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POSITIVE FORMS ON HYPERKÄHLER MANIFOLDS

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Abstract

Let (M, I, J, K, g) be a hyperkähler manifold, $\dim_{\mathbb{R}} M = 4n$. We study positive, ∂-closed $(2p, 0)$-forms on (M, I). These forms are quaternionic analogues of the positive (p, p)-forms, well-known in complex geometry. We construct a monomorphism $\mathcal{V}_{p,p}\colon \Lambda_I^{2p,0}(M) \to \Lambda_I^{n+p,n+p}(M)$, which maps ∂-closed $(2p, 0)$-forms to closed $(n+p, n+p)$-forms, and positive $(2p, 0)$-forms to positive $(n+p, n+p)$-forms. This construction is used to prove a hyperkähler version of the classical Skoda–El Mir theorem, which says that a trivial extension of a closed, positive current over a pluripolar set is again closed. We also prove the hyperkähler version of the Sibony's lemma, showing that a closed, positive $(2p, 0)$-form defined outside of a compact complex subvariety $Z \subset (M, I)$, $\operatorname{codim} Z > 2p$ is locally integrable in a neighbourhood of Z. These results are used to prove polystability of derived direct images of certain coherent sheaves.

Contents

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1. Introduction

1.1. Hypercomplex manifolds and hyperkähler manifolds. Let M be a smooth manifold, equipped with an action of the algebra

$$\mathbb{H} = \langle 1, I, J, K \mid I^2 = J^2 = IJK = -1 \rangle$$

of quaternions on its tangent bundle. Such a manifold is called *an almost hypercomplex manifold*. The operators I, J, K define three almost complex structures on M. If these almost complex structures are integrable, (M, I, J, K) is called *a hypercomplex manifold*.

Hypercomplex manifolds can be defined in terms of complex geometry, using the notion of a twistor space ([16], [26]). A scheme-theoretic definition of a hypercomplex space also exists, allowing one to define hypercomplex varieties, and even hypercomplex schemes ([26]).

Still, in algebraic geometry, the notion of a hyperkähler manifold is much more popular. A hyperkähler manifold is a hypercomplex manifold (M, I, J, K), equipped with a Riemannian form g, in such a way that g is a Kähler metric with respect to I, J and K.

Historically, these definitions were given in opposite order: Calabi defined the hyperkähler structure in 1978, and constructed one on the total space of a cotangent bundle to $\mathbb{C}P^n$ ([9]), and Boyer defined hypercomplex structures and classified compact hypercomplex manifolds in quaternionic dimension 1 in 1988 ([8]). The hyperkähler structures are much more prominent because of Calabi–Yau theorem, [35], which can be used to construct hyperkähler structures on compact, holomorphically symplectic Kähler manifolds ([7]).

Let (M, I, J, K, g) be a hyperkähler manifold. Since g is Kähler with respect to I, J, K, the manifold M is equipped with three symplectic forms:

$$\omega_I(\cdot\,, \cdot) := g(\cdot\,, I\,\cdot), \quad \omega_J(\cdot\,, \cdot) := g(\cdot\,, J\,\cdot), \quad \omega_K(\cdot\,, \cdot) := g(\cdot\,, K\,\cdot).$$

A simple linear-algebraic calculation can be used to show that the form $\Omega := \omega_J + \sqrt{-1}\omega_K$ is of Hodge type $(2, 0)$ with respect to the complex structure I (see e.g. [7]). Since Ω is also closed, it is holomorphic. This gives a holomorphic symplectic structure on a given hyperkähler manifold. Conversely, each holomorphically symplectic, compact, Kähler manifold admits a hyperkähler metric, which is unique in a given Kähler class ([7]).

In algebraic geometry, the words "hyperkähler" and "holomorphically symplectic" are used as synonyms, if applied to projective manifolds. There are papers on "hyperkähler manifolds in characteristic p" dealing with holomorphically symplectic, projective manifolds in characteristic p.

The first occurrence of hyperkähler manifolds precedes the definition given by Calabi by almost 25 years. In his work on classification of irreducible holonomy groups on Riemannian manifolds, [6], M. Berger listed, among other groups, the group of $Sp(n)$ of quaternionic unitary matrices. The holonomy of the Levi–Civita connection of a Kähler manifold preserves its complex structure (this is one of the definitions of a Kähler manifold). Therefore, the holonomy of a hyperkähler manifold preserves I, J, and K. We obtained that the holonomy group of a hyperkähler manifold lies in $Sp(n)$. The converse is also true: if the Levi–Civita connection of a Riemannian manifold M preserves a complex structure, it is Kähler (this is, again, one of the definitions of a Kähler manifold), and if it preserves an action of quaternions, it is hyperkähler.

In physics, this is often used as a definition of a hyperkähler structure.

Summarizing, there are three competing approaches to hyperkähler geometry.

(i) A hyperkähler manifold is a Riemannian manifold (M, g) equipped with almost complex structures I, J, K satisfying $I \circ J = -J \circ I = K$, such that (M, I, g), (M, J, g) and (M, K, g) are Kähler.

(ii) A hyperkähler manifold is a Riemannian manifold with holonomy which is a subgroup of $Sp(n)$.

(iii) (for compact manifolds) A hyperkähler manifold is a compact complex manifold of Kähler type, equipped with a holomorphic symplectic structure.

Returning to hypercomplex geometry, there is no hypercomplex analogue of Calabi–Yau theorem, hence no definition in terms of algebro-geometric data such as in (iii). However, hypercomplex manifold can also be characterized in terms of holonomy.

Recall that *Obata connection* on an almost hypercomplex manifold is a torsion-free connection preserving I, J and K. Obata ([19]) has shown that such a connection is unique, and exists if the almost complex structures I, J and K are integrable. The holonomy of Obata connection obviously lies in $GL(n, \mathbb{H})$. The converse is also true: if a manifold M admits a torsion-free connection preserving operators $I, J, K \in \operatorname{End}(TM)$, generating the quaternionic action,

$$I^2 = J^2 = K^2 = IJK = -\operatorname{Id}_{TM},$$

then the almost complex structure operators I, J, K are integrable. Indeed, an almost complex structure is integrable if it is preserved by some torsion-free connection.

We obtain that a hypercomplex manifold is a manifold equipped with a torsion-free connection ∇ with holonomy $\operatorname{Hol}(\nabla) \subset GL(n, \mathbb{H})$. If, in addition, the holonomy of Obata connection is a compact group, M is hyperkähler.

Some notions of complex geometry have natural quaternionic analogues in hypercomplex geometry, many of them quite useful.

By far, the most useful of these is the notion of HKT-forms, which is a quaternionic analogue of Kähler forms ([13], [3], [1]). Generalizing HKT-forms, one naturally comes across the notion of closed, positive (2, 0)-forms on a hypercomplex manifold.

1.2. Positive (2, 0)-forms on hypercomplex manifolds and quaternionic Hermitian structures. Let (M, I, J, K) be a hypercomplex manifold. We denote the space of (p, q)-forms on (M, I) by $\Lambda_I^{p,q}(M)$. The operators I and J anticommute, and therefore,

$J(\Lambda_I^{p,q}(M)) = \Lambda_I^{q,p}(M)$. The map $\eta \to J(\bar\eta)$ induces an anticomplex endomorphism of $\Lambda_I^{p,q}(M)$. Clearly,

$$J^2|_{\Lambda_I^{p,q}(M)} = (-1)^{p+q}\mathrm{Id}.$$

For $p+q$ even, $\eta \to J(\bar\eta)$ is an anticomplex involution, that is, a real structure on $\Lambda_I^{q,p}(M)$. A $(2p, 0)$-form $\eta \in \Lambda_I^{2p,0}(M)$ is called *real* if $\eta = J(\bar\eta)$. The bundle of real $(2p, 0)$-forms is denoted $\Lambda_I^{2p,0}(M, \mathbb{R})$.

The real (2, 0)-forms are most significant, because they can be interpreted as quaternionic pseudo-Hermitian structures.

Recall that a Riemannian metric g on an almost complex manifold (M, I) is called *Hermitian* if g is $U(1)$-invariant, with respect to the $U(1)$-action on TM defined by

$$t \to \cos t \cdot \mathsf{id}_{TM} + \sin t \cdot I.$$

This is equivalent to $g(I\,\cdot\,, I\,\cdot\,) = g(\,\cdot\,, \,\cdot\,)$.

When M is almost hypercomplex, it is natural to consider a group $G \subset \mathrm{End}(TM)$ generated by $U(1)$-action associated with I, J, K as above. It is easy to see that G is the group of unitary quaternions, naturally identified with $SU(2)$. Thus obtained action of $SU(2)$ on $\Lambda^*(M)$ is fundamental, and plays in hypercomplex and hyperkähler geometry the same role as played by the Hodge structures in complex algebraic geometry.

Recall that bilinear symmetric forms (not necessarily positive definite) on TM are called *pseudo-Riemannian structures*.

A (pseudo-)Riemannian structure g on an almost hypercomplex manifold (M, I, J, K) is called *quaternionic* (*pseudo-*)*Hermitian* if g is $SU(2)$-invariant. In other words, a quaternionic pseudo-Hermitian structure is a bilinear, symmetric, $SU(2)$-invariant form on TM.

Given a real (2, 0)-form $\eta \in \Lambda_I^{2,0}(M, \mathbb{R})$, consider a bilinear form

$$g_\eta(x, y) := \eta(x, Jy)$$

on TM. Since η is a (2, 0)-form, we have

$$\eta(Ix, Iy) = -\eta(x, y)$$

for all $x, y \in TM$ and therefore

$$g_\eta(Ix, Iy) = g_\eta(x, y).$$

Similarly, we obtain $g_\eta(Jx, Jy) = g_\eta(x, y)$, because $\eta(J(\bar x), J(\bar y)) = \eta(x, y)$.

Since η is skew-symmetric, and $J^2 = -1$, g_η is symmetric. We obtained that g_η is a pseudo-Hermitian form on TM. This construction is invertible (see Section 3), and gives an isomorphism between the bundle H of real (2, 0)-forms and the bundle $\Lambda_I^{2,0}(M, \mathbb{R})$ of quaternionic pseudo-Hermitian forms (Claim 3.1). The inverse iso-

$\mathbb{C}$	$\mathbb{H}$
complex manifold	hypercomplex manifold
$\Lambda^{p,p}(M,\mathbb{R})$	$\Lambda_I^{2p,0}(M,\mathbb{R})$
d, d^c	∂, ∂_J
real (1, 1)-forms	real (2, 0)-forms
closed positive definite (1, 1)-forms	HKT-forms
Kähler potentials	HKT-potentials

morphism $H \to \Lambda_I^{2,0}(M,\mathbb{R})$ is given as follows. Starting from a quaternionic pseudo-Hermitian form g, we construct 2-forms ω_I, ω_J, ω_K as in Subsection 1.1. Then $\Omega_g := \omega_J + \sqrt{-1}\omega_K$ is a real (2, 0)-form.

A real $(2,0)$-form η is called *positive definite* if the corresponding symmetric form g_η is positive definite.

There are two differentials on $\Lambda_I^{*,0}(M)$: the standard Dolbeault differential $\partial\colon \Lambda_I^{p,0}(M) \to \Lambda_I^{p+1,0}(M)$, and ∂_J, which is obtained from ∂ by twisting with $\eta \to J(\bar\eta)$. One could define ∂_J as $\partial_J(\eta) := -J\bar\partial(J\eta)$.

The pair of differentials ∂, ∂_J behaves in many ways similarly to the operators d, d^c on a complex manifold. They anticommute, and satisfy $\partial^2 = \partial_J^2 = 0$.

A positive definite (2, 0)-form $\eta \in \Lambda_I^{2,0}(M,\mathbb{R})$ is called *HKT-form* if $\partial\eta = 0$. The corresponding quaternionic Hermitian metric is called *the HKT-metric*. This notion was first defined by string physicists [17], and much studied since then (see [13] for an excellent survey of an early research).

