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ON INDUCED REPRESENTATIONS

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Let G be a locally compact topological group and K a compact subgroup of G . For any irreducible unitary representation σ of K , we denote by $U(\sigma)$ the induced representation generated by σ (see §1). In general, $U(\sigma)$ is not irreducible.

The purpose of this paper is to give a method of extracting the irreducible components of $U(\sigma)$ when G is one of the special types of Lie groups.

1. Let G be a connected non-compact semisimple Lie group with a finite dimensional faithful representation and K a maximal compact subgroup of G . We assume that $\text{rank } G = \text{rank } K$. For any given irreducible unitary representation σ of K on a representation space V , we can construct a unitary representation $U(\sigma)$ of G as follows. Let $\mathfrak{F}(\sigma)$ be the set of all "Haar-measurable" V -valued functions f which satisfy the following conditions;

$$f(kx) = \sigma(k)f(x) \quad (k \in K, x \in G)$$

and

$$\|f\|^2 = \int_G \|f(x)\|_V^2 dx < \infty$$

where $\|\cdot\|_V$ denotes the norm in V .

Then $\mathfrak{F}(\sigma)$ is a Hilbert space if we identify functions which differ only on subsets of G of Haar measure zero. The inner product (\cdot, \cdot) in $\mathfrak{F}(\sigma)$ is given by

$$(f_1, f_2) = \int_G (f_1(x), f_2(x))_V dx \quad (f_1, f_2 \in \mathfrak{F}(\sigma))$$

where $(\cdot, \cdot)_V$ denotes the inner product in V . Finally for any $g \in G$, $U_g(\sigma)$ is defined by

$$(U_g(\sigma)f)(x) = f(xg) \quad (f \in \mathfrak{F}(\sigma), x \in G).$$

Thus we obtained the induced representation $U(\sigma)$ generated by σ (cf. [7] (d)).

Our aim is to find out an irreducible closed subspace of $\mathfrak{F}(\sigma)$.

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2. Let \mathfrak{g} (resp. \mathfrak{k}) be the Lie algebra of G (resp. K) and \mathfrak{p} the orthogonal complement of \mathfrak{k} in \mathfrak{g} with respect to the Killing form of \mathfrak{g} . Then $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition of \mathfrak{g} . Let T be a maximal torus in K and \mathfrak{h} the Lie algebra of T . Then since G has a finite dimensional faithful representation and $\text{rank } G = \text{rank } K$, T is a Cartan subgroup of G ; i.e.

$$T = \{g \in G; \text{Ad}(g)H = H \text{ for all } H \in \mathfrak{h}\}$$

where Ad denotes the adjoint representation of G . Let $\mathfrak{g}^{\mathbb{C}}$ denote the complexification of \mathfrak{g} . Let Σ be the set of all non-zero roots of $\mathfrak{g}^{\mathbb{C}}$ with respect to the Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$ ($\mathfrak{h}^{\mathbb{C}}$ is the subspace of $\mathfrak{g}^{\mathbb{C}}$ spanned by \mathfrak{h}). We denote by Σ_k the set of all compact roots (see for definition [8]). Let \mathcal{F} be the vector space over \mathbf{R} (the field of real numbers) consisting of all purely imaginary complex valued linear forms on \mathfrak{h} . Then $\Sigma \subset \mathcal{F}$. Introduce a linear order in \mathcal{F} and let P (resp. P_k) be the set of all positive roots in Σ (resp. Σ_k). We denote by \mathfrak{G} the universal enveloping algebra of $\mathfrak{g}^{\mathbb{C}}$. As usual we regard elements of \mathfrak{G} as left-invariant differential operators on G . Since V is finite dimensional, it has the canonical structure of an analytic manifold. We denote by $C^0(G, V)$ the vector space of all continuous mappings from G to V . For any $f \in C^0(G, V)$ and $v \in V$, we put

$$f_v(x) = (f(x), v)_V \quad (x \in G).$$

Let $C^\infty(G)$ be the vector space of all infinitely differentiable complex valued functions on G . We denote by $C^\infty(G, V)$ the set of all $f \in C^0(G, V)$ such that

