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Realization of Levine's motives and Chern class map

Sho Ogaki

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Introduction

During the sixties, in his study of the Weil conjecture concerning the zeta functions of smooth projective varieties over a finite field, Grothendieck introduced the notion of a Weil cohomology (which is a cohomology theory on the category of smooth projective varieties over a field, is equipped with the cycle class maps, and satisfies Poincaré duality and the Künneth formula) and also described the theory of *motives*, which is conjecturally universal among Weil cohomology theories. Deligne and Beilinson independently developed the theory of motives further and described a conjectural abelian tensor category $\mathcal{MM}_k$ of *mixed motives* over a field k, in analogy to the category of mixed Hodge structures.

Even now, the existence of the category $\mathcal{MM}_k$ of mixed motives remains conjectural. However, Hanamura [30], Huber [32], Kelly [39], Levine [41], and Voevodsky [52] defined triangulated tensor categories that have many of the structural properties expected of the derived category of $\mathcal{MM}_k$. Moreover, Levine [41] defined such a triangulated tensor category $\mathcal{DM}(S)$ of mixed motives whose base scheme S is not only the spectrum of a field but also a reduced separated Noetherian scheme.

The theme of this thesis is realizations of Levine's motivic triangulated category. This thesis consists of two parts: We study a construction of the realization of Levine's motivic category in Part 1, and study a relation between the realization and the Chern class map in Part 2. Part 1 and Part 2 are based on the papers [46] and [47] respectively.

Part 1: Realization of Levine's Motives. Levine [41] defined a *geometric cohomology theory* on a reduced separated Noetherian scheme S, and constructed the *realization functor* on $\mathcal{DM}(S)$ for this geometric cohomology theory. Roughly speaking, Levine's geometric cohomology theory is a contravariant functor Γ from the product of the category $\mathbf{Sm}_S$ of smooth quasi-projective schemes over S and the category $\mathbf{Z}$ to a category $\mathcal{C}$ of complexes of abelian sheaves on a Grothendieck site,

$$\Gamma\colon \mathbf{Sm}_S^{\mathrm{op}} \times \mathbf{Z} \to \mathcal{C},$$

with a class of cycle classes and with a Künneth map, that satisfies the following conditions: Γ exhibits the compatibility with direct sums, semi-purity, homotopy invariance, the Gysin isomorphism, and excision; and that $\Gamma(S)$ is quasi-isomorphic to the unit object of $\mathcal{C}$.

On the other hand, cohomology theories such as de Rham, étale, etc. possess the canonical additional structures, and such structures are essential to the study of the arithmetic or geometric properties of algebraic varieties. Therefore, it is natural to try to construct a realization functor for a generalized geometric cohomology theory that takes values in a category of complexes in a tensor category. The first main theorem of this thesis, Theorem 3.5, is in this direction. We state only the special case of this theorem here, since it is so complicated.

Theorem 0.1 (Corollary 3.8)**.** *Let $\mathcal{C}$ be an exact tensor category of which there exists the derived category $D^+\mathcal{C}$ of bonded below complexes. Let Γ be a geometric cohomology theory on S that takes values in $\mathcal{C}$. Suppose that the unit object of $\mathcal{C}$ is projective. Then, there exists a triangulated functor $\mathcal{R}_\Gamma$ from $\mathcal{DM}(S)$ to the pseudo-abelian hull $D^+\mathcal{C}_\natural$ of the triangulated tensor category $D^+\mathcal{C}$ that exhibits the following properties*:

(1) *The restriction of the functor $\mathcal{R}_\Gamma$ to the triangulated tensor subcategory $\mathcal{DM}_{\mathfrak{sh}}(S)$, which is equivalent to $\mathcal{DM}(S)$, is a triangulated pseudo-tensor functor.*

(2) *The diagram*

$$\begin{array}{ccc} \mathbf{Sm}_S^{\mathrm{op}} \times \mathbf{Z} & \longrightarrow & \mathcal{DM}_{\mathfrak{sh}}(S) \\ {\scriptstyle\Gamma}\big\downarrow & & {\scriptstyle\mathcal{R}_\Gamma}\big\downarrow \\ C^+\mathcal{C} & \longrightarrow & D^+\mathcal{C}_\sharp \end{array}$$

commutes up to a canonical isomorphism. Here, the top horizontal arrow sends the pair (X, j) *of a scheme in* $\mathbf{Sm}_S$ *and an integer to Levine's motive* $\mathbf{Z}_X(j)$*, and the bottom one is the composite of the canonical functors*

$$C^+\mathcal{C} \to K^+\mathcal{C} \to D^+\mathcal{C} \to D^+\mathcal{C}_\sharp.$$

(3) *The functor* $\mathcal{R}_\Gamma$ *is compatible with the class of cycle classes and the Künneth map of the geometric cohomology theory* Γ*.*

Using the generalization of Theorem 0.1 (Theorem 3.5), we have a realization functor for the geometric cohomology theory associated with the singular cohomology, the étale cohomology, and the de Rham cohomology (see Section 4). We remark that Levine already pointed out the existence of such realization functors and planned their constructions (cf. [40, Chapter V, Section 2]).

We here mention main results on the study of the conditions under which we can extend a cohomology theory to a realization functor on Voevodsky's motivic category (cf. [52]). Huber [33], [34] constructed the realization functor for a cohomology theory that takes values in an abelian tensor category and that exhibits descent for open, proper and Galois covers, and homotopy invariance. Cisinski and Déglise [17] constructed the realization functor for a cohomology theory that takes values in a category of vector spaces and that satisfies milder conditions than ours.

In Part 1, we also study a geometric cohomology theory and a realization functor associated with the p-adic Hodge cohomology, which is a p-adic analog of the Hodge cohomology.

Let $\mathcal{O}$ be a complete discrete valuation ring $\mathcal{O}$ of mixed characteristic with finite residue field. To each smooth scheme X over $\mathcal{O}$, we associate a triple that consists of the following data:

(1) The rigid cohomology of the special fiber of X with the Frobenius automorphism.
(2) The de Rham cohomology of the generic fiber of X with the Hodge filtration.
(3) The map between these cohomology groups, which is called the specialization map.

This triple can be regarded as a p-adic analog of the mixed Hodge structure for the variety X over $\mathcal{O}$. In fact, if X is projective, then this triple forms a (n admissible) filtered ϕ-module that is defined by Fontaine [26]. Combining the studies of these triple, we see that the correspondence from X to its triple is extended to a geometric cohomology theory Γ_{pH} on $\mathbf{Sm}_{\mathcal{O}}$.

Because the unit object of the category of filtered ϕ-modules is not projective, we cannot apply Theorem 0.1 to this geometric cohomology theory Γ_{pH}. However, by forgetting the Frobenius automorphisms of this geometric cohomology theory Γ_{pH}, we have another geometric cohomology theory Γ_{pH_0} that takes values in the exact category pHS_0 (defined in Definition 5.1). Because the unit object of pHS_0 is projective, we can apply Theorem 0.1 to this geometric cohomology theory. We thus have a realization functor on the motivic category $\mathcal{DM}(\operatorname{Spec}\mathcal{O})$ for Γ_{pH_0}. This result is the second main theorem of Part 1.

Theorem 0.2 (Theorem 5.19)**.** *There exists a triangulated functor* $\mathcal{R}_{pHS_0}$ *from the triangulated category* $\mathcal{DM}(\operatorname{Spec}\mathcal{O})$ *to the derived category* D^+pHS_0 *of bounded below complexes of the exact category* pHS_0 *that exhibits the following properties:*

(1) *The restriction of the functor* $\mathcal{R}_{pHS_0}$ *to the subcategory* $\mathcal{DM}_{\mathfrak{sh}}(S)$ *is a triangulated pseudo-tensor functor.*

(2) *The composite*

$$\mathbf{Sm}_{\mathcal{O}}^{\mathrm{op}} \times \mathbf{Z} \to \mathcal{DM}(\operatorname{Spec} \mathcal{O}) \xrightarrow{\mathcal{R}_{pHS_0}} D^+pHS_0$$

is canonically isomorphic to the composite of the functor Γ_{pH_0} *and the one*

$$C^+pHS_0 \to D^+pHS_0$$

.

(3) *The functor* $\mathcal{R}_{pHS_0}$ *is compatible with the cycle classes and the Künneth map of* Γ_{pH_0}.

(4) *For any object* D *of* $\mathcal{DM}(\operatorname{Spec} \mathcal{O})$ *and* $i \in \mathbf{Z}$, *the vector spaces* $H^i(\mathcal{R}_{pHS_0}(D))_0$ *and* $H^i(\mathcal{R}_{pHS_0}(D))_{\mathrm{dR}}$ *are of finite dimension, and they are zero if* $i \gg 0$.

It is natural to expect that there should exist a realization functor for the geometric cohomology theory Γ_{pH}. However, at present, the author cannot offer any proof of its existence.

The organization of Part 1 is as follows. In Section 1, we review the construction of Levine's motivic category $\mathcal{DM}(S)$, which we will use in latter sections. In the following section, we provide the definition of our geometric cohomology theory, which takes values in some tensor category. In Section 3, by assuming certain conditions, we construct the realization functor from Levine's motivic category for any geometric cohomology theory. In Section 4, using the result in the previous section, we construct the realization functors associated with the Hodge cohomology theory and with the étale cohomology theory. In the last section of this part, we construct a geometric cohomology theory associated with the triple of the rigid cohomology, the de Rham cohomology and the specialization map. We finally prove Theorem 0.2 as an example of the application of Theorem 0.1.

Part 2: Chern Class Map. Beilinson [5] constructed the higher regulator ch_{Be} as the Chern character to the Deligne-Beilinson cohomology group, and conjectured that this regulator determines the special value of Hasse-Weil L-function of an algebraic variety over $\mathbf{Q}$ modulo $\mathbf{Q}^\times$. After that, Beilinson and Deligne gave a conjectural motivic interpretation of Beilinson's regulator ch_{Be}, which states that ch_{Be} coincides with the map from the extension group in the conjectural abelian category of mixed motives to the one in the abelian category of mixed Hodge structures induced by the Hodge realization (cf. [45, Section 9]). The main objective of this part is to give such a motivic interpretation for the Hodge realization of Levine's motivic category and give its l-adic étale analog.

For a finite extension field k over $\mathbf{Q}$, there exists the Hodge realization functor

$$\mathcal{R}_{\mathrm{HS}} \colon \mathcal{DM}(k) \to D^+_{\mathcal{H}_{k,\mathbf{R}}}$$

from Levine's triangulated category $\mathcal{DM}(k)$ of motives over k to the derived category $D^+_{\mathcal{H}_{k,\mathbf{R}}}$ of real mixed Hodge complexes over k (cf. Example 4.1). On the other hand, Levine constructed the Chern character

$$ch^{i,j}_{\mathrm{Le}} \colon K_i(X) \to \operatorname{Hom}_{\mathcal{DM}(k)}(1, \mathbf{Z}_X(j)[2j-i])$$

on the higher K-group $K_i(X)$ for a smooth and quasi-projective scheme over k, where the object 1 is the unit. Furthermore, he proved his own Chern character induces an isomorphism

$$(K_i(X) \otimes \mathbf{Q})^{(j)} \cong \operatorname{Hom}_{\mathcal{DM}(k)}(1, \mathbf{Z}_X(j)[2j-i]) \otimes \mathbf{Q},$$

where the left is the j-th eigenspace of $K_i(X) \otimes \mathbf{Q}$ with respect to the Adams operators (cf. Proposition 8.1). The first main result of Part 2 is the following theorem.

Theorem 0.3. (Corollary 9.13). *For a smooth and quasi-projective scheme X over k, and $i, j \in \mathbf{Z}$, the diagram*

$$\begin{tikzcd} K_i(X) \arrow[r, "(-1)^j ch^{i,j}_{\mathrm{Le}}"] \arrow[d, "ch^{i,j}_{\mathrm{Be}}"'] & \mathrm{Hom}_{\mathcal{DM}(k)}(1, \mathbf{Z}_X(j)[2j-i]) \arrow[d, "\mathcal{R}_{\mathrm{HS}}"] \\ H^{2j-i}_{DB}(X \otimes_{\mathbf{Q}} \mathbf{C}, \mathbf{R}(j))^{\mathrm{Gal}(\mathbf{C}/\mathbf{R})} \arrow[r, "\cong"] & \mathrm{Hom}_{D^+_{\mathcal{H}_{k,\mathbf{R}}}}(1, \mathcal{R}_{\mathrm{HS}}(\mathbf{Z}_X(j))[2j-i]) \end{tikzcd}$$

commutes, where the bottom arrow is the canonical isomorphism, and the map $ch^{i,j}_{\mathrm{Be}}$ is Beilinson's regulator.

We will prove this theorem along the flow below:

(1) Construct a Chern character for a certain cohomology theory (Section 6).
(2) Prove that Levine's Chern character coincides with ours for a cohomology theory associated with his motivic category $\mathcal{DM}(k)$ (Section 8).
(3) Prove that Beilinson's regulator coincides with our Chern character for the Deligne-Beilinson cohomology theory up to sign (Section 9).
(4) Prove that naturalities of our Chern character implies Theorem 0.3 (Section 7).

In Section 10, after proving Theorem 0.3, we will prove an l-adic étale analog of this theorem. We have the l-adic étale realization functor

$$\mathcal{R}_{\acute{e}t} \colon \mathcal{DM}(k) \to D^+_{l,k}$$

from Levin's motivic category to the derived category $D^+_{l,k}$ of bounded below complexes of continuous $\mathbf{Z}_l[G_k]$-modules, where G_k is the absolute Galois group of a field k (cf. Section 4.2). Soulé [50] constructed the Chern class map

$$c^{2j-i,j}_{\mathrm{So}} \colon K_i(A) \to H^{2j-i}_{\acute{e}t}(\operatorname{Spec} A, \mathbf{Z}_l(j))$$

for a commutative ring A with unit. (It is necessary that A is a Dedekind domain if $i = 1$.) The associated Chern character $ch^{i,j}_{\mathrm{So}}$ is the l-adic étale analog of Beilinson's regulator. The following theorem is the second main result of Part 2.

Theorem 0.4. (Corollary 10.2). *With notations as above, we assume that X is an affine scheme. Then, the diagram*

$$\begin{tikzcd} K_i(X) \otimes \mathbf{Q} \arrow[r, "(-1)^j ch^{i,j}_{\mathrm{Le}}"] \arrow[d, "ch^{i,j}_{\mathrm{So}}"'] & \mathrm{Hom}_{\mathcal{DM}(k)}(1, \mathbf{Z}_X(j)[2j-i]) \otimes \mathbf{Q} \arrow[d, "\mathcal{R}_{\acute{e}t}"] \\ H^{2j-i}_{\acute{e}t}(X, \mathbf{Z}_l(j)) \otimes \mathbf{Q} & \mathrm{Hom}_{D^+_{l,k}}(1, \mathcal{R}_{\acute{e}t}(\mathbf{Z}_X(j))[2j-i]) \otimes \mathbf{Q} \arrow[l] \end{tikzcd}$$

commutes. Here, the bottom arrow is the map that is induced by identifying its domain with Jannsen's continuous étale cohomology $H^{2j-i}_{\mathrm{cont}}(X, \mathbf{Z}_l(j))$ tensored by $\mathbf{Q}$.