In [3] (see also [1]), it was shown that HKT-forms locally always have a real-valued potential φ, known as HKT-potential: $\eta = \partial\partial_J\varphi$. This function is a quaternionic analogue of the Kähler potential.

We obtain the dictionary as in the above table of parallels between the complex and hypercomplex manifolds. This analogy can be built upon, to obtain the notion of positive $(2p, 0)$-forms.

1.3. Positive (2*p*, 0)-forms on hypercomplex manifolds.

DEFINITION 1.1 ([1]). A real $(2p, 0)$-form $\eta \in \Lambda_I^{2,0}(M,\mathbb{R})$ on a hypercomplex manifold is called *weakly positive* if

$$\eta(x_1, J(\bar x_1), x_2, J(\bar x_2), \ldots, x_p, J(\bar x_p)) \geqslant 0$$

for any $x_1, \ldots, x_p \in T_I^{1,0}M$, and *closed* if $\partial\eta = 0$.

In modern complex geometry, the positive, closed (p, p) forms and currents play a central role, due to several by now classical theorems, which were proven in 1960–1980-ies, building upon the ideas of P. Lelong (see [11] for an elementary exposition of the theory of positive currents).

The hypercomplex analogue of these results could be just as significant.

In [1], a hypercomplex version of the classical Chern–Levine–Nirenberg theorem was obtained. In the present paper, we prove quaternionic versions of two classical theorems, both of them quite important in complex geometry.

Theorem 1.2 ("Sibony's Lemma"). *Let (M, I, J, K, g) be a hyperkähler manifold,* $\dim_{\mathbb{R}} M = 4n$*, and $Z \subset (M, I)$ a compact complex subvariety,* $\operatorname{codim} Z > 2p$*. Consider a weakly positive, closed form $\eta \in \Lambda_I^{2p,0}(M\backslash Z, \mathbb{R})$. Then η is locally integrable around Z.*

Proof. See Theorem 5.5. □

The classical version of this theorem states that a closed, positive (p, p)-form defined outside of a complex subvariety of codimension $> p$ is integrable in a neighbourhood of this subvariety. Its proof can be obtained by slicing.

In hypercomplex geometry, the slicing is possible only on a flat manifold, because a typical hypercomplex manifold has no non-trivial hypercomplex subvarieties, even locally. In earlier versions of [28], Theorem 1.2 was proven for flat hypercomplex manifold using slicing, and then extended to non-flat manifold by approximation. The approximation argument was very unclear and ugly. In 2007, a new proof of Sibony's lemma was found ([33]), using the emerging theory of plurisubharmonic functions on calibrated manifolds ([14], [15]) instead of slicing. In Section 5, we adapt this argument to hyperkähler geometry, obtaining a relatively simple and clean proof of Theorem 1.2.

Theorem 1.2 was used in [28] to prove results about stability of certain coherent sheaves on hyperkähler manifolds (Subsection 1.4). Theorem 1.2 was used to show that the form representing $c_1(F)$ for such a sheaf is integrable. To prove theorems about stability, we need also to show that the corresponding current is closed. Then the integral of the form representing $c_1(F)$ can be interpreted in terms of the cohomology.

Given a form η on $M\backslash Z$, locally integrable everywhere on M, we can interpret η as a current on M,

$$\alpha \to \int_{M\backslash Z} \eta \wedge \alpha.$$

This current is called *a trivial extension of η to M*. A priori, it can be non-closed. However, in complex geometry, a trivial extension of an integrable, closed and positive form is again closed. This fundamental result is known as Skoda–El Mir theorem (Theorem 6.2). In Section 6, we prove a hypercomplex analogue of Skoda–El Mir theorem.

Recall that hypercomplex manifolds can be defined in terms of holonomy (Subsection 1.1), as manifolds equipped with a torsion-free connection ∇, with $\mathrm{Hol}(\nabla) \subset GL(n, \mathbb{H})$. A hypercomplex manifold (M, I, J, K) is called *an* $SL(n, \mathbb{H})$*-manifold* if its holonomy lies in $SL(n, \mathbb{H}) \subset GL(n, \mathbb{H})$. Such manifolds were studied in [31] and [4]. It was shown that (M, I, J, K) is an $SL(n, \mathbb{H})$-manifold if and only if M admits a holomorphic, real $(2n, 0)$-form. In particular, all hyperkähler manifolds satisfy $\mathrm{Hol}(\nabla) \subset SL(n, \mathbb{H})$.

Theorem 1.3. *Let (M, I, J, K) be an $SL(n, \mathbb{H})$-manifold, and $Z \subset (M, I)$ a closed complex subvariety. Consider a closed, positive form*

$$\eta \in \Lambda_I^{2p,0}(M\backslash Z, \mathbb{R}),$$

and assume that η is locally integrable around Z. Let $\tilde{\eta}$ be the current obtained as a trivial extension of η to M. Then $\partial\tilde{\eta} = 0$.

Proof. Theorem 6.3. □

Theorem 1.3 is deduced from the classical Skoda–El Mir theorem. In Subsection 4.3, we construct a map $\mathcal{V}_{p,q}\colon \Lambda_I^{p+q,0}(M) \to \Lambda_I^{n+p,n+q}(M)$, which has the following properties.

Claim 1.4. *Let $\eta \in \Lambda_I^{2p,0}(M)$ be a $(2p, 0)$-form on an $SL(n, \mathbb{H})$-manifold. Then the $(n + p, n + p)$-form $(\sqrt{-1})^p \mathcal{V}_{p,p}(\eta)$ is real (in the usual sense) if and only if η is real, positive if and only if η is positive, and closed if and only if $\partial_J \eta = \partial\eta = 0$.*

Proof. Follows immediately from Proposition 4.10. □

To prove Theorem 1.3, take $\eta \in \Lambda_I^{2p,0}(M)$ which is closed and positive. As follows from Claim 1.4, the $(n + p, n + p)$-form $(\sqrt{-1})^p \mathcal{V}_{p,p}(\eta)$ is closed and positive, in the usual complex-analytic sense. Its trivial extension is closed and positive, by the Skoda–El Mir theorem. Then $(\sqrt{-1})^p \mathcal{V}_{p,p}(\tilde{\eta})$ is closed. Applying Claim 1.4 again, we find that closedness of $(\sqrt{-1})^p \mathcal{V}_{p,p}(\tilde{\eta})$ implies that $\partial\tilde{\eta} = 0$.

1.4. Hyperholomorphic bundles and reflexive sheaves. The results about positive $(2, 0)$-forms on hypercomplex manifolds are especially useful in hyperkähler geometry. In [28], we used this notion to prove theorems about stability of direct images of coherent sheaves. The earlier arguments were unclear and flawed, and the machinery of positive $(2p, 0)$-forms was developed in order to obtain clear proofs of these results. Here we give a short sketch of main arguments used in [28]. Throughout this paper, stability of coherent sheaves is understood in Mumford–Takemoto sense.

Let (M, I, J, K) be a compact hyperkähler manifold, and B a holomorphic Hermitian bundle on (M, I). Denote the Chern connection on B by ∇. We say that B is *hyperholomorphic* if its curvature $\Theta_B \in \Lambda^2(M) \otimes \operatorname{End} B$ is $SU(2)$-invariant, with respect to the natural action of $SU(2)$ on $\Lambda^2(M)$. This notion was defined in [24], and much studied since then.

It is easy to check that $SU(2)$-invariant 2-forms are pointwise orthogonal to the Kähler form ω_I. Therefore, (B, ∇) satisfies the Yang–Mills equation $\Lambda\Theta_B = 0$.[1] In other words, ∇ is Hermitian–Einstein.

One can easily prove that Yang–Mills bundles are always *polystable*, that is, obtained as a direct sum of stable bundles of the same slope. The converse is also true: as follows from Donaldson–Uhlenbeck–Yau theorem [34], a Yang–Mills connection exists on any polystable bundle, and is unique.

In [24], it was shown that a polystable bundle on (M, I) admits a hyperholomorphic connection if and only if the Chern classes $c_1(B)$ and $c_2(B)$ are $SU(2)$-invariant.

In [25], it was shown that for any compact hyperkähler manifold (M, I, J, K) there exists a countable set

$$P \subset S^2 = \{a, b, c \mid a^2 + b^2 + c^2 = 1\}$$

with the following property. For any $(a, b, c) \notin P$, let $L := aI + bJ + cK$ be the corresponding complex structure on M induced by the quaternionic action. Then all integer (p, p)-classes on (M, L) are $SU(2)$-invariant. In particular, all stable bundles on (M, L) are hyperholomorphic.

Many of these results can be extended to reflexive coherent sheaves. Recall that a coherent sheaf F on a complex manifold X is called *reflexive* if the natural map $F \to F^{**}$ is an isomorphism. Here, F^* denotes the dual sheaf, $F^* := \operatorname{Hom}(F, \mathcal{O}_X)$. The following properties of reflexive sheaves are worth mention (see [20]).

- Holomorphic vector bundles are obviously reflexive.
- Let $Z \subset X$ be a closed complex subvariety, $\operatorname{codim} Z \geqslant 2$, and $j\colon X\backslash Z \to X$ the natural embedding. Then $j_* F$ is reflexive, for any reflexive sheaf F on $X\backslash Z$.
- The sheaf F^* is reflexive, for any coherent sheaf F.
- For any torsion-free coherent sheaf F, the natural map $F \to F^{**}$ is a monomorphism, and F^{**} is reflexive. Moreover, F^{**} is a minimal reflexive sheaf containing F.
- For any torsion-free coherent sheaf F, the singular set $\operatorname{Sing}(F)$ has codimension $\geqslant 2$. If F is reflexive, $\operatorname{Sing}(F)$ has codimension $\geqslant 3$.
- A torsion-free sheaf of rank 1 is always reflexive.
- A torsion-free sheaf F is stable if and only if F^{**} is stable.

In [27], the definition of a hyperholomorphic connection was extended to reflexive coherent sheaves, using the notion of admissible connection defined by Bando and Siu in a fundamental work [2].

---

[1] Here

$$\Lambda_I^{p,q}(M) \otimes \operatorname{End} B \to \Lambda_I^{p,q}(M) \otimes \operatorname{End} B$$

is the standard Hodge operator, which is Hermitian adjoint to $L(\eta) = \omega_I \wedge \eta$.

Let us recall what Bando and Siu did.

DEFINITION 1.5. Let (X,ω) be a Kähler manifold, $Z \subset X$ a closed complex subvariety, $\operatorname{codim} Z \geqslant 2$, and F a holomorphic vector bundle on $X\backslash Z$. Given a Hermitian metric h on F, denote by ∇ the corresponding Chern connection, and let Θ_F be its curvature. The metric h and the connection ∇ are called *admissible* if
(i) $\Lambda\Theta_F$ is uniformly bounded, where $\Lambda\colon \Lambda^{1,1}_I(M) \otimes \operatorname{End} B \to \operatorname{End} B$ is the Hodge operator, which is Hermitian adjoint to $L(\eta) = \omega_I \wedge \eta$.
(ii) The curvature Θ_F is locally L^2-integrable everywhere on M.

Bando and Siu proved the following.

- Let (X,ω) be a Kähler manifold, $Z \subset X$ a closed complex subvariety, $\operatorname{codim} Z \geqslant 2$, and F a holomorphic vector bundle on $X\backslash Z \stackrel{j}{\hookrightarrow} X$. Assume that F is equipped with an admissible connection. Then j_*F is a reflexive coherent sheaf. Conversely, any coherent sheaf admits an admissible connection outside of its singularities. Such a connection is called *an admissible connection on F*.
- A version of Donaldson–Uhlenbeck–Yau theorem is valid for stable reflexive sheaves. Let F be a reflexive sheaf on a compact Kähler manifold X. The admissible connection on F is called *Yang–Mills* if $\Lambda\Theta_F = c\mathrm{Id}_F$, where Θ_F is its curvature, and c some constant. Bando and Siu proved that a Yang–Mills connection is unique, and exists if and only if F is polystable.