$$f_v \in C^\infty(G) \text{ for all } v \in V.$$

We often call $f \in C^\infty(G, V)$ a V -valued C^∞ -function. Define

$$(U_X(\sigma)f)(g) = \lim_{t \rightarrow 0} \frac{1}{t} (f(g \exp tX) - f(g)) \quad (g \in G)$$

for $X \in \mathfrak{g}$ and $f \in C^\infty(G, V)$. Then we have a representation $X \rightarrow U_X(\sigma)$ of \mathfrak{g} on $C^\infty(G, V)$. This extends uniquely to a representation of \mathfrak{G} . It is obvious that

$$(U_u(\sigma)f)_v = uf_v \text{ for all } u \in \mathfrak{G}.$$

In the following, we shall simply write uf instead of $U_u(\sigma)f$. Let \mathfrak{Z} be the center of \mathfrak{G} and Ω the Casimir operator of \mathfrak{g} . Then $\Omega \in \mathfrak{Z}$. For any $g \in G$, we define

$$(R(g)f)(x) = f(xg) \quad (x \in G, f \in C^\infty(G)).$$

Then an element u of \mathfrak{G} belongs to \mathfrak{Z} if and only if $R(g) \circ u = u \circ R(g)$ for all $g \in G$. Fix a subalgebra \mathfrak{A} of \mathfrak{Z} such that $\Omega \in \mathfrak{A}$. We denote by $\text{Hom}(\mathfrak{A}, \mathbf{C})$ the set of all homomorphisms of \mathfrak{A} into \mathbf{C} . For any $\chi \in \text{Hom}(\mathfrak{A}, \mathbf{C})$ we put

$$\mathfrak{H}(\sigma, \chi) = \{f \in \mathfrak{H}(\sigma) \cap C^\infty(G, V); zf = \chi(z)f \text{ for all } z \in \mathfrak{A}\}.$$

Then we have

Proposition 1. $\mathfrak{H}(\sigma, \chi)$ is a closed invariant subspace of $\mathfrak{H}(\sigma)$. Moreover, every element of $\mathfrak{H}(\sigma, \chi)$ is an analytic mapping from G into V .

Proof. Let B be the Killing form of \mathfrak{g}^c and put

$$\langle X, Y \rangle = -B(X, \theta(Y)) \quad (X, Y \in \mathfrak{g}^c)$$

where θ denotes the conjugation of \mathfrak{g}^c with respect to the compact real form $\mathfrak{g}_\mu = \mathfrak{k} + \sqrt{-1}\mathfrak{p}$. Then \langle, \rangle is an inner product in \mathfrak{g}^c . Select orthonormal bases (X_1, \dots, X_m) and (Y_1, \dots, Y_n) for \mathfrak{k} and \mathfrak{p} , respectively. Then, it follows from the definition of the Casimir operator Ω that

$$\Omega = -(X_1^2 + \dots + X_m^2) + Y_1^2 + \dots + Y_n^2.$$

We put

$$\Omega_{\mathfrak{k}} = X_1^2 + \dots + X_m^2, \quad \Omega_{\mathfrak{p}} = Y_1^2 + \dots + Y_n^2.$$

For any $X \in \mathfrak{g}$, let X' denote the right invariant vector field on G given by

$$(X'f)(x) = \left[\frac{d}{dt} f(\exp(tX)x) \right]_{t=0} \quad (x \in G, f \in C^\infty(G)).$$

Then the mapping $X \rightarrow X'$ ($X \in \mathfrak{g}$) can be extended uniquely to an anti-homomorphism of \mathfrak{G} onto the algebra of right-invariant differential operators on G . It is easy to see that $\Omega' = \Omega$ as differential operators on G . For any $\lambda \in \mathcal{F}$, we shall denote as usual by H_λ an element of \mathfrak{h}^c such that $\lambda(H) = B(H_\lambda, H)$ for all $H \in \mathfrak{h}$; the inner product $\langle \lambda, \mu \rangle$ of two linear forms $\lambda, \mu \in \mathcal{F}$ means the value $\langle H_\lambda, H_\mu \rangle$. We denote by the same notation the infinitesimal representation of σ . Let $\Lambda \in \mathcal{F}$ be the highest weight of σ . Then it is well known that

