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Osaka University
Let $G$ be a locally compact topological group and $K$ a compact subgroup of $G$. For any irreducible unitary representation $\sigma$ of $K$, we denote by $U(\sigma)$ the induced representation generated by $\sigma$ (see §1). In general, $U(\sigma)$ is not irreducible.

The purpose of this paper is to give a method of extracting the irreducible components of $U(\sigma)$ when $G$ is one of the special types of Lie groups.

1. Let $G$ be a connected non-compact semisimple Lie group with a finite dimensional faithful representation and $K$ a maximal compact subgroup of $G$. We assume that $\text{rank } G = \text{rank } K$. For any given irreducible unitary representation $\sigma$ of $K$ on a representation space $V$, we can construct a unitary representation $U(\sigma)$ of $G$ as follows. Let $\mathcal{H}(\sigma)$ be the set of all “Haar-measurable” $V$-valued functions $f$ which satisfy the following conditions:

$$f(kx) = \sigma(k)f(x) \quad (k \in K, x \in G)$$

and

$$\|f\|^2 = \int_G \|f(x)\|_V^2 \, dx < \infty$$

where $\| \|_V$ denotes the norm in $V$.

Then $\mathcal{H}(\sigma)$ is a Hilbert space if we identify functions which differ only on subsets of $G$ of Haar measure zero. The inner product $(,)$ in $\mathcal{H}(\sigma)$ is given by

$$(f_1, f_2) = \int_G (f_1(x), f_2(x))_V \, dx \quad (f_1, f_2 \in \mathcal{H}(\sigma))$$

where $(,)_V$ denotes the inner product in $V$. Finally for any $g \in G$, $U_g(\sigma)$ is defined by

$$(U_g(\sigma)f)(x) = f(xg) \quad (f \in \mathcal{H}(\sigma), \ x \in G).$$

Thus we obtained the induced representation $U(\sigma)$ generated by $\sigma$ (cf. [7] (d)).

Our aim is to find out an irreducible closed subspace of $\mathcal{H}(\sigma)$.

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2. Let \( \mathfrak{g} \) (resp. \( \mathfrak{h} \)) be the Lie algebra of \( G \) (resp. \( K \)) and \( \mathfrak{p} \) the orthogonal complement of \( \mathfrak{h} \) in \( \mathfrak{g} \) with respect to the Killing form of \( \mathfrak{g} \). Then \( \mathfrak{g} = \mathfrak{t} + \mathfrak{p} \) is a Cartan decomposition of \( \mathfrak{g} \). Let \( T \) be a maximal torus in \( K \) and \( \mathfrak{h} \) the Lie algebra of \( T \). Then since \( G \) has a finite dimensional faithful representation and rank \( G = \text{rank } K \), \( T \) is a Cartan subgroup of \( G \); i.e.

\[
T = \{ g \in G; \ Ad(g)H = H \quad \text{for all } H \in \mathfrak{h} \}
\]

where \( Ad \) denotes the adjoint representation of \( G \). Let \( \mathfrak{g}^C \) denote the complexification of \( \mathfrak{g} \). Let \( \Sigma \) be the set of all non-zero roots of \( \mathfrak{g}^C \) with respect to the Cartan subalgebra \( \mathfrak{h}^C \) (\( \mathfrak{h}^C \) is the subspace of \( \mathfrak{g}^C \) spanned by \( \mathfrak{h} \)). We denote by \( \Sigma_\mathfrak{k} \) the set of all compact roots (see for definition [8]). Let \( \mathcal{F} \) be the vector space over \( \mathbf{R} \) (the field of real numbers) consisting of all purely imaginary valued linear forms on \( \mathfrak{h} \). Then \( \Sigma \subset \mathcal{F} \). Introduce a linear order in \( \mathcal{F} \) and let \( \mathcal{P} \) (resp. \( \mathcal{P}_\mathfrak{k} \)) be the set of all positive roots in \( \Sigma \) (resp. \( \Sigma_\mathfrak{k} \)). We denote by \( \mathfrak{S} \) the universal enveloping algebra of \( \mathfrak{g}^C \). As usual we regard elements of \( \mathfrak{S} \) as left-invariant differential operators on \( G \). Since \( V \) is finite dimensional, it has the canonical structure of an analytic manifold. We denote by \( C^\infty(G, V) \) the vector space of all continuous mappings from \( G \) to \( V \). For any \( f \in C^\infty(G, V) \) and \( v \in V \), we put

\[
 f_v(x) = (f(x), v) \quad (x \in G).
\]