In [27], these results were applied to coherent sheaves on a hyperkähler manifold (M,I,J,K,g). A *hyperholomorphic connection* on a reflexive sheaf F on (M,I) is an admissible connection with $SU(2)$-invariant curvature. Since any $SU(2)$-invariant form Θ_F satisfies $\Lambda\Theta_F = 0$, a hyperholomorphic connection is always Yang–Mills. In [27], it was shown that any polystable reflexive sheaf with $SU(2)$-invariant Chern classes $c_1(F)$, $c_2(F)$ admits a hyperholomorphic connection.

In [28], this formalism was used to prove polystability of derived direct images of hyperholomorphic bundles. Let M_1, M_2 be compact hyperkähler manifolds, and B a hyperholomorphic bundle on $M_1 \times M_2$. Denote the natural projection $M_1 \times M_2 \to M_2$ by π. It was shown that the derived direct image sheaves $R^i\pi_*B$ admit a hyperholomorphic connection, outside of their singularities. Were this connection admissible, Bando–Siu theorem would imply polystability of $R^i\pi_*B$ outright. However, L^2-integrability of its curvature is difficult to establish. In [28], we proposed a roundabout argument to prove polystability of $F := (R^i\pi_*B)^{**}$.

Let (M,I,J,K,g) be a compact hyperkähler manifold, $\dim_{\mathbb{R}} M = 4n$, and F a reflexive coherent sheaf on (M,I). Assume that outside of its singularities, F is equipped with a metric, and its Chern connection has $SU(2)$-invariant curvature. Consider a subsheaf $F_1 \subset F$. Then, outside of singularities of F, F_1, the class $-c_1(F)$ is represented by a form ν with $\nu - J(\nu)$ positive, and vanishing only if $F = F_1 \oplus F_2$. This follows from an argument which is similar to one that proves that holomorphic subbundles of a flat bundle have negative c_1: the $SU(2)$-invariance of the curvature Θ_F is equivalent to $\Theta_F - J(\Theta_F) = 0$. From positivity and non-vanishing of $\nu - J(\nu)$, one needs to infer that $\deg c_1(F_1) < 0$, which would suffice to show that F is polystable.

The expression

$$\deg c_1(F_1) = -\int_M \nu \wedge \omega_I^{2n-1} = -\frac{1}{2}\int_M (\nu - J(\nu)) \wedge \omega_I^{2n-1} \tag{1.1}$$

would have been true were the form $\nu - J(\nu)$ integrable, and closed as a current on M. However, the $(2,0)$-form Ω_ν corresponding to ν as in Section 3 is ∂-closed, because ν is closed. This form is positive, because $\nu - J(\nu)$ is positive, and Ω_ν satisfies $2\Omega_\nu = \Omega_{\nu - J\nu}$, which is clear from its construction. This form is defined outside of the set $S \subset M$ where the sheaves F, F_1 are not locally trivial. Since these sheaves are reflexive, $\operatorname{codim} S > 2$, and we could apply the hyperkähler version of Sibony's lemma (Theorem 1.2) to obtain that Ω_ν is integrable. Now, the hypercomplex version of Skoda–El Mir theorem (Theorem 1.3) implies that the trivial extension of Ω_ν is a ∂-closed current. Therefore, $\deg F_1$ can be computed through the integral (1.1). Since $\nu - J(\nu)$ is positive, this integral is negative, and strictly negative unless $F = F_1 \oplus F_2$. Therefore, F is polystable. We gave a sketch of an argument showing that $F = (R^i\pi_* B)^{**}$ is polystable. For a complete proof, please see [28].

2. Quaternionic Dolbeault complex

In this Section, we introduce the quaternionic Dolbeault complex

$$\Big(\bigoplus \Lambda^{p,q}_{I,+}, d_+\Big),$$

used further on in this paper. We follow [29].

2.1. Weights of $SU(2)$-representations. It is well-known that any irreducible representation of $SU(2)$ over $\mathbb{C}$ can be obtained as a symmetric power $S^i(V_1)$, where V_1 is a fundamental 2-dimensional representation. We say that a representation W *has weight* i if it is isomorphic to $S^i(V_1)$. A representation is said to be *pure of weight* i if all its irreducible components have weight i. If all irreducible components of a representation W_1 have weight $\leqslant i$, we say that W_1 *is a representation of weight* $\leqslant i$. In a similar fashion one defines representations of weight $\geqslant i$.

REMARK 2.1. The Clebsch–Gordan formula (see [18]) claims that the weight is *multiplicative*, in the following sense: if $i \leqslant j$, then

$$V_i \otimes V_j = \bigoplus_{k=0}^{i} V_{i+j-2k},$$

where $V_i = S^i(V_1)$ denotes the irreducible representation of weight i.

A subspace $W \subset W_1$ is *pure of weight* i if the $SU(2)$-representation $W' \subset W_1$ generated by W is pure of weight i.

2.2. Quaternionic Dolbeault complex: a definition. Let M be a hypercomplex (e.g. a hyperkähler) manifold, $\dim_{\mathbb{H}} M = n$. There is a natural multiplicative action of $SU(2) \subset \mathbb{H}^*$ on $\Lambda^*(M)$, associated with the hypercomplex structure.

REMARK 2.2. The space $\Lambda^*(M)$ is an infinite-dimensional representation of $SU(2)$, however, all its irreducible components are finite-dimensional. Therefore it makes sense to speak of *weight* of $\Lambda^*(M)$ and its sub-representations. Clearly, $\Lambda^1(M)$ has weight 1. From Clebsch–Gordan formula (Remark 2.1), it follows that $\Lambda^i(M)$ is an $SU(2)$-representation of weight $\leqslant i$. Using the Hodge $*$-isomorphism $\Lambda^i(M) \cong \Lambda^{4n-i}(M)$, we find that for $i > 2n$, $\Lambda^i(M)$ is a representation of weight $\leqslant 2n - i$.

Let $V^i \subset \Lambda^i(M)$ be a maximal $SU(2)$-invariant subspace of weight $< i$. The space V^i is well defined, because it is a sum of all irreducible representations $W \subset \Lambda^i(M)$ of weight $< i$. Since the weight is multiplicative (Remark 2.1), $V^* = \bigoplus_i V^i$ is an ideal in $\Lambda^*(M)$. We also have $V^i = \Lambda^i(M)$ for $i > 2n$ (Remark 2.2).

It is easy to see that the de Rham differential d increases the weight by 1 at most. Therefore, $dV^i \subset V^{i+1}$, and $V^* \subset \Lambda^*(M)$ is a differential ideal in the de Rham DG-algebra $(\Lambda^*(M), d)$.

DEFINITION 2.3. Denote by $(\Lambda^*_+(M), d_+)$ the quotient algebra $\Lambda^*(M)/V^*$ It is called *the quaternionic Dolbeault algebra of* M, or *the quaternionic Dolbeault complex* (qD-algebra or qD-complex for short).

The space $\Lambda^i_+(M)$ can be identified with the maximal subspace of $\Lambda^i(M)$ of weight i, that is, a sum of all irreducible sub-representations of weight i. This way, $\Lambda^i_+(M)$ can be considered as a subspace in $\Lambda^i(M)$; however, this subspace is not preserved by the multiplicative structure and the differential.

REMARK 2.4. The complex $(\Lambda^*_+(M), d_+)$ was constructed much earlier by Salamon, in a different (and much more general) situation, and much studied since then ([21], [10], [5]).

2.3. The Hodge decomposition of the quaternionic Dolbeault complex. Let (M, I, J, K) be a hypercomplex manifold, and L a complex structure induced by the quaternionic action, say, I, J or K. Consider the $U(1)$-action on $\Lambda^1(M)$ provided by $\varphi \xrightarrow{\rho_L} \cos\varphi \mathrm{Id} + \sin\varphi \cdot L$. We extend this action to a multiplicative action on $\Lambda^*(M)$. Clearly, for a (p, q)-form $\eta \in \Lambda^{p,q}(M, L)$, we have

$$\rho_L(\varphi)\eta = e^{\sqrt{-1}(p-q)\varphi}\eta. \tag{2.1}$$

This action is compatible with the weight decomposition of $\Lambda^*(M)$, and gives a Hodge decomposition of $\Lambda^*_+(M)$ ([29]).

$$\Lambda^i_+(M) = \bigoplus_{p+q=i} \Lambda^{p,q}_{+,I}(M).$$

The following result is implied immediately by the standard calculations from the theory of $SU(2)$-representations.

Proposition 2.5. *Let (M, I, J, K) be a hypercomplex manifold and*

$$\Lambda^i_+(M) = \bigoplus_{p+q=i} \Lambda^{p,q}_{+,I}(M)$$

the Hodge decomposition of qD-complex defined above. Then there is a natural isomorphism

$$\Lambda^{p,q}_{+,I}(M) \cong \Lambda^{p+q,0}(M, I). \tag{2.2}$$

Proof. See [29]. □

This isomorphism is compatible with a natural algebraic structure on $\bigoplus_{p+q=i} \Lambda^{p+q,0}(M, I)$, and with the Dolbeault differentials, in the following way.

Let (M, I, J, K) be a hypercomplex manifold. We extend

$$J : \Lambda^1(M) \to \Lambda^1(M)$$

to $\Lambda^*(M)$ by multiplicativity. Recall that

$$J(\Lambda^{p,q}(M, I)) = \Lambda^{q,p}(M, I)$$

because I and J anticommute on $\Lambda^1(M)$. Denote by

$$\partial_J : \Lambda^{p,q}(M, I) \to \Lambda^{p+1,q}(M, I)$$

the operator $J \circ \bar\partial \circ J$, where $\bar\partial : \Lambda^{p,q}(M, I) \to \Lambda^{p,q+1}(M, I)$ is the standard Dolbeault operator on (M, I), that is, the (0.1)-part of the de Rham differential. Since $\bar\partial^2 = 0$, we have $\partial_J^2 = 0$. In [29] it was shown that ∂ and ∂_J anticommute:

$$\{\partial_J, \partial\} = 0. \tag{2.3}$$

Consider the quaternionic Dolbeault complex $(\Lambda^*_+(M), d_+)$ constructed in Subsection 2.2. Using the Hodge decomposition, we can represent this complex as

$$
\begin{array}{ccccc}
 & & \Lambda^0_{+,I}(M) & & \\
 & \swarrow^{d^{1,0}_{+,I}} & & \searrow^{d^{0,1}_{+,I}} & \\
 & \Lambda^{1,0}_{+,I}(M) & & \Lambda^{0,1}_{+,I}(M) & \\
 \swarrow^{d^{1,0}_{+,I}} & & \searrow^{d^{0,1}_{+,I}} \swarrow^{d^{1,0}_{+,I}} & & \searrow^{d^{0,1}_{+,I}} \\
 \Lambda^{2,0}_{+,I}(M) & & \Lambda^{1,1}_{+,I}(M) & & \Lambda^{0,2}_{+,I}(M)
\end{array}
\tag{2.4}
$$

where $d^{1,0}_{+,I}$, $d^{0,1}_{+,I}$ are the Hodge components of the quaternionic Dolbeault differential d_+, taken with respect to I.

