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BRAUER GROUPS OF ALGEBRAIC FUNCTION FIELDS AND THEIR ADÈLE RINGS

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Introduction. Let K be an algebraic number field and $\{\mathfrak{p}\}$ be the valuations of K, then related to Takagi-Artin's class field theory, the following exact sequence is well-known (c.f. Hasse [5]);

$$0 \to Br(K) \to \bigoplus_{\mathfrak{p}} Br(K_{\mathfrak{p}}) \to \boldsymbol{Q}/\boldsymbol{Z} \to 0 \tag{1}$$

where $K_{\mathfrak{p}}$ is the completion of K with respect to $\mathfrak{p}$. In the Seminar 1966 at Bowdoin College, G. Azumaya [4] showed that the middle term of (1) is isomorphic to the Brauer group of the adèle ring A_K of K and that the following diagram with canonical arrows is commutative;

$$\begin{tikzcd} & \bigoplus_{\mathfrak{p}} Br(K_{\mathfrak{p}}) \arrow[dr] \arrow[dd, phantom, "\wr"] & \\ 0 \to Br(K) \arrow[ur] \arrow[dr] & & \boldsymbol{Q}/\boldsymbol{Z} \to 0 \\ & Br(A_K) \arrow[ur] & \end{tikzcd} \tag{2}$$

But on an algebraic function field, the class field theory does not hold except the case of finite constant field (Artin-Whalpe [1]), so the analogies of (1), (2) must have fallen.

The purpose of this paper is to clarify the relations of the Brauer group of the adèle ring of a function field, to the Brauer group of a function field and to Galois cohomologies.

We use the following notations:

k : a perfect field

$\bar{k}$: the algebraic closure of k

F : an algebraic function field of one variable over k i.e. F/k is finitely generated, k is algebraically closed in F and the degree of transcendency of F/k is one

$\bar{F}=F\cdot\bar{k}$: the field theoretic compositum of F and $\bar{k}$

$\mathfrak{p}$: a prime divisor of F over k

$F_{\mathfrak{p}}$: the completion of F with respect to $\mathfrak{p}$

$\mathfrak{O}_{\mathfrak{p}}$: the valuation ring of $F_{\mathfrak{p}}$

$k_{\mathfrak{p}}$: the residue class field of $F_{\mathfrak{p}}$ i.e. $\mathfrak{O}_{\mathfrak{p}}/\mathfrak{p}$
G : the Galois group of $\bar{k}$ over k and we shall identify G with the Galois group of $\bar{F}$ over F
$G_{\mathfrak{p}}$: the decomposition group of $\mathfrak{p}$
$\chi(*)$: the character group of the group $*$
$A_F=A_{F/k}$: the adèle ring of F i.e. the restricted direct product of $F_{\mathfrak{p}}$ with respect to $\mathfrak{O}_{\mathfrak{p}}$
$Br(*)$: the Brauer group of the ring $*$.

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1. The homomorphism of $Br(A_F)$ to $\bigoplus_{\mathfrak{p}}\chi(G_{\mathfrak{p}})$

It is well-known that $\mathfrak{O}_{\mathfrak{p}}$ coincides with the formal power series ring $k_{\mathfrak{p}}[[\pi_{\mathfrak{p}}]]$ with respect to some prime element $\pi_{\mathfrak{p}}$ and $F_{\mathfrak{p}}$ coincides with $k_{\mathfrak{p}}((\pi_{\mathfrak{p}}))$ (c.f. Serre [8] II, §4). Witt [10] and Shuen Yuan [11] showed the sequence

$$(3)\qquad 0\longrightarrow Br(k_{\mathfrak{p}})\xrightarrow{\theta_{\mathfrak{p}}} Br(F_{\mathfrak{p}})\xrightarrow{\eta_{\mathfrak{p}}}\chi(G_{\mathfrak{p}})\longrightarrow 0$$

is exact, where $\theta_{\mathfrak{p}}$ is the one induced by the ring homomorphism $k_{\mathfrak{p}}\rightarrow F_{\mathfrak{p}}$. Azumaya [3] and Auslander-Goldman [2] showed that $Br(k_{\mathfrak{p}})$ is isomorphic to $Br(\mathfrak{O}_{\mathfrak{p}})$. From the sequence (3), considering the direct product for all $\mathfrak{p}$, we have the following exact sequence;

$$(4)\qquad 0\longrightarrow \prod_{\mathfrak{p}}Br(k_{\mathfrak{p}})\xrightarrow{\Pi\theta_{\mathfrak{p}}}\prod_{\mathfrak{p}}Br(F_{\mathfrak{p}})\xrightarrow{\Pi\eta_{\mathfrak{p}}}\prod_{\mathfrak{p}}\chi(G_{\mathfrak{p}})\longrightarrow 0$$

Proposition 1. *There exists the epimorphism φ of $Br(A_F)$ to the direct sum $\bigoplus_{\mathfrak{p}}\chi(G_{\mathfrak{p}})$ of $\chi(G_{\mathfrak{p}})$.*

