)$ defined by $U\colon \varphi(r) \mapsto \sqrt{g(r)}\varphi(r)$. U takes $C_0^\infty((0, \infty))$ into itself and

$$UL_m\, U^{-1} = -\frac{1}{2}\frac{\partial^2}{\partial r^2} + V(r)$$

where

$$V(r) = \frac{1}{4}\frac{g''(r)}{g(r)} - \frac{1}{8}\frac{g'(r)^2}{g(r)^2} + \frac{1}{2}\left(\frac{m-k(r)}{g(r)}\right)^2. \tag{3.2}$$

Then $V(r)$ is in the limit point case at infinity since $V(r) \geqq -\frac{1}{8r^2}$ (see [16], Theorem X. 8), and $V(r)$ is in the limit point case at zero since $V(r) \geqq \frac{3}{8r^2}$ near zero. (see [16], Theorem X.10). Hence $-\frac{1}{2}\frac{\partial^2}{\partial r^2} + V(r)$ is essentially self-adjoint on $C_0^\infty((0, \infty))$ (cf. [16]) and so is L_m on $C_0^\infty((0, \infty))$.

Next, we consider the case of $m=0$. If $\varphi \in \mathrm{Dom}(L_0)$, then there exists $u \in C_0^\infty(\boldsymbol{R}^2)^C$ with $\varphi(r) = \int_0^{2\pi} u(r\cos\theta, r\sin\theta)\, d\theta$. Clearly

$$\varphi(0) = u(0, 0)\,, \tag{3.3}$$

and

$$\varphi'(r) = \pi r\, \Delta\, u(0, 0) + O(r^2) \tag{3.4}$$

which implies that $\lim_{r\to 0} \varphi'(r)/r = \pi\Delta u(0, 0)$. Furthermore it is easy to see by (3.4) that

$$(\varphi, L_0\, \psi)_{\mathcal{L}^2} = (L_0\, \varphi, \psi)_{\mathcal{L}^2}\,,$$

where $\mathcal{L}^2=L^2((0,\infty), g(r)\,dr)$, which means that L_0 is symmetric. Now we show that L_0 is essentially self-adjoint. Let L_0^* be the adjoint operator of L_0 and $\mathrm{Dom}(L_0^*)$ be the domain of L_0^*([27]). Since L_0 is a real operator, we need only prove that if $(L_0+1)^*\varphi=0$ and φ is a real function in $\mathrm{Dom}(L_0^*)$, then $\varphi=0$. Since $(L_0+1)^*\varphi=0$ implies that $(\varphi, (L_0+1)\psi)_{\mathcal{L}^2}=0$ for every $\psi\in\mathrm{Dom}(L_0)$, we can show that $(L_0+1)\varphi=0$ in the sense of distribution. By the hypoellipticity of L_0+1, we obtain that $\varphi\in C^\infty((0,\infty))$, where $C^\infty((0,\infty))$ is the space of infinitely differentiable functions on $(0,\infty)$. Therefore $(L_0+1)\varphi=0$. Using the formula for integration by parts, we get

(3.5)
$$0=\varphi'(\varepsilon)\varphi(\varepsilon)g(\varepsilon)-\varphi'(R)\varphi(R)g(R)+\int_\varepsilon^R\{\varphi'(r)^2+\left(\frac{k(r)^2}{g(r)^2}+1\right)\varphi(r)^2\}g(r)\,dr\,.$$

We set

$$c=\lim_{R\to\infty}\{\varphi'(\varepsilon)\varphi(\varepsilon)g(\varepsilon)+\int_\varepsilon^R\{\varphi'(r)^2+\left(\frac{k(r)^2}{g(r)^2}+1\right)\varphi(r)^2\}g(r)\,dr\}\,,$$

(this limit may be infinite). To prove $c=0$, we first assume that $c\neq 0$.

If $c>0$, then there exists a positive number $R_0\geqq\max\{r_2, 1\}$ such that

(3.6)
$$\varphi'(\varepsilon)\varphi(\varepsilon)g(\varepsilon)+\int_\varepsilon^R\{\varphi'(r)^2+\left(\frac{k(r)^2}{g(r)^2}+1\right)\varphi(r)^2\}g(r)\,dr\geqq\frac{c}{2}\quad\text{for every } R\geqq R_0\,.$$

Therefore, by (3.5) and (3.6),

$$\frac{d}{dR}(\varphi(R)^2g(R))\geqq c+\varphi(R)^2g'(R)\quad\text{for every } R\geqq R_0$$

which implies

$$\varphi(R)^2g(R)-\varphi(R_0)^2g(R_0)\geqq c(R-R_0)+\int_{R_0}^R\varphi(r)^2g'(r)\,dr\,.$$

Taking the limit as $R\to\infty$, we have

$$\lim_{R\to\infty}\varphi(R)^2g(R)=\infty$$

which contradicts the fact that $\int_0^\infty\varphi^2(r)g(r)\,dr<\infty$.

Similarly, if $c<0$, then there also exists a positive number $R_0\geqq\max\{r_2, 1\}$ such that

$$\varphi'(\varepsilon)\varphi(\varepsilon)g(\varepsilon)+\int_\varepsilon^R\{\varphi'(r)^2+\left(\frac{k(r)^2}{g(r)^2}+1\right)\varphi(r)^2\}g(r)\,dr\leqq\frac{c}{2}\quad\text{for every } R\geqq R_0$$

and then we have

$$\varphi(R)^2 g(R)-\varphi(R_0)^2 g(R_0) \leqq c(R-R_0)+\int_{R_0}^{R} \varphi(r)^2 g'(r)\, dr \quad \text{for every } R \geqq R_0 .$$

Hence, we have

$$\lim_{R\to\infty} \varphi(R)^2 g(R) = -\infty$$

which contradicts the fact that $\int_{R_0}^{\infty} \varphi^2(r)\, g(r)\, dr>0$. Combining above results, we have

$$\text{(3.7)} \quad \begin{aligned} &\lim_{R\to\infty} \varphi(R)\,\varphi'(R)\, g(R) = 0 \\ &\varphi(\varepsilon)\,\varphi'(\varepsilon)\, g(\varepsilon)+\int_{\varepsilon}^{\infty} \{\varphi'(r)^2+\Big(\frac{k(r)^2}{g(r)^2}+1\Big)\varphi(r)^2\}\, g(r)\, dr = 0 \quad \text{for every } \varepsilon>0 . \end{aligned}$$

We now take a $\psi \in \mathrm{Dom}(L_0)$. Since $(L_0+1)^* \varphi=0$, it holds that

$$\text{(3.8)} \quad \begin{aligned} &\lim_{\varepsilon\downarrow 0} \{\varphi(\varepsilon)\,\overline{\psi'(\varepsilon)}\, g(\varepsilon)+\int_{\varepsilon}^{\infty} \{\varphi'(r)\,\overline{\psi'(r)}+\Big(\frac{k(r)^2}{g(r)^2}+1\Big)\varphi(r)\,\overline{\psi(r)}\}\, g(r)\, dr\} \\ &\quad = 2\,(\varphi, (L_0+1)\,\psi)_{\mathcal{L}^2} = 0 \end{aligned}$$

and for every $\varepsilon>0$,

$$\text{(3.9)} \quad 0 = \varphi'(\varepsilon)\,\overline{\psi(\varepsilon)}\, g(\varepsilon)+\int_{\varepsilon}^{\infty} \{\varphi'(r)\,\overline{\psi'(r)}+\Big(\frac{k(r)^2}{g(r)^2}+1\Big)\varphi(r)\,\overline{\psi(r)}\}\, g(r)\, dr .$$

Combining (3.6) and (3.7), we have

$$\text{(3.10)} \quad \lim_{\varepsilon\downarrow 0} \{\varphi(\varepsilon)\,\overline{\psi'(\varepsilon)}\, g(\varepsilon)-\varphi'(\varepsilon)\,\overline{\psi(\varepsilon)}\, g(\varepsilon)\} = 0 .$$

Since by (3.3) and (3.4), there exists a function $\psi \in \mathrm{Dom}(L_0)$ such that $\lim_{\varepsilon\downarrow 0} \psi(\varepsilon)$ and $\lim_{\varepsilon\downarrow 0} \psi'(\varepsilon)/\varepsilon$ exist and are not zero, it follows from (3.10) that

$$\text{(3.11)} \quad \lim_{\varepsilon\downarrow 0} \varphi(\varepsilon)\, g(\varepsilon)\, \varepsilon = 0 \quad \text{and} \quad \lim_{\varepsilon\downarrow 0} \varphi'(\varepsilon)\, g(\varepsilon) = 0 .$$

Moreover, in view of (3.7) we know that $\varphi'(\varepsilon)\,\varphi(\varepsilon)\, g(\varepsilon) \leqq 0$. First, we consider the case where for some $\varepsilon>0$, $\varphi'(\varepsilon)\,\varphi(\varepsilon)\, g(\varepsilon)=0$. Then we have, by (3.7),

$$\int_{\varepsilon}^{\infty} \{\varphi'(r)^2+\Big(\frac{k(r)^2}{g(r)^2}+1\Big)\varphi(r)^2\}\, g(r)\, dr = 0$$

Hence by suing $(L_0+1)\,\varphi(r)=0$, we obtain $\varphi \equiv 0$ on $[0, \infty)$.

