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BRAUER GROUPS OF ALGEBRAIC FUNCTION FIELDS
AND THEIR ADÉLE RINGS

KENJI YOKOGAWA

(Received November 25, 1970)

Introduction. Let $K$ be an algebraic number field and $\{\mathfrak{p}\}$ be the valuations of $K$, then related to Takagi-Artin's class field theory, the following exact sequence is well-known (c.f. Hasse [5]);

$$0 \rightarrow Br(K) \rightarrow \bigoplus_{\mathfrak{p}} Br(K_\mathfrak{p}) \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$$

where $K_\mathfrak{p}$ is the completion of $K$ with respect to $\mathfrak{p}$. In the Seminar 1966 at Bowdoin College, G. Azumaya [4] showed that the middle term of (1) is isomorphic to the Brauer group of the adèle ring $A_K$ of $K$ and that the following diagram with canonical arrows is commutative;

$$0 \rightarrow Br(K) \rightarrow \bigoplus_{\mathfrak{p}} Br(K_\mathfrak{p}) \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$$

But on an algebraic function field, the class field theory does not hold except the case of finite constant field (Artin-Whalpe [1]), so the analogies of (1), (2) must have fallen.

The purpose of this paper is to clarify the relations of the Brauer group of the adèle ring of a function field, to the Brauer group of a function field and to Galois cohomologies.

We use the following notations:

- $k$: a perfect field
- $\bar{k}$: the algebraic closure of $k$
- $F$: an algebraic function field of one variable over $k$ i.e. $F/k$ is finitely generated, $k$ is algebraically closed in $F$ and the degree of transcendency of $F/k$ is one
- $F=F_{\bar{k}}$: the field theoretic compositum of $F$ and $\bar{k}$
- $\mathfrak{p}$: a prime divisor of $F$ over $k$
- $F_{\mathfrak{p}}$: the completion of $F$ with respect to $\mathfrak{p}$
- $\mathcal{O}_{F_{\mathfrak{p}}}$: the valuation ring of $F_{\mathfrak{p}}$
$k_p$ : the residue class field of $F_p$ i.e. $\mathcal{O}_p/p$

$G$ : the Galois group of $\bar{k}$ over $k$ and we shall identify $G$ with the Galois group of $F$ over $F$

$G_p$ : the decomposition group of $p$

$\chi(*)$ : the character group of the group $\ast$

$A_{F} = A_{F/k}$ : the adele ring of $F$ i.e. the restricted direct product of $F_p$ with respect to $\mathcal{O}_p$

$Br(*)$ : the Brauer group of the ring $\ast$.

The author would like to express his gratitude to Dr. Y. Watanabe for suggesting this problem, for helpful advices and for continuing encouragement.

1. The homomorphism of $Br(A_F)$ to $\bigoplus_p \chi(G_p)$

It is well-known that $\mathcal{O}_p$ coincides with the formal power series ring $k_p[[\pi_p]]$ with respect to some prime element $\pi_p$ and $F_p$ coincides with $k_p((\pi_p))$ (c.f. Serre [8] II, §4). Witt [10] and Shuen Yuan [11] showed the sequence

$0 \longrightarrow Br(k_p) \xrightarrow{\theta_p} Br(F_p) \xrightarrow{\eta_p} \chi(G_p) \longrightarrow 0$ (3)

is exact, where $\theta_p$ is the one induced by the ring homomorphism $k_p \rightarrow F_p$. Azumaya [3] and Auslander-Goldman [2] showed that $Br(k_p)$ is isomorphic to $Br(\mathcal{O}_p)$. From the sequence (3), considering the direct product for all $p$, we have the following exact sequence;

$0 \longrightarrow \prod_p Br(k_p) \xrightarrow{\prod \theta_p} \prod_p Br(F_p) \xrightarrow{\prod \eta_p} \prod_p \chi(G_p) \longrightarrow 0$ (4)

Proposition 1. There exists the epimorphism $\varphi$ of $Br(A_F)$ to the direct sum $\bigoplus_p \chi(G_p)$ of $\chi(G_p)$.