Proof. Let Λ be a central separable algebra over A_F and $\lambda_1=1, \lambda_2, \cdots, \lambda_m$ be a set of generators of Λ over A_F. Since Λ is separable over A_F, there exist the elements $u_1, \cdots, u_n; v_1, \cdots, v_n$ in Λ satisfying the relations;

$$(*)\quad \begin{cases} \sum u_iv_i=1 \\ \sum_i xu_i\otimes v_i^0=\sum_i u_i\otimes(v_ix)^0 \text{ in the enveloping algebra } \Lambda^e=\Lambda\otimes_{A_F}\Lambda^0 \text{ for any } \\ x \text{ in } \Lambda. \end{cases}$$

Let us set $u_i=\sum_h a_{ih}\lambda_h$, $v_i=\sum_h b_{ih}\lambda_h$, $\lambda_i\lambda_j=\sum_h c_{ijh}\lambda_h$ where a_{ih}, b_{ih}, c_{ijh} are in A_F. Since a_{ih}, b_{ih}, c_{ijh} are adèles, $\{a_{ih}^{\mathfrak{p}}\}$, $\{b_{ih}^{\mathfrak{p}}\}$, $\{c_{ijh}^{\mathfrak{p}}\}$ are in $\mathfrak{O}_{\mathfrak{p}}$ for almost all $\mathfrak{p}$ where $x^{\mathfrak{p}}$ is the $F_{\mathfrak{p}}$-component of an element x in A_F. We shall set $\Lambda_{\mathfrak{p}}=\Lambda\otimes_{A_F}F_{\mathfrak{p}}$ and $u_i^{\mathfrak{p}}=u_i\otimes 1$, $v_i^{\mathfrak{p}}=v_i\otimes 1$, $\lambda_i^{\mathfrak{p}}=\lambda_i\otimes 1$ in $\Lambda_{\mathfrak{p}}$, and let $\Gamma_{\mathfrak{p}}$ be the $\mathfrak{O}_{\mathfrak{p}}$-module generated by $\lambda_1^{\mathfrak{p}}, \cdots, \lambda_m^{\mathfrak{p}}$. Then $u_1^{\mathfrak{p}}, \cdots, u_n^{\mathfrak{p}}; v_1^{\mathfrak{p}}, \cdots, v_n^{\mathfrak{p}}$ satisfy the similar relations as $(*)$. Thus $\Gamma_{\mathfrak{p}}$ is a separable $\mathfrak{O}_{\mathfrak{p}}$-order in $\Lambda_{\mathfrak{p}}$ for almost all $\mathfrak{p}$ since $u_1^{\mathfrak{p}}, \cdots, u_n^{\mathfrak{p}}, v_1^{\mathfrak{p}}, \cdots, v_n^{\mathfrak{p}}$ are contained in $\Gamma_{\mathfrak{p}}$ for almost all $\mathfrak{p}$ and $\Gamma_{\mathfrak{p}}$ forms a ring with identity for almost all $\mathfrak{p}$. Therefore, defining $\varphi_0\colon Br(A_F)\to\prod_{\mathfrak{p}} Br(F_{\mathfrak{p}})$ to be the homomorphism induced by the projection $A_F\to F_{\mathfrak{p}}$, the image of φ_0 is contained in the restricted direct product of $Br(F_{\mathfrak{p}})$ with respect to $Br(\mathfrak{O}_{\mathfrak{p}})\cong Br(k_{\mathfrak{p}})$. We define φ to be the composite $\prod \eta_{\mathfrak{p}}\circ\varphi_0$, then the image of φ is in $\bigoplus_{\mathfrak{p}} \mathfrak{X}(G_{\mathfrak{p}})$.

To see φ is an epimorphism we need the following

Lemma 2. *Let Λ be an algebra over A_F which is a finitely generated free A_F-module with the free basis $w_1, \cdots, w_m$. If $\Lambda_{\mathfrak{p}}=\Lambda\otimes_{A_F} F_{\mathfrak{p}}$ is a central separable algebra over $F_{\mathfrak{p}}$ for all $\mathfrak{p}$ and the $\mathfrak{O}_{\mathfrak{p}}$-module generated by $w_1^{\mathfrak{p}}=w_1\otimes 1, \cdots, w_m^{\mathfrak{p}}=w_m\otimes 1$ in $\Lambda_{\mathfrak{p}}$ is a separable $\mathfrak{O}_{\mathfrak{p}}$-order for almost all $\mathfrak{p}$, then Λ is a central separable algebra over A_F.*