We remark that Ivorra proved a similar result for his l-adic étale realization functor from Voevodsky's triangulated motivic category and for Gillet's Chern character associated with the l-adic étale cohomology theory (cf. [36, Section 3.3, Proposition 5]).

In Section 11 and 12, we will prove that our Chern character coincides with Asakura-Sato's [1], Besser's [11], Gillet's [27], and Huber's [32] up to sign.

In the last section, we will construct another p-adic Hodge realization functor on a full subcategory of mixed Tate motives over a finite extension field that is derived from the p-adic Hodge theory. Using this p-adic Hodge realization, we will give a motivic interpretation of Besser's regulator (Theorem 13.1). This interpretation tells us an another proof of that Besser's regulator sends Huber-Wildeshaus' motivic polylogarithm class [35] to a special value of Coleman's p-adic polylogarithm [19] up to some explicit constant.

We close the introduction with a remark about Theorem 0.3 and 0.4. These theorems have already been used in papers by several authors. However, the present author could not find any references where their proofs are given, and thinks that their statements are natural but the proofs are not obvious. In fact, because we work in the level of dg categories (not only of derived categories) to construct these Chern characters or regulators, we have to study in the same level to compare these maps. Especially, to prove these two theorems, we have to work with Levine's motivic categories and with the realization functors on them in the level of dg categories. The author thus believes that giving their proofs is beneficial to researchers around this field.

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Notation and Preliminaries

Throughout the paper, the term *ring* will refer to a commutative ring with unit. The term *scheme* will refer to a Noetherian and separated scheme. We will denote the category of R-modules by $\mathbf{Mod}_R$ for a ring R, and set $\mathbf{Ab} = \mathbf{Mod}_{\mathbf{Z}}$.

The category of schemes over a scheme S will be denoted by $\mathbf{Sch}_S$. The category of smooth and quasi-projective schemes over S will be denoted by $\mathbf{Sm}_S$. We denote by $\mathbf{Sm}_S^{\mathrm{ess}}$ the full subcategory of $\mathbf{Sch}_S$ whose objects are localizations of schemes in $\mathbf{Sm}_S$. $\mathcal{V}$ will denote a strictly full subcategory of $\mathbf{Sm}_S^{\mathrm{ess}}$ that contains S and is closed under finite products over S and finite disjoint unions.

For a set S, we will denote $\mathbf{Z}S$ the free $\mathbf{Z}$-module generated by S.

For a $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$-module M, we will denote its invariant submodule by M^{∞}.

For a simplicial (resp. cosimplicial) object A in an additive category, the associated chain (resp. cochain) complex will be denoted by A_* (resp. A^*). For a chain complex A_*, its normalization NA_* is the cochain complex defined to be $(NA_*)^i = A_{-i}$.

We will regard $\mathbf{Z}$ as the category whose objects consist of all integers and whose morphisms are the identities.

For a category $\mathcal{C}$, we will denote $\mathcal{C}(-,-)$ the set of morphisms in $\mathcal{C}$ and denote $s.\mathcal{C}$ the category of simplicial objects in $\mathcal{C}$.

A subcategory $\mathcal{C}$ of a category will be called a *strictly full subcategory* if $\mathcal{C}$ is a full subcategory and is closed under isomorphisms.

For an object X of $\mathbf{Sm}_S^{\mathrm{ess}}$ and $q \geq 0$, $\mathcal{Z}^q(X)$ will denote the group of universally integral relative cycles of X over S (cf. [41, Appendix A, 2.1.2]).

The term *tensor category* will refer to a linear symmetric monoidal category with unit (cf. [41, Part II, Chapter I, 1.3.6]). We will denote 1 the unit object of a monoidal category. All semi-monoidal categories will be strictly associative (cf. [41, Part II, Chapter I, 1.1]). We regard 1 as the unit object of a monoidal category.

An exact category $\mathcal{C}$ with a tensor structure will be called an *exact tensor category* if the class of short exact sequences in $\mathcal{C}$ is closed under the operation of taking tensor products for all objects.

An object C of an exact category $\mathcal{C}$ will be called projective if all quasi-isomorphisms from each complex of $\mathcal{C}$ to C admit right inverses.

Let $\mathcal{A}$ be an additive category. Then, $C^+_{\mathrm{dg}}\mathcal{A}$ will denote the dg category of bounded below complexes in $\mathcal{A}$ (cf. [41, Part II, Chapter II, 1.2.2]). If $\mathcal{A}$ is a tensor category (in the sense of [41, Part II, Chapter I, 1.3.6]), then the dg category $C^+_{\mathrm{dg}}\mathcal{A}$ has a dg tensor structure (cf. [41, Part II, Chapter II, 1.2.2]) and its homotopy category $K^+\mathcal{A}$ has a triangulated tensor structure (cf. [41, Part II, Chapter II, 2.1.6.4]).

Let $\mathcal{C}$ be a dg category. Then, $Z^0\mathcal{C}$ (resp. $H^0\mathcal{C}$) will denote the zeroth cocycle (resp. cohomology) category of $\mathcal{C}$, (cf. [41, Part II, Chapter II, 1.2.1]) and $C^b_{\mathrm{dg}}\mathcal{C}$ will denote the dg category of bounded complexes in $\mathcal{C}$ (cf. [41, Part II, Chapter II, 1.2.7]). Now, we set $K^b\mathcal{C} = H^0C^b_{\mathrm{dg}}\mathcal{C}$. If $\mathcal{C}$ is a dg tensor category, then $C^b_{\mathrm{dg}}\mathcal{C}$ (resp. $K^b\mathcal{C}$) has a dg tensor (resp. triangulated tensor) structure (cf. [41, Part II, Chapter II, 1.2.11, 2.1.6.4]). For a dg (resp. dg tensor) functor $F\colon \mathcal{C} \to \mathcal{C}'$ of dg (resp. dg tensor) categories, F can be canonically extended to the triangulated (resp. triangulated tensor) functor $K^bF\colon K^b\mathcal{C} \to K^b\mathcal{C}'$ (cf. [41, Part II, Chapter II, 2.1.7]). For an additive category $\mathcal{A}$, we note that $Z^0C^+_{\mathrm{dg}}\mathcal{A}$ is the usual category $C^+\mathcal{A}$ of bounded below complexes in $\mathcal{A}$ and that $H^0C^+_{\mathrm{dg}}\mathcal{A}$ is the usual homotopy category $K^+\mathcal{A}$ of bounded below complexes in $\mathcal{A}$.

Part 1. Realization of Levine's Motives

1. Levine's Motivic Category

In this section, we recall the construction of Levine's motivic category $\mathcal{DM}(\mathcal{V})$. In later sections, we not only will use this construction but also will need to refer to various steps of the process. For details regarding the discussion presented in this section, we refer to [41, p. 9–19] and to the references quoted in the paragraphs below.

1.1. Flow of the construction of $\mathcal{DM}(\mathcal{V})$.

Since the construction of $\mathcal{DM}(\mathcal{V})$ is complicated, we mention that the flow of its construction before recalling.

We firstly construct the categories in the sequence

$$\mathcal{V} \to \mathcal{L}(\mathcal{V})^* \to \mathcal{A}_1(\mathcal{V}) \to \mathcal{A}_2(\mathcal{V}) \to \mathcal{A}_3(\mathcal{V}) \to \mathcal{A}_4(\mathcal{V}) \to \mathcal{A}_5(\mathcal{V}) \supset \mathcal{A}_{\mathrm{mot}}(\mathcal{V})$$

from left to right. Next, we have the triangulated tensor category $D^b_{\mathrm{mot}}(\mathcal{V})$ by localing the homotopy category $K^b\mathcal{A}_{\mathrm{mot}}(\mathcal{V})$ of bonded complexes of the dg category $\mathcal{A}_{\mathrm{mot}}(\mathcal{V})$ with respect to some multiplicative set of morphisms. Finally, taking the pseudo-abelian hull (see Definition 1.33) of the category $D^b_{\mathrm{mot}}(\mathcal{V})$, we obtain Levine's motivic category $\mathcal{DM}(\mathcal{V})$. The construction is quite technical; however, his conceptualization of this construction is described in [41, p. 7–9].

In parallel with this flow of the construction, we also construct the other categories with the index $\mathfrak{sh}$, such as $\mathcal{A}^{\mathfrak{sh}}_2(\mathcal{V})$, $\mathcal{A}^{\mathfrak{sh}}_3(\mathcal{V})$, etc. These categories will be used when we construct the realization functor in Section 3.

1.2. Construction of $\mathcal{L}(\mathcal{V})^*$.

In this section, we will extend the symmetric monoidal category $\mathcal{V}$ to the one $\mathcal{L}(\mathcal{V})^*$. This extension will assist the cycle class maps that will be adjoined latter (Definition 1.18) in performing their tasks (cf. [41, p. 8]).

Definition 1.1. Let $\mathcal{L}(\mathcal{V})$ denote the category defined as follows:

(1) The objects of $\mathcal{L}(\mathcal{V})$ are the equivalence classes of pairs (X, f), where X is an object of $\mathcal{V}$ and $f\colon X' \to X$ is a morphism in $\mathbf{Sm}^{\mathrm{ess}}_S$ such that there exists a smooth morphism $s\colon X \to X'$ that is a section of f; two pairs $(X, f\colon X' \to X)$, $(Y, g\colon Y' \to Y)$ are equivalent if $X = Y$ and there exists an isomorphism $h\colon X' \to Y'$ with $f = g \circ h$.

(2) For any objects $(X, f\colon X' \to X)$ and $(Y, g\colon Y' \to Y)$ in $\mathcal{L}(\mathcal{V})$, the set of morphisms from (X, f) to (Y, g) is the subset of $\mathcal{V}(X, Y)$ that is defined by the following condition: a

morphism $\phi\colon X \to Y$ in $\mathcal{V}$ yields a morphism $\phi\colon (X, f) \to (Y, g)$ in $\mathcal{L}(\mathcal{V})$ if there exists a flat S-morphism $\psi\colon X' \to Y'$ with $\phi \circ f = \psi \circ g$.

(3) The composition of morphisms is induced from the one in $\mathbf{Sch}_S$.

Definition 1.2. In the category $\mathcal{L}(\mathcal{V})^{\mathrm{op}} \times \mathbf{Z}$, we denote the object $((X, f), q)$ by $X(q)_f$. For each morphism $\phi\colon (X, f) \to (Y, g)$ in $\mathcal{L}(\mathcal{V})$ and for $q \in \mathbf{Z}$, we denote the morphism $\phi^{\mathrm{op}} \times \mathrm{id}_q : X(q)_f \to Y(q)_g$ by ϕ^*.

Let $X(q)_f$ and $Y(r)_g$ be objects of $\mathcal{L}(\mathcal{V})^{\mathrm{op}} \times \mathbf{Z}$. We define the product of $X(q)_f$ and $Y(r)_g$ in $\mathcal{L}(\mathcal{V})^{\mathrm{op}} \times \mathbf{Z}$ as $(X \times_S Y)(q + r)_{f \times_S g}$. This product provides a symmetric monoidal structure on $\mathcal{L}(\mathcal{V})^{\mathrm{op}} \times \mathbf{Z}$.

The integer q of an object $X(q)_f$ corresponds to the degree of Tate twist or the Adams degree. The next lemma is essential for constructing Levine's motivic category and for constructing realization functors from this category.

Lemma 1.3 ([41, Part II, Chapter I, 2.1.2])**.** *Let $\mathscr{C}$ denote the category of symmetric monoidal categories, of tensor categories, or of dg tensor categories without unit.*

(1) (*Adjoining morphisms*). *Let $\mathcal{C}$ be an object of $\mathscr{C}$, and let $\{X_\phi \mid \phi \in \Phi\}$ and $\{Y_\phi \mid \phi \in \Phi\}$ be sets of objects of $\mathcal{C}$ indexed by a set Φ. Then, there exist an object $\mathcal{C}_\Phi$ of $\mathscr{C}$ and a morphism $\iota_\Phi\colon \mathcal{C} \to \mathcal{C}_\Phi$ in $\mathscr{C}$ that uniquely satisfy the following conditions:*
   (1) *For all $\phi \in \Phi$, these ϕ are elements of $\mathcal{C}_\Phi(\iota_\Phi(X_\phi), \iota_\Phi(Y_\phi))$.*
   (2) *For each morphism $F\colon \mathcal{C} \to \mathcal{D}$ in $\mathscr{C}$, let Ψ be a subset of the morphisms of $\mathcal{D}$ indexed by Φ such that the $\psi_\phi \in \Psi$ are morphisms from $F(X_\phi)$ to $F(Y_\phi)$ for all $\phi \in \Phi$, i.e.,*

$$\Psi = \{\psi_\phi\colon F(X_\phi) \to F(Y_\phi) \mid \phi \in \Phi\}.$$

   Then, there exists a unique morphism $F_\Psi\colon \mathcal{C}_\Phi \to \mathcal{D}$ in $\mathscr{C}$ such that $F_\Psi \circ \iota_\Phi = F$ and $F_\Psi(\phi) = \psi_\phi$.