We next consider the case where $\varphi'(\varepsilon)\,\varphi(\varepsilon)\, g(\varepsilon)<0$ for every $\varepsilon>0$. Without loss of generality, we may assume that $\varphi>0$, $\varphi'<0$, and then φ is a monotone decreasing real valued smooth function. Now choose a function $\phi_n \in C^\infty(\boldsymbol{R})$ satisying $0 \leqq \phi_n' \leqq 1$ with

$$\phi_n(x) = \begin{cases} x & \text{if } x \leqq n \\ n+\dfrac{1}{2} & \text{if } x \geqq n+1 \end{cases}$$

and set $\varphi_n(x)=\phi_n(\varphi(x))$. Then we have

$$0 = -\varphi'(\varepsilon)\,\varphi_n(\varepsilon)\,g(\varepsilon)+\varphi'(R)\,\varphi_n(R)\,g(R) - \int_\varepsilon^R \{\varphi'(r)\,\varphi_n'(r)+\left(\frac{k(r)^2}{g(r)^2}+1\right)\varphi(r)\,\varphi_n(r)\}\,g(r)\,dr\,.$$

Since $\lim_{R\to\infty}\varphi'(R)\,\varphi_n(R)\,g(R)=0$ and $\lim_{\varepsilon\downarrow 0}\varphi'(\varepsilon)\,\varphi_n(\varepsilon)\,g(\varepsilon)=0$ by (3.11), we have

$$\begin{aligned} 0 &= \int_0^\infty \{\varphi'(r)\,\varphi_n'(r)+\left(\frac{k(r)^2}{g(r)^2}+1\right)\varphi(r)\,\varphi_n(r)\}\,g(r)\,dr \\ &= \int_0^\infty \{\varphi'(r)^2\,\phi_n'(\varphi(x))+\left(\frac{k(r)^2}{g(r)^2}+1\right)\varphi(r)\,\varphi_n(r)\}\,g(r)\,dr\,. \end{aligned} \tag{3.12}$$

Taking $n\to\infty$ in (3.12), we get

$$\int_0^\infty \{\varphi'(r)^2+\left(\frac{k(r)^2}{g(r)^2}+1\right)\varphi(r)^2\}\,g(r)\,dr = 0$$

and therefore $\varphi\equiv 0$ on $[0, \infty)$. So L_0 is essentially self-adjoint.

4. The point spectrum and the absolute continuous spectrum

In this Section, we first follow the method of Miller-Simon (see [14]) to assert that if $\lim_{r\to\infty}k'(r)/g(r)=0$, then $H(\alpha)$ has the spectrum $[0, \infty)$. To prove this we first recall Weyl's criterion for the spectrum, i.e., λ is a spectrum of $H(\alpha)$ if and only if there exists a sequence $\{\varphi_n\}_{n=1}^\infty$ with $||\varphi_n||=1$ and

$$\lim_{n\to\infty}||(H(\alpha)-\lambda)\,\varphi_n|| = 0$$

where $||\cdot||$ denotes the norm on $L_C^2(\boldsymbol{R}^2; dm)$ (see cf. [15]). Since $\lim_{t\to\infty}k'(r)/g(r)=0$, we can find a positive increasing sequence $\{r_n\}_{n=1}^\infty$ with $r_n^{-1}=o(n^{-1})$ as $n\to\infty$ and

$$\frac{k'(r)}{g(r)} \leqq \frac{1}{n\cdot 2^n} \quad \text{for every } r\geqq r_n\,. \tag{4.1}$$

Take a fixed smooth function ϕ with $||\phi||=1$, vanishing outside the interval $(0, 1)$. For a fixed real number ν, we define a sequence of functions $\{\varphi_n\}_{n=1}^\infty$ in $L_C^2(\boldsymbol{R}^2; dm)$ by

$$\varphi_n(r, \theta) = \phi\left(\frac{r-r_n}{n}\right)e^{\sqrt{-1}(\sqrt{2}\nu r-[k(r_n)]\theta)}/||\phi\left(\frac{r-r_n}{n}\right)||$$

where $[\cdot]$ denotes Gauss's symbol, i.e. $[k(r)]$ is the greatest integer which is less than or equal to $k(r)$. Then it is not difficult to show that

$$(H(\alpha)-\nu^2)\,\varphi_n(r,\theta)=-\frac{1}{2}\frac{\frac{1}{n^2}\frac{\partial^2\phi}{\partial r^2}\left(\frac{r-r_n}{n}\right)e^{-\sqrt{-1}[k(r_n)]\theta+\sqrt{-2}\nu r}}{\|\phi\left(\frac{r-r_n}{n}\right)\|}$$

$$-\frac{1}{2}\frac{\left(\frac{2\sqrt{-2}\,\nu}{n}-\frac{g'(r)}{ng(r)}\right)\frac{\partial\phi}{\partial r}\left(\frac{r-r_n}{n}\right)e^{-\sqrt{-1}[k(r_n)]\theta+\sqrt{-2}\nu r}}{\|\phi\left(\frac{r-r_n}{n}\right)\|}$$

$$-\frac{1}{2}\left\{\frac{[k(r_n)]-k(r)}{g(r)}\right\}^2\varphi_n(r,\theta)-\frac{\sqrt{-2}}{2}\frac{\nu g'(r)}{g(r)}\varphi_n(r,\theta)\,.$$

Hence we have

$$\|(H(\alpha)-\nu^2)\varphi_n\|^2\leqq\frac{\|\frac{1}{n^2}\frac{\partial^2\phi}{\partial r^2}\left(\frac{r-r_n}{n}\right)\|^2}{4\|\phi\left(\frac{r-r_n}{n}\right)\|^2}+2\frac{\|\frac{\nu}{n}\frac{\partial\phi}{\partial r}\left(\frac{r-r_n}{n}\right)\|^2}{\|\phi\left(\frac{r-r_n}{n}\right)\|^2}$$

$$(4.2)\qquad +\frac{\|\frac{g'}{ng}\frac{\partial\phi}{\partial r}\left(\frac{r-r_n}{n}\right)\|^2}{4\|\phi\left(\frac{r-r_n}{n}\right)\|^2}+\frac{\|\frac{\nu g'}{g}\phi\left(\frac{r-r_n}{n}\right)\|^2}{2\|\phi\left(\frac{r-r_n}{n}\right)\|^2}+\frac{\|h_n^2\phi\left(\frac{r-r_n}{n}\right)\|^2}{4\|\phi\left(\frac{r-r_n}{n}\right)\|^2},$$

where $h_n(r)=\{[k(r_n)]-k(r)\}/g(r)$. Now write these terms by $I_1(n)$, $I_2(n)$, $I_3(n)$, $I_4(n)$, $I_5(n)$ in the right hand side of (4.2). We will prove that $I_k(n)\to 0$, $k=1,2,3,4,5$, as $n\to\infty$. First, since

$$I_1(n)=\frac{\int_0^{2\pi}d\theta\int_0^1\left|\frac{\partial^2\phi}{\partial r^2}(r)\right|^2 g(r_n+nr)\,dr}{4n^4\int_0^{2\pi}d\theta\int_0^1|\phi(r)|^2 g(r_n+nr)\,dr}$$

and $\lim_{n\to\infty} g(r_n+nr)/g(r_n)=1$, we see that $I_1(n)\to 0$ as $n\to\infty$. In the same way, we can prove that $I_2(n)\to 0$ as $n\to\infty$. Next, consider $I_3(n)$, Since we have

$$I_3(n)=\frac{\int_0^{2\pi}d\theta\int_0^1\left|\frac{g'(r_n+nr)}{g(r_n+nr)}\right|^2\left|\frac{\partial\phi}{\partial r}(r)\right|^2 g(r_n+nr)\,dr}{4n^2\int_0^{2\pi}d\theta\int_0^1|\phi(r)|^2 g(r_n+nr)\,dr}$$

and $\lim_{n\to\infty} g'(r_n+nr)/g(r_n+nr)=0$, it follows that $I_3(n)\to 0$ as $n\to\infty$. Similarly, we can show that $I_4(n)\to 0$ as $n\to\infty$. At last, consider $I_5(n)$. For every r in $[r_n, r_n+n]$, it is easy to see that

$$|h_n(r)| \leqq \frac{1}{g(r)} + \left|\frac{k(r_n)-k(r)}{g(r)}\right| \leqq \frac{c}{r_n} + \frac{1}{2^n}, \quad c\text{: constant,}$$

which implies that $I_5(n) \to 0$ as $n \to \infty$. Then we obtain

$$\|(H(\alpha)-\nu^2)\varphi_n\|^2 \to 0 \quad \text{as } n \to \infty$$

and hence ν^2 is a spectrum of $H(\alpha)$. Since ν is arbitrary real number, this show that the spectrum of $H(\alpha)$ equals to $[0, \infty)$.

To analyze the spectrum further, we note that

$$(4.3) \qquad H(\alpha)(v(r)e^{-\sqrt{-1}m\theta}) = (H_m v(r))e^{-\sqrt{-1}m\theta}$$

for every fixed integer m, where

$$(4.4) \qquad H_m = -\frac{1}{2}\left[\frac{\partial^2}{\partial r^2} + \frac{g'(r)}{g(r)}\frac{\partial}{\partial r}\right] + \frac{1}{2}\left(\frac{m-k(r)}{g(r)}\right)^2$$

which is the closed extention of L_m given by (3.1) above.

Let U be a unitary operator from $L^2((0, \infty), g(r)\,dr)$ into $L^2((0, \infty), dr)$ given by $\varphi \mapsto \varphi\sqrt{g}$. Then, as in Section 3, we have

$$UH_m U^{-1} = -\frac{1}{2}\frac{\partial^2}{\partial r^2} + V(r),$$

where $V(r)$ is the function given by (3.2).

At first, we consider the case of $\lim_{r\to\infty} k(r)/g(r) = \infty$. Then $UH_m U^{-1}$ has only discrete spectrum and so H_m discrete spectrum, since $V(r) \to \infty$ as $r \to \infty$ (see [24]). This implies that there exist eigenfunctions $\{\varphi_m^n\}_{n=1}^{\infty}$ of H_m which can form a complete orthonormal system in $L^2((0, \infty); g(r)\,dr)$, (see cf. [15]). In other words, the eigenfunctions $\{\varphi_m^n(r)e^{-\sqrt{-1}m\theta}\}_{m=0,\pm1,\pm2,\cdots}^{n=1,2,\cdots}$ of $H(\alpha)$ can form a complete orthonormal system in $L_C^2(\boldsymbol{R}^2; dm)$. Let λ_m^n be the eigenvalues of $H(\alpha)$ corresponding to $\varphi_m^n(r)e^{-\sqrt{-1}m\theta}$, $m=0, \pm1, \pm2, \cdots, n=1, 2, \cdots$, respectively. Assume that $\bigcup_{m=-\infty}^{\infty}\bigcup_{n=1}^{\infty}\{\lambda_m^n\}$ is not a dense in $[0, \infty)$. Then there exists an open interval $(a, b) \subset [0, \infty)$ such that $(a, b) \cap (\bigcup_{m=-\infty}^{\infty}\bigcup_{n=1}^{\infty}\{\lambda_m^n\}) = \phi$. Let E be the spectral measure associated with $H(\alpha)$ i.e., $H(\alpha) = \int_0^\infty \lambda dE(\lambda)$. Then $E((a, b))$ and $E(\{\lambda_m^n\})$ are orthogonal for every $m=0, \pm1, \pm2, \cdots, n=1, 2, \cdots$ and so $E((a, b)) = 0$. This shows that (a, b) is contained in the resolvent set of $H(\alpha)$, which contradicts the fact that $(a, b) \subset [0, \infty) = \sigma(H(\alpha))$, where $\sigma(H(\alpha))$ denotes the spectrum of $H(\alpha)$ (cf. [22]). Hence $H(\alpha)$ has a dense point spectrum $\bigcup_{m=-\infty}^{\infty}\bigcup_{n=1}^{\infty}\{\lambda_m^n\}$ in $[0, \infty)$.