Proof of Lemma 2. As $\Lambda_{\mathfrak{p}}$ is separable over $F_{\mathfrak{p}}$, there are the elements $u_1^{\mathfrak{p}}, \cdots, u_{n_{\mathfrak{p}}}^{\mathfrak{p}}; v_1^{\mathfrak{p}}, \cdots, v_{n_{\mathfrak{p}}}^{\mathfrak{p}}$ in $\Lambda_{\mathfrak{p}}$ satisfying the similar relations as $(*)$. Since $n_{\mathfrak{p}}\leqq m$, we may assume without loss of generalities that $n_{\mathfrak{p}}=n$ is independent of $\mathfrak{p}$. Let us set $u_i^{\mathfrak{p}}=\sum_h a_{ih}^{\mathfrak{p}} w_h^{\mathfrak{p}}$, $v_i^{\mathfrak{p}}=\sum_h b_{ih}^{\mathfrak{p}} w_h^{\mathfrak{p}}$ where $a_{ih}^{\mathfrak{p}}$, $b_{ih}^{\mathfrak{p}}$ are in $F_{\mathfrak{p}}$, then from the hypothesis we may assume that $a_{ih}^{\mathfrak{p}}$, $b_{ih}^{\mathfrak{p}}$ are in $\mathfrak{O}_{\mathfrak{p}}$ for almost all $\mathfrak{p}$. We shall put $u_i=\sum_h a_{ih}w_h$, $v_i=\sum_h b_{ih}w_h$ where $a_{ih}=(\cdots, a_{ih}^{\mathfrak{p}}, \cdots)$, $b_{ih}=(\cdots, b_{ih}^{\mathfrak{p}}, \cdots)$ are in A_F, then the fact that $u_1, \cdots, u_n; v_1, \cdots, v_n$ satisfy the same relations as $(*)$ is readily verified. Thus Λ is a separable algebra over A_F. The statement about the centrality is easily verified and we omit the proof.

Now let us return to the proof of Proposition 1. For any $\bigoplus_{\mathfrak{p}} \mathfrak{X}_{\mathfrak{p}}\in \bigoplus_{\mathfrak{p}} \mathfrak{X}(G_{\mathfrak{p}})$, we can find a central separable algebra $\Lambda_{\mathfrak{p}}$ over $F_{\mathfrak{p}}$ such that the class of $\Lambda_{\mathfrak{p}}$ in $Br(F_{\mathfrak{p}})$ is mapped to $\mathfrak{X}_{\mathfrak{p}}$ by $\eta_{\mathfrak{p}}$ in (3). For $\mathfrak{X}_{\mathfrak{p}}=0$, we can take such that $\Lambda_{\mathfrak{p}}$ is similar to $F_{\mathfrak{p}}$, hence we may assume $(\Lambda_{\mathfrak{p}}\colon F_{\mathfrak{p}})=m$ is independent of $\mathfrak{p}$. For $\mathfrak{p}$ such that $\Lambda_{\mathfrak{p}}$ is similar to $F_{\mathfrak{p}}$, let $w_1^{\mathfrak{p}}, \cdots, w_m^{\mathfrak{p}}$ be matrix units and for another $\mathfrak{p}$ let $w_1^{\mathfrak{p}}, \cdots, w_m^{\mathfrak{p}}$ be an arbitrary basis of $\Lambda_{\mathfrak{p}}$ over $F_{\mathfrak{p}}$. We shall set $w_i^{\mathfrak{p}}w_j^{\mathfrak{p}}=\sum_h c_{ijh}^{\mathfrak{p}} w_h^{\mathfrak{p}}$, $c_{ijh}^{\mathfrak{p}}\in F_{\mathfrak{p}}$. We, now, construct an algebra Λ over A_F as follows; Let Λ be an A_F-algebra with an A_F-free basis $w_1, \cdots, w_m$ and with the structure coefficients $c_{ijh}=(\cdots, c_{ijh}^{\mathfrak{p}}, \cdots)\in A_F$, i.e. $w_iw_j=\sum_h c_{ijh}w_h$. Then by Lemma 2, Λ is a central separable algebra over A_F and the class of Λ in $Br(A_F)$ is mapped to the given $\bigoplus_{\mathfrak{p}} \mathfrak{X}_{\mathfrak{p}}$ by φ. Thus we have proved that φ is an epimorphism of $Br(A_F)$ to $\bigoplus_{\mathfrak{p}} \mathfrak{X}(G_{\mathfrak{p}})$.

This construction of the epimorphism φ is essentially due to Azumaya [4].

REMARK 1. For any element $\prod_{\mathfrak{p}} cl(\Lambda_{\mathfrak{p}})$ in the restricted direct product of $Br(F_{\mathfrak{p}})$ with respect to $Br(k_{\mathfrak{p}})$ such that the set of Schur indexes of $cl(\Lambda_{\mathfrak{p}})$ is bounded, we can construct, by the similar argument as in the proof of Proposition 1, a central separable algebra over A_F whose class in $Br(A_F)$ is mapped to the given $\prod_{\mathfrak{p}} cl(\Lambda_{\mathfrak{p}})$ by φ_0, where "cl" means the algbera class.

REMARK 2. Let $\prod_{\iota\in I} K_\iota$ be the direct product of fields K_ι, $\iota\in I$. For any element $\prod_{\iota\in I} cl(\Gamma_\iota)$ in the direct product of $Br(K_\iota)$ such that the set of Schur indexes of $cl(\Gamma_\iota)$ is bounded, we can construct a central separable algebra over $\prod_{\iota\in I} K_\iota$ whose class in $Br(\prod_{\iota\in I} K_\iota)$ is mapped to the given $\prod_{\iota\in I} cl(\Gamma_\iota)$ by ψ', where ψ' is the homomorphism of $Br(\prod_{\iota\in I} K_\iota)$ to $\prod_{\iota\in I} Br(K_\iota)$ induced by the projection $\prod_{\iota\in I} K_\iota \to K_\iota$.