(2) (*Adjoining relations*). *Let $\mathcal{C}$ be an object of $\mathscr{C}$, and let $R = \{(f_a, g_a) \mid a \in A\}$ be a set of pairs of morphisms of $\mathcal{C}$ with the same domain and range indexed by a set A. Then, there exist an object $\mathcal{C}_R$ of $\mathscr{C}$ and a morphism $\iota_R\colon \mathcal{C} \to \mathcal{C}_R$ in $\mathscr{C}$ that uniquely satisfy the following conditions:*
   (1) *For all $a \in A$, $\iota_R(f_a) = \iota_R(g_a)$.*
   (2) *For each morphism $F\colon \mathcal{C} \to \mathcal{D}$ in $\mathscr{C}$ that satisfies $F(f_a) = F(g_a)$ for all $a \in A$, there exists a unique morphism $F_R\colon \mathcal{C}_R \to \mathcal{D}$ in $\mathscr{C}$ such that $F_R \circ \iota_R = F$.*

Definition 1.4. Using Lemma 1.3, we define the symmetric monoidal category $\mathcal{L}(\mathcal{V})^*$ by adjoining morphisms and relations to $\mathcal{L}(\mathcal{V})^{\mathrm{op}} \times \mathbf{Z}$ as follows:

Let (X, f) and (Y, g) be objects of $\mathcal{V}$, and let i be the natural inclusion $X \to X \coprod Y$. Then, we adjoin the morphism $i_*\colon X(q)_f \to (X \coprod Y)(q)_{f \coprod g}$ for all $q \in \mathbf{Z}$. The relations among morphisms that are thus imposed are as follows:

(1) Let $i\colon X \to X \coprod Y$ and $j\colon X \coprod Y \to X \coprod Y \coprod Z$ be the natural inclusions. Then, $(j \circ i)_* = j_* \circ i_*$.
(2) Let $\phi_i\colon X_i \to Y_i$, $i = 1, 2$, and let $i\colon X_1 \to X_1 \coprod X_2$ and $j\colon Y_1 \to Y_1 \coprod Y_2$ be the natural inclusions. Then, $i_* \circ \phi_1^* = (\phi_1 \coprod \phi_2)^* \circ j_*$.
(3) Let $i\colon X \to X \coprod \emptyset$ be the canonical isomorphism. Then, we have $i^* \circ i_* = \mathrm{id}$.

To be precise, using Lemma 1.3(1) for the case in which

$$\mathcal{C} = \mathcal{L}(\mathcal{V})^{\mathrm{op}} \times \mathbf{Z},\ \Phi = \{i_* \mid i_*\colon X(q)_f \to (X \coprod Y)(q)_{f \coprod g}\},$$

we obtain the symmetric monoidal category $\mathcal{C}_\Phi$. Next, using Lemma 1.3(2) for the case in which $\mathcal{C} = \mathcal{C}_\Phi$ and the set $R = \{(f_a, g_a)\}$ consists of $((j \circ i)_*, j_* \circ i_*)$ in (1), $(i_* \circ \phi_1^*, (\phi_1 \coprod \phi_2)^* \circ j_*)$ in (2), and $(i^* \circ i_*, \mathrm{id})$ in (3), we obtain the category $\mathcal{C}_R$. Then, we define the category $\mathcal{L}(\mathcal{V})^*$ to be this category $\mathcal{C}_R$.

1.3. **Construction of $\mathcal{A}_1(\mathcal{V})$.** We will extend the symmetric monoidal category $\mathcal{L}(\mathcal{V})^*$ to the tensor category $\mathcal{A}_1(\mathcal{V})$.

Definition 1.5. Let $\mathcal{C}$ be a category. Then, we define the additive category $\mathbf{Z}\mathcal{C}$ as follows:

(1) The objects of $\mathbf{Z}\mathcal{C}$ are the finite direct sums of objects of $\mathcal{C}$.
(2) Let C and C' be objects of $\mathbf{Z}\mathcal{C}$. If both C and C' consist of a single object of $\mathcal{C}$, then we define $\mathbf{Z}\mathcal{C}(C, C')$ to be the free abelian group generated by the set $\mathcal{C}(C, C')$. Otherwise, we define $\mathbf{Z}\mathcal{C}(C, C')$ bilinearly considering the above case.

Remark 1.6. If $\mathcal{C}$ is a symmetric monoidal category, then $\mathbf{Z}\mathcal{C}$ is endowed with a natural tensor structure.

Definition 1.7. We define the tensor category $\mathcal{A}_1(\mathcal{V})$ by adjoining the following relations to $\mathbf{Z}\mathcal{L}(\mathcal{V})^*$ by Lemma 1.3(2) (cf. Definition 1.4):

(1) Let $\emptyset$ be the empty scheme. For each object L of $\mathcal{L}(\mathcal{V})$ of the form $\emptyset(q)_f$, the canonical morphism from L to 0 is an isomorphism.
(2) Let (X, f) and (Y, g) be objects of $\mathcal{L}(\mathcal{V})$, and let $i\colon X \to X \coprod Y$ and $j\colon Y \to X \coprod Y$ be the natural inclusions. Then, $i_* \circ i^* + j_* \circ j^*$ is the identity of $(X \coprod Y)(q)_{f \coprod g}$ for each $q \in \mathbf{Z}$.

For each object $X(q)_f$ of $\mathcal{L}(\mathcal{V})^*$, we define the object $\mathbf{Z}_X(q)_f$ of $\mathcal{A}_1(\mathcal{V})$ to be $X(q)_f$, and we denote $\mathbf{Z}_X(q)_{\mathrm{id}_X}$ by $\mathbf{Z}_X(q)$ and the unit object $\mathbf{Z}_S(0)$ by 1.

1.4. **Construction of $\mathcal{A}_2(\mathcal{V})$.** This section begins with the definition of *commutative external products.* This notion arises when we consider the Künneth maps

$$H^\bullet(X) \otimes H^\bullet(Y) \to H^\bullet(X \times Y)$$

of cohomology theories $H^\bullet$ on the level of cochain complexes.

Definition 1.8 ([41, Part II, Chapter I, 2.4])**.** Let $\mathscr{C}$ be the category of symmetric monoidal categories without unit, of tensor categories without unit, of dg tensor categories without unit, or of triangulated tensor categories. Furthermore, let $\mathcal{C}$ and $\mathcal{D}$ be objects of $\mathscr{C}$. A *commutative external product on* $\mathcal{C}$ is a pair (F, θ), where $F\colon \mathcal{C} \to \mathcal{D}$ is a functor in $\mathscr{C}$ and θ is a natural transformation from $\otimes_{\mathcal{D}} \circ (F \times F)$ to $F \circ \otimes_{\mathcal{C}}$ in $\mathscr{C}$ such that θ is associative and commutative.

For any commutative external product (F, θ) on $\mathcal{C}$, the natural transformation θ is said to be a *commutative external product of* F.

For any tensor category $\mathcal{C}$ without unit, the tensor category $\mathcal{C}^{\otimes, c}$ without unit, the additive functor $i^c_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}^{\otimes, c}$, and the commutative external product $\boxtimes^c_{\mathcal{C}}$ of $i^c_{\mathcal{C}}$ are defined in [41, Part II, Chapter I, 2.4.3]. The category $\mathcal{C}^{\otimes, c}$ is called the *category of the universal commutative external product on* $\mathcal{C}$. See [41, Part I, Chapter I, 1.4.3] for a simple explanation of this category $\mathcal{C}^{\otimes, c}$. These satisfy the following properties.

Proposition 1.9 ([41, Part II, Chapter I, 2.4.4])**.** *Let $\mathcal{C}$ be a tensor category. Then, the following properties hold*:

(1) *The pair $(i^c_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}^{\otimes, c}, \boxtimes^c_{\mathcal{C}})$ is the universal commutative external product on $\mathcal{C}$; i.e., for each commutative external product $(F\colon \mathcal{C} \to \mathcal{D}, \theta)$ on $\mathcal{C}$, there exists a unique additive functor $G\colon \mathcal{C}^{\otimes, c} \to \mathcal{D}$ such that $G \circ i^c_{\mathcal{C}} = F$ and $G(\boxtimes^c_{\mathcal{C}}) = \theta$.*
(2) *The functor $i^c_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}^{\otimes, c}$ is fully faithful and is injective on the objects.*

As previously stated in Section 1.1, we must also construct the categories with the index $\mathfrak{sh}$. Now, we prepare the dg tensor category $\mathcal{C}^{\otimes, \mathfrak{sh}}$ without unit, which differs from $\mathcal{C}^{\otimes, c}$ for a tensor category $\mathcal{C}$ without unit. First, however, we define the notion of a *homotopy equivalence* of dg categories.

Definition 1.10. A dg functor $F\colon \mathcal{C}\to\mathcal{D}$ of dg categories is called a homotopy equivalence if F is bijective on objects up to isomorphism and if the maps

$$F_{C,C'}\colon \mathcal{C}(C,C')\to\mathcal{D}(F(C),F(C'))$$

are quasi-isomorphisms for all objects C and C' of $\mathcal{C}$.

For each tensor category $\mathcal{C}$ without unit, the dg tensor category $\mathcal{C}^{\otimes,\mathfrak{sh}}$ without unit, the additive functor $i^{\mathfrak{sh}}_{\mathcal{C}}\colon \mathcal{C}\to\mathcal{C}^{\otimes,\mathfrak{sh}}$, the commutative external product $\boxtimes^{\mathfrak{sh}}_{\mathcal{C}}$ of $i_{\mathcal{C}}$, and the dg tensor functor $\mathfrak{c}_{\mathcal{C}}\colon \mathcal{C}^{\otimes,\mathfrak{sh}}\to\mathcal{C}^{\otimes,c}$ are defined in [41, Part II, Chapter III, 2.1.5, 2.1.6]. They have the following properties.

Proposition 1.11 ([41, Part II, Chapter III, 2.1.7])**.** *Let $\mathcal{C}$ be a tensor category without unit. Then, the following properties hold*:

1. $i^{c}_{\mathcal{C}}=\mathfrak{c}_{\mathcal{C}}\circ i^{\mathfrak{sh}}_{\mathcal{C}}$.
2. *Let C_1 and C_2 be objects of $\mathcal{C}$. Then, $\boxtimes^{c}_{\mathcal{C},C_1,C_2}=\mathfrak{c}_{\mathcal{C}}(\boxtimes^{\mathfrak{sh}}_{\mathcal{C},C_1,C_2})$, where we regard C_1 and C_2 as objects of $\mathcal{C}^{\otimes,c}$ (resp. $\mathcal{C}^{\otimes,\mathfrak{sh}}$) by the functor $i^{c}_{\mathcal{C}}$ (resp. $i^{\mathfrak{sh}}_{\mathcal{C}}$).*
3. *The dg tensor functor $\mathfrak{c}_{\mathcal{C}}\colon \mathcal{C}^{\otimes,\mathfrak{sh}}\to\mathcal{C}^{\otimes,c}$ is a homotopy equivalence.*

Remark 1.12. According to the above proposition, the category $\mathcal{C}^{\otimes,\mathfrak{sh}}$ is close to the category $\mathcal{C}^{\otimes,c}$. Furthermore, Lemma 3.3 describes the property on $\mathcal{C}^{\otimes,\mathfrak{sh}}$ that corresponds to the universality of $\mathcal{C}^{\otimes,c}$. When we construct the realization functor that is compatible with the tensor structures in Section 3, the difference between $\mathcal{C}^{\otimes,c}$ and $\mathcal{C}^{\otimes,\mathfrak{sh}}$ is stated in Proposition 3.4.

Definition 1.13. We define the tensor category $\mathcal{A}_2(\mathcal{V})$ to be the category of the universal commutative external product on $\mathcal{A}_1(\mathcal{V})$, and we define the dg tensor category $\mathcal{A}^{\mathfrak{sh}}_2(\mathcal{V})$ without unit for $(\mathcal{A}_1(\mathcal{V}))^{\otimes,\mathfrak{sh}}$, i.e.,

$$\mathcal{A}_2(\mathcal{V})=(\mathcal{A}_1(\mathcal{V}))^{\otimes,c},\ \mathcal{A}^{\mathfrak{sh}}_2(\mathcal{V})=(\mathcal{A}_1(\mathcal{V}))^{\otimes,\mathfrak{sh}}.$$

We denote the functor $\mathfrak{c}_{\mathcal{A}_1(\mathcal{V})}\colon \mathcal{A}^{\mathfrak{sh}}_2(\mathcal{V})\to\mathcal{A}_2(\mathcal{V})$ by $\mathfrak{c}_2$.

Remark 1.14. In the remainder of this paper, we regard the category $\mathcal{A}_1(\mathcal{V})$ as the subcategories of $\mathcal{A}_2(\mathcal{V})$ and $\mathcal{A}^{\mathfrak{sh}}_2(\mathcal{V})$ induced by the functors $i^{c}_{\mathcal{A}(\mathcal{V})}$ and $i^{\mathfrak{sh}}_{\mathcal{A}_1(\mathcal{V})}$.

1.5. **Construction of $\mathcal{A}_3(\mathcal{V})$.** For $?=\emptyset$ or $\mathfrak{sh}$, we construct the dg tensor category $\mathcal{A}^{?}_3(\mathcal{V})$ without unit by adjoining the cycle class maps to the category $\mathcal{A}^{?}_2(\mathcal{V})$. We begin by introducing the category $\mathbb{E}$ below.

Let $\mathbb{E}$ denote the *homotopy one point dg tensor category*, and let $\mathfrak{e}$ denote the generating object of $\mathbb{E}$. They are defined in [41, Part II, Chapter II, 3.1.11] and satisfy the properties below.

Proposition 1.15 ([41, Part II, Chapter II, 3.1.12])**.** *$\mathbb{E}$ is a dg tensor category without unit. The object $\mathfrak{e}$ generates the objects of $\mathbb{E}$ through finite direct sums and finite tensor powers.*

Proposition 1.16 ([41, Part II, Chapter II, 3.1.13])**.** *Let $\mathcal{D}$ be a dg tensor category without unit, and let $\mathfrak{d}$ be an object of $\mathcal{D}$ such that the following statements hold:*

1. *For each $n>0$ and each $\sigma\in\mathfrak{S}_n$, we have*
$$\tau_\sigma=\mathrm{id}_{\mathfrak{d}^{\otimes n}}\ \textit{in}\ H^0(\mathcal{D}(\mathfrak{d}^{\otimes n},\mathfrak{d}^{\otimes n})),$$
where τ_σ is the symmetry automorphism of $\mathfrak{d}^{\otimes n}$.
2. *For $q<0$ and $n>0$, $H^q(\mathcal{D}(\mathfrak{d}^{\otimes n},\mathfrak{d}^{\otimes n}))=0$ and $\mathcal{D}^q(\mathfrak{d}^{\otimes n},\mathfrak{d}^{\otimes n})$ is 2-torsion free.*

Then, there exists a dg tensor functor $\mathbb{E}\to\mathcal{D}$ that sends $\mathfrak{e}^{\otimes n}$ to $\mathfrak{d}^{\otimes n}$ for $n>0$. In addition, such a dg tensor functor is unique up to homotopy.

Definition 1.17. Let $(X, f\colon X' \to X)$ be an object of $\mathcal{L}(\mathcal{V})$. Then, we define the subgroup $\mathscr{Z}^q(X)_f$ of $\mathscr{Z}^q(X)$ that consists of the cycles C in $\mathscr{Z}^q(X)$ such that $f^*(C)$ is defined, i.e.,

$$\mathrm{codim}_{X'}(f^{-1}(\mathrm{supp}(C))) \geq q,$$

where $\mathrm{supp}(C)$ is the support of the cycle C.