Next, we consider the case of $\lim_{r\to\infty} k(r)/g(r) = c$. In this case, since $V(r) \to \frac{1}{2}c^2$

as $r\to\infty$, the spectrum $UH_m U^{-1}$ is discrete in $[0, \frac{1}{2}c^2]$ and absolutely continuous in $[\frac{1}{2}c^2, \infty)$ (cf. [18], [23]), and so is H_m. Let $\sigma_{\text{dis}}(H_m)$ be the discrete spectrum of H_m. Then by using (4.3), we know that $H(\alpha)$ has a point spectrum $\bigcup_{m=-\infty}^{\infty}\sigma_{\text{dis}}(H_m)$ in $[0, \frac{1}{2}c^2]$ and absolutely continuous in $[\frac{1}{2}c^2, \infty)$. Moreover, by the same proof as above argument, $\bigcup_{m=-\infty}^{\infty}\sigma_{\text{dis}}(H_m)$ is a dense set in $[0, \frac{1}{2}c^2]$.

At last, if we consider the case of $\lim_{r\to\infty} k(r)/g(r)=0$, then the spectrum of $UH_m U^{-1}$ is absolutely continuous in $[0, \infty)$ and so is the spectrum of H_m, since $V(r)\to 0$ as $r\to\infty$ (cf. [18], [23]). Consequently, $H(\alpha)$ has absolutely continuous spectrum in $[0, \infty)$.

5. The discrete spectrum

In this Section, we give the proof of Theorem 2.2. Throughout this Section, we always assume that $\lim_{r\to\infty} k'(r)/g(r)=\infty$. For every $\varphi\in C_0^\infty(\boldsymbol{R}^2)^{\boldsymbol{C}}$, we have

$$\varphi(r, \theta) = \sum_{m=-\infty}^{\infty} \phi(m, r)\, e^{-\sqrt{-1}m\theta}$$

where $\phi(m, r)$, is a Fourier coefficient of φ, i.e.,

$$\phi(m, r) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(r, \theta)\, e^{\sqrt{-1}m\theta}\, d\theta\,.$$

Moreover, letting

$$v_m(r) = \exp\Big\{\int_1^r \frac{m-k(\xi)}{g(\xi)}\, d\xi\Big\}, \quad m = 0, \pm 1, \pm 2, \cdots,$$

we obtain

$$\begin{aligned}
&\Big(\Big(H(\alpha)-\frac{1}{2}\frac{k'(r)}{g(r)}\Big)\varphi, \varphi\Big)\\
&= \sum_{m=-\infty}^{\infty}\int_{(0,\infty)\times S^1}\Big(L_m-\frac{1}{2}\frac{k'(r)}{g(r)}\Big)\phi(m, r)\cdot\overline{\phi(m, r)}\, g(r)\, dr\, d\theta\\
&= \frac{1}{2}\sum_{m=-\infty}^{\infty}\int_{(0,\infty)\times S^1}\Big|\frac{\partial}{\partial r}(v_m^{-1}(r)\,\phi(m, r))\Big|^2\, |v_m(r)|^2 g(r)\, dr\, d\theta \geqq 0
\end{aligned}$$

where we used the formula for integration by parts in the second equality. Hence, setting

$$\lambda(r) = \frac{1}{2}\frac{k'(r)}{g(r)} \quad r\in(0, \infty),$$

we obtain

$$\lim_{r\to\infty}\lambda(r) = \infty$$

and

$$(5.1)\quad (H(\alpha)\varphi,\varphi)\geqq\int_{(0,\infty)\times S^1}\lambda(r)|\varphi(r,\theta)|^2 g(r)\,dr\,d\theta \quad \text{for every } \varphi\in C_0^\infty(\boldsymbol{R}^2)^C.$$

Following an idea of A. Iwatsuka [10] and using (5.1), we now prove that $H(\alpha)$ has only discrete sepctrum. First we note that for every $s\in\boldsymbol{R}$, setting $\tilde{\lambda}(r)=\lambda(r)\wedge s$, we obtain that

$$\tilde{\lambda}(r)\to s \quad \text{as} \quad r\to\infty$$

$$(H(\alpha)\varphi,\varphi)\geqq\int_{\boldsymbol{R}^2}\tilde{\lambda}(r)|\varphi(r,\theta)|^2 g(r)\,dr\,d\theta \quad \text{for every } \varphi\in C_0^\infty(\boldsymbol{R}^2)^C.$$

Hence we obtain

$$(5.2)\qquad H(\alpha)\geqq s+K$$

where K denotes the operator of multiplication by $\tilde{\lambda}(r)-s$. As we will see in the Lemma 5.1 below, the operator K is relatively compact operator with respect to $H(\alpha)$. For any $\nu\in\sigma_{\text{ess}}(H(\alpha))$ there exists a sequence $\{u_k\}_{k=1}^\infty$ in Dom $(H(\alpha))$ such that

$$\|u_k\|=1 \quad \text{and} \quad u_k\to 0 \quad \text{weakly as } k\to\infty$$
$$\|H(\alpha)u_k-\nu u_k\|\to 0 \quad \text{as } k\to\infty$$

where $\sigma_{\text{ess}}(H(\alpha))$ denotes the essential spectrum of $H(\alpha)$. Furthermore, $Ku_k\to 0$ strongly as $k\to\infty$ (see [11]). On the other hand, by (5.2) we have

$$(H(\alpha)u_k,u_k)\geqq s+(Ku_k,u_k).$$

Hence we have

$$\nu\geqq s$$

which implies that

$$(5.3)\qquad \inf\sigma_{\text{ess}}(H(\alpha))\geqq s.$$

Since s is arbitrary, it follows from (5.3) that $H(\alpha)$ has only discrete spectrum.

Therefore it only remains to show that K is relatively compact operator with respect to $H(\alpha)$. To do this we first note that for all $\varphi\in L_C^2(\boldsymbol{R}^2;dm)$, $x\in\boldsymbol{R}^2$ and $\mu>0$, we have, by (2.2)

$$|(H(\alpha)+\mu)^{-1}\varphi(x)|\leqq\int_0^\infty e^{-t\mu}E_x[|\varphi|(X_t)]\,dt=(\mu-\Delta/2)^{-1}|\varphi|(x).$$

Hence if we can prove the following lemma, then the proof of Theorem 2.2 is complete.

Lemma 5.1. *Let K be an operator of multiplication by a bounded mea-*

surable function $k(r, \theta)$ *on* $\boldsymbol{R}^2$ *with* $\lim_{r\to\infty} k(r, \theta)=0$. *Then for every positive number* μ, $K(\mu-\Delta/2)^{-1}$ *is a compact operator in* $L^2(\boldsymbol{R}^2; dm)$, (*the space of all real square integrable functions on* $\boldsymbol{R}^2$ *with respect to the Riemannian volume* dm).

Before proceeding to the proof of this lemma, we give several remarks. Let $\{K_t\}$ be the semi-group of the diffusion process $(X(t), P_x)$ introduced in Section 2. Then $\{K_t\}$ is a family of symmetric operators on $L^2(\boldsymbol{R}^2; dm)$ with respect to dm and there exists a positive continuous kernel function $k(t, (a, \theta_1), (b, \theta_2))$ such that

$$K_t f(a, \theta_1) = \int_{(0,\infty)\times S^1} k(t, (a, \theta_1), (b, \theta_2)) f(b, \theta_2) g(b)\, db\, d\theta_2 \quad \text{for} \quad (a, \theta_1)\in(0, \infty)\times S^1$$

(see for example [5]). Hence it is easy to see that for $\mu>0$

$$(\mu-\Delta/2)^{-1} f(a, \theta_1) = \int_{(0,\infty)\times S^1} G(\mu, (a, \theta_1), (b, \theta_2)) f(b, \theta_2) g(b)\, db\, d\theta_2,$$

where

$$G(\mu, (a, \theta_1), (b, \theta_2)) = \int_0^\infty e^{-\mu t} k(t, (a, \theta_1), (b, \theta_2))\, dt\,. \tag{5.4}$$

By using (5.4), it is easy to see that

$$\int_{(0,\infty)\times S^1} G^2(\mu, (a, \theta_1), (b, \theta_2)) g(b)\, db\, d\theta_2 = \int_0^\infty e^{-\mu t}\cdot t\cdot k(t, (a, \theta_1), (a, \theta_1))\, dt\,. \tag{5.5}$$

On the other hand, as we stated in Section 2, the boundary 0 and ∞ are entrance and natural with respect to the operator A given by (2.2) in the sence of Feller, respectively. Hence there exists a unique diffusion process $(r(t), P_a)$ on $(0, \infty)$ (see [7]). Furthermore, there exists a positive continuous kernel function $p(t, a, b)$ such that

$$E_a[f(r(t))] = \int_0^\infty f(b) p(t, a, b) g(b)\, db \quad \text{for every} \quad f\in L^2((0, \infty), g(b)\, db)$$

where $L^2((0, \infty), g(b)\, db)$ is the space of all real square-integrable functions on $(0, \infty)$ with respect to $g(b)\, db$, and E_a denotes the expectation with respect to the probability P_a, ([7], [13]). Setting

$$\Upsilon(t) = \int_0^t \frac{1}{g(r(s))^2}\, ds\,,$$

we obtain

$$k(t, (a, \theta_1), (b, \theta_2)) = p(t, a, b)\, E_a[q(\Upsilon(t), \theta_1, \theta_2)/r(t) = b] \tag{5.6}$$

where $q(t, \theta_1, \theta_2)$ is the transition density of the Brownian motion on S^1 with respect to uniform measure on S^1, (see [7]).