The proof of this fact is also similar to that of Proposition 1.

To define the epimorphism φ, we used the homomorphism φ_0: $Br(A_F)\to \prod_{\mathfrak{p}} Br(F_{\mathfrak{p}})$. As to φ_0 we have the following

Proposition 3.a. *The homomorphism φ_0 is a monomorphism.*

Proof. Let Λ be a central separable algebra over A_F such that its class in $Br(A_F)$ is contained in the kernel of φ_0, i.e. $\Lambda_{\mathfrak{p}} = \Lambda \otimes_{A_F} F_{\mathfrak{p}} \sim F_{\mathfrak{p}}$ (similar) for all $\mathfrak{p}$. Let $\lambda_1 = 1, \lambda_2, \cdots, \lambda_m$ be a set of generators of Λ as an A_F-module. By the proof of Proposition 1, for almost all $\mathfrak{p}$, the $\mathfrak{O}_{\mathfrak{p}}$-module $\Gamma_{\mathfrak{p}}$ generated by $\lambda_1\otimes 1 = \lambda_1^{\mathfrak{p}}, \cdots, \lambda_m \otimes 1 = \lambda_m^{\mathfrak{p}}$ in $\Lambda_{\mathfrak{p}}$ is a separable $\mathfrak{O}_{\mathfrak{p}}$-order, which is of split type since $\mathfrak{O}_{\mathfrak{p}}$ is a Dedekind domain, i.e. there exists a finitely generated free $\mathfrak{O}_{\mathfrak{p}}$-module $E'_{\mathfrak{p}}$ such that $\Gamma_{\mathfrak{p}}$ is algebra-isomorphic to $\mathrm{Hom}_{\mathfrak{O}_{\mathfrak{p}}}(E_{\mathfrak{p}}', E_{\mathfrak{p}}')$. So we identify $\Gamma_{\mathfrak{p}}$ with $\mathrm{Hom}_{\mathfrak{O}_{\mathfrak{p}}}(E', E')$. We shall set $E_{\mathfrak{p}} = E_{\mathfrak{p}}' \otimes_{\mathfrak{O}_{\mathfrak{p}}} F_{\mathfrak{p}}$, then $\Lambda_{\mathfrak{p}} = \Gamma_{\mathfrak{p}} \otimes_{\mathfrak{O}_{\mathfrak{p}}} F_{\mathfrak{p}} = \mathrm{Hom}_{F_{\mathfrak{p}}}(E_{\mathfrak{p}}, E_{\mathfrak{p}})$. For another $\mathfrak{p}$, $\Lambda_{\mathfrak{p}}$ is algebra-isomorphic to $\mathrm{Hom}_{F_{\mathfrak{p}}}(E_{\mathfrak{p}}, E_{\mathfrak{p}})$ for some finitely generated $F_{\mathfrak{p}}$-module $E_{\mathfrak{p}}$. So we identify $\Lambda_{\mathfrak{p}}$ with $\mathrm{Hom}_{F_{\mathfrak{p}}}(E_{\mathfrak{p}}, E_{\mathfrak{p}})$. Let $E_{\mathfrak{p}}'$ be an arbitrary $\mathfrak{O}_{\mathfrak{p}}$-lattice of $E_{\mathfrak{p}}$, then $E_{\mathfrak{p}}'$ is a finitely generated free $\mathfrak{O}_{\mathfrak{p}}$-module since $\mathfrak{O}_{\mathfrak{p}}$ is a discrete valuation ring. Let E be the restricted direct product of $E_{\mathfrak{p}}$ with respect to $E_{\mathfrak{p}}'$, then one can easily check that E is a finitely generated projective faithful A_F-module with the canonical A_F-module structure on E, since $1 \leqq \mathrm{rank}_{\mathfrak{O}_{\mathfrak{p}}} E_{\mathfrak{p}}' = \mathrm{rank}_{F_{\mathfrak{p}}} E_{\mathfrak{p}} \leqq \sqrt{m}$. We define the Λ-module structure on E *via* the canonical homomorphism $\Lambda \to \prod_{\mathfrak{p}} \Lambda_{\mathfrak{p}}$. Then we obtain an A_F-algebra homomorphism α: $\Lambda \to \mathrm{Hom}_{A_F}(E, E)$ by the homothety. To see α is an epimorphism, it suffices to show that $\mathrm{Hom}_{A_F}(E, E)$ is generated by $\prod_{\mathfrak{p}} \lambda_1^{\mathfrak{p}}, \cdots, \prod_{\mathfrak{p}} \lambda_m^{\mathfrak{p}}$ as an A_F-module. For any $f \in \mathrm{Hom}_{A_F}(E, E)$, we shall denote the restriction of f to $E_{\mathfrak{p}}$ by $f_{\mathfrak{p}}$. Then $f_{\mathfrak{p}}$ has the form $\sum_h a_h^{\mathfrak{p}} \lambda_h^{\mathfrak{p}}$ with $a_h^{\mathfrak{p}}$ in $F_{\mathfrak{p}}$ since $\mathrm{Hom}_{F_{\mathfrak{p}}}(E_{\mathfrak{p}}, E_{\mathfrak{p}})$ is generated by $\lambda_1^{\mathfrak{p}}, \cdots, \lambda_m^{\mathfrak{p}}$ as an $F_{\mathfrak{p}}$-module. But $f_{\mathfrak{p}}$ sends $E_{\mathfrak{p}}'$ into $E_{\mathfrak{p}}'$ for almost all $\mathfrak{p}$, so $a_h^{\mathfrak{p}}$ must belong to $\mathfrak{O}_{\mathfrak{p}}$ for such a $\mathfrak{p}$. Thus $a_h=(\cdots, a_h^{\mathfrak{p}}, \cdots)$ is in fact an adèle. Hence f can be expressed by the form $\sum_h a_h \prod_{\mathfrak{p}} \lambda_h^{\mathfrak{p}}$, $a_h \in A_F$. Therefore α is an epimorphism, so an isomorphism by Corollary 3.2 of [2].