Definition 1.18. Let ? be $\emptyset$ or $\mathfrak{sh}$. When $? = \emptyset$, we regard $\mathcal{A}_2(\mathcal{V})$ as a dg tensor category without unit, where all morphisms are of degree zero. The dg tensor category $\mathcal{A}_2^?(\mathcal{V})[\mathbb{E}]$ without unit is defined as the coproduct of $\mathcal{A}_2^?(\mathcal{V})$ and $\mathbb{E}$ as dg tensor categories without unit.

We define the dg tensor category $\mathcal{A}_3^?(\mathcal{V})$ without unit by adjoining morphisms and relations to $\mathcal{A}_2^?(\mathcal{V})[\mathbb{E}]$ as follows: For each object (X, f) of $\mathcal{L}(\mathcal{V})$ and each $q \in \mathbf{Z}$, let C be a cycle in $\mathscr{Z}^q(X)_f$. Then, we adjoin the morphism $[C]^?\colon \mathfrak{e} \to \mathbf{Z}_X(q)_f$ of degree $2q$. When $? = \mathfrak{sh}$, we set $d[C]^{\mathfrak{sh}} = 0$. If C is the zero element of $\mathscr{Z}^q(X)_f$, then $[C]^?$ is defined to be the zero morphism.

We extend the dg tensor functor $\mathfrak{c}_2[\mathbb{E}]\colon \mathcal{A}_2^{\mathfrak{sh}}(\mathcal{V})[\mathbb{E}] \to \mathcal{A}_2(\mathcal{V})[\mathbb{E}]$ to the dg tensor functor $\mathfrak{c}_3\colon \mathcal{A}_3^{\mathfrak{sh}}(\mathcal{V}) \to \mathcal{A}_3(\mathcal{V})$ by setting $\mathfrak{c}_3([C]^{\mathfrak{sh}}) = [C]$.

Lemma 1.19. *The dg functor $\mathfrak{c}_3$ is a homotopy equivalence.*

Proof. This lemma follows from [41, Part II, Chapter II, 2.2.4, and Chapter III, 3.1.12]. □

1.6. **Construction of $\mathcal{A}_4(\mathcal{V})$.** For $? = \emptyset$ or $\mathfrak{sh}$, we will enlarge the dg modules of morphisms in the category $\mathcal{A}_3^?(\mathcal{V})$ to obtain the dg tensor category $\mathcal{A}_4^?(\mathcal{V})$ without unit by adjoining the relations (up to homotopy) that the cycle class maps should satisfy.

Definition 1.20. For $? = \emptyset$ or $\mathfrak{sh}$, we define the dg tensor category $\mathcal{A}_4^?(\mathcal{V})$ without unit by adjoining morphisms and relations to $\mathcal{A}_3^?(\mathcal{V})$ as follows:

(1) Let $X(q)_f$ be an object of $\mathcal{L}(\mathcal{V})^*$, and let C and D be cycles in $\mathscr{Z}^q(X)_f$. We adjoin the morphisms $h^?_{m,n,C,D}\colon \mathfrak{e} \to \mathbf{Z}_X(q)_f$ of degree $2q-1$ via the relations
$$dh^?_{m,n,C,D} = [mC + nD]^? - [mC]^? - [nD]^?$$
for all $m, n \in \mathbf{Z}$.

(2) Let (X, f) and (Y, g) be objects of $\mathcal{L}(\mathcal{V})$, let $\phi\colon \mathbf{Z}_X(q)_f \to \mathbf{Z}_Y(q)_g$ be a morphism in $\mathcal{A}_1(\mathcal{V})$, and let C be a non-zero cycle in $\mathscr{Z}^q(X)_f$. Then, we adjoin the morphism $h^?_{X,Y,C,\phi}\colon \mathfrak{e} \to \mathbf{Z}_Y(q)_g$ of degree $2q-1$ via the relation $dh^?_{X,Y,C,\phi} = \phi \circ [C]^? - [\phi^* C]^?$.

(3) Let $X(q)_f$ and $Y(r)_g$ be objects of $\mathcal{L}(\mathcal{V})^*$, and let C and D be cycles of $\mathscr{Z}^q(X)_f$ and $\mathscr{Z}^r(Y)_g$, respectively. Then, we adjoin the morphisms
$$h^{l,?}_{X,Y,C,D}\colon \mathfrak{e} \otimes \mathfrak{e} \to \mathbf{Z}_{X \times_S Y}(q+r)_{f \times_s g},$$
$$h^{r,?}_{X,Y,C,D}\colon \mathfrak{e} \otimes \mathfrak{e} \to \mathbf{Z}_{X \times_S Y}(q+r)_{f \times_s g}$$
of degree $2(q+r)-1$ via the relations
$$dh^{l,?}_{X,Y,C,D} = \boxtimes^? \circ ([C]^? \otimes [D]^?) - \boxtimes^? \circ ([C \times_S D]^? \otimes [S]^?),$$
$$dh^{r,?}_{X,Y,C,D} = \boxtimes^? \circ ([C]^? \otimes [D]^?) - \boxtimes^? \circ ([S]^? \otimes [C \times_S D]^?),$$
where $C \times_S D \in \mathscr{Z}^{q+r}(X \times_S Y)_{f \times_s g}$ is the product of the two cycles C and D (cf. [41, Appendix A, 2.2.3(i)]) and $\boxtimes^?$ is the commutative external product on $\mathcal{A}_1^?(\mathcal{V})$ (cf. Propositions 1.9 and 1.11).

We extend the dg tensor functor $\mathfrak{c}_3\colon \mathcal{A}_3^{\mathfrak{sh}}(\mathcal{V}) \to \mathcal{A}_3(\mathcal{V})$ to the dg tensor $\mathfrak{c}_4\colon \mathcal{A}_4^{\mathfrak{sh}}(\mathcal{V}) \to \mathcal{A}_4(\mathcal{V})$ by setting $\mathfrak{c}_4(h^{\mathfrak{sh}}) = h$ for all morphisms $h^?$ that are adjoined to $\mathcal{A}_3^{\mathfrak{s}}(\mathcal{V})$ above.

Remark 1.21. In the previous definition, we constructed the category $\mathcal{A}_4(\mathcal{V})$ by adjoining the desired relations only up to homotopy. Levine also constructed a variant of the category $\mathcal{A}_4(\mathcal{V})$ by adjoining to $\mathcal{A}_3(\mathcal{V})$ the same relations but not up to homotopy in [41, Part I, Chapter I, 1.4.12], and he studied the latter category.

Lemma 1.22. *The dg functor $\mathfrak{c}_4$ is a homotopy equivalence.*

Proof. From the fact that $dh^{\mathfrak{sh}} = 0$ for all morphisms $h^{\mathfrak{sh}}$ adjoined in Definition 1.20, [41, Part I, Chapter V, 1.3.5], and [41, Part II, Chapter III, 3.1.12], we find that the functor $\mathfrak{c}_4$ satisfies the hypothesis of [41, Part II, Chapter II, 2.2.5]. Hence, $\mathfrak{c}_4$ is a homotopy equivalence. □

1.7. **Construction of $\mathcal{A}_5(\mathcal{V})$.** In this section, we construct the dg tensor category $\mathcal{A}_5^?(\mathcal{V})$ without unit by adjoining the higher homotopies of the cycle class maps to $\mathcal{A}_4^?(\mathcal{V})$ for $? = \emptyset$ or $\mathfrak{sh}$.

Definition 1.23. Let ? be $\emptyset$ or $\mathfrak{sh}$, and let k and r be positive integers. We will recursively define the dg tensor categories $\mathcal{A}_5^?(\mathcal{V})^{(r)}$ and $\mathcal{A}_5^?(\mathcal{V})^{(r,k)}$ without unit. Consider $\mathcal{A}_5^?(\mathcal{V})^{(0)} = \mathcal{A}_4^?(\mathcal{V})$. Suppose that we have formed the dg tensor category $\mathcal{A}_5^?(\mathcal{V})^{(r-1)}$ without unit; we denote $\mathcal{A}_5^?(\mathcal{V})^{(r-1)}$ by $\mathcal{A}_5^?(\mathcal{V})^{(r,0)}$:

$$\mathcal{A}_5^?(\mathcal{V})^{(0)} = \mathcal{A}_4^?(\mathcal{V}),\ \mathcal{A}_5^?(\mathcal{V})^{(r,0)} = \mathcal{A}_5^?(\mathcal{V})^{(r-1)}.$$

Suppose further that we have formed the dg tensor category $\mathcal{A}_5^?(\mathcal{V})^{(r,k-1)}$ without unit; we define the dg tensor category $\mathcal{A}_5^?(\mathcal{V})^{(r,k)}$ without unit by adjoining morphisms and relations to $\mathcal{A}_5^?(\mathcal{V})^{(r,k-1)}$ as follows:

For each object $X(q)_f$ of $\mathcal{L}(\mathcal{V})^*$, let $g\colon \mathfrak{e}^{\otimes k} \to \mathbf{Z}_X(q)_f$ be a non-zero morphism of degree $2q-r$ in $\mathcal{A}_5^?(\mathcal{V})^{(r,k-1)}$ with $dg = 0$. Then, we adjoin the morphism $h_g\colon \mathfrak{e}^{\otimes k} \to \mathbf{Z}_X(q)_f$ of degree $2q-r-1$ with $dh_g = g$. We define the dg tensor category $\mathcal{A}_5^?(\mathcal{V})^{(r)}$ as the direct limit of $\{\mathcal{A}_5^?(\mathcal{V})^{(r,k)}\}_{k\geq 0}$, and we define the dg tensor category $\mathcal{A}_5^?(\mathcal{V})$ as the direct limit of $\{\mathcal{A}_5^?(\mathcal{V})^{(r)}\}_{r\geq 0}$, i.e.,

$$\mathcal{A}_5^?(\mathcal{V})^{(r)} = \varinjlim_k \mathcal{A}_5^?(\mathcal{V})^{(r,k)},\ \mathcal{A}_5^?(\mathcal{V}) = \varinjlim_r \mathcal{A}_5^?(\mathcal{V})^{(r)}.$$

Similarly, let $\mathfrak{c}_5^{(1,0)}$ be the dg tensor functor $\mathfrak{c}_4$. Suppose that we have formed the dg tensor functor $\mathfrak{c}_5^{(r-1)}\colon \mathcal{A}_5^{\mathfrak{sh}}(\mathcal{V})^{(r,0)} \to \mathcal{A}_5(\mathcal{V})^{(r,0)}$; we denote $\mathfrak{c}_5^{(r-1)}$ by $\mathfrak{c}_5^{(r,0)}$:

$$\mathfrak{c}_5^{(1,0)} = \mathfrak{c}_4,\ \mathfrak{c}_5^{(r,0)} = \mathfrak{c}_5^{(r-1)}.$$

Suppose that we have formed the dg tensor functor $\mathfrak{c}_5^{(r,k-1)}\colon \mathcal{A}_5^{\mathfrak{sh}}(\mathcal{V})^{(r,k-1)} \to \mathcal{A}_5(\mathcal{V})^{(r,k-1)}$; we extend the functor $\mathfrak{c}_5^{(r,k-1)}$ to $\mathfrak{c}_5^{(r,k)}\colon \mathcal{A}_5^{\mathfrak{sh}}(\mathcal{V})^{(r,k)} \to \mathcal{A}_5(\mathcal{V})^{(r,k)}$ by setting

$$\mathfrak{c}_5^{(r,k)}(h_g) = h_{(\mathfrak{c}_5^{(r,k-1)}(g))}$$

for all morphisms h_g that are adjoined to $\mathcal{A}_5^{\mathfrak{sh}}(\mathcal{V})^{(r,k-1)}$ above. We define the dg tensor functor $\mathfrak{c}_5\colon \mathcal{A}_5^{\mathfrak{sh}}(\mathcal{V}) \to \mathcal{A}_5(\mathcal{V})$ as the direct limit of $\mathfrak{c}_5^{(r,k)}$ with respect to positive integers k and r:

$$\mathfrak{c}_5^{(r)} = \varinjlim_k \mathfrak{c}_5^{(r,k)},\ \mathfrak{c}_5 = \varinjlim_r \mathfrak{c}_5^{(r)}.$$

Lemma 1.24. *The dg functor $\mathfrak{c}_5$ is a homotopy equivalence.*

Proof. If $\mathfrak{c}_5^{(r,k-1)}$ is a homotopy equivalence, then $\mathfrak{c}_5^{(r,k)}$ is a homotopy equivalence by [41, Part I, Chapter V, 1.3.5], [41, Part II, Chapter II, 2.2.5], and [41, Part II, Chapter III, 3.1.12]. Hence, $\mathfrak{c}_5$ is a homotopy equivalence. □

1.8. **Construction of $\mathcal{A}_{\mathrm{mot}}(\mathcal{V})$.**

Definition 1.25. Let ? be $\emptyset$ or $\mathfrak{sh}$. We define the dg tensor category $\mathcal{A}_{\mathrm{mot}}^?(\mathcal{V})$ to be the dg tensor full subcategory of $\mathcal{A}_5^?(\mathcal{V})$, which is generated by objects of the form $\mathbf{Z}_X(q)_f$ or $\mathfrak{e}^{\otimes k} \otimes \mathbf{Z}_X(q)_f$, and we define the dg functor $\mathfrak{c}_{\mathrm{mot}}\colon \mathcal{A}_{\mathrm{mot}}^{\mathfrak{sh}}(\mathcal{V}) \to \mathcal{A}_{\mathrm{mot}}(\mathcal{V})$ as the restriction of $\mathfrak{c}_5$.

Lemma 1.26. *The dg functor $\mathfrak{c}_{\mathrm{mot}}$ is a homotopy equivalence.*

Proof. This lemma follows from [41, Part I, Chapter V, 1.3.5] and [41, Part II, Chapter II, 2.2.5 and Chapter III, 3.1.12]. □

1.9. **Construction of $\mathcal{DM}(\mathcal{V})$.** We first extend the coefficient of $\mathcal{A}^{?}_{\mathrm{mot}}(\mathcal{V})$ and consider the homotopy category of this extended category.

Definition 1.27. For any dg tensor category $\mathcal{A}$ and any commutative ring R, the dg tensor category $\mathcal{A}_R$ is defined as

$$\mathrm{Ob}(\mathcal{A}_R) = \mathrm{Ob}\mathcal{A}\,,\ \mathcal{A}_R(X,Y) = \mathcal{A}(X,Y) \otimes_{\mathbf{Z}} R.$$

Definition 1.28. Let R be a commutative ring. Then, we define the triangulated tensor category $K^b_{\mathrm{mot}}(\mathcal{V})_R$ as the zeroth cohomology category of $C^b_{\mathrm{dg}}(\mathcal{A}_{\mathrm{mot}}(\mathcal{V})_R)$ and define the tensor category $K^b_{\mathrm{mot}\mathfrak{sh}}$ as the full image of the functor

$$K^b\mathfrak{c}_{\mathrm{mot}} \colon K^b(\mathcal{A}^{\mathfrak{sh}}_{\mathrm{mot}}(\mathcal{V})_R) \to K^b_{\mathrm{mot}}(\mathcal{V})_R.$$

Definition 1.29 ([41, Part II, Chapter I, 1.3.7])**.** A triangulated functor $F\colon \mathcal{A} \to \mathcal{B}$ between triangulated tensor categories is a *triangulated pseudo-tensor functor* if there exists a commutative external product θ of F such that θ is an isomorphism and both F and θ are unital.