Proof of Lemma 5.1. (1) 1-st step. We first consider the case where $g(r)=r^{\alpha}$. In this particular case, the diffusion on $\boldsymbol{R}^2$ with generator $\Delta/2$ can be represented as the skew product of its Bessel part BES$(\alpha+1)$ with generator

$$A=\frac{1}{2}\left(\frac{\partial^2}{\partial r^2}+\frac{\alpha}{r}\frac{\partial}{\partial r}\right) \quad r\in(0,\infty)$$

and an independent circular Brownian motion BM(S^1) run with the clock $\Upsilon(t)=\int_0^t r(s)^{-2\alpha}\,ds$, therefore, we obtain

$$k(t,(a,\theta_1),(a,\theta_1))=p(t,a,a)\,E_a[q(\Upsilon(t),\theta_1,\theta_1)/r(t)=a]\,. \tag{5.7}$$

By using the eigenfunction expansion of $q(t,\theta_1,\theta_1)$, we obtain

$$\begin{aligned}E_a[q(\Upsilon(t),\theta_1,\theta_1)/r(t)=a] &= E_a[\sum_{n\in\boldsymbol{Z}}\exp\{-n^2\,\Upsilon(t)\}/r(t)=a]\\ &\leqq E_a[\int_{-\infty}^{\infty}\exp\{-x^2\,\Upsilon(t)\}\,dx/r(t)=a]\\ &=\sqrt{\pi}\,E_a[(\Upsilon(t))^{-1/2}/r(t)=a]\,.\end{aligned}$$

By combining this with Schwarz' inequality, we have

$$\begin{aligned}&E_a[q(\Upsilon(t),\theta_1,\theta_1)/r(t)=a]\leqq\sqrt{\pi}\,E_a[(\Upsilon(t))^{-1/2}/r(t)=a]\\ &=\sqrt{\pi}\,E_a[\left\{\int_0^t r(s)^{-2\alpha}\,ds\right\}^{-1/2}/r(t)=a]\\ &\leqq\sqrt{\pi}\,t^{-1}\,E_a[\left\{\int_0^t r(s)^{2\alpha}\,ds\right\}^{1/2}/r(t)=a]\\ &\leqq\sqrt{\pi}\,t^{-1}\left\{E_a[\int_0^t r(s)^{2\alpha}\,ds/r(t)=a]\right\}^{1/2}\\ &=\sqrt{\pi}\,t^{-1}\left\{\int_0^t E_a[r(s)^{2\alpha}/r(t)=a]\,ds\right\}^{1/2}\quad\text{(Fubini's theorem)}\\ &=\sqrt{\pi}\,t^{-1}\left\{p(t,a,a)^{-1}\int_0^t\int_0^{\infty}r^{3\alpha}\,p(s,a,r)\,p(t-s,r,a)\,dr\,ds\right\}^{1/2}.\end{aligned}$$

Furthermore, by using the formula $I_\nu(x)\leqq e^x/\sqrt{2\pi x}$, we obtain

$$\begin{aligned}&\int_0^{\infty}r^{3\alpha}\,p(s,a,r)\,p(t-s,r,a)\,dr\\ &=\int_0^{\infty}r^{3\alpha}\frac{e^{-(t(a^2+r^2))/2s(t-s)}}{s(t-s)\,(ra)^{\alpha-1}}\,I_{(\alpha-1)/2}\left(\frac{ra}{s}\right)I_{(\alpha-1)/2}\left(\frac{ra}{t-s}\right)dr\\ &\leqq\int_0^{\infty}r^{3\alpha}\frac{e^{-(t(a^2+r^2))/2s(t-s)}}{s(t-s)\,(ra)^{\alpha-1}}\,\frac{e^{ra/s}}{\sqrt{2\pi\frac{ra}{s}}}\,\frac{e^{ra/(t-s)}}{\sqrt{2\pi\frac{ra}{t-s}}}\,dr\end{aligned}$$

$$
\begin{aligned}
&= \int_{-a}^{\infty} (x+a)^{2\alpha} e^{-tx^2/2s(t-s)} dx/2\pi a^\alpha \sqrt{s(t-s)} \\
&\leqq 2^{2\alpha} \int_{a}^{\infty} (x)^{2\alpha} e^{-tx^2/2s(t-s)} dx/2\pi a^\alpha \sqrt{s(t-s)} \\
&\quad +2^{2\alpha} a^\alpha \int_{-a}^{a} e^{-tx^2/2s(t-s)} dx/2\pi \sqrt{s(t-s)} \\
&\leqq 2^{2\alpha} \int_{0}^{\infty} (x)^{2\alpha} e^{-tx^2/2s(t-s)} dx/2\pi a^\alpha \sqrt{s(t-s)} \\
&\quad +2^{2\alpha} a^\alpha \int_{-\infty}^{\infty} e^{-tx^2/2s(t-s)} dx/2\pi \sqrt{s(t-s)} \\
&\leqq 2^{2\alpha-1} \int_{0}^{\infty} (y)^{\alpha-1/2} e^{-ty/2s(t-s)} dy/2\pi a^\alpha \sqrt{s(t-s)} \\
&\quad +2^{2\alpha} a^\alpha \sqrt{2\pi} \sqrt{\frac{s(t-s)}{t}} \Big/ 2\pi \sqrt{s(t-s)} \\
&= 2^{2\alpha-1} \left\{\frac{2s(t-s)}{t}\right\}^{\alpha+1/2} \Gamma(\alpha+1/2)/2\pi a^\alpha \sqrt{s(t-s)} + 2^{2\alpha} a^\alpha/\sqrt{2\pi} t^{1/2} \\
&= 2^{3\alpha-1/2} \{s(t-s)\}^\alpha \Gamma(\alpha+1/2)/2\pi a^\alpha t^{\alpha+1/2} + 2^{2\alpha} a^\alpha/\sqrt{2\pi} t^{1/2} .
\end{aligned}
$$

Combining this with (5.8), we can find

$$
\begin{aligned}
&E_a[q(\Upsilon(t), \theta_1, \theta_1)/r(t) = a] \\
&\quad \leqq \sqrt{\pi} t^{-1} \{p(t, a, a)^{-1} \int_0^t \int_0^\infty r^{3\alpha} p(s, a, r) p(t-s, r, a) \, dr \, ds\}^{1/2} \\
&\quad \leqq t^{-1} \left\{p(t, a, a)^{-1} \int_0^t \left\{\frac{2^{3\alpha-3/2} \{s(t-s)\}^\alpha \Gamma(\alpha+1/2)}{a^\alpha t^\alpha} + \frac{\sqrt{\pi} 2^{2\alpha-1/2} a^\alpha}{t^{1/2}}\right\} ds\right\}^{1/2} \\
&\quad \leqq t^{-1} \left\{p(t, a, a)^{-1} \left\{\frac{2^{2\alpha-3/2} t^{\alpha+1/2} \Gamma(\alpha+1/2)}{a^\alpha} + \sqrt{\pi} 2^{2\alpha-1/2} a^\alpha t^{1/2}\right\}\right\}^{1/2}
\end{aligned}
$$

which implies the following inequality.

$$
\begin{aligned}
&k(t, (a, \theta_1), (a, \theta_1)) \\
&\quad \leqq t^{-1} \left\{p(t, a, a) \left\{\frac{2^{\alpha-3/2} t^{\alpha+1/2} \Gamma(\alpha+1/2)}{a^\alpha} + \sqrt{\pi} 2^{2\alpha-1/2} a^\alpha t^{1/2}\right\}\right\}^{1/2} .
\end{aligned}
$$

Hence by (5.5), we have

(5.9)

$$
\begin{aligned}
&\int_{(0,\infty)\times S^1} G^2(\mu, (a, \theta_1), (b, \theta_2)) b^\alpha \, db d\theta_2 \\
&\quad \leqq \int_0^\infty e^{-\mu t} \left\{p(t, a, a) \left\{\frac{2^{\alpha-3/2} t^{\alpha+1/2} \Gamma(\alpha+1/2)}{a^\alpha} + \sqrt{\pi} 2^{2\alpha-1/2} a^\alpha t^{1/2}\right\}\right\}^{1/2} dt \\
&\quad = \int_0^\infty e^{-\mu t} \left\{\frac{e^{-a^2/t}}{t a^{\alpha-1}} I_{(a-1)/2}\left(\frac{a^2}{t}\right) \left\{\frac{2^{\alpha-3/2} t^{\alpha+1/2} \Gamma(\alpha+1/2)}{a^\alpha} + \sqrt{\pi} 2^{2\alpha-1/?} a^\alpha t^{1/2}\right\}\right\}^{1/2} dt \\
&\quad \leqq \int_0^\infty e^{-\mu t} \left\{\frac{2^{\alpha-2} t^\alpha \Gamma(\alpha+1/2)}{\sqrt{\pi} a^{2\alpha}} + 2^{2\alpha-1}\right\}^{1/2} dt .
\end{aligned}
$$

To complete the proof, let K_n be the operator of multiplication by $k_n(r, \theta) = I_n(r, \theta)\, k(r, \theta)$ on $\boldsymbol{R}^2$, where I_n denotes the characteristic function of $B_n = \{(r, \theta) \in (0, \infty) \times S^1; r \leqq n\}$, i.e.

$$I_n(r, \theta) = \begin{cases} 1 & \text{if } r \leqq n \\ 0 & \text{if } r > n . \end{cases}$$