$$\sigma(\Omega_k) = -\langle \Lambda + 2\rho_k, \Lambda \rangle I$$

where $\rho_k = \frac{1}{2} \sum_{\alpha \in P_k} \alpha$ and I denotes the identity operator on V . Fix any $f \in \mathfrak{H}(\sigma, \chi)$ and $v \in V$. Then

$$\Omega'_{\mathfrak{k}} f_v(x) = (\sigma(\Omega_{\mathfrak{k}})f(x), v)_V = -\langle \Lambda + 2\rho_k, \Lambda \rangle f_v(x)$$

where $f_v(x) = (f(x), v)_V$. It follows that

$$\Omega' f_v(x) = 2\langle \Lambda + 2\rho_k, \Lambda \rangle f_v(x) + (\Omega'_{\mathfrak{k}} + \Omega'_{\mathfrak{p}})f_v(x).$$

On the other hand,

$$\Omega f_v(x) = (\Omega f(x), v)_V = \chi(\Omega) f_v(x)$$

Therefore, we have

$$(*) \quad (\Omega'_\mathfrak{r} + \Omega'_\mathfrak{p})f_v = (\chi(\Omega) - 2\langle \Lambda + 2\rho_\mathfrak{r}, \Lambda \rangle)f_v.$$

Since $\Omega'_\mathfrak{r} + \Omega'_\mathfrak{p}$ is obviously an elliptic differential operators on G , we conclude that f_v is an analytic function. This shows that f is an analytic mapping from G to V . Moreover, owing to the ellipticity of $\Omega'_\mathfrak{r} + \Omega'_\mathfrak{p}$, all solutions of the above equation (*) in the distribution sense are analytic. It is an immediate consequence of this fact that $\mathfrak{H}(\sigma, \chi)$ is closed in $\mathfrak{H}(\sigma)$. Since $R(g) \circ z = z \circ R(g)$ for all $g \in G$, we see that $U_g(\sigma)U_z(\sigma) = U_z(\sigma)U_g(\sigma)$. It follows immediately that $\mathfrak{H}(\sigma, \chi)$ is an invariant subspace of $\mathfrak{H}(\sigma)$. This completes the proof of the proposition.

We denote by $U(\sigma, \chi)$ the subrepresentation of $U(\sigma)$ obtained by restricting $U(\sigma)$ on $\mathfrak{H}(\sigma, \chi)$. In the following, we shall discuss when $U(\sigma, \chi)$ is non-trivial and irreducible.

3. Let \mathcal{E} (resp. \mathcal{E}_K) be the set of all equivalence classes of irreducible unitary representations of G (resp. K). For any irreducible unitary representation π of G , let $\pi|K$ denote the restriction of the representation π to the subgroup K . For any $\mathfrak{d} \in \mathcal{E}_K$, we denote by $(\pi|K: \mathfrak{d})$ the multiplicity with which the representation \mathfrak{d} occurs in $\pi|K$. $(\pi|K: \mathfrak{d})$ depends only on the equivalence class ω which contains π . In this case, we also write $(\omega|K: \mathfrak{d})$ instead of $(\pi|K: \mathfrak{d})$. Let ξ_σ be the character of σ . We define a projection operator E_σ by

$$E_\sigma = d(\sigma) \int_K \overline{\xi_\sigma(k)} U_k(\sigma, \chi) dk$$

where $d(\sigma)$ denotes the degree of σ and dk is the normalized Haar measure of K . We denote by $[\sigma]$ the class in \mathcal{E}_K to which σ belongs.