Proposition 1.30. *Let R be a commutative ring that is flat over* $\mathbf{Z}$*.*

(1) *The functor $K^b\mathfrak{c}_{\mathrm{mot}} \colon K^b(\mathcal{A}^{\mathfrak{sh}}_{\mathrm{mot}}(\mathcal{V})_R) \to K^b_{\mathrm{mot}}(\mathcal{V})_R$ is an equivalence of triangulated categories.*
(2) *The tensor category $K^b_{\mathrm{mot}\mathfrak{sh}}(\mathcal{V})_R$ is a full triangulated tensor subcategory of $K^b_{\mathrm{mot}}(\mathcal{V})_R$, and the functor $K^b\mathfrak{c}_{\mathrm{mot}} \colon K^b(\mathcal{A}^{\mathfrak{sh}}_{\mathrm{mot}}(\mathcal{V})_R) \to K^b_{\mathrm{mot}\mathfrak{sh}}(\mathcal{V})_R$ is a pseudo-tensor equivalence of triangulated tensor categories.*

Proof. See [41, Part II, Chapter II, 2.2.2]. □

We define the objects of $C^b_{\mathrm{dg}}(\mathcal{A}_{\mathrm{mot}}(\mathcal{V})_R)$ that correspond to *local cohomology groups.*

Definition 1.31. For any object $(X, f\colon X' \to X)$ of $\mathcal{L}(\mathcal{V})$ and any closed subscheme W of X, let $j\colon U \to X$ be the inclusion of the complement of W in X, and let $j^*f\colon U \times_X X' \to U$ be the projection. Suppose that the morphisms $j\colon U \to X$ and $j^*f\colon U \times_X X' \to U$ are in $\mathcal{V}$. We define the object $\mathbf{Z}_{X,W}(q)_f$ of $C^b_{\mathrm{dg}}(\mathcal{A}_{\mathrm{mot}}(\mathcal{V})_R)$ as

$$\mathbf{Z}_{X,W}(q)_f = \mathrm{Cone}(j^*\colon \mathbf{Z}_X(q)_f \to \mathbf{Z}_U(q)_{j^*f})[-1].$$

The next step is to localize the triangulated tensor category $K^b_{\mathrm{mot}}(\mathcal{V})_R$ with respect to some class of morphisms, which should be isomorphisms in the derived category of the expected abelian category of mixed motives.

Definition 1.32. Let R be a commutative ring that is flat over $\mathbf{Z}$. We assume that $\mathcal{V}$ satisfies the following three conditions:

(a) If X is an object of $\mathcal{V}$ and U is an open subscheme of X, then U is an object of $\mathcal{V}$.
(b) If X is an object of $\mathcal{V}$ and P is a projective bundle over X, then P is an object of $\mathcal{V}$.
(c) If X is an object of $\mathcal{V}$ and Z is a closed subscheme of X, then the blowup of X along Z is an object of $\mathcal{V}$.

Then, we define the triangulated tensor category $D^b_{\mathrm{mot}}(\mathcal{V})_R$ to be the localization of $K^b_{\mathrm{mot}}(\mathcal{V})_R$ obtained by inverting the following morphisms, i.e., $D^b_{\mathrm{mot}}(\mathcal{V})_R$ is the localization of $K^b_{\mathrm{mot}}(\mathcal{V})_R$ with respect to the saturated tensor multiplicative system of morphisms generated by the following morphisms (cf. [41, Part II, Chapter II, 2.3.2, 2.3.4]):

(1) *Homotopy equivalence.* Let $i\colon (X,f) \to (Y,g)$ be a morphism in $\mathcal{L}(\mathcal{V})$ such that $i\colon X \to Y$ is the inclusion of a closed codimension-one subscheme in $\mathcal{V}$, and let Z be a closed subscheme of Y. We let W denote the inverse image of Z via i. Suppose that Z is isomorphic to the affine line $\mathbf{A}^1_W$ over W and that, via this isomorphism, $i\colon W \to Z$ is the inclusion of $0 \times W$ into $\mathbf{A}^1_W$. Suppose further that W is an object of $\mathbf{Sm}^{\mathrm{ess}}_S$. Then, we invert the morphisms $i^*\colon \mathbf{Z}_{X,W}(q)_f \to \mathbf{Z}_{Y,Z}(q)_g$ for all $q \in \mathbf{Z}$.

(2) *Künneth formula.* Let X and Y be objects of $\mathcal{A}_1(\mathcal{V})$. Then, we invert the morphism induced by the commutative external product

$$\boxtimes_{X,Y}\colon X \otimes Y \to X \times Y,$$

where $\times$ (resp. $\otimes$) is the tensor product on $\mathcal{A}_1(\mathcal{V})$ (resp. $K^b_{\mathrm{mot}}(\mathcal{V})_R$).

(3) *Gysin isomorphism.* Let $p\colon (X,f) \to (W,g)$ be a morphism in $\mathcal{L}(\mathcal{V})$. Suppose that $p\colon X \to W$ is a smooth morphism of relative dimension q with a section $i\colon W \to X$. We regard W as a closed subscheme of X via i. Let

$$\alpha\colon \mathfrak{e} \otimes \mathbf{Z}_W(r-q)_g[-2q] \to \mathbf{Z}_{X\times_S X, W\times_X}(r)_{f\times f}$$

denote the composite

$$\mathfrak{e} \otimes \mathbf{Z}_W(r-q)_g[-2q] \xrightarrow{[W]\otimes p^*} \mathbf{Z}_{X,W}(q)_g \otimes \mathbf{Z}_X(r-q)_f \xrightarrow{\boxtimes} \mathbf{Z}_{X\times_S X, W\times_X}(r)_{f\times f},$$

and let ρ denote the morphism

$$\mathbf{Z}_{X\times_S X, W\times_X}(r)_{(f\times f)\coprod\Delta} \to \mathbf{Z}_{X\times_S X, W\times_X}(r)_{f\times f}$$

induced by the following morphism in $\mathcal{L}(\mathcal{V})$:

$$(X \times_S X, f\times f) \to (X \times_S X, (f\times f)\coprod \Delta),$$

where Δ is the diagonal morphism $X \to X \times_S X$. Then, we invert the morphisms

$$\begin{pmatrix} \alpha & -\rho \\ 0 & \Delta^* \end{pmatrix} \colon \mathfrak{e} \otimes \mathbf{Z}_W(r-q)_g[-2q] \oplus \mathbf{Z}_{X\times_S X, W\times_X}(r)_{(f\times f)\coprod\Delta}$$

$$\to \mathbf{Z}_{X\times_S X, W\times_X}(r)_{f\times f} \oplus \mathbf{Z}_{X\times_S X, W\times_X}(r)_{f\times f},$$

for all $r \in \mathbf{Z}$.

(4) *Excision.* Let (X,f) be an object of $\mathcal{L}(\mathcal{V})$, let W be a closed subscheme of X, and let $j\colon U \to X$ be the inclusion of an open subscheme of X containing W. Then, for all $q \in \mathbf{Z}$, we invert the morphisms

$$j^*\colon \mathbf{Z}_{X,W}(q) \to \mathbf{Z}_{U,W}(q).$$

(5) *Unit.* We invert the morphism

$$[S] \otimes \mathrm{id}\colon \mathfrak{e} \otimes \mathbf{Z}_S(0) \to \mathbf{Z}_S(0) \otimes \mathbf{Z}_S(0).$$

(6) *Moving lemma.* For each object $(X, f\colon X' \to X)$ of $\mathcal{L}(\mathcal{V})$ and each morphism $g\colon Y \to X$ in $\mathcal{V}$, let ϕ be the canonical morphism

$$(X, f\colon X' \to X) \to (X, f \coprod g\colon X' \coprod Y \to X)$$

in $\mathcal{L}(\mathcal{V})$. Then, for all $q \in \mathbf{Z}$, we invert the morphisms

$$\phi^*\colon \mathbf{Z}_X(q)_{f\coprod g} \to \mathbf{Z}_X(q)_f.$$

We define the triangulated tensor category $D^b_{\mathrm{mot}\mathfrak{sh}}(\mathcal{V})_R$ as the full image of $K^b_{\mathrm{mot}\mathfrak{sh}}(\mathcal{V})_R$ in $D^b_{\mathrm{mot}}(\mathcal{V})_R$.

Finally, by adjoining projectors to $D^b_{\mathrm{mot}}(\mathcal{V})_R$, we obtain Levine's triangulated category $\mathcal{DM}(\mathcal{V})_R$ of mixed motives with coefficients in the ring R.

Definition 1.33 ([3, 1.2], [41, Part II, Chapter II, 2.4.1])**.** Let $\mathcal{C}$ be an additive category. Then, the category $\mathcal{C}_\sharp$ is defined as follows:

(1) The objects of $\mathcal{C}_\sharp$ are pairs of the form (X, p), where X is an object of $\mathcal{C}$ and p is a morphism in $\mathcal{C}(X, X)$ with $p^2 = p$.
(2) A morphism $f\colon (X, p) \to (Y, q)$ in $\mathcal{C}_\sharp$ is given by a morphism $f\colon X \to Y$ in $\mathcal{C}$ such that $f = q \circ f \circ p$. The composition of morphisms in $\mathcal{C}_\sharp$ is that induced by the composition in $\mathcal{C}$.

We can treat $\mathcal{C}$ as a full subcategory of $\mathcal{C}_\sharp$ by sending X to (X, id). The category $\mathcal{C}_\sharp$ is called the *pseudo-abelian hull* (or *Karoubi envelope*, or *idempotent completion*) *of* $\mathcal{C}$.

Proposition 1.34 ([3, 1.5], [41, Part II, Chapter II, 2.4.7])**.** *For each triangulated* (*resp. triangulated tensor*) *category* $\mathcal{T}$*, the pseudo-abelian hull* $\mathcal{T}_\sharp$ *of* $\mathcal{T}$ *admits a unique triangulated* (*resp. triangulated tensor*) *category structure such that the inclusion is a triangulated* (*resp. triangulated tensor*) *functor. Moreover, each triangulated functor from* $\mathcal{T}$ *to* $\mathcal{T}'$ *is canonically extended to a triangulated functor from* $\mathcal{T}_\sharp$ *to* $\mathcal{T}'_\sharp$*.*

Proposition 1.35 ([41, Part II, Chapter II, 2.4.8.2])**.** *Let* $\mathcal{T}$ *be a triangulated* (*resp. triangulated tensor*) *category that contains the countable self-direct sum for each object of* $\mathcal{T}$*. Then, the inclusion of* $\mathcal{T}$ *in* $\mathcal{T}_\sharp$ *is an equivalence of triangulated* (*resp. triangulated tensor*) *categories.*

Definition 1.36. Let R be a commutative ring that is flat over $\mathbf{Z}$. Then, we define the triangulated tensor category $\mathcal{DM}(\mathcal{V})_R$ as the pseudo-abelian hull of $D^b_{\mathrm{mot}}(\mathcal{V})_R$. $\mathcal{DM}(\mathcal{V})_R$ is called the *triangulated motivic category of* $\mathcal{V}$ *with coefficients in* R. We denote the category $\mathcal{DM}(\mathbf{Sm}_S)_R$ by $\mathcal{DM}(S)_R$. If S is the spectrum of a commutative ring A, then we denote $\mathcal{DM}(\mathbf{Sm}_S)_R$ by $\mathcal{DM}(A)_R$.

Remark 1.37. For any perfect field k that admits the resolution of singularities for all schemes of finite type over k, Voevodsky [52] defined his own triangulated tensor category of motives over k. Levine proved that this motivic category defined by Voevodsky and his own $\mathcal{DM}(k)$ are equivalent (cf. [41, Part I, Chapter VI, 2.5.5]).

2. Geometric Cohomology Theory

As mentioned in the Introduction, Levine defined a geometric cohomology theory on $\mathcal{V}$ with values in the category of complexes of abelian sheaves on some Grothendieck site and constructed the realization functor associated with his geometric cohomology theory. In this section, we will define a geometric cohomology theory on $\mathcal{V}$ with values in some tensor category. Our geometric cohomology theory is a natural generalization of Levine's. In fact, upon taking the Godement resolution and the global sections, Levine's theory becomes equivalent to ours. This generalization is important for the constructions of realization functors associated with the mixed Hodge cohomology theory and the étale cohomology theory (see Section 4).

First, we define a *triangulated tensor system*, which is a generalization of an exact category that admits its derived category and on which our geometric cohomology theory takes its values.

Definition 2.1. A *triangulated tensor system* is a triple of the form $(\mathcal{C}, \mathcal{T}, F)$, where $\mathcal{C}$ is a tensor category, $\mathcal{T}$ is a triangulated tensor category, and F is a triangulated tensor functor from $K^+\mathcal{C}$ to $\mathcal{T}$.

Remark 2.2. For a triangulated tensor system $(\mathcal{C}, \mathcal{T}, F)$, we often treat objects or morphisms in $C^+\mathcal{C}$ as those in $\mathcal{T}$ obtained via the functor F.

Example 2.3. For a weakly idempotent complete exact category $\mathcal{C}$ (e.g., $\mathcal{C}$ is an abelian category, cf. [15, Section 7]), let $\mathbf{Ac}^+\mathcal{C}$ denote the full subcategory of the bounded below homotopy category $K^+(\mathcal{C})$ that consists of all acyclic objects. Because $\mathcal{C}$ is weakly idempotent complete,

$\mathbf{Ac}^+\mathcal{C}$ is a thick subcategory of the triangulated category $K^+\mathcal{C}$ (cf. [15, 10.14]). Then, we obtain the derived category $D^+\mathcal{C} := K^+\mathcal{C}/\mathbf{Ac}^+\mathcal{C}$ of bounded below complexes over $\mathcal{C}$. Now, the triangulated functor F is defined to be the natural functor $K^+\mathcal{C} \to D^+\mathcal{C}$. Assume further that $\mathcal{C}$ is an exact tensor category. Then, the triangulated category $D^+\mathcal{C}$ has a natural tensor structure induced by that of $\mathcal{C}$, and F is a triangulated tensor functor (cf. [41, Part II, Chapter II, 2.3.4]). Hence, the triple $(\mathcal{C}, D^+\mathcal{C}, F)$ forms a triangulated tensor system.