A central separable algebra Λ over A_F is a finitely generated A_F-module so Schur indexes of $\{\Lambda \otimes_{A_F} F_{\mathfrak{p}}\}_{\mathfrak{p}}$ are bounded. Combining this fact and the proof of Proposition 1, Remark 1 and Proposition 3.1, we get the following

Corollary 4.a. *By the monomorphism φ_0, $Br(A_F)$ can be identified with the subgroup of the restricted direct product of $Br(F_{\mathfrak{p}})$ with respect to $Br(k_{\mathfrak{p}})$ consisting of the elements whose Schur indexes of $\mathfrak{p}$-components are bounded.*

By the similar argument to the proof of Proposition 3.a, we get

Proposition 3.b. *Let $\prod_{\iota \in I} K_\iota$ be the direct product of fields K_ι (the cardinality of the index set $I=\{\iota\}$ is utterly arbitrary). Then the canonical homomorphism ψ_0': $Br(\prod_{\iota \in I} K_\iota) \to \prod_{\iota \in I} Br(K_\iota)$ is a monomorphism.*

Similarly to the proof of Corollary 4.a, we get

Corollary 4.b. *By the monomorphism ψ_0', $Br(\prod_{\iota \in I} K_\iota)$ can be identified with the subgroup of $\prod_{\iota \in I} Br(K_\iota)$ consisting of the elements whose Schur indexes of ι-components are bounded.*

As mentioned at the beginning of this section, $k_{\mathfrak{p}}$ can be imbedded to $\mathfrak{O}_{\mathfrak{p}}$. So $\prod_{\mathfrak{p}} k_{\mathfrak{p}}$ can be imbedded to $\prod_{\mathfrak{p}} \mathfrak{O}_{\mathfrak{p}} \subset A_F$, using this imbedding we shall define the homomorphism ψ: $Br(\prod k_{\mathfrak{p}}) \to Br(A_F)$.

Theorem 5. *The following sequence is exact.*

$$0 \longrightarrow Br(\prod_{\mathfrak{p}} k_{\mathfrak{p}}) \xrightarrow{\psi} Br(A_F) \xrightarrow{\varphi} \bigoplus_{\mathfrak{p}} \chi(G_{\mathfrak{p}}) \longrightarrow 0$$

Proof. Let us consider the following diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Br(\prod_{\mathfrak{p}} k_{\mathfrak{p}}) & \xrightarrow{\psi} & Br(A_F) & \xrightarrow{\varphi} & \bigoplus_{\mathfrak{p}} \chi(G_{\mathfrak{p}}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \psi_0} & & \downarrow{\scriptstyle \varphi_0} & & \downarrow & & \\
0 & \longrightarrow & \prod_{\mathfrak{p}} Br(k_{\mathfrak{p}}) & \xrightarrow{\prod \theta_{\mathfrak{p}}} & \prod_{\mathfrak{p}} Br(F_{\mathfrak{p}}) & \xrightarrow{\prod \eta_{\mathfrak{p}}} & \prod_{\mathfrak{p}} \chi(G_{\mathfrak{p}}) & \longrightarrow & 0
\end{array}$$

Then by the definitions of arrows, the above diagram is commutative with the exact lower row. From the commutativity of the above diagram, it follows that