Now, by using the method of Example 6 in p. 117 of [17], we will show that $K_n(\mu - \Delta/2)^{-1}$ is a Hilbert-Schmidt operator. In other words, it is sufficient to prove that

$$\text{(5.10)} \quad \int_{(0,\infty)\times S^1} k_n(a, \theta_1)^2 \{\int_{(0,\infty)\times S^1} G^2(\mu, (a, \theta_1), (b, \theta_2))\, b^\alpha\, db\, d\theta_2\}\, a^\alpha\, da\, d\theta_1 < \infty$$

is valid. For this purpose, we can use (5.9) to get

$$\begin{aligned}
&\int_{(0,\infty)\times S^1} k_n(a, \theta_1)^2 \{\int_{(0,\infty)\times S^1} G^2(\mu, (a, \theta_1), (b, \theta_2))\, b^\alpha\, db d\theta_2\}\, a^\alpha\, da d\theta_1 \\
&\leqq \int_{(0,\infty)\times S^1} k_n(a, \theta_1)^2 \Big\{\int_0^\infty e^{-\mu t} \Big\{\frac{2^{\alpha-2}\, t^\alpha\, \Gamma(\alpha+1/2)}{\sqrt{\pi}\, a^{2\alpha}} + 2^{2\alpha-1}\Big\}^{1/2} dt\Big\}\, a^\alpha\, da d\theta_1 \\
&\leqq \int_{(0,\infty)\times S^1} k_n(a, \theta_1)^2 \Big\{\int_0^\infty e^{-\mu t} \Big\{\frac{2^{\alpha-2}\, t^\alpha\, \Gamma(\alpha+1/2)}{\sqrt{\pi}}\Big\}^{1/2} dt\Big\}\, da d\theta_1 \\
&\quad + \int_{(0,\infty)\times S^1} k_n(a, \theta_1)^2 \{\int_0^\infty e^{-\mu t}\, 2^{\alpha-1/2}\, dt\}\, a^\alpha\, da d\theta_1 \\
&= \Big\{\frac{2^{\alpha-2}\, \Gamma(\alpha+1/2)}{\sqrt{\pi}}\Big\}^{1/2} \int_{(0,\infty)\times S^1} k_n(a, \theta_1)^2 \{\int_0^\infty e^{-\mu t}\, t^{\alpha/2}\, dt\}\, da d\theta_1 \\
&\quad + 2^{\alpha-1/2} \int_{(0,\infty)\times S^1} k_n(a, \theta_1)^2 \{\int_0^\infty e^{-\mu t}\, dt\}\, a^\alpha\, da d\theta_1 \\
&< \infty
\end{aligned}$$

which proves (5.10). Since $\lim_{x\to\infty} \|K_n - K\|_\infty = 0$, $K_n(\mu - \Delta/2)^{-1}$ converges in norm to $K(\mu - \Delta/2)^{-1}$ and so $K(\mu - \Delta/2)^{-1}$ is a compact operator.

(2) 2-nd step. We now return to the general case in which the metric g is given by (2.1). We first note that for every compact set S in $\boldsymbol{R}^2 \setminus \{0\}$, the Laplace-Beltrami operator Δ restricted on S is uniformly elliptic. We now take a compact set S in $\boldsymbol{R}^2 \setminus \{(r, \theta); 0 < r < r_3\}$, where $0 < r_3 < r_1$. Then it is known that there exist positive constants c_1, c_2 such that

$$\begin{aligned}
&k(t, (a, \theta_1), (b, \theta_2)) \\
&\leqq \frac{c_1}{2\pi t} \cdot \exp\Big\{-\frac{c_2 \cdot \rho((a, \theta_1), (b, \theta_2))^2}{2t}\Big\} \quad \text{for every } (a, \theta_1), (b, \theta_2) \in S,
\end{aligned}$$

where $\rho((a, \theta_1), (b, \theta_2))$ denotes the Riemannian distance between (a, θ_1) and (b, θ_2), (cf. [25]). Therefore, by combining this with (5.5), we can show that there exists a positive constant C satistying the following inequality

$$\int_{(0,\infty)\times S^1} G^2(\mu, (a, \theta_1), (b, \theta_2))\, g(b)\, dbd\theta_2 \leqq C \quad \text{for every } (a, \theta_1)\in S\,.$$

Hence for the proof of Lemma 5.1, it is sufficient to show that for $r_4\in(r_3, r_1)$,

$$\int_{S(r_4)}\int_{(0,\infty)\times S^1} G^2(\mu, (a, \theta_1), (b, \theta_2))\, g(b)\, dbd\theta_2 < \infty, \tag{5.11}$$

where $S(r_4)=\{(r, \theta)\in(0, \infty)\times S^1; 0<r\leqq r_4\}$. Now we fix a positive constant r_5 such that $r_4<r_5<r_1$, and set

$$S(r_5) = \{(r, \theta)\in \boldsymbol{R}^2; 0<r\leqq r_5\}\ .$$

Let $(\tilde{X}(t), \tilde{P}_{(a,\theta_1)})$ be the diffusion process generated by the operator

$$\frac{1}{2}\Delta = \frac{1}{2}\left(\frac{\partial^2}{\partial r^2}+\frac{\alpha}{r}\frac{\partial}{\partial r}+\frac{1}{r^{2\alpha}}\frac{\partial^2}{\partial\theta^2}\right)$$

discussed in the 1-st step, we set

$$\tau = \inf\{t; \tilde{X}(t)\in\partial S(r_5)\}\ ,$$

and we consider the probability measure $\mu_{(a,\theta_1)}(dsd\theta)$ on $(0, \infty)\times\partial S(r_5)$ defined by

$$\mu_{(a,\theta_1)}(dsd\theta) = P_{(a,\theta_1)}[\tau\in ds, \tilde{X}(\tau)\in d\theta]\ .$$

Let $k_0(t, (a, \theta_1), (b, \theta_2))$ be the probability density function of the minimal diffusion process $\{\tilde{X}(t); 0\leqq t<\tau\}$, i.e. for every bounded continuous function f,

$$\begin{aligned}&\tilde{E}_{(a,\theta_1)}[f(\tilde{X}(t); 0<t<\tau]\\ &\quad=\int_{S(r_5)} k_0(t, (a, \theta_1), (b, \theta_2))\, f((b, \theta_2))\, b^\alpha\, dbd\theta_2 \quad \text{for every } (b, \theta_2)\in S(r_5)\end{aligned}$$

where $\tilde{E}_{(a,\theta_1)}$ denotes the expectation with resepct to $\tilde{P}_{(a,\theta_1)}$. Then by (2.1), the transition density function $k(t, (a, \theta_1), (b, \theta_2))$ in the general case can be represented in the following form:

$$k(t, (a, \theta_1), (b, \theta_2)) = k_0(t, (a, \theta_1), (b, \theta_2)) + \tilde{k}(t, (a, \theta_1), (b, \theta_2))$$

where

$$\tilde{k}(t, (a, \theta_1), (b, \theta_2)) = \int_0^t\int_{\partial S(r_5)} k(t-s, (r_5, \theta), (b, \theta_2))\, \mu_{(a,\theta_1)}(dsd\theta)\,.$$

It is easy to see that

$$\int_{S(r_4)}\left\{\int_0^\infty e^{-\mu t}\, t\cdot\tilde{k}(t, (a, \theta_1), (a, \theta_1))dt\right\} a^\alpha\, dad\theta_1 < \infty\ .$$

Furthermore, the results obtained in the 1-st step implies that

$$\int_{S(r_4)}\left\{\int_0^\infty e^{-\mu t}\, t\cdot k_0(t,(a,\theta_1),(a,\theta_1))\,dt\right\} a^\alpha\, da d\theta_1<\infty\,.$$

Combining these and (5.5), we obtain (5.11). Hence the proof of Lemma 5.1 is now complete.

6. Asymptotic distribution

By using a usual method in the Malliavin calculus and the probabilistic representation of the semi-group Q_t, we can show that there exists the C^∞-density $h(t, x, x)$ of the kernel of the operator Q_t, $t>0$, with respect to the Riemannian volume dm. For details of the proof, see [5], [6] and [26]. We next note that

$$m(\{x;\ |x|<\lambda\})\sim\frac{2\pi}{\alpha_2+1}\lambda^{\alpha_2+1}\quad\text{as } \lambda\to\infty \tag{6.1}$$

and

$$b(x)=b(r,\theta)=\frac{k'(r)}{g(r)}=pr^{p-1-\alpha_2}\qquad\text{if}\quad r\geqq r_0\,. \tag{6.2}$$

As in [2], setting

$$N_0(\lambda)=\int_{R^2}\nu_{b(x)}(\lambda)m(dx)\quad \lambda>0\,,$$

we have

$$\int_0^\infty e^{-\lambda t}\,dN_0(\lambda)=\frac{1}{2\pi t}\int_{R^2}\frac{t\frac{b(x)}{2}}{\sinh t\frac{b(x)}{2}}m(dx)\,,\quad t>0\,. \tag{6.3}$$

Hence combining (6.1), (6.2), (6.3) with the Tauberian theorem, it is sufficient for the proof of Theorem 2.3 to show that as $t\downarrow 0$

$$\int_0^\infty e^{-\lambda t}\,dN(\lambda)\sim\frac{1}{2\pi t}\int_{R^2}\frac{t\frac{b(x)}{2}}{\sinh t\frac{b(x)}{2}}m(dx)\,. \tag{6.4}$$

Since $\int_0^\infty e^{-\lambda t}\,dN(\lambda)=\int_{R^2}h(t,x,x)m(dx)$, the relationship (6.4) is reduced to the following asymptotic formula

$$\int_{R^2}h(t,x,x)m(dx)\sim\frac{1}{2\pi t}\int_{R^2}\frac{t\frac{b(x)}{2}}{\sinh t\frac{b(x)}{2}}m(dx)\,,\qquad\text{as}\quad t\downarrow 0\,. \tag{6.5}$$

Next, on the two-dimensional standard Wiener space (W_0^2, P) we consider the following stochastic differential equation: for $w=(w^1, w^2)\in W_0^2$,

$$(6.6)\qquad \begin{cases} dr(t) = \varepsilon dw^1(t)+\varepsilon^2\dfrac{g'(r(t))}{g(r(t))}dt \\ d\theta(t) = \varepsilon\dfrac{dw^2(t)}{g(r(t))} \\ r(0) = a>0 \\ \theta(0) = \theta_1 . \end{cases}$$

It is known (cf. [6]) that there exists a unique solution $(r^\varepsilon(t, a, w^1), \theta^\varepsilon(t, (a, \theta_1), w))$, $t>0$ of (6.6). Furthermore, it is easily seen that

(6.7)
1) $r^\varepsilon(t, a, w^1)$ is independent of $w^2(t)$.
2) $\theta^\varepsilon(t, (a, \theta_1), w) = \theta_1+\varepsilon \int_0^t \dfrac{dw^2(s)}{g(r^\varepsilon(s, a, w^1))}$, for $w(t) = (w^1(t), w^2(t))$.