For simplicity, we denote this triangulated tensor system, which is associated with $\mathcal{C}$, by $D^+\mathcal{C}$.

For each object X of $\mathbf{Sm}_S^{\mathrm{ess}}$ and a closed subscheme W of X, let $\mathscr{Z}^q_W(X)$ denote the subgroup of $\mathscr{Z}^q(X)$ that consists of the cycles in $\mathscr{Z}^q(X)$ whose supports are contained by W. For each object $X(q)_f$ of $\mathcal{L}(\mathcal{V})^*$, let $(X,f)^q$ denote the set of closed subschemes W of X that is the support of some effective cycle in $\mathscr{Z}^q(X)_f$. The lemma below will be used in Section 3.

Lemma 2.4 ([41, Part I, Chapter V, 1.2.2])**.** *Let (X,f) and (Y,g) be objects of $\mathcal{L}(\mathcal{V})$.*

(1) *Let $\phi\colon (X,f) \to (Y,g)$ be a morphism in $\mathcal{L}(\mathcal{V})$. Then, for all $q \geq 0$ and all elements W of $(Y,g)^q$, the $\phi^{-1}(W)$ are in $(X,f)^q$.*
(2) *Let $q, r \geq 0$. Then, for all $W \in (X,f)^q$ and all $Z \in (Y,g)^r$, the subschemes $W \times_S Z$ of $X \times_S Y$ are in $(X \times_S Y, f \times g)^{q+r}$.*

Let $\mathcal{C}$ be an additive category, and let Γ be a functor from $\mathcal{V}^{\mathrm{op}} \times \mathbf{Z}$ to $C^+\mathcal{C}$. Now, we define the concept that corresponds to a *local cohomology* of Γ. Furthermore, we define the functors $\Gamma_{\mathcal{L}}$ and Γ_s as well as the natural transformation $\Gamma_s \to \Gamma_{\mathcal{L}}$. We will use them to construct the realization functor in the next section.

Definition 2.5. Using the above notation, for an object X of $\mathcal{V}$ and a closed subscheme $W \subset X$ of codimension q, let $j\colon X \setminus W \to X$ be the inclusion of the complement of $W \subset X$. Then, we define

$$\Gamma_W(X,q) = \mathrm{Cone}(\Gamma(j,q)\colon \Gamma(X,q) \to \Gamma(X \setminus W, q))[-1],$$

and we define the functor $\Gamma_s\colon \mathcal{L}(\mathcal{V})^{\mathrm{op}} \times \mathbf{Z} \to C^+\mathcal{C}$ to be

$$\Gamma_s((X,f),q) = \begin{cases} \varinjlim\limits_{W \in (X,f)^q} \Gamma_W(X,q) & \text{for } q \geq 0 \\ \Gamma(X,q) & \text{otherwise} \end{cases}$$

if it exists. Furthermore, we define the functor $\Gamma_{\mathcal{L}}\colon \mathcal{L}(\mathcal{V})^{\mathrm{op}} \times \mathbf{Z} \to C^+\mathcal{C}$ to be $\Gamma_{\mathcal{L}}(X,f,q) = \Gamma(X,q)$ for all objects $X(q)_f$ of $\mathcal{L}(\mathcal{V})^{\mathrm{op}} \times \mathbf{Z}$.

From the definitions of $\Gamma_{\mathcal{L}}$ and Γ_s, the natural morphisms $\Gamma_W(X,q) \to \Gamma(X,q)$ induce a natural transformation $\Gamma_s \to \Gamma_{\mathcal{L}}$. Let ν be this natural transformation $\Gamma_s \to \Gamma_{\mathcal{L}}$.

Remark 2.6. In the remainder of this paper, we denote $\Gamma(f,q)$ by f^* for any morphism f in $\mathcal{V}$ if the integer q is clear from the context.

The definition of our geometric cohomology theory is as follows (see [41, Part I, Chapter V, 1.1.6] for the corresponding definition of Levine's theory).

Definition 2.7. Let $(\mathcal{C}, \mathcal{T}, F)$ be a triangulated tensor system with a commutative external product $\boxtimes$ of a functor $\Gamma\colon \mathcal{V}^{\mathrm{op}} \times \mathbf{Z} \to C^+\mathcal{C}$. Furthermore, for each object X of $\mathcal{V}$ and each effective cycle C in $\mathscr{Z}^q(X)$, let $\mathrm{cl}^q_{W,X}$ be a homomorphism from $\mathscr{Z}^q_W(X)$ to $\mathcal{T}(1, \Gamma_W(X,q)[2q])$, where W is the support of C.

Then, the triple $(\Gamma, \boxtimes, \mathrm{cl}^q_{W,X})$ is said to be a $(\mathcal{C}, \mathcal{T}, F)$*-valued geometric cohomology theory on* $\mathcal{V}$ if it satisfies the following axioms:

(1) *Direct sum.* For each of the objects X and Y of $\mathcal{V}$ and for $q \in \mathbf{Z}$, the canonical morphism $\Gamma(X \coprod Y, q) \to \Gamma(X,q) \oplus \Gamma(Y,q)$ is an isomorphism in $C^+(\mathcal{C})$.
(2) *Cycle classes.* The morphisms $\mathrm{cl}^q_{W,X}$ satisfy the following conditions:

(a) Let $f : X \to Y$ be a morphism in $\mathcal{V}$. If $f^{-1}(W)$ is contained in the support Z of some effective cycle in $\mathcal{Z}^q(Y)$, then the diagram

$$\begin{tikzcd} \mathcal{Z}^q_W(X) \arrow[r, "\mathrm{cl}^q_{W,X}"] \arrow[d, "f^*"'] & \mathcal{T}(1, \Gamma_W(X,q)[2q]) \arrow[d, "f^*"] \\ \mathcal{Z}^q_Z(Y) \arrow[r, "\mathrm{cl}^q_{Z,Y}"'] & \mathcal{T}(1, \Gamma_Z(Y,q)[2q]) \end{tikzcd}$$

commutes.

(b) Let Z be the support of an effective cycle in $\mathcal{Z}^r(Y)$. Then,

$$\mathrm{cl}^q_{W,X}(C) \boxtimes \mathrm{cl}^r_{Z,Y}(D) = \mathrm{cl}^{q+r}_{W\times_S Z, X\times_S Y}(C \times_S D)$$

for each $C \in \mathcal{Z}^q_W(X)$ and each $D \in \mathcal{Z}^r_Z(Y)$, where the morphism

$$\mathrm{cl}^q_{W,X}(C) \boxtimes \mathrm{cl}^r_{Z,Y}(D)$$

is the composite of

$$1 \xrightarrow{\mathrm{cl}^q_{W,X}(C)\otimes \mathrm{cl}^r_{Z,Y}(D)} \Gamma_W(X,q) \otimes \Gamma_Z(Y,r)[2q+2r] \text{ and}$$

$$\Gamma_W(X,q) \otimes \Gamma_Z(Y,r) \xrightarrow{\boxtimes} \Gamma_{W\times_S Z}(X \times_S Y, q+r)[2q+2r].$$

(3) *Semi-purity.* Let (X, f) be an object of $\mathcal{L}(\mathcal{V})$ and $q \in \mathbf{Z}$. Then,

$$\mathcal{T}(1, \Gamma_W(X,q)[2q-p]) = 0$$

for all $W \in (X, f)^q$, and $p \geq 1$.

(4) *Homotopy equivalence.* Let $i : X \to Y$ be the inclusion of a closed subscheme of codimension one in $\mathcal{V}$. Let Z be a closed subscheme of Y, and let W be the inverse image of Z via i. Suppose that Z is isomorphic to the affine line $\mathbf{A}^1_W$ over W and that, via this isomorphism, $i : W \to Z$ is the inclusion of $0 \times W$ into $\mathbf{A}^1_W$. Suppose further that W is in $\mathbf{Sm}^{\mathrm{ess}}_S$. Then, $i^* : \Gamma_W(X,q) \to \Gamma_Z(Y,q)$ is an isomorphism in $\mathcal{T}$ for $q \in \mathbf{Z}$.

(5) *Künneth formula.* For all objects X and Y of $\mathcal{V}$ and for all $q, r \in \mathbf{Z}$, the morphisms induced by the commutative external product

$$\boxtimes^{q,r}_{X,Y} : \Gamma(X,q) \otimes \Gamma(Y,r) \to \Gamma(X \times_S Y, q+r)$$

are isomorphisms in $\mathcal{T}$.

(6) *Gysin isomorphism.* Let $p\colon X \to W$ be a smooth morphism in $\mathcal{V}$ of relative dimension q with a section $i\colon W \to X$. We regard W as a closed subscheme of X via i. Then, the composite

$$\Gamma(W,r) \xrightarrow{p^*} \Gamma(X,r) \xrightarrow{\cup \mathrm{cl}^r_{W,X}(W)} \Gamma_W(X, q+r)[2q]$$

is an isomorphism in $\mathcal{T}$ for each $r \in \mathbf{Z}$, where $\cup\,\mathrm{cl}^r_{W,X}(W)$ is the composite

$$\Gamma(X,r) \xrightarrow{\mathrm{id}\otimes \mathrm{cl}^q_{W,X}(W)} \Gamma(X,r) \otimes \Gamma_W(X,q)[2q]$$

$$\xrightarrow{\boxtimes} \Gamma_{X\times_S W}(X \times_S X, q+r)[2q] \xrightarrow{\Delta^*} \Gamma_W(X, q+r)[2q].$$

Here, Δ is the diagonal morphism $X \to X \times_S X$.

(7) *Excision.* Let X be an object of $\mathcal{V}$, let W be a closed subscheme of X, and let $j\colon U \to X$ be the inclusion of an open subscheme U of X containing Z. Then, the morphism $j^*\colon \Gamma_W(X,q) \to \Gamma_W(U,q)$ is an isomorphism in $\mathcal{T}$ for each $q \in \mathbf{Z}$.

(8) *Unit.* The morphism $\mathrm{cl}^0_{S,S}(S)\colon 1 \to \Gamma(S,0)$ is an isomorphism in $\mathcal{T}$ when we regard S as a cycle in $\mathcal{Z}^0(S)$.

Remark 2.8. For a geometric cohomology theory $(\Gamma, \boxtimes, \mathrm{cl}^q_{W,X})$, we will often omit one or more of the subscripts q, W or X of the cycle class map $\mathrm{cl}^q_{W,X}$ when it or they are clear from the context.

3. Realization of Levine's Motivic Category

In this section, the category $\mathcal{V}$ will satisfy the conditions (a),(b) and (c) of Definition 1.32. We will construct a realization functor from the motivic category $\mathcal{DM}(\mathcal{V})$ associated with a given geometric cohomology theory on $\mathcal{V}$ that satisfies certain assumptions. The flow of our construction is similar to that of Levine's for the realization functor associated with his geometric cohomology theory (cf. [41, Chapter V, Section 1]). The difference between the two constructions is primarily derived from the difference in the definitions of the two geometric cohomology theories. As mentioned at the beginning of Section 2, upon taking the Godement resolution and the global sections, Levine's geometric cohomology theory becomes equivalent to ours. This section begins with the preparations for formulating our first main theorem (Theorem 3.5).

Let Δ denote the category whose objects are the ordered sets $[n] := \{0 < \cdots < n\}$ for all $n \geq 0$ and whose morphisms are non-decreasing maps. For a category $\mathcal{A}$, we let c.s.$\mathcal{A}$ denote the category of cosimplicial objects of $\mathcal{A}$; i.e., c.s.$\mathcal{A}$ is the category of functors from Δ to $\mathcal{A}$.

Definition 3.1. For an additive category $\mathcal{A}$, the functor $\iota\colon C^+_{\mathrm{dg}}\mathcal{A} \to \mathrm{c.s.}C^+_{\mathrm{dg}}\mathcal{A}$ is defined as $\iota(A)([n]) = A$ for all objects A of $\mathcal{A}$ and as $\iota(f)([n]) = f$ for all morphisms f of $\mathcal{A}$.

We define the dg functor G as the composite functor

$$C^+_{\mathrm{dg}}\mathcal{A} \xrightarrow{\iota} \mathrm{c.s.}C^+_{\mathrm{dg}}\mathcal{A} \xrightarrow{\mathrm{cc}} C^+_{\mathrm{dg}}C^+_{\mathrm{dg}}\mathcal{A} \xrightarrow{\mathrm{Tot}} C^+_{\mathrm{dg}}\mathcal{A},$$

where the central arrow cc is defined to represent taking the associated cochain complex (cf. [41, Part II, Chapter III, 1.2.1]) and Tot is the total complex functor defined in [44, 2.18] (see also [43, 2.29]).

Lemma 3.2. *Using the above notation, the canonical morphism* $\mathrm{id} \to G$ *of endofunctors of* $C^+_{\mathrm{dg}}\mathcal{A}$ *is homotopy equivalent.*

Proof. For an object M of $C^+_{\mathrm{dg}}\mathcal{A}$, the complex $cc \circ \iota(M)$ is $(M^\bullet, d^\bullet)$, which is defined as

$$M^i = \begin{cases} M & \text{if } i \geq 0 \\ 0 & \text{otherwise} \end{cases}, \quad d^i = \begin{cases} \mathrm{id}_M & \text{if } i \text{ is odd} \\ 0 & \text{otherwise} \end{cases} : M^i \to M^{i+1}.$$

Hence, it is evident that the map $M \to cc \circ \iota(M)$ is homotopy equivalent. Upon applying the total complex functor Tot, we see that $M \to G(M)$ is also homotopy equivalent, as Tot is a dg functor (cf. [43, 2.29], [44, 2.18]). $\square$

Lemma 3.3 ([41, Part II, Chapter III, 2.2.4])**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be tensor categories without unit, and let F be an additive functor from $\mathcal{A}$ to c.s.$\mathcal{B}$ with a commutative external product $\boxtimes$. Then, F can be naturally extended to a dg tensor functor $F^{\mathfrak{sh}}\colon \mathcal{A}^{\otimes,\mathfrak{sh}} \to C^+_{\mathrm{dg}}\mathcal{B}$; i.e., the following diagram*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathrm{c.s.}\mathcal{B} \\ {\scriptstyle i^{\mathfrak{sh}}}\downarrow & & \downarrow{\scriptstyle \mathrm{cc}} \\ \mathcal{A}^{\otimes,\mathfrak{sh}} & \xrightarrow{F^{\mathfrak{sh}}} & C^+_{\mathrm{dg}}\mathcal{B} \end{array}$$

commutes, and $F^{\mathfrak{sh}}$ is compatible with the external product $\boxtimes$ of F.