Let \( C^\infty(G) \) be the vector space of all infinitely differentiable complex valued functions on \( G \). We denote by \( C^\infty(G, V) \) the set of all \( f \in C^\infty(G, V) \) such that

\[
f_v \in C^\infty(G) \quad \text{for all } v \in V.
\]

We often call \( f \in C^\infty(G, V) \) a \( V \)-valued \( C^\infty \)-function. Define

\[
(U_\chi(\sigma)f)(g) = \lim_{t \to 0} \frac{1}{t} \left( f(g \exp t\chi) - f(g) \right) \quad (g \in G)
\]

for \( \chi \in \mathfrak{g} \) and \( f \in C^\infty(G, V) \). Then we have a representation \( X \rightarrow U_\chi(\sigma) \) of \( \mathfrak{g} \) on \( C^\infty(G, V) \). This extends uniquely to a representation of \( \mathfrak{S} \). It is obvious that

\[
(U_\omega(\sigma)f)_v = uf_v \quad \text{for all } u \in \mathfrak{S}.
\]

In the following, we shall simply write \( uf \) instead of \( U_\omega(\sigma)f \). Let \( \mathfrak{B} \) be the center of \( \mathfrak{S} \) and \( \Omega \) the Casimir operator of \( \mathfrak{g} \). Then \( \Omega \in \mathfrak{B} \). For any \( g \in G \), we define

\[
(R(g)f)(x) = f(xg) \quad (x \in G, f \in C^\infty(G)).
\]

Then an element \( u \) of \( \mathfrak{S} \) belongs to \( \mathfrak{B} \) if and only if \( R(g)u = uR(g) \) for all \( g \in G \). Fix a subalgebra \( \mathfrak{A} \) of \( \mathfrak{B} \) such that \( \Omega \in \mathfrak{A} \). We denote by \( \text{Hom}(\mathfrak{A}, C) \) the set of all homomorphisms of \( \mathfrak{A} \) into \( C \). For any \( \chi \in \text{Hom}(\mathfrak{A}, C) \) we put...
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\[ \Phi(\sigma, X) = \{ f \in \mathfrak{g}(\sigma) \cap C^\infty(G, V); zf = \chi(z)f \text{ for all } z \in \mathbb{H} \}. \]

Then we have

**Proposition 1.** \( \Phi(\sigma, X) \) is a closed invariant subspace of \( \mathfrak{g}(\sigma) \). Moreover, every element of \( \Phi(\sigma, X) \) is an analytic mapping from \( G \) into \( V \).

Proof. Let \( B \) be the Killing form of \( g^c \) and put

\[ \langle X, Y \rangle = -B(X, \theta(Y)) \quad (X, Y \in g^c) \]

where \( \theta \) denotes the conjugation of \( g^c \) with respect to the compact real form \( g_0 = \mathfrak{k} + \sqrt{-1} \mathfrak{p} \). Then \( \langle , \rangle \) is an inner product in \( g^c \). Select orthonormal bases \((X_1, \ldots, X_m)\) and \((Y_1, \ldots, Y_n)\) for \( \mathfrak{k} \) and \( \mathfrak{p} \), respectively. Then, it follows from the definition of the Casimir operator \( \Omega \) that

\[ \Omega = -(X_1^2 + \cdots + X_m^2) + Y_1^2 + \cdots + Y_n^2. \]

We put

\[ \Omega_t = X_1^2 + \cdots + X_m^2, \quad \Omega_p = Y_1^2 + \cdots + Y_n^2. \]

For any \( X \in \mathfrak{g} \), let \( X' \) denote the right invariant vector field on \( G \) given by

\[ (X'f)(x) = \left[ \frac{d}{dt} f(\exp(tX)x) \right]_{t=0} \quad (x \in G, f \in C^\infty(G)). \]

Then the mapping \( X \rightarrow X' \) \((X \in \mathfrak{g})\) can be extended uniquely to an anti-homomorphism of \( \mathfrak{g} \) onto the algebra of right-invariant differential operators on \( G \).