Theorem 2.6. *Under the isomorphism*

$$\Lambda^{p,q}_{+,I}(M) \cong \Lambda^{p+q,0}(M, I)$$

constructed in Proposition 2.5, $d^{1,0}_+$ *corresponds to* ∂ *and* $d^{0,1}_+$ *to* ∂_J:

$$
\begin{array}{ccccc}
 & & \Lambda^0_{+}(M) & & \\
 & \swarrow^{d^{0,1}_{+}} & & \searrow^{d^{1,0}_{+}} & \\
 & \Lambda^{1,0}_{+}(M) & & \Lambda^{0,1}_{+}(M) & \\
 \swarrow^{d^{0,1}_{+}} & & \searrow^{d^{1,0}_{+}} \swarrow^{d^{0,1}_{+}} & & \searrow^{d^{1,0}_{+}} \\
 \Lambda^{2,0}_{+}(M) & & \Lambda^{1,1}_{+}(M) & & \Lambda^{0,2}_{+}(M)
\end{array}
\quad \cong \quad
\begin{array}{ccccc}
 & & \Lambda^{0,0}_{I}(M) & & \\
 & \swarrow^{\partial} & & \searrow^{\partial_J} & \\
 & \Lambda^{1,0}_{I}(M) & & \Lambda^{1,0}_{I}(M) & \\
 \swarrow^{\partial} & & \searrow^{\partial_J} \swarrow^{\partial} & & \searrow^{\partial_J} \\
 \Lambda^{2,0}_{I}(M) & & \Lambda^{2,0}_{I}(M) & & \Lambda^{2,0}_{I}(M)
\end{array}
\tag{2.5}
$$

Proof. See [29] or [32]. For another proof Theorem 2.6, please see Claim 4.2. □

3. Quaternionic pseudo-Hermitian structures

Further on in this paper, we shall use some results about diagonalization of certain (2, 0)-forms associated to quaternionic pseudo-Hermitian structures. The results of this section are purely linear-algebraic and elementary. We follow [29], [30] and [1].

Let (M, I, J, K) be a hypercomplex manifold. A quaternionic pseudo-Hermitian form on M is a bilinear symmetric real-valued form g which is $SU(2)$-invariant. Equivalently,

g is quaternionic pseudo-Hermitian if

$$g(\cdot\,,\ \cdot\,) = g(I\,\cdot\,,\, I\,\cdot\,) = g(J\,\cdot\,,\, J\,\cdot\,) = g(K\,\cdot\,,\, K\,\cdot\,).$$

If g is in addition positive definite, g is called *quaternionic Hermitian*. Notice that a quaternionic Hermitian structure exists, globally, on any hypercomplex manifold. Indeed, one could take any Riemannian form, and average it with $SU(2)$

As in Subsection 1.1, we can associate three 2-forms ω_I, ω_J and ω_K with g,

$$\omega_I(\cdot\,,\ \cdot\,) = g(\cdot\,,\, I\,\cdot\,),\quad \omega_J(\cdot\,,\ \cdot\,) = g(\cdot\,,\, J\,\cdot\,),\quad \omega_K(\cdot\,,\ \cdot\,) = g(\cdot\,,\, K\,\cdot\,).$$

An easy linear-algebraic calculation shows that $\Omega_g := \omega_J + \sqrt{-1}\omega_K$ has Hodge type $(2, 0)$ under I:

$$\Omega_g \in \Lambda_I^{2,0}(M).$$

The involution $\eta \to J(\bar\eta)$ gives a real structure on $\Lambda_I^{2,0}(M)$. A $(2, 0)$-form η is called *real* if $\eta = J(\bar\eta)$. The bundle of real $(2, 0)$-forms is denoted $\Lambda_I^{2,0}(M, \mathbb{R})$. It is easy to see that the form Ω_g is real. In [30], it was shown that the converse is also true: any real $(2, 0)$-form η is obtained from a quaternionic pseudo-Hermitian form, which is determined uniquely from η.

Claim 3.1. *Let (M, I, J, K) be a hypercomplex manifold, H the bundle of quaternionic pseudo-Hermitian forms, and $\Lambda_I^{2,0}(M,\mathbb{R})$ the bundle of real $(2,0)$-forms. Consider the map $H \xrightarrow{\nu} \Lambda_I^{2,0}(M,\mathbb{R})$ constructed above, $\nu(g) = \Omega_g$. Then ν is an isomorphism, and the inverse map is determined by $g(x, \bar y) = \Omega_g(x,\, J(\bar y))$, for any $x, y \in T_I^{1,0}(M)$.*

Proof. This is Lemma 2.10, [1]. □

The standard diagonalization arguments, applied to quaternionic pseudo-Hermitian forms, give similar results about real $(2, 0)$-forms on hypercomplex manifolds.

Proposition 3.2. *Let (M, I, J, K) be a hypercomplex manifold, $\dim_{\mathbb{R}} M = 4n$, and $\eta, \eta' \in \Lambda_I^{2,0}(M,\mathbb{R})$ two real $(2,0)$-forms. Then, locally around each point, η and η' can be diagonalized simultaneously: there exists a frame $\xi_1, J(\bar\xi_1), \xi_2, J(\bar\xi_2), \ldots, \xi_n, J(\bar\xi_n) \in \Lambda_I^{1,0}(M)$, such that*

$$\eta = \sum_i \alpha_i \xi_i \wedge J(\bar\xi_i),\quad \eta' = \sum_i \beta_i \xi_i \wedge J(\bar\xi_i),$$

with α_i, β_i real-valued functions.

Proof. Follows from Claim 3.1 and a standard argument which gives a simultaneous diagonalization of two pseudo-Hermitian forms. □

In a similar spirit, the Gram–Schmidt orthogonalization procedure brings the following statement.

A real form $\eta \in \Lambda^{2,0}_I(M, \mathbb{R})$ is called *strictly positive*, if it satisfies $\eta(x, J(\bar{x})) > 0$ for any non-zero vector $x \in T^{1,0}_I(M)$.

Let $x_1, \dots, x_n \in T^{1,0}_I(M)$ be a set of vector fields. The set $\{x_i\}$ is called *orthogonal with respect to* η if

$$\eta(x_i, x_j) = \eta(x_i, J(\bar{x}_j)) = 0$$

whenever $i \neq j$.

Proposition 3.3 (Gram–Schmidt orthogonalization procedure). *Let $\eta \in \Lambda^{2,0}_I(M, \mathbb{R})$ be a real, strictly positive form on a hypercomplex manifold, and $x_1, \dots, x_n \in T^{1,0}_I(M)$ a set of vector fields, which are linearly independent everywhere. Then there exists functions $\alpha_{i,j}$, $i > j$, such that the vector fields*

$$\begin{aligned}
y_1 &:= x_1,\\
y_2 &:= x_2 + \alpha_{2,1} y_1,\\
y_3 &:= x_3 + \alpha_{3,2} y_2 + \alpha_{3,1} y_1,\\
&\dots\\
y_k &:= x_k + \sum_{i<k} \alpha_{k,i} y_i\\
&\dots
\end{aligned}$$

are orthogonal.

Proof. Use Claim 3.1 and apply the Gram–Schmidt orthogonalization to the quaternionic Hermitian form associated with η. □

4. Positive, closed $(2p, 0)$-forms

4.1. The isomorphism $\Lambda^{p+q,0}_I(M) \xrightarrow{\mathcal{R}_{p,q}} \Lambda^{p,q}_{+,I}(M)$. Let (M, I, J, K) be a hypercomplex manifold. In Proposition 2.5, an isomorphism

$$\bigoplus \Lambda^{p+q,0}_I(M) \xrightarrow{\Psi} \bigoplus \Lambda^{p,q}_{+,I}(M)$$

was constructed. As shown in [29], this isomorphism is multiplicative. It is uniquely determined by the values it takes on $\Lambda^1(M)$: on $\Lambda^{1,0}_I(M)$, Ψ is tautological, and on

$\Lambda^{0,1}_I(M)$, we have $\Psi(x) = J(x)$. This isomorphism has an explicit construction, which is given as follows.

Claim 4.1. *Let (M, I, J, K) be a hypercomplex manifold, and*

$$\mathcal{R}_{p,q} \colon \Lambda_I^{p+q,0}(M) \to \Lambda_I^{p,q}(M)$$

map a form $\eta \in \Lambda_I^{p+q,0}(M)$ to $\mathcal{R}_{p,q}(\eta)$, which is defined by

$$\mathcal{R}_{p,q}(\eta)(x_1, \ldots, x_p, \bar{y}_1, \ldots, \bar{y}_q) := \eta(x_1, \ldots, x_p, J\bar{y}_1, \ldots, J\bar{y}_q)$$

Then $\mathcal{R}_{p,q}$ is multiplicative, in the following sense:

$$\mathcal{R}_{p,q}(\eta_1 \wedge \eta_2) = \sum_{\substack{p_1+p_2=p,\\ q_1+q_2=q}} \mathcal{R}_{p_1,q_1}(\eta_1) \wedge \mathcal{R}_{p_2,q_2}(\eta_1).$$

Moreover, $\mathcal{R}_{p,q}$ induces the isomorphism

$$\bigoplus \Lambda_I^{p+q,0}(M) \xrightarrow{\psi} \bigoplus \Lambda_{+,I}^{p,q}(M)$$

constructed above.

Proof. The multiplicativity of $\mathcal{R}_{p,q}$ is clear from its definition. The isomorphism $\mathcal{R}$ is uniquely determined by the values it takes on $\Lambda^1(M)$ and multiplicativity, hence it coinsides with $\mathcal{R}_{p,q}$. □

This map also agrees with the differentials, and the anticomplex involution $\eta \to J\bar{\eta}$ acting on $\Lambda_I^{p+q,0}(M)$.

Claim 4.2. *Let (M, I, J, K) be a hypercomplex manifold, and*

$$\mathcal{R}_{p,q} \colon \Lambda_I^{p+q,0}(M) \to \Lambda_{I,+}^{p,q}(M)$$

the map constructed in Claim 4.1. *Then*

(i) $\mathcal{R}_{p,q}(J\bar{\eta}) = (-1)^{pq}\overline{\mathcal{R}_{q,p}(\eta)}$,

(ii) $\mathcal{R}_{p,q}(\partial\eta) = d_+^{1,0}\mathcal{R}_{p-1,q}(\eta)$,

(iii) $\mathcal{R}_{p,q}(\partial_J\eta) = d_+^{0,1}\mathcal{R}_{p,q-1}(\eta)$.

Proof. Claim 4.2 (i) is clear from the definition. Using Leibniz identity, we find that it suffices to check Claim 4.2 (ii) and (iii) on some set of multiplicative generators of $\bigoplus_{p,q} \Lambda_I^{p+q,0}(M)$. For functions, these identities are clear. For ∂-exact 1-forms, Claim 4.2 (ii) is clear, because $\partial^2 = 0$ and $(d_+^{1,0})^2 = 0$, hence

$$0 = \mathcal{R}_{p,q}(\partial\partial f), \quad \text{and} \quad d_+^{1,0}\mathcal{R}_{p-1,q}(\partial f) = (d_+^{1,0})^2 f = 0.$$

For a ∂-exact 1-form $\eta = \partial\psi$, with ψ a holomorphic function, Claim 4.2 (iii) follows from

$$\mathcal{R}_{p,q}(\partial_J \partial\psi) = -\mathcal{R}_{p,q}(\partial\partial_J \psi) = -\mathcal{R}_{p,q}(\partial J \bar{\partial}\psi) = 0.$$

The functions, together with 1-forms $\eta = \partial\psi$, with ψ a holomorphic function, generate the algebra $\Lambda_I^{*,0}(M)$ multiplicatively. Now, the Leibniz identity can be used to prove that Claim 4.2 (ii) and (iii) is true on the whole $\Lambda_I^{*,0}(M)$.

Please notice that we just gave a proof of Theorem 2.6. □

4.2. Strongly positive, weakly positive and real (2*p*, 0)-forms. The notion of positive $(2p, 0)$-forms on hypercomplex manifolds was developed in [1] and in ongoing collaboration with S. Alesker.