Proof. Let $\Lambda$ be a central separable algebra over $A_F$ and $\lambda_1=1$, $\lambda_2$, $\cdots$, $\lambda_m$ be a set of generators of $\Lambda$ over $A_F$. Since $\Lambda$ is separable over $A_F$, there exist the elements $u_1, \cdots, u_n$; $v_1, \cdots, v_m$ in $\Lambda$ satisfying the relations;

$$\sum_i u_i \psi_i = 1$$

$$\sum_i x u_i \otimes v_i^p = \sum_i u_i \otimes (v_i x)^p$$ in the enveloping algebra $\Lambda^e = \Lambda \otimes A_F^e$ for any $x$ in $\Lambda$.

The $\mathcal{O}_p$-module generated
by $\lambda_1^p, \ldots, \lambda_n^p$. Then $u_1^p, \ldots, u_n^p; v_1^p, \ldots, v_n^p$ satisfy the similar relations as $(*)$. Thus $\Gamma_p$ is a separable $\mathcal{O}_p$-order in $\Lambda_p$ for almost all $p$ since $u_1^p, \ldots, u_n^p; v_1^p, \ldots, v_n^p$ are contained in $\Gamma_p$ for almost all $p$ and $\Gamma_p$ forms a ring with identity for almost all $p$. Therefore, defining $\varphi_p : Br(A_F) \rightarrow \prod_p Br(F_p)$ to be the homomorphism induced by the projection $A_F \rightarrow F_p$, the image of $\varphi_p$ is contained in the restricted direct product of $Br(F_p)$ with respect to $Br(\mathcal{O}_p) \simeq Br(k_p)$. We define $\varphi$ to be the composite $\prod_p \eta_p \circ \varphi_0$, then the image of $\varphi$ is in $\bigoplus_p \mathcal{X}(G_p)$.

To see $\varphi$ is an epimorphism we need the following

**Lemma 2.** Let $\Lambda$ be an algebra over $A_F$ which is a finitely generated free $A_F$-module with the free basis $w_1, \ldots, w_m$. If $\Lambda_p = \Lambda \otimes F_p$ is a central separable algebra over $F_p$ for all $p$ and the $\mathcal{O}_p$-module generated by $w_1^p = w_1 \otimes 1, \ldots, w_m^p = w_m \otimes 1$ in $\Lambda_p$ is a separable $\mathcal{O}_p$-order for almost all $p$, then $\Lambda$ is a central separable algebra over $A_F$.

Proof of Lemma 2. As $\Lambda_p$ is separable over $F_p$, there are the elements $u_1^p, \ldots, u_n^p; v_1^p, \ldots, v_n^p$ in $\Lambda_p$ satisfying the similar relations as $(*)$. Since $n_p \leq m$, we may assume without loss of generalities that $n_p = n$ is independent of $p$. Let us set $u_i^p = \sum a_{ih}^p w_h, \quad v_i^p = \sum b_{ih}^p w_h^p$ where $a_{ih}^p, b_{ih}^p$ are in $F_p$, then from the hypothesis we may assume that $a_{ih}^p, b_{ih}^p$ are in $\mathcal{O}_p$ for almost all $p$. We shall put $u_i = \sum a_{ih} w_h, \quad v_i = \sum b_{ih}^p w_h$ where $a_{ih} = (\cdots, a_{ih}^p, \cdots), \quad b_{ih} = (\cdots, b_{ih}^p, \cdots)$ are in $A_F$, then the fact that $u_1, \ldots, u_n; v_1, \ldots, v_n$ satisfy the same relations as $(*)$ is readily verified. Thus $\Lambda$ is a separable algebra over $A_F$. The statement about the centrality is easily verified and we omit the proof.