We now denote by δ the Dirac δ-function at y associated with the Riemannian volume dm, i.e., for any C^∞-function f with compact support

$$\int_{R^2} \delta_y(x)f(x)dm(x) = f(y) .$$

Then it is known (cf. [5]) that

$$(6.8)\qquad \begin{aligned} &h(\varepsilon^2, (a, 0), (a, 0)) \\ &\quad = \int_W \exp\{\sqrt{-1}\int_0^1 k(r^\varepsilon(t, a, w^1))\circ d\theta^\varepsilon(t, (a, 0), w)\} \\ &\qquad \cdot\delta_{(a,0)}(r^\varepsilon(1, a, w^1), \theta^\varepsilon(1, (a, 0), w))P(dw) \end{aligned}$$

where the right hand side means the generalized expectation in the sense of Watanabe [26] (also see [6]). Then (6.8) can be rewritten in the form.

$$(6.9)\qquad \begin{aligned} &h(\varepsilon^2, (a, 0), (a, 0)) \\ &\quad = E[\exp\{\sqrt{-1}\int_0^1 k(r^\varepsilon(t, a, w^1))\circ d\theta^\varepsilon(t, (a, 0), w)\} \\ &\qquad /(r^\varepsilon(1, a, w^1), \theta^\varepsilon(1, (a, 0), w)) = (a, 0)]\cdot k(\varepsilon^2, (a, 0), (a, 0)) \end{aligned}$$

where $E[\cdot/(r^\varepsilon(1, a, w^1), \theta^\varepsilon(1, (a, 0), w))=(a, 0)]$ is the conditional expectation with respect to the Wiener measure under the condition $(r^\varepsilon(1, a, w^1), \theta^\varepsilon(1, (a, 0), w))=(a, 0)$. Set

$$\begin{aligned} I(\varepsilon, a) = E[&\exp\{\sqrt{-1}\int_0^1 k(r^\varepsilon(t, a, w^1))\circ d\theta^\varepsilon(t, (a, 0), w)\} \\ &/(r^\varepsilon(1, a, w^1), \theta^\varepsilon(1, (a, 0), w)) = (a, 0)] , \end{aligned}$$

and let $\mathcal{F}^1$ be the σ-field generated by $\{w^1(t); 0\leqq t\leqq 1\}$. Then

$$I(\varepsilon, a) = E[E\,[\exp\{\sqrt{-1}\int_0^1 k(r^\varepsilon(t, a, w^1))\circ d\theta^\varepsilon(t, (a, 0), w)\} \tag{6.10}$$
$$/\{\theta^\varepsilon(1, (a, 0), w)) = 0\} \cap \mathcal{F}^1]/(r^\varepsilon(1, a, w^1) = a]\,.$$

Since $\{w^1(t); 0 \leqq t \leqq 1\}$ and $\{w^2(t); 0 \leqq t \leqq 1\}$ are independent under the Wiener measure, there exists a Brownian motion $B(t)$, $0 \leqq t \leqq 1$ which is independent of $\{w^1(t); 0 \leqq t \leqq 1\}$ such that

$$\theta^\varepsilon(t, (a, 0), w)) = B(\varepsilon^2 \phi^\varepsilon(t, a, w^1))\,, \tag{6.11}$$

where

$$\phi^\varepsilon(t, a, w^1) = \int_0^t \frac{ds}{g(r^\varepsilon(s, a, w^1))^2}\,.$$

Furthermore we can construct a Brownian motion $\tilde{B}(t)$, $0 \leqq t \leqq 1$, on the Wiener space $\{W_0^2, P\}$ such that

1) $\tilde{B}(t)$, $0 \leqq t \leqq 1$, is independent of $\{w^1(t); 0 \leqq t \leqq 1\}$,

2) $$\mathrm{E}[\exp\{\sqrt{-1}\int_0^1 k(r^\varepsilon(t, a, w^1))\circ d\theta^\varepsilon(t, (a, 0), w)\}$$
$$/\{\theta^\varepsilon(1, (a, 0)), w)) = 0\} \cap \mathcal{F}^1]$$
$$= E[\exp\{\sqrt{-1}\int_0^1 k(r^\varepsilon(t, a, w^1))\circ dB(\varepsilon^2\phi^\varepsilon(t, a, w^1))\}$$
$$/\{B(\varepsilon^2\phi^\varepsilon(t, a, w^1)) = 0\} \cap \mathcal{F}^1]$$
$$= E[\exp\{\sqrt{-1}\{\int_0^1 k(r^\varepsilon(t, a, w^1))\cdot d\tilde{B}(\varepsilon^2\phi^\varepsilon(t, a, w^1))$$
$$-\frac{\int_0^1 k(r^\varepsilon(t, a, w^1))\cdot d\phi^\varepsilon(t, a, w^1)}{\phi^\varepsilon(1, a, w^1)}\tilde{B}(\varepsilon^2\phi^\varepsilon(1, a, w^1))\}\}]\,.$$

Therefore we obtain

$$I(\varepsilon, a) = E[\exp\{-\frac{\varepsilon^2}{2}\int_0^1 (k(r^\varepsilon(t, a, w^1)) - \bar{k}(w^1))^2\cdot d\phi^\varepsilon(t, a, w^1)\} \tag{6.12}$$
$$/r^\varepsilon(1, a, w^1) = a]\,.$$

where

$$\bar{k}(w^1) = \frac{1}{\phi^\varepsilon(1, a, w^1)}\int_0^1 k(r^\varepsilon(t, a, w^1))\cdot d\phi^\varepsilon(t, a, w^1)\,. \tag{6.13}$$

For every continuous function f on [0, 1], we set

$$V[f]^2 = \int_0^1 (f(t) - \bar{f})^2\, d\phi^\varepsilon(t, a, w^1), \quad \tilde{V}[f]^2 = \int_0^1 (f(t) - \tilde{\bar{f}})^2\, dt$$

where

$$\bar{f} = \frac{1}{\phi^\varepsilon(1, a, w^1)} \int_0^1 f(t) d\phi^\varepsilon(t, a, w^1), \quad \tilde{\bar{f}} = \int_0^1 f(t) dt .$$

We also use the following notations

$$\begin{aligned} f^\varepsilon(t, a, w^1) &= k(r^\varepsilon(t, a, w^1)) - k(a) \\ f_1^\varepsilon(t, a, w^1) &= k'(a)(r^\varepsilon(t, a, w^1) - a) \\ f_2^\varepsilon(t, a, w^1) &= f^\varepsilon(t, a, w^1) - f_1^\varepsilon(t, a, w^1) . \end{aligned} \tag{6.14}$$

Then we obtain that

$$f_2^\varepsilon(t, a, w^1) = k''(a + \eta(t, a, w)(r^\varepsilon(t, a, w^1) - a))(r^\varepsilon(t, a, w^1) - a)^2 \quad \text{for some } 0 < \eta(t, a, w) < 1 . \tag{6.15}$$

Then (6.12) can be rewritten in the following form

$$I(\varepsilon, a) = E[\exp\{-\frac{\varepsilon^2}{2} V[f^\varepsilon(\cdot, a, w^1)]^2\} / r^\varepsilon(1, a, w^1) = a] .$$

We now note that by (6.6)

$$r^\varepsilon(t, a, w^1) = a + \varepsilon w^1(t) + \varepsilon^2 \int_0^t \frac{g'(r^\varepsilon(s, a, w^1))}{g(r^\varepsilon(s, a, w^1))} ds . \tag{6.16}$$

Next we set, for every fixed $c > 0$,

$$a(\varepsilon, c) = (c/\varepsilon^4)^{1/2(p - \alpha_2 - 1)}.$$

Then

$$\varepsilon^4 a^{2(p - \alpha_2 - 1)} \leqq c \qquad \text{for} \quad a \leqq a(\varepsilon, c) .$$

Next, we set $r^* = r_0 \vee r_2$ and fix constants b_1, b_2 such that $r^* < b_1 < b_2$. We also define $\sigma^\varepsilon(b_1, a, w^1)$ by

$$\sigma^\varepsilon(b_1, a, w^1) = \begin{cases} \inf \{t; 0 \leqq t \leqq 1, \quad r^\varepsilon(t, a, w^1) = b_1\}, & \text{if } \{\ \} \neq \phi , \\ 1, \text{ if } \{\ \} = \phi , \end{cases}$$

and let $A_1[\varepsilon, a] = \{w^1; \sigma^\varepsilon(b_1, a, w^1) = 1\}$. For a fixed positive constant δ satisfying $\frac{2}{3} < \delta < 1$, we set

$$A_2[\varepsilon, a] = \{w^1; \max_{0 \leqq t \leqq 1} |w^1(t)| < \varepsilon^{-(1-\delta)}\} .$$

Then by using (6.14), (6.15) and (6.16), we can show the following results: There exist positive constants $\varepsilon^* > 0$, $K_1 > 0$, $K_2 > 0$ such that if $b_2 \leqq a \leqq a(\varepsilon, c)$ and $w^1 \in A_1[\varepsilon, a] \cap A_2[\varepsilon, a]$ then

$$\varepsilon^2 V[f_1^\varepsilon(\cdot, a, w^1)]^2(1-K_1\varepsilon^\delta)-K_2\varepsilon^{3(\delta-(2/3))}\leqq\varepsilon^2 V[f^\varepsilon(\cdot, a, w^1)]^2$$
$$\leqq\varepsilon^2 V[f_1^\varepsilon(\cdot, a, w^1)]^2(1+K_1\varepsilon^\delta)+K_2\varepsilon^{3(\delta-(2/3))}$$
$$\text{for} \quad 0<\varepsilon<\varepsilon^* .$$