ψ is a monomorphism since ψ_0 is a monomorphism by Proposition 3.b. Also it follows that the image of ψ is contained in the kernel of φ. Conversely, let Λ be a central separable algebra over A_F such that its class in $Br(A_F)$ is contained in the kernel of φ. We shall set $\Lambda_{\mathfrak{p}}=\Lambda\underset{A_F}{\otimes}F_{\mathfrak{p}}$, then the calss of $\Lambda_{\mathfrak{p}}$ in $Br(F_{\mathfrak{p}})$ is contained in the kernel of $\eta_{\mathfrak{p}}$. So by the exactness of the sequence (3), there exists a central separable algebra $\Gamma_{\mathfrak{p}}$ over $k_{\mathfrak{p}}$ such that $\Gamma_{\mathfrak{p}}\underset{k_{\mathfrak{p}}}{\otimes}F_{\mathfrak{p}}$ is similar to $\Lambda_{\mathfrak{p}}$. The Schur indexes of $\Gamma_{\mathfrak{p}}$'s are bounded since those of $\Lambda_{\mathfrak{p}}$'s are bounded. So, according to Remark 2 we can construct a central separable algebra Γ over $\prod_{\mathfrak{p}}k_{\mathfrak{p}}$ in such a way that $cl(\Gamma)\in Br(\prod_{\mathfrak{p}}k_{\mathfrak{p}})$ is mapped to $\prod cl(\Gamma_{\mathfrak{p}})\in\prod_{\mathfrak{p}}Br(k_{\mathfrak{p}})$ by ψ_0. If we set $\Gamma'=\Gamma\underset{\prod_{\mathfrak{p}}k_{\mathfrak{p}}}{\otimes}A_F$, then $cl(\Gamma')\in Br(A_F)$ is mapped to $\prod_{\mathfrak{p}}cl(\Lambda_{\mathfrak{p}})\in\prod_{\mathfrak{p}}Br(F_{\mathfrak{p}})$ by φ_0. Thus Λ and Γ' are similar since φ_0 is a monomorphism by Proposition 3.a. Therefore, the kernel of φ is contained in the image of ψ. And φ is an epimorphism by Proposition 1. This completes the proof of Theorem 5.

Corollary 6. *If k is a finite field, then $Br(A_F)$ is isomorphic to $\bigoplus_{\mathfrak{p}}\chi(G_{\mathfrak{p}})$ and isomorphic to $\bigoplus_{\mathfrak{p}}Br(F_{\mathfrak{p}})$,* i.e. $Br(A_F)\cong\bigoplus_{\mathfrak{p}}\chi(G_{\mathfrak{p}})\cong\bigoplus_{\mathfrak{p}}Br(F_{\mathfrak{p}})$.

REMARK 3. Let A_K be an adèle ring of the algebraic number field K, then, replacing A_F by A_K and $F_{\mathfrak{p}}$ by $K_{\mathfrak{p}}$, Proposition 1 and Proposition 3.a still hold. Thus we get the isomorphisms $Br(A_K)\cong\bigoplus_{\mathfrak{p}}\chi(G_{\mathfrak{p}})\cong\bigoplus_{\mathfrak{p}}Br(K_{\mathfrak{p}})$. So our results contain those of Azumaya [4] essentially.

2. The homomorphism of $Br(F)$ to $Br(A_F)$

The ring homomorphism $F\ni a\mapsto(\cdots, a, \cdots)\in A_F$ (diagonal) induces the homomorphism $\rho\colon Br(F)\to Br(A_F)$. In this section we shall determine the kernel of ρ.

Let K be a finite dimensional Galois extension of k with the Galois group G_K. We identify G_K with the Galois group of FK over F, where FK is the field theoretic composition of F and K. Let J_{FK} be the idèle group of FK over K, i.e. the group consisting of all the units of the adèle ring A_{FK} of FK. We fix a prime divisor $\mathfrak{P}$ of FK over K lying above the prime divisor $\mathfrak{p}$ of F over k. Let $G_{K\mathfrak{p}}$ be the decomposition group of $\mathfrak{P}$, and $\mathfrak{P}_1=\mathfrak{P}, \cdots, \mathfrak{P}_g$ be the complete set of prime divisors of FK over K lying above $\mathfrak{p}$. If we put $FK_{\mathfrak{p}}^*=\{(\cdots, a_{\mathfrak{Q}}, \cdots)\in J_{FK}\mid a_{\mathfrak{Q}}=1$ if $\mathfrak{Q}\neq\mathfrak{P}_1, \cdots, \mathfrak{P}_K\}$ the G_K-subgroup of J_{FK}, then we have $H^q(G_K, FK_{\mathfrak{p}}^*)\cong H^q(G_{K\mathfrak{p}}, FK_{\mathfrak{P}}^*)$ by Shapiro's lemma (c.f. Serre [8], p. 128, Exercices) since $FK_{\mathfrak{p}}^*\cong FK_{\mathfrak{P}_1}^*\times\cdots\times FK_{\mathfrak{P}_g}^*=\prod_{\bar{\sigma}\in G_K/G_{K\mathfrak{p}}}(FK_{\mathfrak{P}}^*)^{\bar{\sigma}}$ where $FK_{\mathfrak{P}_i}^*$ is the group of all the units of $FK_{\mathfrak{P}_i}$. We shall denote the valuation ring of $FK_{\mathfrak{P}_i}$ by