Now let us return to the proof of Proposition 1. For any $\bigoplus \mathcal{X}_p \subseteq \bigoplus \mathcal{X}(G_p)$, we can find a central separable algebra $\Lambda_p$ over $F_p$ such that the class of $\Lambda_p$ in $Br(F_p)$ is mapped to $\chi_p$ by $\eta_p$ in (3). For $\chi_p = 0$, we can take such that $\Lambda_p$ is similar to $F_p$, hence we may assume $(\Lambda_p : F_p) = m$ is independent of $p$. For $p$ such that $\Lambda_p$ is similar to $F_p$, let $w_1^p, \ldots, w_m^p$ be matrix units and for another $p$ let $w_1^p, \ldots, w_m^p$ be an arbitrary basis of $\Lambda_p$ over $F_p$. We shall set $w_i^p w_j^p = \sum c_{ijh}^p w_h$, $c_{ijh}^p \in F_p$. We, now, construct an algebra $\Lambda$ over $A_F$ as follows; Let $\Lambda$ be an $A_F$-algebra with an $A_F$-free basis $w_1, \ldots, w_m$ and with the structure coefficients $c_{ijh} = (\cdots, c_{ijh}, \cdots) \in A_F$, i.e. $w_i w_j = \sum c_{ijh} w_h$. Then by Lemma 2, $\Lambda$ is a central separable algebra over $A_F$ and the class of $\Lambda$ in $Br(A_F)$ is mapped to the given $\bigoplus \chi_p$ by $\varphi$. Thus we have proved that $\varphi$ is an epimorphism of $Br(A_F)$ to $\bigoplus \chi(G_p)$.

This construction of the epimorphism $\varphi$ is essentially due to Azumaya [4].
Remark 1. For any element $\prod_{p} cl(A_p)$ in the restricted direct product of $Br(F_p)$ with respect to $Br(k_p)$ such that the set of Schur indexes of $cl(A_p)$ is bounded, we can construct, by the similar argument as in the proof of Proposition 1, a central separable algebra over $A_F$ whose class in $Br(A_F)$ is mapped to the given $\prod_{p} cl(A_p)$ by $\varphi_0$, where "$cl$" means the algebra class.

Remark 2. Let $\prod_{i \in I} K_i$ be the direct product of fields $K_i, i \in I$. For any element $\prod_{i \in I} cl'(\Gamma_i)$ in the direct product of $Br(K_i)$ such that the set of Schur indexes of $cl'(\Gamma_i)$ is bounded, we can construct a central separable algebra over $HK_{\prod_{i \in I}}$ whose class in $Br(K_{\prod_{i \in I}})$ is mapped to the given $\prod_{i \in I} cl'(\Gamma_i)$ by $\psi'$, where $\psi'$ is the homomorphism of $Br(\prod_{i \in I} K_i)$ to $\prod_{i \in I} Br(K_i)$ induced by the projection $\prod_{i \in I} K_i \to K_i$.

The proof of this fact is also similar to that of Proposition 1.

To define the epimorphism $\varphi$, we used the homomorphism $\varphi_0: Br(A_F) \to \prod_{p} Br(F_p)$. As to $\varphi_0$ we have the following

**Proposition 3.a. The homomorphism $\varphi_0$ is a monomorphism.**

Proof. Let $\Lambda$ be a central separable algebra over $A_F$ such that its class in $Br(A_F)$ is contained in the kernel of $\varphi_0$, i.e. $\Lambda_p = \Lambda \otimes F_p \sim F_p$ (similar) for all $p$.

Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be a set of generators of $\Lambda$ as an $A_F$-module. By the proof of Proposition 1, for almost all $p$, the $\mathcal{O}_p$-module $\Gamma_p$ generated by $\lambda_i \otimes 1 = \lambda_i$ in $\Lambda_p$ is a separable $\mathcal{O}_p$-order, which is of split type since $\mathcal{O}_p$ is a Dedekind domain, i.e. there exists a finitely generated free $\mathcal{O}_p$-module $E_p$ such that $\Gamma_p$ is algebra-isomorphic to $\text{Hom}_{\mathcal{O}_p}(E_p, E_p)$. So we identify $\Gamma_p$ with $\text{Hom}_{\mathcal{O}_p}(E_p, E_p)$.