Let $\eta \in \Lambda_I^{p,q}(M)$ be a differential form. Since I and J anticommute, $J(\eta)$ lies in $\Lambda_I^{q,p}(M)$. Clearly, $J^2|_{\Lambda_I^{p,q}(M)} = (-1)^{p+q}$. For $p+q$ even, $J|_{\Lambda_I^{p,q}(M)}$ is an anticomplex involution, that is, a real structure on $\Lambda_I^{p,q}(M)$. A form $\eta \in \Lambda_I^{2p,0}(M)$ is called *real* if $J(\bar{\eta}) = \eta$. We denote the bundle of real $(2p, 0)$-forms by $\Lambda_I^{2p,0}(M, \mathbb{R})$.

For a real $(2p, 0)$-form,

$$\begin{aligned} &\eta(x_1, J(\bar{x}_1), x_2, J(\bar{x}_2), \ldots, x_p, J(\bar{x}_p)) \\ &= \bar{\eta}(J(x_1), J^2(\bar{x}_1), J(x_2), J^2(\bar{x}_2), \ldots, J(x_p), J^2(\bar{x}_p)) \\ &= \bar{\eta}(\bar{x}_1, J(x_1), \bar{x}_2, J(x_2), \ldots, \bar{x}_p, J(x_p)) \end{aligned} \tag{4.1}$$

for any $x_1, \ldots, x_p \in T_I^{1,0}(M)$. From (4.1), we obtain that the number

$$\eta(x_1, J(\bar{x}_1), x_2, J(\bar{x}_2), \ldots, x_p, J(\bar{x}_p))$$

is always real.

DEFINITION 4.3. Let (M, I, J, K) be a hypercomplex manifold, and $\eta \in \Lambda_I^{2p,0}(M)$ a real $(2p, 0)$-form. It is called *weakly positive*, if

$$\eta(x_1, J(\bar{x}_1), x_2, J(\bar{x}_2), \ldots, x_p, J(\bar{x}_p)) \geqslant 0$$

for any $x_1, \ldots, x_p \in T_I^{1,0}(M)$.

Let $\dim_{\mathbb{R}} M = 4n$. The complex line bundle $\Lambda^{2n,0}(M)$ is equipped with a real structure, hence it is a complexification of a real line bundle $\Lambda_I^{2n,0}(M, \mathbb{R})$. This real line bundle is trivial topologically. To see this, take a quaternionic Hermitian form q on M (such a form always exists: see Section 3). Let $\Omega := \omega_J + \sqrt{-1}\omega_K$ be the corresponding (2, 0)-form. Since $J\omega_J = \omega_J$, $J(\omega_K) = -\omega_K$, the form Ω is real. Then, Ω^n is a nowhere degenerate, real section which trivializes $\Lambda_I^{2n,0}(M, \mathbb{R})$.

The pairing

$$\Lambda_I^{2p,0}(M, \mathbb{R}) \times \Lambda_I^{2n-2p,0}(M, \mathbb{R}) \to \Lambda_{\mathbb{R}}^{2n,0}(M, \mathbb{R})$$

is nowhere degenerate. Denote by $\mathcal{C}_w \subset \Lambda_I^{2*,0}(M,\mathbb{R})$ the cone of weakly positive forms, and $\mathcal{C}_s \subset \Lambda_I^{2*,0}(M, \mathbb{R})$ the dual cone. This cone is called *the cone of strongly positive forms*.

This notion is well known in complex geometry; a complex analogue of the following claim is often used as a definition of strongly positive cone, and then the above definition becomes a (trivial) theorem.

Claim 4.4. *Let M be a hypercomplex manifold. The cone $\mathcal{C}_s \subset \Lambda_I^{2*,0}(M, \mathbb{R})$ of strongly positive real $(2p, 0)$-forms is multiplicatively generated by products of forms $\xi \wedge J(\bar{\xi})$, for $\xi \in \Lambda_I^{1,0}(M)$.*

Proof. A form η is weakly positive if

$$\langle \eta, \xi_1 \wedge J(\bar{\xi}_1) \wedge \xi_2 \wedge J(\bar{\xi}_2) \wedge \cdots \wedge J(\bar{\xi}_p)\rangle \geqslant 0$$

for any $\xi_1, \ldots, \xi_p \in \Lambda_I^{1,0}(M)$. Therefore, weakly positive cone is dual to the cone generated by such products. □

The strong positivity of a form implies its weak positivity. Unlike the complex case, in the quaternionic case this is not immediate from its definition.

For $p = n$, this implication can be seen as follows. For any $\xi_1, \ldots, \xi_p \in \Lambda_I^{1,0}(M)$, we have

$$\xi_1 \wedge J(\bar{\xi}_1) \wedge \xi_2 \wedge J(\bar{\xi}_2) \wedge \cdots \wedge J(\bar{\xi}_n) = \frac{1}{n!}\Omega^n,$$

where $\Omega = \sum \xi_i \wedge J(\bar{\xi}_1)$ is a $(2, 0)$-form, which is obtained from a quaternionic Hermitian form q as in Claim 3.1. The form Ω^n is positive, because for $\{\langle x_i, J(\bar{x}_i)\rangle\}$ pairwise orthogonal with respect to q, we have

$$\Omega^n(x_1, J(\bar{x}_1), \ldots, x_n, J(\bar{x}_n)) = \prod_i q(x_i, \bar{x}_i),$$

and for $\{x_i\}$ non-orthogonal, this set can be orthogonalized, without changing $\eta(x_1, J(\bar{x}_1), \ldots, x_n, J(\bar{x}_n))$, as shown in Proposition 3.3.

This gives

$$\frac{1}{n!}\Omega^n(x_1, J(\bar{x}_1), \dots, x_n, J(\bar{x}_n)) \geqslant 0 \tag{4.2}$$

For $p < n$, we restrict η to a quaternionic subspace generated by $x_1, \dots, x_p$, and find that the positivity of

$$\xi_1 \wedge J(\bar{\xi}_1) \wedge \xi_2 \wedge J(\bar{\xi}_2) \wedge \dots \wedge J(\bar{\xi}_p)(x_1, J(\bar{x}_1), x_2, J(\bar{x}_2), \dots, x_p, J(\bar{x}_p))$$

follows from (4.2).

Recall that a real (p, p)-form ρ on a complex manifold X is called *weakly positive* if

$$(-\sqrt{-1})^p \rho(x_1, \bar{x}_1, \dots, x_p, \bar{x}_p) \geqslant 0$$

for any $x_1, \dots, x_p \in T^{1,0}(X)$.

Claim 4.5. *Let (M, I, J, K) be a hypercomplex manifold, and*

$$\mathcal{R}_{p,p} \colon \Lambda_I^{2p,0}(M) \to \Lambda_I^{p,p}(M)$$

the map constructed in Subsection 4.1. *Consider a $(2p, 0)$-form $\eta \in \Lambda_I^{2p,0}(M)$. Then*

(i) *η is real if and only if $(\sqrt{-1})^p\mathcal{R}_{p,p}(\eta)$ is real (in the usual sense).*

(ii) *η is weakly positive if and only if $(\sqrt{-1})^p\mathcal{R}_{p,p}(\eta)$ is a weakly positive (p, p)-form.*

Proof. Claim 4.5 (i) is clear from the definition. Indeed,

$$\mathcal{R}_{p,p}(\eta)(x_1, \bar{x}_1, \dots, x_p, \bar{x}_p) = \eta(x_1, J(\bar{x}_1), \dots, x_p, J(\bar{x}_p)).$$

It is easy to see that a (p, p)-form ρ is real if and only if $(\sqrt{-1})^p\rho$ satisfies $\rho(x_1, \bar{x}_1, \dots, x_p, \bar{x}_p) \in \mathbb{R}$.

Claim 4.5 (ii) is also clear. Indeed,

$$\begin{aligned}&\eta(x_1, J(\bar{x}_1), x_2, J(\bar{x}_2), \dots, x_p, J(\bar{x}_p))\\ &= (-1)^{p(p-1)}\eta(x_1, x_2, \dots, x_p, J(\bar{x}_1), J(\bar{x}_2), \dots, J(\bar{x}_p)).\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathcal{R}_{p,p}(\eta)(x_1, \bar{x}_1, \dots, x_p, \bar{x}_p)\mathcal{R}_{p,p}(\eta)(x_1, \dots, x_p, \bar{x}_1, \dots, \bar{x}_p)\\ &= \eta(x_1, \dots, x_p, J(\bar{x}_1), \dots, J(\bar{x}_p)) = \eta(x_1, J(\bar{x}_1), x_2, J(\bar{x}_2), \dots, x_p, J(\bar{x}_p)).\end{aligned} \tag{4.3}$$

Then, (4.3) is non-negative if and only if η is weakly positive, and this is equivalent to $(\sqrt{-1})^p\mathcal{R}_{p,p}(\eta)$ being weakly positive, by definition of positive (p, p)-forms. □

4.3. The map $\mathcal{V}_{p,q}\colon \Lambda_I^{p+q,0}(M) \to \Lambda_I^{n+p,n+q}(M)$ on $SL(n,\mathbb{H})$-manifolds. Let (M, I, J, K) be a hypercomplex manifold, $\dim_{\mathbb{R}} M = 4n$, and

$$\mathcal{R}_{p,q}\colon \Lambda_I^{p+q,0}(M) \to \Lambda_{I,+}^{p,q}(M)$$

the isomorphism defined in Subsection 4.1. Consider the projection

$$\Lambda_I^{p,q}(M) \to \Lambda_{I,+}^{p,q}(M), \tag{4.4}$$

and let

$$R\colon \Lambda_I^{p,q}(M) \to \Lambda_I^{p+q,0}(M)$$

denote the composition of (4.4) and $\mathcal{R}_{p,q}^{-1}$.

Lemma 4.6. *In these assumptions,*

$$R(\xi_1 \wedge \cdots \wedge \xi_p \wedge \bar{\xi}_{p+1} \wedge \cdots \wedge \bar{\xi}_{p+q}) = \xi_1 \wedge \cdots \wedge \xi_p \wedge J(\bar{\xi}_{p+1}) \wedge \cdots \wedge J(\bar{\xi}_{p+q}), \tag{4.5}$$

for any $\xi_1, \ldots, \xi_{p+q} \in \Lambda_I^{1,0}(M)$.

Proof. Denote by R' the map defined by the formula (4.5). From the definition of the $SU(2)$-action on $\Lambda^*(M)$ it is apparent that $R'(\eta)$ belongs to the same $SU(2)$-representation as η. Since $R'(\eta)$ lies in $\Lambda_I^{p+q,0}(M)$, it belongs to $\Lambda_+^*(M)$. Therefore, R' vanishes on the kernel of (4.4). By definition, R is the unique map $\Lambda_I^{p,q}(M) \to \Lambda_I^{p+q,0}(M)$ vanishing on the kernel of (4.4) and satisfying

$$R \circ \mathcal{R}_{p,q} = \mathrm{Id}_{\Lambda_I^{p+q,0}(M)}.$$

To prove that $R' = R$ it suffices now to check that $R(\mathcal{R}_{p,q}(\eta)) = \eta$, but this is obvious from the definition. □

REMARK 4.7. The formula (4.5) could be used as a definition of R.

The map R is compatible with Dolbeault differentials, in the following sense.

Lemma 4.8. *Let (M, I, J, K) be a hypercomplex manifold, and*

$$R\colon \Lambda_I^{p,q}(M) \to \Lambda_I^{p+q,0}(M)$$

the map defined above. Then

$$R(\partial\eta) = \partial R(\eta), \quad \textit{and} \quad R(\bar{\partial}\eta) = \partial_J R(\eta). \tag{4.6}$$

Proof. Lemma 4.8 follows immediately from Claim 4.2 and $R\circ\mathcal{R}_{p,q} = \mathrm{Id}_{\Lambda_I^{p+q,0}(M)}$, which is a part of the definition of R. □

Let Φ_I be a nowhere degenerate holomorphic section of $\Lambda_I^{2n,0}(M)$. Assume that Φ_I is real, that is, $J(\Phi_I) = \bar{\Phi}_I$, and positive.