By combining this with several results in the theory of 1-dimensional diffusion processes, (cf. [7]). We obtain that as $\varepsilon \downarrow 0$,

$$\begin{aligned}
&\int_{S^1}\int_{b_2}^{a(\varepsilon,c)} I(\varepsilon, a)k(\varepsilon^2, (a, 0), (a, 0))g(a)da d\theta_1 \\
&= \int_{S^1}\int_{b_2}^{a(\varepsilon,c)} E[\exp\{-\frac{\varepsilon^2}{2}V[f^\varepsilon(\cdot, a, w^1)]^2\} I_{A_1[\varepsilon,a]}(w^1)\, I_{A_2[\varepsilon,a]}(w^1) \\
&\qquad /r^\varepsilon(1, a, w^1) = a]k(\varepsilon^2, (a, 0), (a, 0))g(a)da d\theta_1(1+o(1)) \\
&= \int_{S^1}\int_{b_2}^{a(\varepsilon,c)} E[\exp\{-\frac{\varepsilon^2}{2}V[f^\varepsilon(\cdot, a, w^1)]^2/r^\varepsilon(1, a, w^1) = a] \\
&\qquad k(\varepsilon^2, (a, 0), (a, 0))g(a)da d\theta_1(1+o(1)) \\
&= \frac{1}{2\pi\varepsilon^2}\int_{S^1} d\theta_1 \int_{b_2}^{a(\varepsilon,c)} E[\exp\{-\frac{\varepsilon^4}{2}\frac{k'(a)^2}{g(a)^2}\tilde{V}[w^1]^2\}/w^1(1) = 0] \\
&\qquad g(a)da(1+o(1))
\end{aligned} \tag{6.17}$$

where $I_{A_1[\varepsilon,a]}(w^1)$ and $I_{A_2[\varepsilon,a]}(w^1)$ are the indicator functions of $A_1[\varepsilon, a]$ and $A_2[\varepsilon, a]$ respectively. Furthermore, for a fixed positive constant b_2, as $\varepsilon \downarrow 0$,

$$\begin{aligned}
&\int_{S^1}\int_0^{b_2} I(\varepsilon, a)k(\varepsilon^2, (a, 0), (a, 0))g(a)da d\theta_1 \\
&= o\left(\frac{1}{2\pi\varepsilon^2}\int_{R^2}\frac{\varepsilon^2\frac{b(x)}{2}}{\sinh \varepsilon^2\frac{b(x)}{2}}m(dx)\right)
\end{aligned} \tag{6.18}$$

and

$$\begin{aligned}
&\frac{1}{2\pi\varepsilon^2}\int_{S^1} d\theta_1 \int_0^{b_2} E[\exp\{-\frac{\varepsilon^4}{2}\frac{k'(a)^2}{g(a)^2}\tilde{V}[w^1]^2\}/w^1(1) = 0]g(a)da \\
&= o\left(\frac{1}{2\pi\varepsilon^2}\int_{R^2}\frac{\varepsilon^2\frac{b(x)}{2}}{\sinh \varepsilon^2\frac{b(x)}{2}}m(dx)\right)
\end{aligned} \tag{6.19}$$

On the other hand,

$$J(\varepsilon, a) = E[\exp\{-\frac{\varepsilon^4}{2}\frac{k'(a)^2}{g(a)^2}\tilde{V}[w^1]^2\}/w^1(1) = 0]$$

can be rewritten in the following form:

$$J(\varepsilon, a) = E[\exp\{-\frac{1}{2}E[\{\varepsilon^2\frac{k'(a)}{g(a)}\int_0^1 (w^1(s)-\int_0^1 w^1(t)dt)dw^2(s)\}^2/\mathscr{F}^1]\}/w^1(1)=0].$$

Hence

$$(6.20)\quad \begin{aligned} J(\varepsilon, a) &= E[E[\exp\{\sqrt{-1}\varepsilon^2\frac{k'(a)}{g(a)}\int_0^1 w^1(s)dw^2(s)\}/\mathscr{F}^1\cap\{w^2(1)=0\}]/w^1(1)=0] \\ &= E[E[\exp\{\sqrt{-1}\varepsilon^2\frac{k'(a)}{g(a)}S(1,w)\}/\mathscr{F}^1\cap\{w^2(1)=0\}]/w^1(1)=0] \\ &= E[\exp\{\sqrt{-1}\varepsilon^2\frac{k'(a)}{g(a)}S(1,w)\}/w(1)=0] \end{aligned}$$

where $S(1,w)$ is the stochastic area of Lévy, i.e.,

$$S(1,w) = \frac{1}{2}\int_0^1 (w^1(s)dw^2(s)-w^2(s)dw^1(s)).$$

Combining this with (6.17) (6.18) and (6.19) we obtain that as $\varepsilon \downarrow 0$,

$$\begin{aligned} &\int_{S^1}\int_0^{a(\varepsilon,c)} I(\varepsilon, a)k(\varepsilon^2, (a,0), (a,0))g(a)da d\theta_1 \\ &\sim \frac{1}{2\pi\varepsilon^2}\int_{S^1} d\theta_1 \int_0^{a(\varepsilon,c)} E[\exp\{\sqrt{-1}\varepsilon^2\frac{k'(a)}{g(a)}S(1,w)\}/w(1)=0]g(a)da \\ &\sim \frac{1}{2\pi\varepsilon^2}\int_{S^1} d\theta_1 \int_0^{a(\varepsilon,c)} \frac{\varepsilon^2\frac{b(a,0)}{2}}{\sinh \varepsilon^2\frac{b(a,0)}{2}} g(a)da. \end{aligned}$$

Here we used the following formula: for every $\lambda\in\boldsymbol{R}$,

$$E[\exp\{\sqrt{-1}\lambda S(1,w)\}/w(1)=0] = \frac{\frac{|\lambda|}{2}}{\sinh\frac{|\lambda|}{2}}$$

(for example, see [6]). Hence as $\varepsilon \downarrow 0$,

$$(6.21)\quad \begin{aligned} &\int_{S^1}\int_0^{a(\varepsilon,c)} I(\varepsilon, a)k(\varepsilon^2, (a,0), (a,0))g(a)da d\theta_1 \\ &\sim \frac{1}{2\pi\varepsilon^2}\int_{\boldsymbol{R}^2} \frac{\varepsilon^2\frac{b(x)}{2}}{\sinh \varepsilon^2\frac{b(x)}{2}} m(dx). \end{aligned}$$

We now set

$$M^{\varepsilon}(a, w^1) = \max_{0 \leq t \leq 1} \left(\frac{|r^{\varepsilon}(t, a, w^1) - a|}{\varepsilon}\right).$$

For sufficiently small $\gamma_1 > 0$, we set $\delta(a) = a^{\gamma_1}$.

Next, for sufficiently small $\gamma_2 > 0$, we set

$$A_1(\varepsilon, c, \gamma_2) = \int_{a(\varepsilon,c)}^{\infty} E[\exp\{-\frac{\varepsilon^2}{2} V[f^{\varepsilon}(\cdot, a, w^1)]^2\} ; M^{\varepsilon}(a, w^1) > \frac{\delta(a)}{\varepsilon^{\gamma_2}} / r^{\varepsilon}(1, a, w^1) = a] k(\varepsilon^2, (a, 0), (a, 0)) g(a) da .$$

and

$$A_2(\varepsilon, c, \gamma_2) = \int_{a(\varepsilon,c)}^{\infty} E[\exp\{-\frac{\varepsilon^2}{2} V[f^{\varepsilon}(\cdot, a, w^1)]^2\} ; M^{\varepsilon}(a, w^1) \leqq \frac{\delta(a)}{\varepsilon^{\gamma_2}} / r^{\varepsilon}(1, a, w^1) = a] k(\varepsilon^2, (a, 0), (a, 0)) g(a) da .$$

Then it is easy to see that

$$A_1(\varepsilon, c, \gamma_2) \leqq \int_{a(\varepsilon,c)}^{\infty} P[M^{\varepsilon}(a, w^1) > \frac{\delta(a)}{\varepsilon^{\gamma_2}} / r^{\varepsilon}(1, a, w^1) = a] k(\varepsilon^2, (a, 0), (a, 0)) g(a) da . \tag{6.22}$$

Next we note that

$$M^{\varepsilon}(a, w^1) \leqq \max_{0 \leq t \leq 1} |w^1(t)| + \varepsilon \int_0^1 \frac{g'(r^{\varepsilon}(s, a, w^1))}{g(r^{\varepsilon}(s, a, w^1))} ds .$$

and without loss of generality we may assume

$$a(\varepsilon, c) \geqq 2r^*$$

Hence setting $\sigma^{\varepsilon}(w^1) = \inf\{t; r^{\varepsilon}(t, a, w^1) = r^*\}$, we obtain that for any $a \geqq a(\varepsilon, c)$

$$\begin{aligned} &P[M^{\varepsilon}(a, w^1) > \frac{\delta(a)}{\varepsilon^{\gamma_2}} / r^{\varepsilon}(1, a, w^1) = a] \\ &\leqq P[M^{\varepsilon}(a, w^1) > \frac{\delta(a)}{\varepsilon^{\gamma_2}}, \sigma^{\varepsilon}(w^1) \geqq 1 / r^{\varepsilon}(1, a, w^1) = a] \\ &\quad + P[\sigma^{\varepsilon}(w^1) < 1 / r^{\varepsilon}(1, a, w^1) = a] \\ &\leqq P[\max_{0 \leq t \leq 1} |w^1(t)| > \frac{\delta(a)}{\varepsilon^{\gamma_2}} - \frac{\varepsilon \alpha_2}{\gamma_2} / r^{\varepsilon}(1, a, w^1) = a] \\ &\quad + P[\sigma^{\varepsilon}(w^1) < 1 / r^{\varepsilon}(1, a, w^1) = a] . \end{aligned}$$

Therefore, for sufficicently small $\varepsilon > 0$,

$$A_1(\varepsilon, c, \gamma_2) \leqq \int_{a(\varepsilon,c)}^{\infty} \{P[\max_{0\leqq t\leqq 1} |w^1(t)| > \frac{1}{2}\frac{\delta(a)}{\varepsilon^{\gamma_2}} / r^\varepsilon(1, a, w^1) = a] + P[\sigma^\varepsilon(w^1) < 1 / r^\varepsilon(1, a, w^1) = a]\} k(\varepsilon^2, (a, 0), (a, 0)) g(a) da .$$