We shall set $E_p = E_p \otimes F_p$, then $\Lambda_p = \Gamma_p \otimes F_p = \text{Hom}_{F_p}(E_p, E_p)$.

For another $p$, $\Lambda_p$ is algebra-isomorphic to $\text{Hom}_{F_p}(E_p, E_p)$ for some finitely generated $F_p$-module $E_p$. So we identify $\Lambda_p$ with $\text{Hom}_{F_p}(E_p, E_p)$. Let $E_p'$ be an arbitrary $\mathcal{O}_p$-lattice of $E_p$, then $E_p'$ is a finitely generated free $\mathcal{O}_p$-module since $\mathcal{O}_p$ is a discrete valuation ring. Let $E$ be the restricted direct product of $E_p$ with respect to $E_p'$, then one can easily check that $E$ is a finitely generated projective faithful $A_F$-module with the canonical $A_F$-module structure on $E$, since $1 \leq \text{rank}_{\mathcal{O}_p} E_p' = \text{rank}_{F_p} E_p \leq \sqrt{m}$. We define the $\Lambda$-module structure on $E$ via the canonical homomorphism $\Lambda \to \prod_{p} \Lambda_p$. Then we obtain an $A_F$-algebra homomorphism $\alpha: \Lambda \to \text{Hom}_{A_F}(E, E)$ by the homothety. To see $\alpha$ is an epimorphism, it suffices to show that $\text{Hom}_{A_F}(E, E) = \prod_{p} \text{Hom}_{\Lambda_p}(E_p, E_p)$ as an $A_F$-module. For any $f \in \text{Hom}_{A_F}(E, E)$, we shall denote the restriction of $f$ to $E_p$ by $f_p$. Then $f_p$ has the form $\sum_{i} a_i^p \lambda_i^p$ with $a_i^p$ in $F_p$ since $\text{Hom}_{F_p}(E_p, E_p)$ is generated by $\lambda_1^p, \ldots, \lambda_m^p$ as an $F_p$-module. But $f_p$ sends $E_p'$ into $E_p'$ for almost all $p$,.
so $a^p_k$ must belong to $\mathcal{O}_p$ for such a $p$. Thus $a^p = (\cdots, a^p_k, \cdots)$ is in fact an adèle. Hence $f$ can be expressed by the form $\sum_s a^p_s \prod_p \lambda^p_x$, $a^p_k \in A_F$. Therefore $\alpha$ is an epimorphism, so an isomorphism by Corollary 3.2 of [2].

A central separable algebra $\Lambda$ over $A_F$ is a finitely generated $A_F$-module so Schur indexes of $\{\Lambda \otimes F_p\}_p$ are bounded. Combining this fact and the proof of Proposition 1, Remark 1 and Proposition 3.1, we get the following

**Corollary 4.a.** By the monomorphism $\varphi_\psi$, $\text{Br}(A_F)$ can be identified with the subgroup of the restricted direct product of $\text{Br}(F_p)$ with respect to $\text{Br}(k_p)$ consisting of the elements whose Schur indexes of $\psi$-components are bounded.

By the similar argument to the proof of Proposition 3.a, we get

**Proposition 3.b.** Let $\prod_{\nu \in I} K_{\nu}$ be the direct product of fields $K_{\nu}$ (the cardinality of the index set $I = \{\nu\}$ is utterly arbitrary). Then the canonical homomorphism $\varphi'_{\psi} : \text{Br}(\prod_{\nu \in I} K_{\nu}) \rightarrow \prod_{\nu \in I} \text{Br}(K_{\nu})$ is a monomorphism.

Similarly to the proof of Corollary 4.a, we get

**Corollary 4.b.** By the monomorphism $\varphi'_{\psi}$, $\text{Br}(\prod_{\nu \in I} K_{\nu})$ can be identified with the subgroup of $\prod_{\nu \in I} \text{Br}(K_{\nu})$ consisting of the elements whose Schur indexes of $\psi$-components are bounded.