Existence of such a section is highly non-trivial. When M is hyperkähler, we could take the top power of the holomorphic symplectic form $\Omega = \omega_J + \sqrt{-1}\omega_K$. For a general hypercomplex M, such a form Φ_I is preserved by the Obata connection, and reduces the holonomy of Obata connection to a subgroup of $SL(n, \mathbb{H})$. Such manifolds were studied in [31] and [4].

A manifold with a nowhere degenerate, real, positive form $\Phi_I \in \Lambda_I^{2n,0}(M)$ is called *an $SL(n, \mathbb{H})$-manifold.*

REMARK 4.9. Let (M, I, J, K, Φ_I) be an $SL(n, \mathbb{H})$-manifold. For any section $\eta \in \Lambda_I^{2n,0}(M)$, positivity of η in the quaternionic sense is equivalent to positivity of $\eta \wedge \Phi_I \in \Lambda_I^{2n,2n}(M)$, in the usual sense.

Define the map

$$\mathcal{V}_{p,q}\colon \Lambda_I^{p+q,0}(M) \to \Lambda_I^{n+p,n+q}(M)$$

by the relation

$$\mathcal{V}_{p,q}(\eta) \wedge \alpha = \eta \wedge R(\alpha) \wedge \bar{\Phi}_I \tag{4.7}$$

for any test form $\alpha \in \Lambda_I^{n-p,n-q}(M)$.

The map $\mathcal{V}_{p,p}$ is especially remarkable, because it maps closed, positive $(2p, 0)$-forms to closed, positive $(n + p, n + p)$-forms, as the following proposition implies.

Proposition 4.10. *Let (M, I, J, K, Φ_I) be an $SL(n, \mathbb{H})$-manifold, and*

$$\mathcal{V}_{p,q}\colon \Lambda_I^{p+q,0}(M) \to \Lambda_I^{4n-p,4n-q}(M)$$

be the map defined above. Then

(i) $\mathcal{V}_{p,q}(\eta) = \mathcal{R}_{p,q}(\eta) \wedge \mathcal{V}_{0,0}(1)$.

(ii) *The map $\mathcal{V}_{p,q}$ is injective, for all p, q.*

(iii) $(\sqrt{-1})^{(n-p)^2}\mathcal{V}_{p,p}(\eta)$ *is real if and only $\eta \in \Lambda_I^{2p,0}(M)$ is real, and weakly positive if and only if η is weakly positive.*

(iv) $\mathcal{V}_{p,q}(\partial\eta) = \partial\mathcal{V}_{p-1,q}(\eta)$, *and* $\mathcal{V}_{p,q}(\partial_J\eta) = \bar{\partial}\mathcal{V}_{p,q-1}(\eta)$.

(v) $\mathcal{V}_{0,0}(1) = \lambda\mathcal{R}_{n,n}(\Phi_I)$, *where λ is a positive rational number, depending only on the dimension n.*

Proof. The map $R\colon \Lambda_I^{p,q}(M) \to \Lambda_I^{p+q,0}(M)$ is by construction multiplicative, and satisfies

$$R(\mathcal{R}_{p,q}(\eta)) = \eta \tag{4.8}$$

for all $\eta \in \Lambda_I^{p+q,0}(M)$. This gives

$$\mathcal{V}_{p,q}(\eta) \wedge \alpha = \eta \wedge R(\alpha) \wedge \Phi_I = R(\mathcal{R}_{p,q}(\eta) \wedge \alpha) \wedge \Phi_I = \mathcal{V}_{0,0}(1) \wedge \mathcal{R}_{p,q}(\eta) \wedge \alpha \tag{4.9}$$

(to obtain the last equation, we take the test-form $\alpha' := \mathcal{R}_{p,q}(\eta) \wedge \alpha$ and apply (4.7)). Since α is arbitrary, (4.9) gives

$$\mathcal{V}_{p,q}(\eta) = \mathcal{V}_{0,0}(1) \wedge \mathcal{R}_{p,q}(\eta).$$

This proves Proposition 4.10 (i).

Injectivity of $\mathcal{V}_{p,q}$ is clear, because for any $\eta \in \Lambda_I^{p+q,0}(M)$ there exists χ such that $\eta \wedge \chi \wedge \Phi_I \neq 0$. Using (4.8), we find that

$$\mathcal{V}_{p,q}(\eta) \wedge \mathcal{R}_{n-p,n-q}(\chi) = \eta \wedge R(\mathcal{R}_{n-p,n-q}(\chi)) \wedge \Phi_I = \eta \wedge \chi \wedge \Phi_I \neq 0.$$

We proved Proposition 4.10 (ii).

From Claim 4.2 (i), we obtain that $R(\bar{\alpha}) = (-1)^{pq} R(\alpha)$, for any $\alpha \in \Lambda_I^{p,q}(M)$. Then

$$\mathcal{V}_{p,q}(J\bar{\eta}) = (-1)^{(n-p)(n-q)} \overline{\mathcal{V}_{q,p}(\eta)}$$

as follows from (4.7). Then, $(\sqrt{-1})^p \mathcal{V}_{p,p}(\eta)$ is real if $J\bar{\eta} = \eta$. The “only if” part follows from injectivity of $\mathcal{V}_{p,p}$.

To check the weak positivity of $(\sqrt{-1})^p \mathcal{V}_{p,p}$, take $\alpha = \xi_1 \wedge \bar{\xi}_1 \wedge \cdots \wedge \xi_{n-p} \wedge \xi_{n-p}$, with $\xi_1, \ldots, \xi_{n-p} \in \Lambda_I^{1,0}(M)$. Then $(-\sqrt{-1})^{n-p}\alpha$ is positive. Such forms generate the strongly positive cone. Then $R(\alpha) = \xi_1 \wedge J(\bar{\xi}_1) \wedge \cdots \wedge \xi_{n-p} \wedge J(\bar{\xi}_{n-p})$ is strongly positive by definition, and, moreover, $R(\alpha)$, for all such α, generate the strongly positive cone.

The weak positivity of $(-\sqrt{-1})^{n-p} \mathcal{V}_{p,q}(\eta)$ is equivalent to

$$(-\sqrt{-1})^{n-p} \mathcal{V}_{p,q}(\eta) \wedge \alpha \geqslant 0,$$

and the weak positivity of η is equivalent to

$$\eta \wedge R(\alpha) \wedge \bar{\Phi}_I \geqslant 0.$$

These two inequalities are equivalent by the formula (4.7) which is a definition of $\mathcal{V}_{p,q}(\eta)$. We proved Proposition 4.10 (iii).

Proposition 4.10 (iv) follows from the Stokes' formula

$$\int_M \partial\alpha \wedge \beta = (-1)^{\deg\alpha} \int_M \alpha \wedge \partial\beta,$$

where α or β have compact support.

Take an $(n-q, n-p)$-form α with compact support. By Lemma 4.8,

$$\begin{aligned}\int_M \mathcal{V}_{p,q}(\partial\eta) \wedge \alpha &= \int_M \partial\eta \wedge R(\alpha) \wedge \bar{\Phi}_I = (-1)^{p+q-1} \int_M \eta \wedge \partial R(\alpha) \wedge \bar{\Phi}_I \\ &= (-1)^{p+q-1} \int_M \eta \wedge R(\partial\alpha) \wedge \bar{\Phi}_I \\ &= (-1)^{p+q-1} \int_M \mathcal{V}_{p-1,q}(\eta) \wedge \partial\alpha \\ &= \int_M \partial\mathcal{V}_{p-1,q}(\eta) \wedge \alpha.\end{aligned}$$

Applying complex conjugation to both sides of $\mathcal{V}_{p,q}(\partial\eta) = \partial\mathcal{V}_{p-1,q}(\eta)$ and using

$$\mathcal{V}_{p,q}(J\bar{\eta}) = (-1)^{(n-p)(n-q)}\overline{\mathcal{V}_{q,p}(\eta)}$$

and $J\bar{\partial}\eta = \partial_J J(\bar{\eta})$, we obtain the second equation of Proposition 4.10 (iv).

Proposition 4.10 (v) follows from a direct (but tedious) linear-algebraic calculation. The bundle $\Lambda^{n,n}_{I,+}(M)$ is 1-dimensional, by Proposition 2.5. The form $\mathcal{V}_{0,0}(1)$ lies in $\Lambda^{n,n}_{I,+}(M)$. Indeed,

$$\mathcal{V}_{0,0}(1) \wedge \alpha = R(\alpha) \wedge \bar{\Phi}_I,$$

and therefore $\alpha \to \mathcal{V}_{0,0}(1) \wedge \alpha$ vanishes on all forms of weight less than $2n$. Therefore, $\mathcal{V}_{0,0}(1)$ has weight $2n$, hence belongs to $\Lambda^{n,n}_{I,+}(M)$. The form $\mathcal{R}_{n,n}(\Phi_I)$ is a nowhere degenerate section of $\Lambda^{n,n}_{I,+}(M)$, by construction; therefore, $\mathcal{V}_{0,0}(1)$ is proportional to $\mathcal{R}_{n,n}(\Phi_I)$:

$$\mathcal{V}_{0,0}(1) = \lambda\mathcal{R}_{n,n}(\Phi_I),$$

where λ is a smooth function on M. To prove Proposition 4.10 (v), we need to show that λ is a positive rational number depending only from n. Since $(\sqrt{-1})^n\mathcal{R}_{n,n}(\Phi_I)$ and $(\sqrt{-1})^n\mathcal{V}_{0,0}(1)$ are both real and positive, by Proposition 4.10 (iii) and Claim 4.5, λ is real and positive. Taking $\alpha = \Phi_I$ and applying (4.7), we obtain

$$1 \wedge \Phi_I \wedge \bar{\Phi}_I = R(\mathcal{R}_{n,n}(\Phi_I)) \wedge \bar{\Phi}_I = \mathcal{V}_{0,0}(1) \wedge \mathcal{R}_{n,n}(\Phi_I) = \lambda\mathcal{R}_{n,n}(\Phi_I)) \wedge \mathcal{R}_{n,n}(\Phi_I).$$

This gives an expression for λ:

$$\lambda = \frac{\Phi_I \wedge \bar{\Phi}_I}{\mathcal{R}_{n,n}(\Phi_I) \wedge \mathcal{R}_{n,n}(\Phi_I)}.$$

From this formula, it is clear that λ is independent from the choice of Φ_I. Therefore, we may assume that Φ_I is associated with a quaternionic Hermitian form q as above: $\Phi_I = \Omega^n$, where $\Omega = \omega_J + \sqrt{-1}\omega_K$, and ω_J, ω_K are the Hermitian skew-linear forms of (M, J) and (M, K). From the definition of $\mathcal{R}_{p,q}$, it is clear that $\mathcal{R}_{1,1}(\Omega) = \omega_I$. Using multiplicativity of $\mathcal{R}_{p,p}$, we obtain

$$\mathcal{R}_{n,n}(\Omega^n) = \Pi_+(\mathcal{R}_{1,1}(\Omega)^n) = \Pi_+(\omega_I^n),$$

where Π_+ is the $SU(2)$-invariant projection to the $\Lambda^*_+(M)$-part. Since the metric on $\Lambda^*(M)$ is $SU(2)$-invariant, the weight decomposition of $\Lambda^*(M)$ is orthogonal; therefore, Π_+ is an orthogonal projection to $\Lambda^*_+(M)$.