Proposition 2. *If $(U(\sigma, \chi)|K: [\sigma]) = 1$, then $U(\sigma, \chi)$ is irreducible.*

Proof. It is sufficient to prove that every non-zero closed invariant subspace of $\mathfrak{H}(\sigma, \chi)$ contains $E_\sigma \mathfrak{H}(\sigma, \chi)$. Let \mathfrak{H} be an arbitrary non-zero closed invariant subspace of $\mathfrak{H}(\sigma, \chi)$. Fix a non-zero element $f \in \mathfrak{H}$. Then from Proposition 1, f is analytic. Hence there exists a $g_0 \in G$ such that $f(g_0) \neq 0$. Put $f_0 = U_{g_0} f$. Then it is obvious that $f_0(1) = f(g_0) \neq 0$ (1 is the identity element of G) and that f_0 is analytic on G . Notice that

$$\begin{aligned} (E_\sigma f_0)(1) &= d(\sigma) \int_K \overline{\xi_\sigma(k)} U_k(\sigma, \chi) f_0(1) dk \\ &= d(\sigma) \int_K \overline{\xi_\sigma(k)} f_0(k) dk \\ &= d(\sigma) \int_K \overline{\xi_\sigma(k)} \sigma(k) dk f_0(1) \\ &= f_0(1) \neq 0. \end{aligned}$$

Then since $E_\sigma f_0$ is again analytic, we can conclude that $E_\sigma f_0 \neq 0$. Moreover, since \mathfrak{H} is closed invariant subspace, we have $E_\sigma f_0 \in \mathfrak{H}$. It follows from the assumption $(U(\sigma, \chi) | K: [\sigma]) = 1$ that $E_\sigma \mathfrak{H}(\sigma, \chi) \subset \mathfrak{H}$. This proves the proposition.

We denote by $\text{End}(V)$ the algebra of all linear endomorphisms of V . An $\text{End}(V)$ -valued C^∞ -function φ on G is called a zornal spherical functions of type (σ, χ) if it satisfies the conditions

- (1) $\varphi(k_1 g k_2) = \sigma(k_1) \varphi(g) \sigma(k_2) \quad (k_1, k_2 \in K, g \in G)$
- (2) $z\varphi = \chi(z)\varphi \quad \text{for all } z \in \mathfrak{A}.$

Let φ be a zornal spherical function of type (σ, χ) . We call φ square-integrable if

$$\int_G \|\varphi(g)\|_V^2 dg < +\infty$$

where $\|\cdot\|_V$ is the Hilbert-Schmidt norm of $\text{End}(V)$. Here we mean by the Hilbert-Schmidt norm of an element of $A \in \text{End}(V)$ the square root of the trace of the operator A^*A , where A^* denotes the adjoint operator of A .

Proposition 3. *If there exists a non-zero square-integrable zornal spherical function of type (σ, χ) , then $U(\sigma, \chi)$ is not trivial (i.e. $\mathfrak{H}(\sigma, \chi) \neq (0)$).*

Proof. Let φ be a non-zero square-integrable zornal spherical function of type (σ, χ) . Then there exists $v \in V$ such that $\varphi_v \neq 0$ where $\varphi_v(g) = \varphi(g)v$. It is easy to see that $\varphi_v \in \mathfrak{H}(\sigma, \chi)$. This completes the proof of the proposition.

4. Now we need some results of F.I. Mautner. For any unitary representation π of G or K , we denote by the $[\pi]$ equivalence class to which π belongs. Then it is easy to see that $[U(\sigma_1)] = [U(\sigma_2)]$ if $[\sigma_1] = [\sigma_2] \in \mathcal{E}_K$. In case $\sigma \in \mathfrak{d}$, we shall write $U(\mathfrak{d})$ instead of $[U(\sigma)]$.

Lemma 1. *Put $\mathcal{E}(\sigma) = \{\omega \in \mathcal{E}; (\omega | K: [\sigma]) \neq 0\}$. Then*

$$[U(\sigma)] = \int_{\mathcal{E}(\sigma)} (\omega | K: [\sigma]) \omega d\mu(\omega) \quad (\text{direct integral})$$

where μ is the Plancherel measure for G . This means that the multiplicity with which ω occurs in $U(\sigma)$ coincides with the multiplicity with which $[\sigma]$ occurs in $\omega | K$.