It is easy to see that \( \Omega' = \Omega \) as differential operators on \( G \). For any \( \lambda \in \mathcal{F} \), we shall denote as usual by \( H_\lambda \) an element of \( \mathfrak{h}^c \) such that \( \lambda(H) = B(H_\lambda, H) \) for all \( H \in \mathfrak{h} \); the inner product \( \langle \lambda, \mu \rangle \) of two linear forms \( \lambda, \mu \in \mathcal{F} \) means the value \( \langle H_\lambda, H_\mu \rangle \). We denote by the same notation the infinitesimal representation of \( \sigma \). Let \( \Lambda \in \mathcal{F} \) be the highest weight of \( \sigma \). Then it is well known that

\[ \sigma(\Omega_\mu) = -\langle \Lambda + 2\rho_\mu, \Lambda \rangle I \]

where \( \rho_\mu = \frac{1}{2} \sum_{\alpha \in \Delta^{+}_\mu} \alpha \) and \( I \) denotes the identity operator on \( V \). Fix any \( f \in \Phi(\sigma, X) \) and \( v \in V \). Then

\[ \Omega'_t f_v(x) = (\sigma(\Omega_t) f(x), v)_V = -\langle \Lambda + 2\rho_\mu, \Lambda \rangle f_v(x) \]

where \( f_v(x) = (f(x), v)_V \). It follows that

\[ \Omega'_t f_v(x) = 2\langle \Lambda + 2\rho_\mu, \Lambda \rangle f_v(x) + (\Omega'_t + \Omega'_p)f_v(x). \]

On the other hand,

\[ \Omega f_v(x) = (\Omega f(x), v)_V = \chi(\Lambda) f_v(x) \]

Therefore, we have
Since $\Omega'_t + \Omega'_\nu$ is obviously an elliptic differential operators on $G$, we conclude that $f_\nu$ is an analytic function. This shows that $f$ is an analytic mapping from $G$ to $V$. Moreover, owing to the ellipticity of $\Omega'_t + \Omega'_\nu$, all solutions of the above equation (1) in the distribution sense are analytic. It is an immediate consequence of this fact that $\mathcal{D}(\sigma, \chi)$ is closed in $\mathcal{D}(\sigma)$. Since $R(g) \circ z = z \circ R(g)$ for all $g \in G$, we see that $U_g(\sigma) U_\nu(\sigma) = U_\nu(\sigma) U_g(\sigma)$. It follows immediately that $\mathcal{D}(\sigma, \chi)$ is an invariant subspace of $\mathcal{D}(\sigma)$. This completes the proof of the proposition.

We denote by $U(\sigma, \chi)$ the subrepresentation of $U(\sigma)$ obtained by restricting $U(\sigma)$ on $\mathcal{D}(\sigma, \chi)$. In the following, we shall discuss when $U(\sigma, \chi)$ is non-trivial and irreducible.

3. Let $\mathcal{E}$ (resp. $\mathcal{E}_K$) be the set of all equivalence classes of irreducible unitary representations of $G$ (resp. $K$). For any irreducible unitary representation $\pi$ of $G$, let $\pi | K$ denote the restriction of the representation $\pi$ to the subgroup $K$. For any $\omega \in \mathcal{E}_K$, we denote by $(\pi | K: \omega)$ the multiplicity with which the representation $\omega$ occurs in $\pi | K$. $(\pi | K: \omega)$ depends only on the equivalence class $\omega$ which contains $\pi$. In this case, we also write $(\omega | K: \omega)$ instead of $(\pi | K: \omega)$. Let $\xi_\sigma$ be the character of $\sigma$. We define a projection operator $E_\sigma$ by

$$E_\sigma = d(\sigma) \int_K \overline{\xi_\sigma(k)} U_k(\sigma, \chi) dk$$

where $d(\sigma)$ denotes the degree of $\sigma$ and $dk$ is the normalized Haar measure of $K$. We denote by $[\sigma]$ the class in $\mathcal{E}_K$ to which $\sigma$ belongs.