As mentioned at the beginning of this section, $k_p$ can be imbedded to $\mathcal{O}_p$. So $\prod_p k_p$ can be imbedded to $\prod_p \mathcal{O}_p \subset A_F$, using this imbedding we shall define the homomorphism $\psi : \text{Br}(\prod_p k_p) \rightarrow \text{Br}(A_F)$.

**Theorem 5.** The following sequence is exact.

$$0 \rightarrow \text{Br}(\prod_p k_p) \xrightarrow{\psi} \text{Br}(A_F) \xrightarrow{\varphi} \bigoplus_p \chi(G_p) \rightarrow 0$$

**Proof.** Let us consider the following diagram.

$$0 \rightarrow \text{Br}(\prod_p k_p) \xrightarrow{\psi} \text{Br}(A_F) \xrightarrow{\varphi} \bigoplus_p \chi(G_p) \rightarrow 0$$

$$\downarrow \psi_\psi \quad \downarrow \varphi_\psi \quad \downarrow \psi_\varphi \quad \downarrow \varphi_\varphi$$

$$0 \rightarrow \prod_p \text{Br}(k_p) \xrightarrow{\Pi \theta_p} \prod_p \text{Br}(F_p) \xrightarrow{\Pi \eta_p} \prod_p \chi(G_p) \rightarrow 0$$

Then by the definitions of arrows, the above diagram is commutative with the exact lower row. From the commutativity of the above diagram, it follows that
ψ is a monomorphism since ψ₀ is a monomorphism by Proposition 3.b. Also it follows that the image of ψ is contained in the kernel of φ. Conversely, let Λ be a central separable algebra over A_F such that its class in Br(A_F) is contained in the kernel of φ. We shall set Λ_p= Λ_F F_p, then the class of Λ_p in Br(F_p) is contained in the kernel of ψ_p. So by the exactness of the sequence (3), there exists a central separable algebra Γ_p over k_p such that Γ_p F_p is similar to Λ_p.

The Schur indexes of Γ_p's are bounded since those of Λ_p's are bounded. So, according to Remark 2 we can construct a central separable algebra Γ over k such that the class of Γ in Br(k) is contained in the kernel of ψ. We shall set A_Γ= A®F A then the class of Λ in Br(k) is contained in the kernel of ψ. Thus, by the exactness of the sequence (3), there exists a central separable algebra Γ over k such that Γ F is similar to Λ.

If k is a finite field, then Br(A_F) is isomorphic to ⨁p X(G_p) and isomorphic to ⨁p Br(F_p), i.e. Br(A_F)≈ ⨁p X(G_p)≈ ⨁p Br(F_p).

**Remark 3.** Let A_K be an adèle ring of the algebraic number field K, then, replacing A_F by A_K and F_p by K_p, Proposition 1 and Proposition 3.a still hold. Thus we get the isomorphisms Br(A_K)≈ ⨁p X(G_p)≈ ⨁p Br(K_p). So our results contain those of Azumaya [4] essentially.

**2. The homomorphism of Br(F) to Br(A_F)**

The ring homomorphism F∋a→(⋯, a, ⋯)∈A_F (diagonal) induces the homomorphism ρ: Br(F)→Br(A_F). In this section we shall determine the kernel of ρ.