For a proof, see [7] (c), and notice the following. Let R (resp. r) be the right-regular representation of G (resp. K). Then owing to the Peter-Weyl theorem, one knows that

$$[r] = \sum_{\mathfrak{d} \in \mathcal{E}_K} m(\mathfrak{d}) \mathfrak{d} \quad (\text{direct sum})$$

where $m(\mathfrak{d})$ is the multiplicity with which \mathfrak{d} occurs in r ($m(\mathfrak{d}) = \deg \mathfrak{d}$). It follows from the theorem on inducing a representation “in stages” (see [7] (d)) that

$$[R] = \sum_{\mathfrak{d} \in \mathcal{E}_{\mathcal{K}}} m(\mathfrak{d})U(\mathfrak{d}) \quad (\text{direct sum}).$$

This shows that $[U(\sigma)]$ is a subrepresentation of the regular representation of G .

Now we shall need another lemma due to F.I. Mautner.

Consider the decomposition in Lemma 1. Then there exists a choice of representatives $\tilde{\pi}_{\omega} \in \omega$ ($\omega \in \mathcal{E}(\sigma)$) with the following property. Let $\tilde{\mathfrak{H}}_{\omega}$ denote the representation space of $\tilde{\pi}_{\omega}$. We denote by π_{ω} the $(\omega|K: [\sigma])$ -times direct sum of $\tilde{\pi}_{\omega}$ and let \mathfrak{H}_{ω} be the representation space of π_{ω} . Then we have

$$\mathfrak{H}_{\omega} = \tilde{\mathfrak{H}}_{\omega} \oplus \cdots \oplus \tilde{\mathfrak{H}}_{\omega} \quad ((\omega|K: [\sigma])\text{-times direct sum}).$$

Then we have

$$\mathfrak{H}(\sigma) = \int_{\mathcal{E}(\sigma)} \mathfrak{H}_{\omega} d\mu(\omega) \quad (\text{direct integral}).$$

For any $f \in \mathfrak{H}(\sigma)$, let f_{ω} denote the “component” of f in \mathfrak{H}_{ω} . We denote by the same notations the infinitesimal representations of \mathfrak{G} for $U(\sigma)$ (resp. π_{ω}) on the Gårding subspaces $\mathfrak{H}^0(\sigma)$ (resp. \mathfrak{H}_{ω}^0) where $\omega \in \mathcal{E}(\sigma)$ (cf. [7] (a))

Lemma 2. *For any $f \in \mathfrak{H}^0(\sigma)$ and $u \in \mathfrak{G}$, we have*

$$(U_u(\sigma)f)_{\omega} = \pi_{\omega}(u)f_{\omega}$$

for almost every $\omega \in \mathcal{E}(\sigma)$.

For a proof, see [7] (a), (b).

Let χ_{ω} be the infinitesimal character of $\omega \in \mathcal{E}$. For any $\chi \in \text{Hom}(\mathfrak{Z}, \mathbf{C})$, we denote by $\chi|_{\mathfrak{A}}$ the restriction of χ on \mathfrak{A} . Then $\chi|_{\mathfrak{A}} \in \text{Hom}(\mathfrak{A}, \mathbf{C})$. For any $\chi \in \text{Hom}(\mathfrak{A}, \mathbf{C})$, we put

$$\varepsilon(\chi) = \{\omega \in \mathcal{E}; \chi_{\omega}|_{\mathfrak{A}} = \chi\}.$$

Let \mathcal{E}_d be the set of all discrete classes in \mathcal{E} (see [4] (d)). We denote by L the set of all $\lambda \in \mathcal{F}$ such that

$$\frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbf{Z} \quad \text{for all } \alpha \in \Sigma,$$

where \mathbf{Z} is the set of all integers. Let L' be the set of all $\lambda \in L$ such that $\langle \lambda, \alpha \rangle \neq 0$ for all $\alpha \in \Sigma$. Then owing to the profound result of Harish-Chandra ([4] (d) Theorem 16, p. 96), one has that for any $\lambda \in L'$, there corresponds an element $\omega(\lambda) \in \mathcal{E}_d$ such that

$$\chi_{\omega(\lambda)}(\Omega) = |\lambda|^2 - |\rho|^2$$

where $||^2 = \langle , \rangle$ and $\rho = \frac{1}{2} \sum_{\alpha \in \mathcal{P}} \alpha$. As is easily seen, $\lambda \rightarrow |\lambda|^2 - |\rho|^2$ ($\lambda \in \mathcal{F}$) is a polynomial of degree 2 and its homogeneous part of degree 2 is a positive definite quadratic form. It follows that $\mathcal{E}(\mathcal{X}) \cap \mathcal{E}_d$ is finite set.