**Proposition 2.** If $(U(\sigma, \chi) | K: [\sigma]) = 1$, then $U(\sigma, \chi)$ is irreducible.

Proof. It is sufficient to prove that every non-zero closed invariant subspace of $\mathcal{D}(\sigma, \chi)$ contains $E_\sigma \mathcal{D}(\sigma, \chi)$. Let $\mathcal{D}$ be an arbitrary non-zero closed invariant subspace of $\mathcal{D}(\sigma, \chi)$. Fix a non-zero element $f \in \mathcal{D}$. Then from Proposition 1, $f$ is analytic. Hence there exists a $g_0 \in G$ such that $f(g_0) \neq 0$. Put $f_0 = U_{g_0} f$. Then it is obvious that $f_0(1) = f(g_0) \neq 0$ ($1$ is the identity element of $G$) and that $f_0$ is analytic on $G$. Notice that

$$(E_\sigma f_0)(1) = d(\sigma) \int_K \overline{\xi_\sigma(k)} U_k(\sigma, \chi) f_0(1) dk$$

$$= d(\sigma) \int_K \overline{\xi_\sigma(k)} f_0(k) dk$$

$$= d(\sigma) \int_K \xi_\sigma(k) \sigma(k) d f_0(1)$$

$$= f_0(1) \neq 0.$$
Then since $E_{\sigma}f_0$ is again analytic, we can conclude that $E_{\sigma}f_0 \neq 0$. Moreover, since $\mathcal{S}$ is closed invariant subspace, we have $E_{\sigma}f_0 \in \mathcal{S}$. It follows from the assumption $(U(\sigma, \chi)|K: [\sigma]) = 1$ that $E_{\sigma}\mathcal{S}(\sigma, \chi) \subset \mathcal{S}$. This proves the proposition.

We denote by $\text{End}(V)$ the algebra of all linear endomorphisms of $V$. An $\text{End}(V)$-valued $C^\infty$-function $\varphi$ on $G$ is called a zornal spherical functions of type $(\sigma, \chi)$ if it satisfies the conditions

\begin{align*}
(1) \quad & \varphi(k, g_k) = \sigma(k_1)\varphi(g)\sigma(k_2) \quad (k_1, k_2 \in K, g \in G) \\
(2) \quad & z\varphi = \chi(z)\varphi \quad \text{for all} \quad z \in \mathbb{A}.
\end{align*}

Let $\varphi$ be a zornal spherical function of type $(\sigma, \chi)$. We call $\varphi$ square-integrable if

$$\int_G \|\varphi(g)\|_V^2 \, dg < +\infty$$

where $\| \|_V$ is the Hilbert-Schmidt norm of $\text{End}(V)$. Here we mean by the Hilbert-Schmidt norm of an element of $A \in \text{End}(V)$ the square root of the trace of the operator $A^*A$, where $A^*$ denotes the adjoint operator of $A$.

**Proposition 3.** If there exists a non-zero square-integrable zornal spherical function of type $(\sigma, \chi)$, then $U(\sigma, \chi)$ is not trivial (i.e. $\mathcal{S}(\sigma, \chi) \neq \{0\}$).

**Proof.** Let $\varphi$ be a non-zero square-integrable zornal spherical function of type $(\sigma, \chi)$. Then there exists $v \in V$ such that $\varphi_0 \neq 0$ where $\varphi_0(g) = \varphi(g)v$. It is easy to see that $\varphi_0 \in \mathcal{S}(\sigma, \chi)$. This completes the proof of the proposition.

4. Now we need some results of F.I. Mautner. For any unitary representation $\pi$ of $G$ or $K$, we denote by the $[\pi]$ equivalence class to which $\pi$ belongs. Then it is easy to see that $[U(\sigma_i)] = [U(\sigma_i)]$ if $[\sigma_i] = [\sigma_i] \in \mathcal{C}_K$. In case $\sigma \in \mathfrak{b}$, we shall write $U(b)$ instead of $[U(\sigma)]$.