Consider the algebra $A^* = \bigoplus A^{2i}$ generated by ω_I, ω_J, and ω_K. In [25], this algebra was computed explicitly. It was shown, that, up to the middle degree, A^* is a symmetric algebra with generators ω_I, ω_J, ω_K. The algebra A^* has Hodge bigrading $A^k = \bigoplus_{p+q=k} A^{p,q}$, and its $A^{p,p}$-part is generated by the forms

$$\omega_I^i \wedge (\Omega \wedge \bar{\Omega})^j,$$

$i, j = 0, 1, 2, \ldots$. From the Clebsch–Gordan formula, we obtain that $A^{2i}_+ := \Lambda^{2i}_+(M) \cap A^{2i}$, for $i \leqslant n$, is an orthogonal complement to $Q(A^{2i-4})$, where $Q(\eta) = \eta \wedge (\omega_I^2 + \omega_J^2 + \omega_K^2)$. The space $A^{n,n}_+ = \ker Q^*|_{A^{n,n}}$ is 1-dimensional, as we have shown above, and generated by $\mathcal{R}_{n,n}(\Omega^n)$. Clearly,

$$Q^*(\omega_I^i \wedge (\Omega \wedge \bar{\Omega})^j) = \omega_I^{i-2} \wedge (\Omega \wedge \bar{\Omega})^j + \omega_I^i \wedge (\Omega \wedge \bar{\Omega})^{j-2}.$$

Therefore, $\ker Q^*|_A^{n,n}$ is generated by

$$\Xi := \omega_I^n - \omega_I^{n-2} \wedge (\Omega \wedge \bar{\Omega}) + \omega_I^{n-4} \wedge (\Omega \wedge \bar{\Omega})^2 - \omega_I^{n-6} \wedge (\Omega \wedge \bar{\Omega})^3 + \cdots . \tag{4.10}$$

Since $\mathcal{R}_{n,n}(\Omega^n)$ is equal to the projection of ω_I^n to $\ker Q^*$, this gives

$$\mathcal{R}_{n,n}(\Omega^n) = \Xi \cdot \frac{(\omega_I^n, \Xi)}{(\Xi, \Xi)} = \gamma\,\Xi,$$

where γ is a rational coefficient which can be expressed through binomial coefficients using (4.10). A similar calculation can be used to express

$$\lambda = \frac{\Phi_I \wedge \bar{\Phi}_I}{\mathcal{R}_{n,n}(\Phi_I) \wedge \mathcal{R}_{n,n}(\Phi_I)} = \frac{\Omega^n \wedge \bar{\Omega}^n}{\gamma^2\,\Xi \wedge \Xi}$$

through a combinatorial expression which would take half a page. □

5. Sibony's lemma for positive $(2p, 0)$-forms

5.1. ω^q-positive $(1, 1)$-forms. Recall that a real (p, p)-form η on a complex manifold is called *weakly positive* if for any complex subspace $V \subset T_c M$, $\dim_{\mathbb{C}} V = p$, the restriction $\rho|_V$ is a non-negative volume form. Equivalently, this means that

$$(\sqrt{-1})^p \rho(x_1, \bar{x}_1, x_2, \bar{x}_2, \ldots, x_p, \bar{x}_p) \geqslant 0$$

for any vectors $x_1, \ldots, x_p \in T_x^{1,0} M$. A form is called *strongly positive* if it can be expressed as a sum

$$\eta = (\sqrt{-1})^p \sum_{i_1, \ldots, i_p} \alpha_{i_1, \ldots, i_p} \xi_{i_1} \wedge \bar{\xi}_{i_1} \wedge \cdots \wedge \xi_{i_p} \wedge \bar{\xi}_{i_p},$$

running over some set of p-tuples $\xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_p} \in \Lambda^{1,0}(M)$, with $\alpha_{i_1, \ldots, i_p}$ real and non-negative functions on M.

The strongly positive and the weakly positive forms form closed, convex cones in the space $\Lambda^{p,p}(M, \mathbb{R})$ of real (p, p)-forms. These two cones are dual with respect to the Poincare pairing

$$\Lambda^{p,p}(M, \mathbb{R}) \times \Lambda^{n-p,n-p}(M, \mathbb{R}) \to \Lambda^{n,n}(M, \mathbb{R})$$

where $n = \dim_{\mathbb{C}} M$. For $(1, 1)$-forms and $(n-1, n-1)$-forms, the strong positivity is equivalent to weak positivity.

DEFINITION 5.1. Let (M, ω) be a Kähler manifold. A real $(1, 1)$-form $\eta \in \Lambda^{1,1}(M, \mathbb{R})$ is called ω^q-positive if $\omega^{q-1} \wedge \eta$ is a weakly positive form.

This notion was studied in [33], in connection with plurisubharmonic functions on calibrated manifolds ([14], [15]). In [33], a characterization of ω^q-positivity in terms of the eigenvalues was obtained. At each point $x \in M$, we can find an orthonormal basis $\xi_1, \ldots, \xi_n \in \Lambda_x^{1,0}(M)$, such that

$$\eta = -\sqrt{-1} \sum_i \alpha_i \xi_i \wedge \bar{\xi}_i.$$

The numbers α_i are called *the eigenvalues* of η at x.

The following theorem was proven in [33].

Theorem 5.2. *Let (M, ω) be a Kähler manifold, and $\eta \in \Lambda^{1,1}(M, \mathbb{R})$ a real $(1, 1)$-form. Let $\alpha_1(x), \alpha_2(x), \ldots, \alpha_n(x)$ denote the eigenvalues of η at $x \in M$. Then the following conditions are equivalent.*

(i) *η is ω^q-positive.*

(ii) $\eta \wedge \omega^{q-1}$ *is weakly positive.*
(iii) $\eta \wedge \omega^{q-1}$ *is strongly positive.*
(iv) *The sum of any q eigenvalues of η is positive, for any $x \in M$:*

$$\sum_{k=1}^{q} \alpha_{i_k}(x) \geqslant 0 \tag{5.1}$$

for any q-tuple $\{i_1, \dots, i_q\} \subset \{1, 2, \dots, n\}$.

Proof. This is [33], Theorem 2.4. In [33], this statement was stated for forms $\eta = dd^c\varphi$, but the proof is purely linear-algebraic, and can be extended to arbitrary (1, 1)-forms. □

DEFINITION 5.3. A form η is called *strictly ω^q-positive*, if $\eta - h\omega$ is ω^q-positive, for some continuous, nowhere vanishing, positive function h on M.

5.2. Positive $(2p, 0)$-forms on hypercomplex manifolds. Let (M, I, J, K) be a hypercomplex manifold. In Subsection 4.2, a notion of positivity for $(2p, 0)$-forms on M was defined. We say that a real (2, 0)-form η is Ω^q-positive if $\eta \wedge \Omega^{q-1}$ is positive, and *strictly positive* if $\eta \wedge \Omega^{q-1} - h\Omega^q$ is positive, for some continuous, nowhere vanishing, positive function h on M.

As shown in Claim 3.1, quaternionic pseudo-Hermitian forms are in (1, 1)-correspondence with real (2, 0)-forms. This allows one to diagonalize a given (2, 0)-form η locally in an orthonormal frame (Proposition 3.2).

Given a real (2, 0)-form η on a hyperkähler manifold, at any point $x \in M$ there exists an orthonormal frame $\xi_1, J\bar{\xi}_1, \dots, \xi_n, J\bar{\xi}_n \in \Lambda^{1,0}_I(M)$, such that $\eta|_x$ is written as

$$\eta|_x = \sum_i \alpha_i \xi_1 \wedge J\bar{\xi}_1,$$

with α_i being real-valued functions. The condition of Ω^q-positivity is equivalent to the inequality

$$\sum_{k=1}^{q} \alpha_{i_k}(x) \geqslant 0, \tag{5.2}$$

just like in Theorem 5.2.

Given a (1, 1)-form $\eta \in \Lambda^{1,1}_I(M)$, consider a (2, 0)-form $R(\eta) \in \Lambda^{2,0}_I(M)$,

$$R(\eta)(x, y) := \eta(x, J(y)).$$

Clearly, $R(\eta)$ is real and positive if η is real and positive. It is easy to see that R vanishes on $SU(2)$-invariant forms, and induces an isomorphism $\Lambda^{1,1}_{+,I}(M) \to \Lambda^{2,0}_I(M)$ described in Claim 4.1 (see Lemma 4.6 for a detailed argument).

Lemma 5.4. *Let M be a hyperkähler manifold,* $\dim_{\mathbb{R}} M = 4n$, *and* $\eta \in \Lambda^{1,1}(M, \mathbb{R})$ *a real* $(1, 1)$*-form, which is* ω^{2n-2p}*-positive. Then* $R(\eta)$ *is* Ω^{n-p}*-positive.*

Proof. Denote by η' the $(1,1)$-form $\eta - \eta_{\text{inv}}$, where $\eta_{\text{inv}} = (1/2)(\eta + J(\eta))$ denotes the $SU(2)$-invariant part of η. Clearly,

$$\eta' = \frac{1}{2}(\eta - J(\eta)).$$

Since $-J(\eta)$ has the same eigenvalues as η, by Theorem 5.2 (iv) it is also ω^{2n-2p}-positive. Then η' is ω^{2n-2p}-positive, too.

Using the orthonormal frame as in the proof of (5.2), we find that η' can be written as

$$\eta' = -\sqrt{-1} \sum_i \alpha_i \xi_i \wedge \bar{\xi}_i,$$

with ξ_i an orthonormal basis in $\Lambda_I^{1,0}(M)$ satisfying

$$J(\xi_{2i-1}) = \bar{\xi}_{2i}, \quad J(\xi_{2i}) = -\bar{\xi}_{2i-1}$$

(see Proposition 3.2). Since $J(\eta') = -\eta'$, the eigenvalues of η' occur in pairs:

$$\alpha_{2i-1} = \alpha_{2i}. \tag{5.3}$$

Renumbering the basis, we may assume that $\alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_{2n}$. Now, ω^{2n-2p}-positivity of η' is equivalent to

$$\alpha_1 + \alpha_2 + \cdots + \alpha_{2n-2p} \geqslant 0. \tag{5.4}$$

By definition

$$R(\eta') = 2 \sum_i \alpha_{2i} \xi_{2i-1} \wedge \xi_{2i},$$

hence (5.2) implies that Ω^{n-p}-positivity of $R(\eta')$ is equivalent to $\alpha_2 + \alpha_4 + \cdots + \alpha_{2n-2p} \geqslant 0$. From (5.3), this is equivalent to (5.4). We proved Lemma 5.4. □

5.3. ω^q-positive forms in a neighbourhood of a subvariety. Now we can prove the hypercomplex version of Sibony's lemma.

Theorem 5.5. *Let M be a hyperkähler manifold, $Z \subset (M, I)$ a compact complex subvariety,* $\operatorname{codim}_{\mathbb{C}} Z \geqslant 3$, *and* $\eta \in \Lambda^{2,0}(M \backslash Z, I)$ *a real and positive form, which satisfies* $\partial\eta = 0$. *Then η is locally integrable everywhere in M.*

Proof. We adapt to hypercomplex situation the coordinate-free proof of the complex-analytic version of Sibony's lemma, obtained in [33]. In [33], the following result was proven.