Theorem 1. *Let \mathfrak{A} be an arbitrary subalgebra of \mathfrak{B} such that $\Omega \in \mathfrak{A}$ and let χ be a homomorphism of \mathfrak{A} into \mathbf{C} . Then $\mathcal{E}(\mathcal{X}) \cap \mathcal{E}_d$ is a finite set. Moreover, let σ be an irreducible unitary representation of K such that*

$$(A) \quad \mathcal{E}(\sigma) \cap \mathcal{E}(\mathcal{X}) - \mathcal{E}_d$$

is of measure zero with respect to the Plancherel measure for G . Then we have

$$[U(\sigma, \chi)] = \sum_{\omega} (\omega | K : [\sigma]) \omega \quad (\omega \in \mathcal{E}(\sigma) \cap \mathcal{E}(\mathcal{X}) \cap \mathcal{E}_d).$$

Proof. We have already proved the first assertion. We consider the decompositions in Lemma 1 and 2 and use the notations in Lemma 2. Fix any $f \in \mathfrak{S}(\sigma, \chi) \cap \mathfrak{S}^0(\sigma)$. Then we know that

$$U_z(\sigma)f = \chi(z)f \quad \text{and} \quad \pi_{\omega}(z)f_{\omega} = \chi_{\omega}(z)f_{\omega} \quad \text{for all } z \in \mathfrak{A}.$$

On the other hand, from Lemma 2 we have

$$(U_z(\sigma)f)_{\omega} = \pi_{\omega}(z)f_{\omega}$$

for almost every $\omega \in \mathcal{E}(\sigma)$. Hence, there exists a subset $\mathcal{N} \subset \mathcal{E}(\sigma)$ of measure zero such that

$$(\chi(z) - \chi_{\omega}(z))f_{\omega} = 0 \quad \text{for all } \omega \in \mathcal{E}(\sigma) - \mathcal{N}.$$

In general, \mathcal{N} depends on z as well as f . But one knows that \mathfrak{A} is finitely generated. Therefore, every $\chi \in \text{Hom}(\mathfrak{A}, \mathbf{C})$ is uniquely determined by its values at a finite number of elements of \mathfrak{A} . Hence, we can assume that \mathcal{N} does not depend on z . It follows immediately from the assumption (A) in the theorem that

$$f = \sum_{\omega} f_{\omega} \quad (\omega \in \mathcal{E}(\sigma) \cap \mathcal{E}(\mathcal{X}) \cap \mathcal{E}_d).$$

This completes the proof of the theorem.

REMARK 1. For any real number c , define

$$\mathcal{E}_c = \{\omega \in \mathcal{E}; \chi_{\omega}(\Omega) = c\}.$$

Then in case $\text{rank } G/K = 1$, we can show that $\mathcal{E}_c - \mathcal{E}_d$ is of measure zero with respect to the Plancherel measure for G , using the explicit form of the Plancherel measure given in [4] (c), [8]. We have a conjecture that it holds in general. If this is true, then the condition (A) in Theorem 1 is always satisfied

for all σ .

Now we have assumed that G has a compact Cartan subgroup T . Owing to Harish-Chandra [4] (d), one sees that $\mathcal{E}_d \neq \emptyset$. Fix an $\omega \in \mathcal{E}_d$ and put $\mathcal{X} = \mathcal{X}_\omega | \mathfrak{A}$. Then it is obvious that there exists a $[\sigma] \in \mathcal{E}_K$ such that $\omega \in \mathcal{E}(\sigma) \cap \mathcal{E}(\mathcal{X}) \cap \mathcal{E}_d$. It follows from Theorem 1 that ω is a subrepresentation of $U(\sigma, \mathcal{X})$. If $\pi \in \omega$, we say that “ π is a realization of ω ” or that “ ω is realized by π .”