The property of $\mathcal{A}^{\otimes,\mathfrak{sh}}$ stated in Lemma 3.3 is similar to the universality of $\mathcal{A}^{\otimes,c}$ stated in Lemma 1.9. The primary difference between them is that the extended functor $F^{\mathfrak{sh}}$ in this lemma is a dg *tensor* functor, whereas the extended functor in Lemma 1.9 is only additive. The proposition below corresponds to Step 2 of Levine's construction of the realization functor given in [41, Part I, Chapter 5, 1.3.1]. Using the lemma above, we will extend the additive functor $G \circ \Gamma_{\mathcal{L}}$ to a dg tensor functor from $\mathcal{A}_2^{\mathfrak{sh}}(\mathcal{V})$ to $C^+_{\mathrm{dg}}\mathcal{C}$.

Proposition 3.4. *Let $\mathcal{C}$ be a tensor category. For a functor $\Gamma\colon \mathcal{V}^{\mathrm{op}} \times \mathbf{Z} \to C^+\mathcal{C}$ that admits the functor Γ_s (cf. Definition 2.5), let $\boxtimes$ be a commutative external product of Γ. Assume that Γ satisfies axiom (1) in Definition 2.7. Then, the additive functors $G \circ \Gamma_{\mathcal{L}}$ and $G \circ \Gamma_s$ can be naturally extended to the dg tensor functors Γ_2 and $\Gamma_{s,2}$ from $\mathcal{A}_2^{\mathfrak{sh}}(\mathcal{V})$ to $\mathcal{C}$, which are compatible with the commutative external product $\boxtimes$ of Γ. The natural transformation $\nu\colon \Gamma_{\mathcal{L}} \to \Gamma_s$ (defined in Definition 2.5) can also be naturally extended to $\nu_2\colon \Gamma_2 \to \Gamma_{s,2}$:*

$$\begin{array}{ccccccc} \mathcal{L}(\mathcal{V})^{\mathrm{op}} \times \mathbf{Z} & \xrightarrow{G\circ\Gamma_{\mathcal{L}}} & C^+\mathcal{C} & \qquad & \mathcal{L}(\mathcal{V})^{\mathrm{op}} \times \mathbf{Z} & \xrightarrow{G\circ\Gamma_{s}} & C^+\mathcal{C} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{A}_2^{\mathfrak{sh}}(\mathcal{V}) & \xrightarrow{\Gamma_2} & C^+_{\mathrm{dg}}\mathcal{C}, & & \mathcal{A}_2^{\mathfrak{sh}}(\mathcal{V}) & \xrightarrow{\Gamma_{s,2}} & C^+_{\mathrm{dg}}\mathcal{C}. \end{array}$$

Proof. First, using Lemma 1.3, we extend the functors $\Gamma_{\mathcal{L}}$ and Γ_s to the symmetric monoidal functors $\Gamma_{\mathcal{L}^*}$ and $\Gamma_{s,\mathcal{L}^*}$ from $\mathcal{L}(\mathcal{V})^*$ to $C^+_{\mathrm{dg}}\mathcal{C}$ by setting

$$\Gamma_{\mathcal{L}^*}(i_*)\colon \Gamma(X,q) \xrightarrow{\mathrm{id}\oplus 0} \Gamma(X,q) \oplus \Gamma(Y,q) \xrightarrow{\cong} \Gamma(X \coprod Y, q),$$

$$\Gamma_{s,\mathcal{L}^*}(i_*)\colon \Gamma_s(X,f,q) \xrightarrow{\mathrm{id}\oplus 0} \Gamma_s(X,f,q) \oplus \Gamma_s(Y,g,q) \xrightarrow{\cong} \Gamma(X \coprod Y, f \coprod g, q)$$

for each natural inclusion $i\colon X(q)_f \to (X \coprod Y)(q)_{f\coprod g}$ in $\mathcal{L}(\mathcal{V})^{\mathrm{op}} \times \mathbf{Z}$ using Definition 2.7(1). To be precise, using Lemma 1.3(1) for $\mathcal{C} = \mathcal{L}(\mathcal{V})^{\mathrm{op}} \times \mathbf{Z}$, $\mathcal{D} = C^+_{\mathrm{dg}}\mathcal{C}$, $\Phi = \{i_* \mid i_*\colon X(q)_f \to (X \coprod Y)(q)_{f\coprod g}\}$, $F = \Gamma_{\mathcal{L}}$ (resp. Γ_s), and $\psi_{i_*} = \Gamma_{\mathcal{L}^*}(i_*)$ (resp. $\Gamma_{s,\mathcal{L}^*}(i_*)$), we obtain the symmetric monoidal functor $F_\Psi\colon \mathcal{C}_\Phi \to \mathcal{D}$. Because the morphisms $\psi_{i_*} = \Gamma_{\mathcal{L}^*}(i_*)$ (resp. $\Gamma_{s,\mathcal{L}^*}(i_*)$) satisfy the three relations specified in Definition 1.4, we can apply Lemma 1.3(2) in this case. Thus, we obtain the symmetric monoidal functor $F_R\colon \mathcal{C}_R = \mathcal{L}(\mathcal{V})^* \to C^+_{\mathrm{dg}}\mathcal{C}$, and $\Gamma_{\mathcal{L}^*}(i_*)$ (resp. $\Gamma_{s,\mathcal{L}^*}(i_*)$) is defined to be F_R. We extend the natural transformation $\nu\colon \Gamma_s \to \Gamma_{\mathcal{L}}$ to the natural transformation $\nu_{\mathcal{L}}\colon \Gamma_{s,\mathcal{L}} \to \Gamma_{\mathcal{L}}$ by setting $\nu_{\mathcal{L}}(\Gamma_{s,\mathcal{L}^*}(i_*)) = \Gamma_{\mathcal{L}^*}(i_*)$.

Next, we will extend the functors $\Gamma_{\mathcal{L}^*}$ and $\Gamma_{s,\mathcal{L}^*}$ to the functors Γ_2 and $\Gamma_{s,2}$ from $\mathcal{A}_2^{\mathfrak{sh}}(\mathcal{V})$ to $C^+_{\mathrm{dg}}\mathcal{C}$. Let ? be $\emptyset$ or s. Because $C^+_{\mathrm{dg}}(\mathcal{C})$ is an additive category, we can canonically extend the symmetric monoidal functor $\Gamma_{?,\mathcal{L}^*}$ to the tensor functor from $\mathbf{Z}\mathcal{L}(\mathcal{V})^*$ to $C^+_{\mathrm{dg}}\mathcal{C}$ (cf. Definition 1.5). Let $\Gamma_{?,\mathbf{Z}\mathcal{L}^*}$ denote this functor, $\Gamma_{?,\mathbf{Z}\mathcal{L}^*}\colon \mathbf{Z}\mathcal{L}(\mathcal{V})^* \to C^+_{\mathrm{dg}}\mathcal{C}$.

The commutative external product $\boxtimes$ of Γ can be naturally extended to that of the functor $\Gamma_{?,\mathbf{Z}\mathcal{L}^*}\colon \mathbf{Z}\mathcal{L}(\mathcal{V})^* \to C^+_{\mathrm{dg}}\mathcal{C}$, using Lemma 2.4, for the case of ? $= s$. By applying Lemma 3.3 to the pair $(\iota \circ \Gamma_{?,\mathbf{Z}\mathcal{L}^*}, \iota \circ \boxtimes)$, where ι is the functor defined in Definition 3.1, we obtain the dg tensor functor $(\iota \circ \Gamma_{?,\mathbf{Z}\mathcal{L}^*})^{\mathfrak{sh}}\colon (\mathbf{Z}\mathcal{L}(\mathcal{V})^*)^{\otimes,\mathfrak{sh}} \to C^+_{\mathrm{dg}}C^+_{\mathrm{dg}}\mathcal{C}$. Now, we define the functor $\Gamma_{?,0}$ to be the composite of this functor $(\iota \circ \Gamma_{?,\mathbf{Z}\mathcal{L}^*})^{\mathfrak{sh}}$ and the total complex functor used in Definition 3.1. This total complex functor is a tensor functor (cf. [43, 2.29], [44, 2.18]). Similarly, the natural transformation $\nu_{\mathcal{L}}\colon \Gamma_{s,\mathcal{L}} \to \Gamma_{\mathcal{L}}$ extends to the natural transformation $\Gamma_{s,0} \to \Gamma_0$. Let ν_0 be this natural transformation, $\Gamma_{s,0} \to \Gamma_0$. We remark that the composite of $\mathbf{Z}\mathcal{L}(\mathcal{V})^* \to (\mathbf{Z}\mathcal{L}(\mathcal{V})^*)^{\otimes,\mathfrak{sh}}$ and $\Gamma_{?,0}$ is equal to $G \circ \Gamma_{?,\mathbf{Z}\mathcal{L}^*}$ by Lemma 3.3.

It follows from the construction of $(-)^{\otimes,\mathfrak{sh}}$ (cf. [41, Part II, Chapter III, 2.1.5]) that $\mathcal{A}_2^{\mathfrak{sh}}(\mathcal{V})$ is isomorphic to the dg tensor category obtained by imposing the relations in Definition 1.7 on $(\mathbf{Z}\mathcal{L}(\mathcal{V}))^{\otimes,\mathfrak{sh}}$. Upon applying the functor $\Gamma_{?,0}$ to these relations, we see that similar relations hold in $C^+_{\mathrm{dg}}\mathcal{C}$. Hence, using Lemma 1.3(2), we can extend the dg tensor functor $\Gamma_{?,0}$ to the dg tensor functor from $\mathcal{A}_2^{\mathfrak{sh}}(\mathcal{V})$ to $C^+_{\mathrm{dg}}\mathcal{C}$. We denote this extended tensor functor by $\Gamma_{?,2}$. The natural transformation ν_0 naturally extends the natural transformation $\nu_2 : \Gamma_{s,2} \to \Gamma_2$. $\square$

The following is our first main result, which is a generalization of Levine's theorem ([41, Part I, Chapter V, 1.3.1]).

Theorem 3.5. *Let $(\mathcal{C}, \mathcal{T}, F)$ be a triangulated tensor system, and let R be a commutative ring that is flat over $\mathbf{Z}$ such that the ring $\mathcal{C}(1,1)$ is an R-algebra. Let $\mathfrak{d}$ be an object of $C^+\mathcal{C}$ with an isomorphism $\mathfrak{d} \cong 1$ in $D^+\mathcal{C}$ that satisfies the assumption in Proposition 1.16. Furthermore, let $(\Gamma, \boxtimes, \mathrm{cl}^q_{W,X})$ be a $(\mathcal{C}, \mathcal{T}, F)$-valued geometric cohomology theory on $\mathcal{V}$ that admits the functor Γ_s (defined in 2.5). Now, we assume the following two conditions*:

(1) *The class of the cycle class $\mathrm{cl}(C)$ in $\mathcal{T}(1, \Gamma_s(X,f,q)[2q])$ belongs to the image of the map from $K^+\mathcal{C}(\mathfrak{d}, \Gamma_s(X,f,q)[2q])$, which is induced by the functor F and by the morphism $\mathfrak{d} \cong 1$, for each object $X(q)_f$ of $\mathcal{L}(\mathcal{V})^*$ and each cycle C in $\mathcal{Z}^q(X)_f$, where W is the support of C.*

(2) *The maps*

$$K^+\mathcal{C}(\mathfrak{d}, \Gamma_s(X,f,q)[2q-p]) \to \mathcal{T}(1, \Gamma_s(X,f,q)[2q-p]),$$

which are induced by F and $\mathfrak{d} \cong 1$, are injective for all objects (X,f) of $\mathcal{L}(\mathcal{V})$, $p \geq 0$, and $q \in \mathbf{Z}$.

Then, the functor Γ can be canonically extended to a triangulated functor

$$D^b\Gamma_{\mathrm{mot}} \colon D^b_{\mathrm{mot}}(\mathcal{V})_R \to \mathcal{T},$$

which sends $\mathfrak{e}$ to $\mathfrak{d}$ and satisfies the following conditions:

1. *The composite functor $F \circ G \circ \Gamma$ is equal to the composite*
$$\mathcal{V}^{\mathrm{op}} \times \mathbf{Z} \to D^b_{\mathrm{mot}}(\mathcal{V})_R \xrightarrow{D^b\Gamma_{\mathrm{mot}}} \mathcal{T},$$
where the left arrow is the natural functor that sends (X,q) to $\mathbf{Z}_X(q)$.
2. *The functor $D^b(\Gamma_{\mathrm{mot}})$ is compatible with the cycle class maps cl via $\mathfrak{d} \cong 1$.*
3. *The restriction of $D^b\Gamma_{\mathrm{mot}}$ to $D^b_{\mathrm{mot}\mathfrak{sh}}(\mathcal{V})_R$ is a triangulated pseudo-tensor functor (cf. Definition 1.29) from $D^b_{\mathrm{mot}\mathfrak{sh}}(\mathcal{V})_R$ to $\mathcal{T}$, which is compatible with the commutative external product $\boxtimes$ of Γ.*

Definition 3.6. The triangulated functor $\mathcal{R}_\Gamma \colon \mathcal{DM}(\mathcal{V})_R \to \mathcal{T}_\sharp$ is defined to be the pseudo-abelian hull (cf. Proposition 1.34) of the functor $D^b\Gamma_{\mathrm{mot}}$ in the above theorem. We call this functor $\mathcal{R}_\Gamma$ the *realization functor associated with* Γ. In particular, when the embedding of $\mathcal{T}$ in $\mathcal{T}_\sharp$ is an equivalence (cf. Proposition 1.35), we define the realization functor associated with Γ to be the composite of $\mathcal{R}_\Gamma$ and its quasi-inverse $\mathcal{T}_\sharp \to \mathcal{T}$.

To prove Theorem 3.5, we first prove the proposition below.

Proposition 3.7. *Given the same assumptions as in the theorem above and without assuming that the triple $(\Gamma, \boxtimes, \mathrm{cl})$ satisfies axioms (4), (5), (6) and (7) of Definition 2.7, the functor Γ can be extended to a dg tensor functor*

$$\Gamma_{\mathrm{mot}} \colon \mathcal{A}^{\mathfrak{sh}}_{\mathrm{mot}}(\mathcal{V}) \otimes R \to C^+_{\mathrm{dg}}\mathcal{C},$$

which sends $\mathfrak{e}$ to $\mathfrak{d}$ such that the composite

$$K^b(\mathcal{A}^{\mathfrak{sh}}_{\mathrm{mot}}(\mathcal{V}) \otimes R) \xrightarrow{H^0 C^b_{\mathrm{dg}}\Gamma_{\mathrm{mot}}} K^b C^+_{\mathrm{dg}}\mathcal{C} \xrightarrow{\mathrm{Tot}} K^+\mathcal{C} \xrightarrow{F} \mathcal{T}$$

is compatible with the triple $(\Gamma, \boxtimes, \mathrm{cl})$ via the isomorphism $\mathfrak{d} \cong 1$, where Tot is the total complex functor (cf. [41, Part II, Chapter II, 1.2.9]).