**Lemma 1.** Put $\mathcal{E}(\sigma) = \{\omega \in \mathcal{E}; (\omega|K: [\sigma]) \neq 0\}$. Then

$$[U(\sigma)] = \int_{\mathcal{E}(\sigma)} (\omega|K: [\sigma]) \omega d\mu(\omega) \quad \text{(direct integral)}$$

where $\mu$ is the Plancherel measure for $G$. This means that the multiplicity with which $\omega$ occurs in $U(\sigma)$ coincides with the multiplicity with which $[\sigma]$ occurs in $\omega|K$.

For a proof, see [7] (c), and notice the following. Let $R$ (resp. $r$) be the right-regular representation of $G$ (resp. $K$). Then owing to the Peter-Weyl theorem, one knows that

$$[r] = \sum_{b \in \mathcal{C}_K} m(b)b \quad \text{(direct sum)}$$
where \( m(b) \) is the multiplicity with which \( b \) occurs in \( m(b) = \deg b \). It follows from the theorem on inducing a representation "in stages" (see \([7] (d)\)) that

\[
[R] = \sum_{b \in \mathcal{C}} m(b)U(b) \quad \text{(direct sum)}.
\]

This shows that \([U(\sigma)]\) is a subrepresentation of the regular representation of \( G \).

Now we shall need another lemma due to F.I. Mautner.

Consider the decomposition in Lemma 1. Then there exists a choice of representatives \( \pi_\omega \in \omega \) \((\omega \in \mathcal{E}(\sigma))\) with the following property. Let \( \mathcal{H}_\omega \) denote the representation space of \( \pi_\omega \). We denote by \( \pi_\omega \) the \((\omega | K : [\sigma])\)-times direct sum of \( \pi_\omega \) and let \( \mathcal{H}_\omega \) be the representation space of \( \pi_\omega \). Then we have

\[
\mathcal{H}_\omega = \mathcal{H}_\omega \oplus \cdots \oplus \mathcal{H}_\omega \quad ((\omega | K : [\sigma])\)-times direct sum).
\]

Then we have

\[
\mathcal{H}(\sigma) = \int_{\mathcal{E}(\sigma)} \mathcal{H}_\omega d\mu(\omega) \quad \text{(direct integral)}.
\]

For any \( f \in \mathcal{H}(\sigma) \), let \( f_\omega \) denote the "component" of \( f \) in \( \mathcal{H}_\omega \). We denote by the same notations the infinitesimal representations of \( \mathcal{H} \) for \( U(\sigma) \) (resp. \( \pi_\omega \)) on the Gårding subspaces \( \mathcal{H}(\sigma) \) (resp. \( \mathcal{H}_\omega \)) where \( \omega \in \mathcal{E}(\sigma) \) (cf. \([7] (a)\))

**Lemma 2.** For any \( f \in \mathcal{H}(\sigma) \) and \( u \in \mathcal{G} \), we have

\[
(U(u)(\sigma)f)_\omega = \pi_\omega(u)f_\omega
\]

for almost every \( \omega \in \mathcal{E}(\sigma) \).

For a proof, see \([7] (a), (b)\).

Let \( \chi_\omega \) be the infinitesimal character of \( \omega \in \mathcal{E} \). For any \( \chi \in \text{Hom}(\mathcal{H}, \mathcal{C}) \), we denote by \( \chi|_\mathcal{J} \) the restriction of \( \chi \) on \( \mathcal{J} \). Then \( \chi|_\mathcal{J} \in \text{Hom}(\mathcal{J}, \mathcal{C}) \). For any \( \chi \in \text{Hom}(\mathcal{J}, \mathcal{C}) \), we put

\[
\mathcal{E}(\chi) = \{ \omega \in \mathcal{E}; \chi_\omega|_\mathcal{J} = \chi \}.
\]