Corollary. *Let \mathfrak{A} be a subalgebra of \mathfrak{B} such that $\Omega \in \mathfrak{A}$. Fix an $\omega \in \mathcal{E}_d$ and put $\mathcal{X} = \mathcal{X}_\omega | \mathfrak{A}$. Assume that there exists an irreducible unitary representation σ of K which satisfies the following conditions (A.1)~(A.3).*

$$(A.1) \quad \mathcal{E}(\sigma) \cap \mathcal{E}(\mathcal{X}) \cap \mathcal{E}_d = \{\omega\}.$$

$$(A.2) \quad (\omega | K : \sigma) = 1.$$

(A.3) $\mathcal{E}(\sigma) \cap \mathcal{E}(\mathcal{X}) - \mathcal{E}_d$ is of measure zero with respect to the Plancherel measure for G .

Then ω is realized by $U(\sigma, \mathcal{X})$.

5. Consider the special case that $\mathfrak{A} = \mathfrak{B}$. Then it is known (see [4] (a)) that $\mathcal{E}(\sigma) \cap \mathcal{E}(\mathcal{X})$ is always a finite set. Hence, in case $\mathfrak{A} = \mathfrak{B}$, the assumption (A) in Theorem 1 and the assumption (A.3) in the corollary to Theorem 1 are always satisfied.

Theorem 2. *Fix any $[\sigma] \in \mathcal{E}_K$ and $\mathcal{X} \in \text{Hom}(\mathfrak{B}, \mathbf{C})$. Then $U(\sigma, \mathcal{X})$ is non-trivial and irreducible if and only if σ and \mathcal{X} satisfy the following condition (C).*

$$(C) \quad \mathcal{E}(\sigma) \cap \mathcal{E}(\mathcal{X}) \cap \mathcal{E}_d \text{ consists of only one element } \omega \text{ such that } (\omega | K : \sigma) = 1.$$

Moreover, the condition (C) implies that $U(\sigma, \mathcal{X})$ is a realization of ω .

REMARK 2. Since $K \backslash G$ is simply connected, $\mathfrak{F}(\sigma)$ can be realized as V -valued square-integrable functions on a certain submanifold of G with respect to a certain measure. If the rank of the symmetric space $K \backslash G$ is equal to be one, the radial components of $U_z(\sigma)$ ($z \in \mathfrak{B}$) coincide with ordinary differential equations (see [9] and cf. [4] (b)). It is very cumbersome to calculate the radial components of $U_z(\sigma)$ ($z \in \mathfrak{B}$) even if G is the lower dimensional Lie group such as the universal covering group of De Sitter group. However, R. Takahashi [9] computed the radial component of $U_\Omega(\sigma)$ in a very ingenious manner, making use of the quaternion field. Thus he proved that $U(\sigma, \mathcal{X})$ is non-trivial and irreducible for a certain $[\sigma] \in \mathcal{E}_K$ and $\mathcal{X} \in \text{Hom}(\mathfrak{A}, \mathbf{C})$ in case $\mathfrak{A} = \mathbf{C}[\Omega]$ (the algebra of all polynomials of Ω).

Now we shall give here an another proof of this fact, making use of the corollary to Theorem 1 and the result of J. Dixmier [1] (b). In the following,

we use the notations of [1] (b) and [9]. Let G be the universal covering group of De Sitter group. We consider the irreducible unitary representation $\rho_K^{n,0}$ of the maximal compact subgroup K of G (where $2n \in \mathbf{Z}$ and $n \geq 1$). Put $\sigma_n = \rho_K^{n,0}$. Then it follows immediately from Fig. 2-3-4-5 ([1] (b) p. 24) that

$$\mathcal{E}(\sigma_n) = \left\{ \pi_{n,q}^+; q = n, n-1, \dots, 1 \text{ or } \frac{1}{2} \right\} \cup \{ \nu_{n,s}; s > 0 \}.$$

On the other hand, from (12) (in [1] (b) p. 12) and (53), (55) in ([1] (b) p. 27) one gets that

$$\begin{aligned} \chi_{\pi_{n,q}^+}(\Omega) &= n^2 + n + q^2 - q - 2, \\ \chi_{\nu_{n,s}}(\Omega) &= n^2 + n - s - 2. \end{aligned}$$