Then by using the theory of 1-dimensional diffusion processes (cf. [7]), we can show that the integral in the right hand side converges.
This implies

$$A_1(\varepsilon, c, \gamma_2) \to 0 . \tag{6.23}$$

Next we note that

$$V[f^\varepsilon(\cdot, a, w^1)] \geqq V[f_1^\varepsilon(\cdot, a, w^1)] - V[f_2^\varepsilon(\cdot, a, w^1)] .$$

Hence

$$\begin{aligned} &A_2(\varepsilon, c, \gamma_2) \\ &\leqq \int_{a(\varepsilon,c)}^{\infty} P\Big[V[f_1^\varepsilon(\cdot, a, w^1)] \leqq 4V[f_2^\varepsilon(\cdot, a, w^1)] , \\ &\qquad M^\varepsilon(a, w^1) \leqq \frac{\delta(a)}{\varepsilon^{\gamma_2}} / r^\varepsilon(1, a, w^1) = a\Big] k(\varepsilon^2, (a, 0), (a, 0)) g(a) da \\ &\quad + \int_{a(\varepsilon,c)}^{\infty} E\Big[\exp\{-\frac{\varepsilon^2}{4} V[f_1^\varepsilon(\cdot, a, w^1)]^2\} ; V[f_1^\varepsilon(\cdot, a, w^1)] > 4V[f_2^\varepsilon(\cdot, a, w^1)] , \\ &\qquad M^\varepsilon(a, w^1) \leqq \frac{\delta(a)}{\varepsilon^{\gamma_2}} / r^\varepsilon(1, a, w^1) = a\Big] k(\varepsilon^2, (a, 0), (a, 0)) g(a) da \\ &\equiv B_1(\varepsilon, c, \gamma_2) + B_2(\varepsilon, c, \gamma_2) . \end{aligned}$$

By using same arguments as above, it is easily seen that as $\varepsilon \downarrow 0$,

$$B_2(\varepsilon, c, \gamma_2) \to 0 . \tag{6.24}$$

Furthermore there exist positive constants K_3 and K_4 such that for $a \geqq a(\varepsilon, c)$ on the $\left\{w^1 ; M^\varepsilon(a, w^1) \leqq \frac{\delta(a)}{\varepsilon^{\gamma_2}}\right\}$

$$\begin{aligned} &V[f_2^\varepsilon(\cdot, a, w^1)] \leqq K_3\, \varepsilon^2 \frac{\delta(a)^2}{\varepsilon^{2\gamma_2}} a^{p-2} \phi^\varepsilon(1, a, w^1)^{1/2} \\ &V[f_1^\varepsilon(\cdot, a, w^1)] \geqq \tilde{V}[w^1] |k'(a)| \varepsilon - |k'(a)| V[z^\varepsilon(\cdot, a, w^1)] \varepsilon^2 \\ &V[z^\varepsilon(\cdot, a, w^1)] \leqq K_1 a^{-1} \phi^\varepsilon(1, a, w^1)^{1/2} \end{aligned}$$

where

$$z^\varepsilon(t, a, w^1) = \int_0^t \frac{g'(r^\varepsilon(s, a, w^1))}{g(r^\varepsilon(s, a, w^1))} ds .$$

Combining these we can show that there exists a positive constant K_5 such that

$$\int_{a(\varepsilon,c)}^{\infty} P\Big[V[f_1^\varepsilon(\cdot, a, w^1)] \leqq 4V[f_2^\varepsilon(\cdot, a, w^1)],\ M^\varepsilon(a, w^1) \leqq \frac{\delta(a)}{\varepsilon^{\gamma_2}}$$

$$/r^\varepsilon(1, a, w^1) = a\Big] k(\varepsilon^2, (a, 0), (a, 0)) g(a) da$$

$$\leqq \int_{a(\varepsilon,c)}^{\infty} P\Big[\tilde{V}(w^1) \leqq K_5 a^{-(1-2\gamma_1)} \varepsilon^{1-2\gamma_2},\ M^\varepsilon(a, w^1) \leqq \frac{\delta(a)}{\varepsilon^{\gamma_2}}$$

$$/r^\varepsilon(1, a, w^1) = a\Big] k(\varepsilon^2, (a, 0), (a, 0)) g(a) da$$

$$\leqq \int_{a(\varepsilon,c)}^{\infty} P\Big[\tilde{V}[w^1] \leqq K_5 a^{-(1-2\gamma_2)} \varepsilon^{1-2\gamma_2}$$

$$/r^\varepsilon(1, a, w^1) = a\Big] k(\varepsilon^2, (a, 0), (a, 0)) g(a) da\,.$$

Hence by using same arguments as in the proof of V-(10.22) in [6], we can show that there exist positive constants γ_3 and K_6 such that

$$B_1(\varepsilon, c, \gamma_2) \leqq \int_{a(\varepsilon,c)}^{\infty} \exp\{-K_6(\frac{a}{\delta(a)} \varepsilon^{2\gamma_2 - 1})^{\gamma_3}\} k(\varepsilon^2, (a, 0), (a, 0)) g(a) da$$

which implies that as $\varepsilon \downarrow 0$, $B_1(\varepsilon, c, \gamma_2) \to 0$. Combining this with (6.19), we obtain that as $\varepsilon \downarrow 0$, $A_2(\varepsilon, c, \gamma_2) \to 0$. Therefore we complete the proof of following: for every $c > 0$,

$$\int_{a(\varepsilon,c)}^{\infty} E[\exp\{-\frac{\varepsilon^2}{2} V[f^\varepsilon(\cdot, a, w^1)]^2\} / r^\varepsilon(1, a, w^1) = a] k(\varepsilon^2, (a, 0), (a, 0)) g(a) da \to 0$$

as $\varepsilon \downarrow 0$. Combining this with (6.21) and (6.23), we can complete the proof of Theorem 2.3.

Acknowledgements.

I would like to express my gratitude to my advisor, Professors N. Ikeda and I. Shigekawa who guided me the direction of research and gave me numerous helpful comments. My best thanks also go to T. Mikami, M. Takeda and N. Ueki who made many suggestions in this paper at seminar.

References

[1] J. Avron, I. Herbst and B. Simon: *Schrödinger operators with magnetic fields. I. general interactions,* Duke Math. J. **45** (1978), 847–883.

[2] Y. Colin de Verdiere: *L'asymptotique de Weyl pour les bouteilles magnetiques,* Commun. Math. Phys. **105** (1986), 327–335.

[3] H.L. Cycon, R.G. Froese, W. Kirsch and B. Simon: Schrödinger Operators, Springer-Verlag, Berlin/Heidelberg/New York/London/Paris/Tokyo, 1987.

[4] T. Ikebe: Eigenvalue Problems in Mathematical Physics (in Japanese), Sangyō Tosho, Tokyo, 1976.

[5] N. Ikeda, I. Shigekawa and S. Taniguchi: *The Malliavin calculus and long time asymptotics of certain Wiener integrals*, Proc. Centre for Mathematical Analysis. Australian National Univ. **9** (1985), 46–113.

[6] N. Ikeda and S. Watanabe: Stochastic Differential Equations and Diffusion Processes. 2nd edition, Kodansha/North-Holland, Tokyo/Amsterdam, 1989.

[7] K. Itô and H.P. McKean, Jr.: Diffusion Processess and Their Sample Paths, Springer-Verlag, Berlin/Heidelberg/New York, 1974.

[8] A. Iwatsuka: *The essential spectrum of two-dimensional Schrödinger operators with perturbed constant magnetic fields,* J. Math. Kyoto University **23** (1983), 475–480.

[9] A. Iwatsuka: *Examples of absolutely continuous Schrödinger operators in magnetic fields*, RIMS, Kyoto University **21** (1985), 385–401.

[10] A. Iwatsuka: *Magnetic Schrödinger operators with compact resolvent*, J. Math. Kyoto University **26** (1986), 357–374.

[11] T. Kato: Perturbation Theory for Linear Operators, 2nd ed., Springer-Verlag, Berlin/Heidelberg/New York, 1980.

[12] P. Malliavin: *Sur certaines integrales stochastiques oscillantes*, C.R. Acad. Sci. Paris **295** (1982), 295–300.

[13] H.P. McKean, JR: *Elementary solutions for certain parabolic partial differential equations*, Trans. Amer. Math. Soc. **82** (1956), 519–548.

[14] K. Miller and B. Simon: *Quantum magnetic Hamiltonians with remarkable spectral properties*, Phys. Rev. Lett. **44** (1980), 1706–1707.

[15] M. Reed and B. Simon: Methods of Modern Mathematical Physics, volume 1, Acacemic Press, New York, 1972.

[16] M. Reed and B. Simon: Methods of Modern Mathematical Physics. volume 2, Academic Prres, New York. 1975.

[17] M. Reed and B. Simon: Methods of Modern Mathematical Physics. volume 4, Academic Press, New York, 1978.

[18] M. Schechter: Operator Methods in Quantum Mechanics, North Holland, New York/Oxford, 1981.

[19] I. Shigekawa, *Eigenvalue problems for the Schrödinger operator with the magnetic field on the compact Riemannian manifold,* J. Funct. Anal. **75** (1987), 92–127.

[20] B. Simon: Functional Integration and Quantum Physics. Acdemic Press, 1979.

[21] H. Tamura: *Asymptotic distribution of eigenvalues for Schrödinger operatirs with magnetic fields*, Nagoya Math. J. **105** (1987), 49–69.

[22] H. Tanabe: Functional Analysis (in Japanese), Jikkyo, 1978.

[23] E.C. Titchmarsh: Eigenfunction expansions associated with second-order differential equations, Part 1, Oxford Univ., Oxford, 1962.

[24] E.C. Titchmarsh: Eigenfunction expansions associated with second-order differential equations, Part 2, Clarendon Press, Oxford, 1958.

[25] S.R.S. Varadhan: *On the Behavior of the Fundamental Solution of the Heat Equation with Variable Coefficients*, Communication on pure and applied Mathematics **20** (1967), 431–455.

[26] S. Watanabe: Stochastic Differential Equations and Malliavin Calculus. Tata Institute of Fundamental Research, Springer-Verlag, Berlin/Heidelberg/New York/Tokyo, 1984.

[27] K. Yosida: Functional Analysis, Springer-Verlag, Berlin/Heidelberg/New York, 1971.

Department of Mathematics & Science Education
National Taichung Teachers College
Taichung, Taiwan,
40302 Republic of China.