$\mathfrak{O}_{\mathfrak{P}_i}$ and we shall denote the group of all the units of $\mathfrak{O}_{\mathfrak{P}_i}$ by $U_{\mathfrak{P}_i}$. We shall set $U'_{\mathfrak{p}}=U_{\mathfrak{P}_1}\times\cdots\times U_{\mathfrak{P}_g}$. Then we have $H^q(G_K, U'_{\mathfrak{p}})\cong H^q(G_K, U_{\mathfrak{P}})$ by the above isomorphism. Now, as J_{FK} can be considered to be the restricted direct product of the G_K-subgroup $FK^*_{\mathfrak{p}}$ with respect to $U'_{\mathfrak{p}}$, $H^q(G_K, J_{FK})$ is mapped surjectively to the restricted direct product $\prod'_{\mathfrak{p}} H^q(G_K, FK^*_{\mathfrak{p}})$ of $H^q(G_K, FK^*_{\mathfrak{p}})$ with respect to $H^q(G_K, U'_{\mathfrak{p}})$, since if the $FK^*_{\mathfrak{p}}$-component of $f\in Z^q(G_K, \prod_{\mathfrak{p}} FK^*_{\mathfrak{p}})$ is in $Z^q(G_K, U'_{\mathfrak{p}})$ for almost all $\mathfrak{p}$, then f is a cocycle in $Z^q(G_K, J_{FK})$. The homomorphism $H^q(G_{K\mathfrak{p}}, K^*_{\mathfrak{P}})\to H^q(G_{K\mathfrak{p}}, FK^*_{\mathfrak{P}})$ is a monomorphism for any $\mathfrak{p}$, since any $\mathfrak{p}$ is unramified, where $K^*_{\mathfrak{P}}$ is the group of all units of $K_{\mathfrak{P}}=\mathfrak{O}_{\mathfrak{P}}/\mathfrak{P}$. Now, we suppose that for $f\in Z^q(G_K, J_{FK})$ there exists $g_{\mathfrak{p}}\in C^{q-1}(G_K, FK^*_{\mathfrak{p}})$ such that $\partial g_{\mathfrak{p}}=f_{\mathfrak{p}}$ for all the $FK^*_{\mathfrak{p}}$-component $f_{\mathfrak{p}}$ of f. Then we may assume that $g_{\mathfrak{p}}$ is in $C^{q-1}(G_K, U'_{\mathfrak{p}})$ for almost all $\mathfrak{p}$, since $H^q(G_{K\mathfrak{p}}, K^*_{\mathfrak{P}})$ is isomorphic to $H^q(G_K, U^*_{\mathfrak{p}})$. We define $g\in C^{q-1}(G_K, J_{FK})$ that its $FK^*_{\mathfrak{p}}$-component is equal to $g_{\mathfrak{p}}$. Then we get $f=\partial g$. Thus we have proved that $H^q(G_K, J_{FK})$ is isomorphic to $\prod'_{\mathfrak{p}} H^q(G_K, FK^*_{\mathfrak{p}})$. Passing to direct limit and using the well-known isomorphisms: $Br(FK_{\mathfrak{P}}/F_{\mathfrak{p}})\cong H^2(G_{K\mathfrak{p}}, FK^*_{\mathfrak{P}})$, $H^2(G_{K\mathfrak{p}}, U_{\mathfrak{P}})\cong H^2(G_{K\mathfrak{p}}, K^*_{\mathfrak{P}})\cong Br(K_{\mathfrak{P}}/k_{\mathfrak{p}})$, we get the following

Proposition 7.a. *$H^2(G, \bar{J})$ is isomorphic to the subgroup of the restricted direct product of $Br(F_{\mathfrak{p}})$ with respect to $Br(k_{\mathfrak{p}})$, which consists of the elements $\prod_{\mathfrak{p}} cl(\Lambda_{\mathfrak{p}})$ satisfying the following condition: There exists a finite dimensional Galois extension K of k such that $FK_{\mathfrak{P}}$ splits the $\mathfrak{p}$-component $cl(\Lambda_{\mathfrak{p}})$ for every $\mathfrak{p}$, where $\bar{J}$ is the idèle group of $\bar{F}=F\cdot\bar{k}$ over $\bar{k}$.*

Similarly, we get the following

Proposition 7.b. *$H^2(G, \bar{U})$ is isomorphic to the subgroup of the direct product $\prod_{\mathfrak{p}} Br(k_{\mathfrak{p}})$ of $Br(k_{\mathfrak{p}})$, which consists of the elements $\prod_{\mathfrak{p}} cl(\Gamma_{\mathfrak{p}})$ satisfying the same condition of Proposition 7.a, where $\bar{U}$ is the group of idèle units in $\bar{J}$.*