We denote by $\chi_{n,p}$ the unique element of $\text{Hom}(\mathfrak{A}, \mathbf{C})$ such that $\chi_{n,p}(\Omega) = n^2 + n + p^2 - p - 2$. Then it is clear that $\mathcal{E}(\sigma_n) \cap \mathcal{E}(\chi_{n,p}) = \{ \pi_{n,p}^+ \}$ for any p such that $2p, n-p \in \mathbf{Z}$ and $n \geq p \geq 1$. Since every $\mathfrak{d} \in \mathcal{E}_K$ is contained at most once in each $\omega|K$ ($\omega \in \mathcal{E}$), it follows from the Corollary to Theorem 1 that $[U(\sigma_n; \chi_{n,p})] = \pi_{n,p}^+$. This shows that $U(\sigma_n, \chi_{n,p})$ is non-trivial and irreducible. If we take $\sigma_n = \rho_K^{0,n}$, similarly we have $[U(\sigma_n, \chi_{n,p})] = \pi_{n,p}^-$. These facts together with Theorem 1 and 2 in [1] (b) prove the following.

Proposition 4. (R. Takahashi) *Let G be the universal covering group of De Sitter group. Then every irreducible unitary representation of discrete class $\omega \in \mathcal{E}_d$ can be realized by $U(\sigma, \chi)$ for some $[\sigma] \in \mathcal{E}_K$ and $\chi \in \text{Hom}(\mathfrak{A}, \mathbf{C})$ where $\mathfrak{A} = \mathbf{C}[\Omega]$ (the algebra of all polynomials of Ω). More precisely, ω is realized by*

$$\begin{aligned} &U(\rho_K^{n,0}, \chi_{n,p}) \text{ (resp. } U(\rho_K^{0,n}, \chi_{n,p})) \\ &\text{if } \omega = \pi_{n,p}^+ \text{ (resp. } \omega = \pi_{n,p}^-) \end{aligned}$$

where $\chi_{n,p}$ is the unique element of $\text{Hom}(\mathfrak{A}, \mathbf{C})$ such that

$$\chi_{n,p}(\Omega) = n^2 + n + p^2 - p - 2.$$

REMARK 3. It is interesting to observe the fact that the theory of unitary representations has an application to the theory of partially differential equations; i.e. the differential equation (31) on page 399 in [9] has non trivial solutions in $H_0^{p,p}$ (for the notations, see [9]).

6. Finally, we shall apply the above theory to the group $SU(m, 1)$ and the universal covering group of $SO_0(2m, 1)$ where m is an arbitrary positive integer (for the notations, see [5]). Let G be any one of these groups. Then it is known that every $\mathfrak{d} \in \mathcal{E}_K$ is contained at most once in each $\omega|K$ ($\omega \in \mathcal{E}$) (cf. [1] (b), [2], [3]). We fix an element χ of $\text{Hom}(\mathfrak{A}, \mathbf{C})$. If $\omega_1, \omega_2 \in \mathcal{E}(\chi) \cap \mathcal{E}_d$, then

$\omega_1|K$ and $\omega_2|K$ are disjoint, that is, $\omega_1|K$ and $\omega_2|K$ have no irreducible components in common (see for proof, [2], [6]). Therefore, if $\omega \in \mathcal{E}(\sigma) \cap \mathcal{E}_d$, then we have $\mathcal{E}(\sigma) \cap \mathcal{E}(\mathcal{X}_\omega) \cap \mathcal{E}_d = \{\omega\}$. Making use of Theorem 2 we obtain the following proposition.

Proposition 5. *Let G be either $SU(m, 1)$ or the universal covering group of $SO_0(2m, 1)$ where m is an arbitrary positive integer. Then every $\omega \in \mathcal{E}_d$ is realized by $U(\sigma, \mathcal{X}_\omega)$ for any $[\sigma] \in \mathcal{E}_K$ such that $(\omega|K: [\sigma]) \neq 0$.*

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