Let \( \mathcal{E}_d \) be the set of all discrete classes in \( \mathcal{E} \) (see \([4] (d)\)). We denote by \( L \) the set of all \( \lambda \in \mathcal{E} \) such that

\[
\frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z} \quad \text{for all } \alpha \in \Sigma,
\]

where \( \mathbb{Z} \) is the set of all integers. Let \( L' \) be the set of all \( \lambda \in L \) such that \( \langle \lambda, \alpha \rangle \neq 0 \) for all \( \alpha \in \Sigma \). Then owing to the profound result of Harish-Chandra \(([4] (d) \text{ Theorem 16, p. 96})\), one has that for any \( \lambda \in L' \), there corresponds an element \( \omega(\lambda) \in \mathcal{E}_d \) such that

\[
\chi_{\omega(\lambda)}(\Omega) = |\lambda|^2 - |\rho|^2
\]
where \(|^2=\langle , \rangle \) and \(\rho=\frac{1}{2} \sum_{\alpha} \alpha \). As is easily seen, \(\lambda \rightarrow |\lambda|^2 - |\rho|^2 \) \((\lambda \in \mathcal{F})\) is a polynomial of degree 2 and its homogeneous part of degree 2 is a positive definite quadratic form. It follows that \(\mathcal{E}(\chi) \cap \mathcal{E}_d \) is finite set.

**Theorem 1.** Let \(\mathcal{A}\) be an arbitrary subalgebra of \(\mathfrak{B}\) such that \(\Omega \subseteq \mathcal{A}\) and let \(\chi\) be a homomorphism of \(\mathcal{A}\) into \(\mathbb{C}\). Then \(\mathcal{E}(\chi) \cap \mathcal{E}_d \) is a finite set. Moreover, let \(\sigma\) be an irreducible unitary representation of \(K\) such that

\[
(A) \quad \mathcal{E}(\sigma) \cap \mathcal{E}(\chi) - \mathcal{E}_d
\]

is of measure zero with respect to the Plancherel measure for \(G\). Then we have

\[
[U(\sigma, \chi)] = \sum_{\omega} (\omega | K : \sigma) \omega \quad (\omega \in \mathcal{E}(\sigma) \cap \mathcal{E}(\chi) \cap \mathcal{E}_d).
\]

**Proof.** We have already proved the first assertion. We consider the decompositions in Lemma 1 and 2 and use the notations in Lemma 2. Fix any \(f \in \mathfrak{S}(\sigma, \chi) \cap \mathfrak{S}(\sigma)\). Then we know that

\[
U_\sigma(\sigma)f = \chi(z)f \quad \text{and} \quad \pi_\sigma(z)f_\omega = \chi_\omega(z)f_\omega \quad \text{for all} \quad z \in \mathcal{A}.
\]

On the other hand, from Lemma 2 we have

\[
(U_\sigma(\sigma)f)_\omega = \pi_\omega(z)f_\omega
\]

for almost every \(\omega \in \mathcal{E}(\sigma)\). Hence, there exists a subset \(\mathcal{N} \subseteq \mathcal{E}(\sigma)\) of measure zero such that

\[
(\chi(z) - \chi_\omega(z))f_\omega = 0 \quad \text{for all} \quad \omega \in \mathcal{E}(\sigma) - \mathcal{N}.
\]

In general, \(\mathcal{N}\) depends on \(z\) as well as \(f\). But one knows that \(\mathcal{A}\) is finitely generated. Therefore, every \(\chi \in \text{Hom} (\mathcal{A}, \mathbb{C})\) is uniquely determined by its values at a finite number of elements of \(\mathcal{A}\). Hence, we can assume that \(\mathcal{N}\) does not depend on \(z\). It follows immediately from the assumption \((A)\) in the theorem that

\[
f = \sum_\omega f_\omega \quad (\omega \in \mathcal{E}(\sigma) \cap \mathcal{E}(\chi) \cap \mathcal{E}_d).
\]

This completes the proof of the theorem.

**Remark 1.** For any real number \(c\), define

\[
\mathcal{E}_c = \{\omega \in \mathcal{E}; \chi_\omega(\Omega) = c\}.
\]

Then in case \(\text{rank } G/K = 1\), we can show that \(\mathcal{E}_c - \mathcal{E}_d\) is of measure zero with respect to the Plancherel measure for \(G\), using the explicit form of the Plancherel measure given in [4] (c), [8]. We have a conjecture that it holds in general. If this is true, then the condition \((A)\) in Theorem 1 is always satisfied.
for all $\sigma$.