Let K be a finite dimensional Galois extension of k with the Galois group G_K. We identify G_K with the Galois group of FK over F, where FK is the field theoretic composition of F and K. Let J_{FK} be the idèle group of FK over K, i.e. the group consisting of all the units of the adèle ring A_{FK} of FK. We fix a prime divisor ν of FK over K lying above the prime divisor ν of F over k. Let G_{Kp} be the decomposition group of ν over K, and G{ν}= G{ν}, ⋯, G{ν} be the complete set of prime divisors of FK over K lying above ν. If we put FK{ν}= (⋯, a_ν, ⋯)∈J_{FK}|a_ν= 1 if ν ⊆ G{ν}, ⋯, G{ν} the G_K-subgroup of J_{FK}, then we have H*(G_K, FK{ν})≈ H*(G_{Kp}, FK{ν}) by Shapiro's lemma (c.f. Serre [8], p. 128, Exercises) since FK{ν}= FK{ν}_1 × ⋯ × FK{ν}_k = (FK{ν})σ= G_{Kp}/G_{Kp} for FK{ν}_1 is the group of all the units of FK{ν}_1. We shall denote the valuation ring of FK{ν}_1 by
and we shall denote the group of all the units of $\mathcal{O}_E$ by $U_E$. We shall set $U'_p = U_{E_p} \times \cdots \times U_{E_p}$. Then we have $H^q(G_K, U'_p) \approx H^q(G_K, U_E)$ by the above isomorphism. Now, as $J_{FK}$ can be considered to be the restricted direct product of the $G_K$-subgroup $FK^*_p$ with respect to $U'_p$, $H^q(G_K, J_{FK})$ is mapped surjectively to the restricted direct product $\prod_p H^q(G_K, FK^*_p)$ with respect to $H^q(G_K, U'_p)$, since if the $FK^*_p$-component of $f \in Z^q(G_K, IFK^*_p)$ is in $Z^q(G_K, U'_p)$ for almost all $p$, then $f$ is a cocycle in $Z^q(G_K, J_{FK})$. The homomorphism $H^q(G_K, K^*_p) \to H^q(G_K, FK^*_p)$ is a monomorphism for any $p$, since any $p$ is unramified, where $K^*_p$ is the group of all units of $K_p$. Now, we suppose that for $f \in Z^q(G_K, J_{FK})$ there exists $g_p \in C^q-1(G_K, FK^*_p)$ such that 

$\partial g_p = f_p$ for all the $FK^*_p$-component $f_p$ of $f$. Then we may assume that $g_p$ is in $C^q-1(G_K, U'_p)$ for almost all $p$, since $H^q(G_K, K^*_p)$ is isomorphic to $H^q(G_K, U'_p)$. We define $g \in C^q-1(G_K, J_{FK})$ that its $FK^*_p$-component is equal to $g_p$. Then we get $f = \partial g$. Thus we have proved that $H^q(G_K, J_{FK})$ is isomorphic to $\prod_p H^q(G_K, FK^*_p)$. Passing to direct limit and using the well-known isomorphisms:

$$Br(\mathbb{F}_p/F_p) \approx H^q(G_{K_p}, FK^*_p)$$,

$$H^q(G_{K_p}, U_p) \approx H^q(G_{K_p}, K^*_p) \approx Br(\mathbb{K}_p/k_p),$$

we get the following

**Proposition 7.a.** $H^q(G, J)$ is isomorphic to the subgroup of the restricted direct product of $Br(F_p)$ with respect to $Br(k_p)$, which consists of the elements $\prod_p \text{cl}(\Lambda_p)$ satisfying the following condition: There exists a finite dimensional Galois extension $K$ of $k$ such that $FK_{\mathbb{K}}$ splits the $p$-component $\text{cl}(\Lambda_p)$ for every $p$, where $J$ is the idèle group of $F = F \cdot \tilde{k}$ over $\tilde{k}$.

Similarly, we get the following

**Proposition 7.b.** $H^q(G, \bar{U})$ is isomorphic to the subgroup of the direct product $\prod_p Br(k_p)$ of $Br(k_p)$, which consists of the elements $\prod_p \text{cl}(\Gamma_p)$ satisfying the same condition of Proposition 7.a, where $\bar{U}$ is the group of idèle units in $\bar{J}$.