Proposition 5.6. *Let M be a Kähler manifold, and $Z \subset M$ a complex subvariety, $\dim_{\mathbb{C}} Z < p$. Then there exists an open neighbourhood U of Z, and a sequence $\{\rho_i\}$ of ω^p-positive, exact, smooth $(1,1)$-forms on U satisfying the following.*
(i) *For any open subset $V \subset U$, with the closure $\bar{V}$ compact and not intersecting Z, the restriction $\rho_i|_V$ stabilizes as $i \to \infty$. Moreover, $\rho_i|_V$ is strictly ω^p-positive for $i \gg 0$.*
(ii) *For all i, $\rho_i = 0$ in some neighbourhood of Z.*
(iii) *The limit $\rho = \lim \rho_i$ is a strictly ω^p-positive current on U.*
(iv) *The forms ρ_i can be written as $\rho_i = dd^c\varphi_i$, where φ_i are smooth functions on U. On any compact set not intersecting Z, the sequence $\{\varphi_i\}$ stabilizes as $i \to \infty$.*

Proof. This is [33], Proposition 5.3. □

We apply Proposition 5.6 to prove Theorem 5.5. Let φ_i be the sequence of functions defined in a neighbourhood $U \supset Z$ and satisfying conditions of Proposition 5.6. From Lemma 4.8, we obtain

$$R(\partial\bar{\partial}\varphi_i) = \partial J(\bar{\partial}\varphi_i). \tag{5.5}$$

Therefore, $R(\rho_i)$ is ∂-closed. By Lemma 5.4, this form is also Ω^{n-1}-positive. Since η is positive, to show that η is locally integrable on an open set $U \subset M$, it suffices to prove that the integral

$$\int_D \eta \wedge \Omega^{n-1} \wedge \bar{\Omega}^n \tag{5.6}$$

is universally bounded, for any compact subset $D \subset U \backslash Z$. Indeed,

$$\int_D \eta \wedge \Omega^{n-1} \wedge \bar{\Omega}^n = \sum_i \int_D \alpha_i \operatorname{Vol}_M,$$

where $\{\alpha_i\}$ are the eigenvalues of η considered as functions on M. In (5.6), we may replace Ω^{n-1} by any strictly positive real $(n-1)$-form, and if this integral us bounded, (5.6) is also bounded. Therefore, Theorem 5.5 would follow from a universal bound on

$$\int_D \eta \wedge \rho \wedge \Omega^{n-2} \wedge \bar{\Omega}^n,$$

where $\rho = \lim R(\rho_i)$ is the form constructed in Proposition 5.6 (it is smooth outside of Z, because $\{\rho_i\}$ stabilizes). Now, a universal bound on $\int_D \eta \wedge \rho \wedge \Omega^{n-2} \wedge \bar{\Omega}^n$ would obviously follow from a universal bound on the integral

$$\int_D \eta \wedge R(\rho_i) \wedge \Omega^{n-2} \wedge \bar{\Omega}^n;$$

this integral is bounded by

$$\int_U \eta \wedge R(\rho_i) \wedge \Omega^{n-2} \wedge \bar{\Omega}^n,$$

because the forms η and $R(\rho_i) \wedge \Omega^{n-2}$ are positive.[2]

The last integral can be expressed by Stokes' theorem as

$$\int_U \eta \wedge R(\rho_i) \wedge \Omega^{n-2} \wedge \bar{\Omega}^n = \int_{\partial U} \eta \wedge J(\bar{\varphi}_i) \wedge \Omega^{n-2} \wedge \bar{\Omega}^n \tag{5.7}$$

(see (5.5)). However, the integral $\int_{\partial U} \eta \wedge J(\bar{\varphi}_i) \wedge \Omega^{n-2} \wedge \bar{\Omega}^n$ stabilizes as $i \to \infty$, because φ_i stabilizes in a neighbourhood of ∂U. This shows that (5.6) is universally bounded. We proved Theorem 5.5. □

6. Skoda–El Mir theorem for hyperkähler manifolds

We are going to prove a hypercomplex analogue of the classical Skoda–El Mir theorem ([12], [23], [22], [11]).

DEFINITION 6.1. Let M be a connected complex manifold, and $Z \subset M$ a closed subset. Assume that there exists a nonconstant plurisubharmonic function $\varphi\colon M \to [-\infty, \infty[$, such that $Z \subset \varphi^{-1}(-\infty)$. Then Z is called *pluripolar*.

Skoda–El Mir theorem is a result about extending a closed positive current over a pluripolar set Z.

Theorem 6.2 ([12], [23], [22], [11]). *Let X be a complex manifold, and Z a closed pluripolar set in X. Consider a closed positive current Θ on $X \backslash Z$ which is locally integrable around Z. Then the trivial extension of Θ to X is closed on X.*

The hypercomplex analogue of this theorem goes as follows.

Theorem 6.3. *Let M be a $SL(n, \mathbb{H})$-manifold, $Z \subset (M, I)$ a pluripolar set, and $\eta \in \Lambda^{2p,0}(M\backslash Z, I)$ a form satisfying the following properties.*
(i) $\eta = J(\bar{\eta})$ (*reality*),

[2] The product $\eta \wedge R(\rho_i) \wedge \Omega^{n-2}$ is well defined on the whole U, because $R(\rho_i)$ vanishes in a neighbourhood of Z.

(ii) $\eta(x_1, J(\bar{x}_1), x_2, J(\bar{x}_2), \ldots, x_p, J(\bar{x}_p)) \geqslant 0$ (*weak positivity*),
(iii) $\partial\eta = 0$ (*closedness*).
Assume that η is integrable around each point $z \in Z$. Then the trivial extension of η to M is a ∂-closed $(2p, 0)$-current.

Proof. To prove Theorem 6.3, we could repeat the argument proving the Skoda–El Mir theorem in the hypercomplex setting. However, it is much easier to deduce Theorem 6.3 from the classical Skoda–El Mir. Consider the (p, p)-form $\mathcal{R}_{p,p}(\eta) \in \Lambda_I^{p,p}(M)$ obtained as

$$\mathcal{R}_{p,p}(\eta)(x_1, \bar{y}_1, \ldots, x_p, \bar{y}_p) = \eta(x_1, J(\bar{y}_1), \ldots, x_p, J(\bar{y}_p)).$$

where $x_i, y_i \in T^{1,0}(M)$ (see Subsection 4.1).

From Proposition 4.10, it follows that the $(n+p, n+p)$-form $\mathcal{R}_{p,p}(\Omega^n) \wedge \mathcal{R}_{p,p}(\eta)$ is positive in the usual sense if and only if η is positive in the quaternionic sense, and closed if and only if $\partial\eta = 0$. Now, η is closed and positive on $M \backslash Z$, hence $\mathcal{R}_{p,p}(\Omega^n) \wedge \mathcal{R}_{p,p}(\eta)$ is closed and positive on $M \backslash Z$ (in the usual sense). Applying the Skoda–El Mir theorem, we obtain that a trivial extension of $\mathcal{R}_{p,p}(\Omega^n) \wedge \mathcal{R}_{p,p}(\eta)$ is closed on M. Applying Proposition 4.10 again, we find that the trivial extension of η to M is ∂-closed. We proved Theorem 6.3. □

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References

[1] S. Alesker and M. Verbitsky: *Plurisubharmonic functions on hypercomplex manifolds and HKT-geometry*, J. Geom. Anal. **16** (2006), 375–399.

[2] S. Bando and Y.-T. Siu: *Stable sheaves and Einstein-Hermitian metrics*; in Geometry and Analysis on Complex Manifolds, World Sci. Publ., River Edge, NJ., 1994, 39–50

[3] B. Banos and A. Swann: *Potentials for hyper-Kähler metrics with torsion*, Classical Quantum Gravity **21** (2004), 3127–3135.

[4] M.L. Barberis, I.G. Dotti and M. Verbitsky: *Canonical bundles of complex nilmanifolds, with applications to hypercomplex geometry*, Math. Res. Lett. **16** (2009), 331–347.

[5] R.J. Baston: *Quaternionic complexes*, J. Geom. Phys. **8** (1992), 29–52.

[6] M. Berger: *Sur les groupes d'holonomie homogène des variétés à connexion affine et des variétés riemanniennes*, Bull. Soc. Math. France **83** (1955), 279–330.

[7] A.L. Besse: Einstein Manifolds, Springer, Berlin, 1987.

[8] C.P. Boyer: *A note on hyperhermitian four-manifolds*, Proc. Amer. Math. Soc. **102** (1988), 157–164.

[9] E. Calabi: *Métriques kählériennes et fibrés holomorphes*, Ann. Sci. École Norm. Sup. (4) **12** (1979), 269–294.

[10] M.M. Capria and S.M. Salamon: *Yang-Mills fields on quaternionic spaces*, Nonlinearity **1** (1988), 517–530.

[11] J.-P. Demailly: *L^2 vanishing theorems for positive line bundles and adjunction theory*; in Transcendental Methods in Algebraic Geometry (Cetraro, 1994), Lecture Notes in Math. **1646**, Springer, Berlin, 1996, 1–97.
[12] H. El Mir: *Sur le prolongement des courants positifs fermés*, Acta Math. **153** (1984), 1–45.
[13] G. Grantcharov and Y.S. Poon: *Geometry of hyperkähler connections with torsion*, Comm. Math. Phys. **213** (2000), 19–37.
[14] F.R. Harvey and H.B. Lawson, Jr.: *An introduction to potential theory in calibrated geometry*, Amer. J. Math. **131** (2009), 893–944.
[15] F.R. Harvey and H.B. Lawson, Jr.: *Duality of positive currents and plurisubharmonic functions in calibrated geometry*, Amer. J. Math. **131** (2009), 1211–1239.
[16] N.J. Hitchin, A. Karlhede, U. Lindström and M. Roček: *Hyperkähler metrics and supersymmetry*, Comm. Math. Phys. **108** (1987), 535–589.
[17] P.S. Howe and G. Papadopoulos: *Twistor spaces for hyperkähler manifolds with torsion*, Phys. Lett. B **379** (1996), 80–86.
[18] J.E. Humphreys: Introduction to Lie Algebras and Representation Theory, Springer, New York, 1972.
[19] M. Obata: *Affine connections on manifolds with almost complex, quaternion or Hermitian structure*, Jap. J. Math. **26** (1956), 43–77.
[20] C. Okonek, M. Schneider and H. Spindler: Vector Bundles on Complex Projective Spaces, Progr. Math. **3**, Birkhäuser, Boston, Mass., 1980.
[21] S.M. Salamon: Quaternionic Manifolds, Inst. Nazionale di Alta Matematica Francesco Severi Symposia Math. **26**, 1982.
[22] N. Sibony: *Quelques problèmes de prolongement de courants en analyse complexe*, Duke Math. J. **52** (1985), 157–197.
[23] H. Skoda: *Prolongement des courants, positifs, fermés de masse finie*, Invent. Math. **66** (1982), 361–376.
[24] M. Verbitsky: *Hyperholomorphic bundles over a hyperkähler manifold*, J. Algebraic Geom. **5** (1996), 633–669.
[25] M. Verbitsky: *Hyperkähler embeddings and holomorphic symplectic geometry* II, GAFA **5** (1995), 92–104.
[26] M. Verbitsky: *Hypercomplex varieties*, Comm. Anal. Geom. **7** (1999), 355–396.
[27] M. Verbitsky: *Hyperholomorphic sheaves and new examples of hyperkähler manifolds*; in M. Verbitsky and D. Kaledin: Hyperkähler manifolds, International Press., Boston, 1999, 15–127.
[28] M. Verbitsky: *Hyperholomorphic connections on coherent sheaves and stability*, 40 pages, `math.AG/0107182`.
[29] M. Verbitsky: *Hyperkähler manifolds with torsion, supersymmetry and Hodge theory*, Asian J. Math. **6** (2002), 679–712.
[30] M. Verbitsky: *Hyperkähler manifolds with torsion obtained from hyperholomorphic bundles*, Math. Res. Lett. **10** (2003), 501–513.
[31] M. Verbitsky: *Hypercomplex manifolds with trivial canonical bundle and their holonomy*; in Moscow Seminar on Mathematical Physics. II, Amer. Math. Soc. Transl. Ser. **2**, **221**, Amer. Math. Soc., Providence, RI, 2007, 203–211.
[32] M. Verbitsky: *Quaternionic Dolbeault complex and vanishing theorems on hyperkähler manifolds*, Compos. Math. **143** (2007), 1576–1592.
[33] M. Verbitsky: *Plurisubharmonic functions in calibrated geometry and q-convexity*, arXiv: `0712.4036`, 27 pages, to appear in Math. Z.
[34] K. Uhlenbeck and S.-T. Yau: *On the existence of Hermitian Yang–Mills connections in stable vector bundles*, Comm. Pure Appl. Math. **39** (1986), 257–293.
[35] S.T. Yau: *On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation*. I, Comm. Pure Appl. Math. **31** (1978), 339–411.

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