Now we have assumed that $G$ has a compact Cartan subgroup $T$. Owing to Harish-Chandra [4] (d), one sees that $E_d \neq \emptyset$. Fix an $\omega \in E_d$ and put $\chi = \chi_\sigma | \mathfrak{A}$. Then it is obvious that there exists a $[\sigma] \in E_K$ such that $\omega \in E(\sigma) \cap E(\chi) \cap E_d$. It follows from Theorem 1 that $\omega$ is a subrepresentation of $U(\sigma, \chi)$. If $\pi \in \omega$, we say that "$\pi$ is a realization of $\omega$" or that "$\omega$ is realized by $\pi$.

**Corollary.** Let $\mathfrak{A}$ be a subalgebra of $\mathfrak{B}$ such that $\Omega \in \mathfrak{A}$. Fix an $\omega \in E_d$ and put $\chi = \chi_\omega | \mathfrak{A}$. Assume that there exists an irreducible unitary representation $\sigma$ of $K$ which satisfies the following conditions (A1)-(A3).

(A1) $E(\sigma) \cap E(\chi) \cap E_d = \{ \omega \}$.

(A2) $(\omega | K: \sigma) = 1$.

(A3) $E(\sigma) \cap E(\chi) \cap E_d$ is of measure zero with respect to the Plancherel measure for $G$.

Then $\omega$ is realized by $U(\sigma, \chi)$.

5. Consider the special case that $\mathfrak{A} = \mathfrak{B}$. Then it is known (see [4] (a)) that $E(\sigma) \cap E(\chi)$ is always a finite set. Hence, in case $\mathfrak{A} = \mathfrak{B}$, the assumption (A) in Theorem 1 and the assumption (A.3) in the corollary to Theorem 1 are always satisfied.

**Theorem 2.** Fix any $[\sigma] \in E_K$ and $\chi \in \text{Hom} (\mathfrak{B}, C)$. Then $U(\sigma, \chi)$ is non-trivial and irreducible if and only if $\sigma$ and $\chi$ satisfy the following condition (C).

(C) $E(\sigma) \cap E(\chi) \cap E_d$ consists of only one element $\omega$ such that $(\omega | K: \sigma) = 1$.

Moreover, the condition (C) implies that $U(\sigma, \chi)$ is a realization of $\omega$.

**Remark 2.** Since $K \setminus G$ is simply connected, $\mathfrak{S}(\sigma)$ can be realized as $V$-valued square-integrable functions on a certain submanifold of $G$ with respect to a certain measure. If the rank of the symmetric space $K \setminus G$ is equal to be one, the radial components of $U_\mathfrak{d}(\sigma) (z \in \mathfrak{B})$ coincide with ordinary differential equations (see [9] and cf. [4] (b)). It is very cumbersome to calculate the radial components of $U_\mathfrak{d}(\sigma) (z \in \mathfrak{B})$ even if $G$ is the lower dimensional Lie group such as the universal covering group of De Sitter group. However, R. Takahashi [9] computed the radial component of $U_\mathfrak{d}(\sigma)$ in a very ingenious manner, making use of the quaternion field. Thus he proved that $U(\sigma, \chi)$ is non-trivial and irreducible for a certain $[\sigma] \in E_K$ and $\chi \in \text{Hom} (\mathfrak{A}, C)$ in case $\mathfrak{A} = C[\Omega]$ (the algebra of all polynomials of $\Omega$).

Now we shall give here another proof of this fact, making use of the corollary to Theorem 1 and the result of J. Dixmier [1] (b). In the following,
we use the notations of [1] (b) and [9]. Let $G$ be the universal covering group of De Sitter group. We consider the irreducible unitary representation $\rho_K^0$ of the maximal compact subgroup $K$ of $G$ (where $2n \in \mathbb{Z}$ and $n \geq 1$). Put $\sigma_n = \rho_K^0$. Then it follows immediately from Fig. 2–3–4–5 ([1] (b) p. 24) that

$$\mathcal{E}(\sigma_n) = \left\{ \pi_{n,q}^+; q = n, n-1, \ldots, 1 \text{ or } \frac{1}{2} \right\} \cup \{ \nu_{n,s}; s > 0 \}.$$ 