Now we are ready to prove the following

**Theorem 8.** The kernel of $\rho: Br(F) \to Br(A_F)$ is isomorphic to $H^1(G, \bar{CJ})$. More precisely, the following diagram with canonical arrows is commutative with exact rows and columns.
\[\begin{array}{c}
0 \\
\downarrow \\
\vdots \\
\downarrow \quad H_1(G, H) \quad 0 \\
\downarrow \\
0 \to H_1(G, \overline{CU}) \to Br(k) \to Br(\prod k_p) \\
\downarrow \\
0 \to H_1(G, \overline{CJ}) \to Br(F) \to Br(A_F) \\
\downarrow \\
0 \to H_1(G, \overline{CD}) \to H_1(G, \overline{H}) \to \bigoplus \chi(G_p) \to \ldots \\
\downarrow \\
\vdots \\
\downarrow \\
0
\end{array}\]

where we use the following notations:

\(D\): the divisor group of \(F/k\)

\(H\): the group of principal divisors in \(D\)

\(\overline{CD}=D/H\): the divisor class group

\(\overline{CJ}=\mathcal{J}/F^*\): the group of idèle classes

\(\overline{CU}=U/\mathcal{U}^*\): the group of idèle unit classes.

Proof. The commutativity is easy so we omit it. The exactness of columns is clear by Theorem 5 and by the following commutative diagram of \(G\)-modules and \(G\)-homomorphisms with exact rows and columns (c.f. Roquette [6], Scharlau [7]).

\[\begin{array}{c}
0 \\
\downarrow \\
0 \to k^* \to \mathcal{U} \to \overline{CU} \to 0 \\
\downarrow \\
0 \to F^* \to \mathcal{J} \to \overline{CJ} \to 0 \\
\downarrow \\
0 \to H \to \overline{D} \to \overline{CD} \to 0 \\
\downarrow \\
0 \to 0 \\
\end{array}\]

The exactness of the lower row is clear by the above diagram since \(H_1(G, \overline{D})=\bigoplus \chi(G_p)\) by Shapiro's lemma. Also the homomorphisms \(H_1(G, \overline{CJ})\to Br(F)\), \(H_1(G, \overline{CU})\to Br(k)\) are monomorphisms since \(H_1(G, \mathcal{U})=0\), \(H_1(G, \mathcal{J})=0\). By the following commutative diagrams,
and by Proposition 7.a, 7.b, we can easily see that the upper row and the middle row are exact. This completes the proof of Theorem 10.

By Tate [9], we have immediately

**Corollary 9.** If k is a \( \mathfrak{p} \)-adic number field, then the homomorphism \( \rho: \text{Br}(F) \to \text{Br}(A_F) \) is a monomorphism.

**Remark 4.** By Remark 1, 2 and Proposition 7a, 7b, we can define \( \beta: H^2(G, \mathcal{J}) \to \text{Br}(A_F), \gamma: H^2(G, \mathcal{U}) \to \text{Br}(\prod k_p) \). With these homomorphisms, the following diagram is commutative with exact rows.

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & H^2(G, \mathcal{J}) & \longrightarrow & H^2(G, \mathcal{U}) & \longrightarrow & \bigoplus_p \chi(G_p) & \longrightarrow & \cdots \\
\downarrow & & \downarrow \gamma & & \downarrow \beta & & \downarrow \varphi & & \\
0 & \longrightarrow & \text{Br}(\prod k_p) & \longrightarrow & \text{Br}(A_F) & \longrightarrow & \bigoplus_p \chi(G_p) & \longrightarrow & 0
\end{array}
\]

The homomorphism \( \beta \) is given by the crossed product. In fact, for any finite Galois extension \( K \) of \( k \), \( A_{FK} \) is nothing else \( A_F \otimes K \). So \( A_{FK}/A_K \) is Galois extension of rings. Since \( \varphi_0 \) is a monomorphism, the homomorphism \( \beta \) is completely determined by the composite \( \varphi_0 \circ \beta: H^2(G, \mathcal{J}) \to \prod \text{Br}(F_p) \). And the composite \( \varphi_0 \circ \beta \) is obtained by the componentwise crossed product. So our assertion follows immediately. But the author does not know whether \( \beta \) is an epimorphism or not.

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**References**