Now we are ready to prove the following

Theorem 8. *The kernel of $\rho\colon Br(F)\to Br(A_F)$ is isomorphic to $H^1(G, \overline{CJ})$. More precisely, the following diagram with canonical arrows is commutative with exact rows and columns.*

$$
\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & \vdots \arrow[d] & H^1(G,\bar{H}) \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & H^1(G,\overline{CU}) \arrow[r] \arrow[d] & Br(k) \arrow[r] \arrow[d] & Br(\prod_{\mathfrak{p}} k_{\mathfrak{p}}) \arrow[d, "\psi"] & \\
0 \arrow[r] & H^1(G,\overline{CJ}) \arrow[r] \arrow[d] & Br(F) \arrow[r, "\rho"] \arrow[d] & Br(A_F) \arrow[d, "\varphi"] & \\
0 \arrow[r] & H^1(G,\overline{CD}) \arrow[r] \arrow[d] & H^2(G,\bar{H}) \arrow[r] \arrow[d] & \bigoplus_{\mathfrak{p}} \chi(G_{\mathfrak{p}}) \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots & 0 &
\end{tikzcd}
$$

where we use the following notations:

$\bar{D}$: *the divisor group of* $\bar{F}/\bar{k}$

$\bar{H}$: *the group of principal divisors in* $\bar{D}$

$\overline{CD}=\bar{D}/\bar{H}$: *the divisor class group*

$\overline{CJ}=\bar{J}/\bar{F}^*$: *the group of idèle classes*

$\overline{CU}=\bar{U}/\bar{k}^*$: *the group of idèle unit classes.*

Proof. The commutativity is easy so we omit it. The exactness of columns is clear by Theorem 5 and by the following commutative diagram of G-modules and G-homomorphisms with exact rows and columns (c.f. Roquette [6], Scharlau [7]).

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \bar{k}^* \arrow[r] \arrow[d] & \bar{U} \arrow[r] \arrow[d] & \overline{CU} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bar{F}^* \arrow[r] \arrow[d] & \bar{J} \arrow[r] \arrow[d] & \overline{CJ} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bar{H} \arrow[r] \arrow[d] & \bar{D} \arrow[r] \arrow[d] & \overline{CD} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

The exactness of the lower row is clear by the above diagram since $H^2(G,\bar{D})\cong\bigoplus_{\mathfrak{p}}\chi(G_{\mathfrak{p}})$ by Shapiro's lemma. Also the homomorphisms $H^1(G,\overline{CJ})\to Br(F)$, $H^1(G,\overline{CU})\to Br(k)$ are monomorphisms since $H^1(G,\bar{U})=0$, $H^1(G,\bar{J})=0$. By the following commutative diagrams,

$$
\begin{tikzcd}
0 \arrow[d] & & 0 \arrow[d] & \\
H^1(G, \overline{CJ}) \arrow[d] & & H^1(G \overline{CU}) \arrow[d] & \\
Br(F) \arrow[r, "\rho"] \arrow[d] & Br(A_F) \arrow[d, "\varphi_0\text{: mono.}"] & Br(k) \arrow[r] \arrow[d] & Br(\prod_{\mathfrak{p}} k_{\mathfrak{p}}) \arrow[d, "\psi_0\text{: mono.}"] \\
H^2(G, \bar{J}) \arrow[r, "\text{mono.}"'] & \prod_{\mathfrak{p}} Br(F_{\mathfrak{p}}) & H^2(G, \bar{U}) \arrow[r, "\text{mono.}"'] & \prod_{\mathfrak{p}} Br(k_{\mathfrak{p}})
\end{tikzcd}
$$

and by Proposition 7.a, 7.b, we can easily see that the upper row and the middle row are exact. This completes the proof of Theorem 10.

By Tate [9], we have immediately

Corollary 9. *If k is a $\mathfrak{p}$-adic number field, then the homomorphism $\rho: Br(F) \to Br(A_F)$ is a monomorphism.*

REMARK 4. By Remark 1, 2 and Proposition 7a, 7b, we can define β: $H^2(G, \bar{J}) \to Br(A_F)$, $\gamma: H^2(G, \bar{U}) \to Br(\prod k_{\mathfrak{p}})$. With these homomorphisms, the following diagram is commutative with exact rows.

$$
\begin{tikzcd}
0 \arrow[r] & H^2(G, \bar{U}) \arrow[r] \arrow[d, "\gamma"] & H^2(G, \bar{J}) \arrow[r] \arrow[d, "\beta"] & \bigoplus_{\mathfrak{p}} \chi(G_{\mathfrak{p}}) \arrow[r] \arrow[d, equal] & \cdots \\
0 \arrow[r] & Br(\prod_{\mathfrak{p}} k_{\mathfrak{p}}) \arrow[r, "\psi"] & Br(A_F) \arrow[r, "\varphi"] & \bigoplus_{\mathfrak{p}} \chi(G_{\mathfrak{p}}) \arrow[r] & 0
\end{tikzcd}
$$

The homomorphism β is given by the crossed product. In fact, for any finite Galois extension K of k, A_{FK} is nothing else $A_F \otimes_k K$. So A_{FK}/A_K is Galois extension of rings. Since φ_0 is a monomorphism, the homomorphism β is completely determined by the composite $\varphi_0 \circ \beta$: $H^2(G, \bar{J}) \to \prod_{\mathfrak{p}} Br(F_{\mathfrak{p}})$. And the composite $\varphi_0 \circ \beta$ is obtained by the componentwise crossed product. So our assertion follows immediately. But the author does not know whether β is an epimorphism or not.

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