On the other hand, from (12) (in [1] (b) p. 12) and (53), (55) in ([1] (b) p. 27) one gets that

$$\chi_{n,q}^+ (\Omega) = n^2 + n + q^2 - q - 2,$$

$$\chi_{n,s} (\Omega) = n^2 + n - s - 2.$$ 

We denote by $\chi_{n,p}$ the unique element of Hom ($\mathfrak{g}, \mathbb{C}$) such that $\chi_{n,p}(\Omega) = n^2 + n + p^2 - p - 2$. Then it is clear that $\mathcal{E}(\sigma_n) \cap \mathcal{E}(\chi_{n,p}) = \{ \pi_{n,q}^+ \}$ for any $p$ such that $2p$, $n-p \in \mathbb{Z}$ and $n \geq p \geq 1$. Since every $b \in \mathcal{E}_K$ is contained at most once in each $\omega \in \mathcal{E}_d$, it follows from the Corollary to Theorem 1 that $[U(\sigma_n; \chi_{n,p})] = \pi_{n,p}^+$. This shows that $U(\sigma_n, \chi_{n,p})$ is non-trivial and irreducible. If we take $\sigma_n = \rho_K^0$, similarly we have $[U(\sigma_n; \chi_{n,p})] = \pi_{n,p}^-$ These facts together with Theorem 1 and 2 in [1] (b) prove the following.

**Proposition 4.** (R. Takahashi) Let $G$ be the universal covering group of De Sitter group. Then every irreducible unitary representation of discrete class $\omega \in \mathcal{E}_d$ can be realized by $U(\sigma, \chi)$ for some $[\sigma] \in \mathcal{E}_K$ and $\chi \in \text{Hom} (\mathfrak{g}, \mathbb{C})$ where $\mathfrak{g} = \mathbb{C}[\Omega]$ (the algebra of all polynomials of $\Omega$). More precisely, $\omega$ is realized by

$$U(\rho_K^0, \chi_{n,p}) \text{ (resp. } U(\rho_K^0, - \chi_{n,p}) \text{)}$$

if $\omega = \pi_{n,p}^+$ (resp. $\omega = \pi_{n,p}^-$)

where $\chi_{n,p}$ is the unique element of Hom ($\mathfrak{g}, \mathbb{C}$) such that

$$\chi_{n,p}(\Omega) = n^2 + n + p^2 - p - 2.$$ 

**Remark 3.** It is interesting to observe the fact that the theory of unitary representations has an application to the theory of partially differential equations; i.e. the differential equation (31) on page 399 in [9] has non trivial solutions in $H_0^p$ (for the notations, see [9]).

6. Finally, we shall apply the above theory to the group $SU(m, 1)$ and the universal covering group of $SO_o(2m, 1)$ where $m$ is an arbitrary positive integer (for the notations, see [5]). Let $G$ be any one of these groups. Then it is known that every $b \in \mathcal{E}_K$ is contained at most once in each $\omega \in \mathcal{E}_d$ (cf. [1] (b), [2], [3]). We fix an element $\chi$ of Hom ($\mathfrak{g}, \mathbb{C}$). If $\omega_1, \omega_2 \in \mathcal{E}(\chi) \cap \mathcal{E}_d$, then
$\omega_1|K$ and $\omega_2|K$ are disjoint, that is, $\omega_1|K$ and $\omega_2|K$ have no irreducible components in common (see for proof, [2], [6]). Therefore, if $\omega \in \mathcal{E}(\sigma) \cap \mathcal{E}_d$, then we have $\mathcal{E}(\sigma) \cap \mathcal{E}(\chi_{\omega}) \cap \mathcal{E}_d = \{\omega\}$. Making use of Theorem 2 we obtain the following proposition.

**Proposition 5.** Let $G$ be either $SU(m, 1)$ or the universal covering group of $SO_{d}(2m, 1)$ where $m$ is an arbitrary positive integer. Then every $\omega \in \mathcal{E}_d$ is realized by $U(\sigma, \chi_{\omega})$ for any $[\sigma] \in \mathcal{E}_K$ such that $\langle \omega | K : [\sigma] \rangle \neq 0$.

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