Proof. We will extend the functor Γ to the functor from $\mathcal{A}^{\mathfrak{sh}}_{\mathrm{mot}}(\mathcal{V})$ step by step along the flow of the construction of $\mathcal{A}^{\mathfrak{sh}}_{\mathrm{mot}}(\mathcal{V})$ (cf. Section 1.1). First, by Proposition 3.4, we extend the natural transformation $\nu \colon \Gamma_{\mathcal{L}} \to \Gamma_s$ to the natural transformation $\nu_2 \colon \Gamma_2 \to \Gamma_{2,s}$ of functors from $\mathcal{A}^{\mathfrak{sh}}_2(\mathcal{V})$ to $C^+_{\mathrm{dg}}\mathcal{C}$.

Step 1. The extension to $\mathcal{A}_3(\mathcal{V})$.

By Proposition 1.16, we choose a dg tensor functor $I\colon \mathbb{E} \to C^+_{\mathrm{dg}}\mathcal{C}$ that satisfies $I(\mathfrak{e}^{\otimes n}) = \mathfrak{d}^{\otimes n}$ for all $n \geq 1$. Such a functor is unique up to homotopy (cf. loc. cit.). Let $\Gamma_{?,2}[I]\colon \mathcal{A}^{\mathfrak{sh}}_2(\mathcal{V})[\mathbb{E}] \to C^+_{\mathrm{dg}}\mathcal{C}$ denote the dg tensor functor that is the coproduct of I and $\Gamma_{?,2}$. The natural transformation ν_2 extends to the natural transformation $\nu_2[I]\colon \Gamma_{s,2}[I] \to \Gamma_2[I]$.

Using Lemma 1.3, we extend the dg tensor functor $\Gamma_{?,2}[I]$ to the dg tensor functor $\Gamma_{?,3}$ from $\mathcal{A}^{\mathfrak{sh}}_3(\mathcal{V})$ to $C^+_{\mathrm{dg}}\mathcal{C}$ as follows: For each object $X(q)_f$ of $\mathcal{L}(\mathcal{V})^*$ and each non-zero cycle C in $\mathscr{Z}^q(X)_f$, we choose a morphism $\gamma\colon \mathfrak{d} \to G \circ \Gamma_s(X, f, q)[2q]$ in $C^+_{\mathrm{dg}}\mathcal{C}$, which represents the cycle class $\mathrm{cl}^q_{W,X}(C)\colon \mathfrak{d} \to \Gamma_W(X, q)[2q]$ of C, using assumption (1) of this theorem, where W is the support of C. Then, we consider the morphism

$$\Gamma_{s,3}([C]^{\mathfrak{sh}})\colon \mathfrak{d} \to \Gamma_{s,2}[I](X, f, q) = G \circ \Gamma_s(X, f, q)$$

of degree $2q$ for the composite morphism

$$\mathfrak{d} \xrightarrow{\gamma} \Gamma_s(X, f, q)[2q] \to G \circ \Gamma_s(X, f, q)[2q],$$

where the right arrow is the canonical morphism.

We define the morphism $\Gamma_3([C]^{\mathfrak{sh}})\colon \mathfrak{d} \to \Gamma_3[I](X, f, q)$ of degree $2q$ to be the composite of $\Gamma_{s,3}([C]^{\mathfrak{sh}})$ and the morphism $\Gamma_{s,2}(X, f, q) \to \Gamma_2(X, f, q)$ that is induced by the natural transformation $\nu_2\colon \Gamma_{s,2} \to \Gamma_2$. The natural transformation $\nu_2[I]$ is naturally extended to the natural transformation $\nu_3\colon \Gamma_{s,3} \to \Gamma_3$.

Step 2. The extension to $\mathcal{A}_4(\mathcal{V})$.

We extend the dg tensor functor $\Gamma_{s,3}$ to the dg tensor functor $\Gamma_{s,4}$ from $\mathcal{A}^{\mathfrak{sh}}_4(\mathcal{V})$ to $C^+_{\mathrm{dg}}\mathcal{C}$ using Lemma 1.3 as follows: Let $X(q)_f$ be an object of $\mathcal{L}(\mathcal{V})^*$, and let C and D be cycles in $\mathscr{Z}^q(X)_f$. Then, the morphism

$$\Gamma_s([mC + nD]^{\mathfrak{sh}} - [mC]^{\mathfrak{sh}} - [nD]^{\mathfrak{sh}})\colon \mathfrak{d} \to \Gamma_s(X, f, q)[2q]$$

is zero in $\mathcal{T}$ by the additivity of the cycle class maps $\mathrm{cl}^q_{W,X}$. This morphism is already zero in $K^+\mathcal{C}$ by Lemma 3.2 and assumption (2) of this theorem. Then, we choose a morphism $\eta\colon \mathfrak{d} \to \Gamma_s(X, f, q)$ of degree $2q$ in $C^+_{\mathrm{dg}}\mathcal{C}$ such that

$$d\eta = \Gamma_s([mC + nD]^{\mathfrak{sh}} - [mC]^{\mathfrak{sh}} - [nD]^{\mathfrak{sh}}).$$

Here, we consider the morphism $\Gamma_{s,4}(h^{\mathfrak{sh}}_{m,n,C,D})\colon \mathfrak{d} \to \Gamma_{s,3}(X, f, q)$ for this morphism η. Similarly, for each morphism h that is adjoined to $\mathcal{A}^{\mathfrak{sh}}_3(\mathcal{V})$ in Definition 1.20(2) and (3), there exists a morphism η_h in $C^+_{\mathrm{dg}}\mathcal{C}$ such that $d\eta_h = \Gamma_{s,3}(h)$, by axioms (2)(a) and (2)(b) of Γ in Definition 2.7 and by the assumptions of this theorem. We choose such a morphism η_h and set $\Gamma_{s,4}(h) = \eta_h$. Then, we obtain the functor $\Gamma_{s,4}$.

We extend the dg tensor functor Γ_3 to that denoted by Γ_4 from $\mathcal{A}^{\mathfrak{sh}}_4(\mathcal{V})$ to $C^+_{\mathrm{dg}}\mathcal{C}$, in a manner similar to that used in the construction of Γ_3, by means of the functor $\Gamma_{s,4}$ and the natural transformation ν_3. We also naturally extend ν_3 to the natural transformation $\nu_4\colon \Gamma_{s,4} \to \Gamma_4$.

Step 3. The extension to $\mathcal{A}_5(\mathcal{V})$.

Let ? be $\emptyset$ or s. We set $\Gamma^{(0)}_{?,5} = \Gamma_{?,4}\colon \mathcal{A}^{\mathfrak{sh}}_4(\mathcal{V}) \to C^+_{\mathrm{dg}}\mathcal{C}$ and $\nu^{(0)}_5 = \nu_4\colon \Gamma_{s,4} \to \Gamma_4$. Let r be a positive integer. Suppose that we have formed the dg tensor functor $\Gamma^{(r-1)}_{?,5}$ from $\mathcal{A}^{\mathfrak{sh}}_5(\mathcal{V})^{(r-1)}$ to $C^+_{\mathrm{dg}}\mathcal{C}$ and the natural transformation $\nu^{(r-1)}_5\colon \Gamma^{(r-1)}_{s,5} \to \Gamma^{(r-1)}_5$. We denote $\Gamma^{(r-1)}_{?,5}$ and $\nu^{(r-1)}_5$ by $\Gamma^{(r,0)}_{?,5}$ and $\nu^{(r,0)}_5$, respectively. Furthermore, let k be a positive integer, and suppose that we have formed the dg tensor functor $\Gamma^{(r,k-1)}_{?,5}$ and the natural transformation $\nu^{(r,k-1)}_5$:

$$\Gamma^{(r,k-1)}_{?,5}\colon \mathcal{A}^{\mathfrak{sh}}_5(\mathcal{V})^{(r,k-1)} \to C^+_{\mathrm{dg}}\mathcal{C},\ \nu^{(r,k-1)}_5\colon \Gamma^{(r,k-1)}_{s,5} \to \Gamma^{(r,k-1)}_5.$$

Then, we extend the dg tensor functor $\Gamma^{(r,k-1)}_{?,5}$ to the dg tensor functor

$$\Gamma^{(r,k)}_{?,5}\colon \mathcal{A}^{\mathfrak{sh}}_5(\mathcal{V})^{(r,k)} \to C^+_{\mathrm{dg}}\mathcal{C}$$

and extend the natural transformation $\nu^{(r,k-1)}_5$ to the natural transformation

$$\nu^{(r,k)}_5\colon \Gamma^{(r,k)}_{s,5} \to \Gamma^{(r,k)}_5$$

in a manner similar to that used in the previous step:

For each $q \in \mathbf{Z}$, each object (X,f) of $\mathcal{L}(\mathcal{V})$ and each non-zero morphism $g\colon \mathfrak{e}^{\otimes k} \to \mathbf{Z}_X(q)_f$ of degree $2q-r$ in $\mathcal{A}^{\mathfrak{sh}}_5(\mathcal{V})^{(r,k-1)}$ with $dg=0$, the morphism

$$\Gamma^{(r,k-1)}_{s,5}(g)\colon \mathfrak{d} \to \Gamma_{s,5}(X,f,q)[2q-r] = \Gamma_s(X,f,q)[2q-r]$$

is zero in $\mathcal{T}$ because of the semi-purity in Γ (cf. Definition 2.7(3)). By assumption (2) of this theorem, the map induced by F and $\mathfrak{d} \cong 1$,

$$K^+\mathcal{C}(\mathfrak{d}, \Gamma_s(X,f,q)[2q-r]) \to \mathcal{T}(1, \Gamma_s(X,f,q)[2q-r]),$$

is injective; therefore, the morphism $\Gamma^{(r,k-1)}_{s,5}(g)$ is also zero in $K^+\mathcal{C}$. Hence, there exists a morphism $\eta\colon \mathfrak{d} \to \Gamma_s(X,f,q)$ of degree $2q-r-1$ in $C^+_{\mathrm{dg}}\mathcal{C}$ such that $d\eta = \Gamma^{(r,k-1)}_{s,5}(g)$ because the morphism $d\Gamma^{(r,k-1)}_{s,5}(g)$ is zero in $C^+_{\mathrm{dg}}\mathcal{C}$. We choose such a morphism η and set $\Gamma^{(r,k)}_{s,5}(h_g) = \eta$, where h_g is the morphism adjoined to $\mathcal{A}^{\mathfrak{sh}}_5(\mathcal{V})^{(r,k-1)}$ (cf. Definition 1.23). Using Lemma 1.3, the assignment $h_g \mapsto \eta$ extends the dg tensor functor $\Gamma^{(r,k-1)}_{s,5}$ to the dg tensor functor $\mathcal{A}^{\mathfrak{sh}}_5(\mathcal{V})^{(r,k)} \to C^+_{\mathrm{dg}}\mathcal{C}$. Now, we denote this extended functor by $\Gamma^{(r,k)}_{s,5}$.

We define the dg tensor functor $\Gamma^{(r,k)}_5$ in a manner similar to that used in the previous step, and we define the dg tensor functor $\Gamma_5\colon \mathcal{A}^{\mathfrak{sh}}_5(\mathcal{V}) \to C^+_{\mathrm{dg}}\mathcal{C}$ to be the limit of the direct system of $\{\Gamma^{(r,k)}_5\}_{k,r}$ with respect to positive integers k and r:

$$\Gamma^{(r)}_5 = \varinjlim_k \Gamma^{(r,k)}_5, \ \Gamma_5 = \varinjlim_r \Gamma^{(r)}_5.$$

Step 4. The extension to $\mathcal{A}^{\mathfrak{sh}}_{\mathrm{mot}}(\mathcal{V})$.

We define the dg tensor functor Γ_{mot} to be the restriction of the functor Γ_5 to the dg tensor subcategory $\mathcal{A}^{\mathfrak{sh}}_{\mathrm{mot}}(\mathcal{V})$. From the construction, it is evident that this functor exhibits the desired properties. □

Proof of Theorem 3.5. Let Γ_{mot} be a dg tensor functor constructed as in the previous proposition. Then, we define the triangulated tensor functor $K^b\Gamma^{\mathfrak{sh}}_{\mathrm{mot}}$ to be the composite

$$K^b(\mathcal{A}^{\mathfrak{sh}}_{\mathrm{mot}}(\mathcal{V}) \otimes R) \xrightarrow{H^0C^b_{\mathrm{dg}}\Gamma_{\mathrm{mot}}} K^bC^+_{\mathrm{dg}}\mathcal{C} \xrightarrow{\mathrm{Tot}} K^+\mathcal{C}.$$

By Proposition 1.30, the triangulated functor

$$K^b\mathfrak{c}_{\mathrm{mot}}\colon K^b(\mathcal{A}^{\mathfrak{sh}}_{\mathrm{mot}}(\mathcal{V}) \otimes R) \to K^b_{\mathrm{mot}}(\mathcal{V})_R := K^b(\mathcal{A}_{\mathrm{mot}}(\mathcal{V}) \otimes R)$$

is an equivalence of triangulated categories; therefore, the functor $K^b\Gamma^{\mathfrak{sh}}_{\mathrm{mot}}$ can be extended to the triangulated functor

$$K^b\Gamma_{\mathrm{mot}}\colon K^b_{\mathrm{mot}}(\mathcal{V})_R \to K^+\mathcal{C}.$$

It follows from axioms (4)–(8) of Definition 2.7 that the composite

$$K^b_{\mathrm{mot}}(\mathcal{V})_R \xrightarrow{K^b\Gamma_{\mathrm{mot}}} K^+\mathcal{C} \xrightarrow{F} \mathcal{T}$$

of the functors $K^b\Gamma_{\mathrm{mot}}$ and $F\colon K^+\mathcal{C}\to\mathcal{T}$ factors through the localization $D^b_{\mathrm{mot}}(\mathcal{V})_R$ of $K^b_{\mathrm{mot}}(\mathcal{V})_R$. Let $D^b\Gamma_{\mathrm{mot}}$ denote this triangulated functor from $D^b_{\mathrm{mot}}(\mathcal{V})_R$ to $\mathcal{T}$. Thus, the diagram

$$\begin{tikzcd} K^b_{\mathrm{mot}}(\mathcal{V})_R \arrow[r, "K^b\Gamma_{\mathrm{mot}}"] \arrow[d] & K^+\mathcal{C} \arrow[d, "F"] \\ D^b_{\mathrm{mot}}(\mathcal{V})_R \arrow[r, "D^b\Gamma_{\mathrm{mot}}"'] & \mathcal{T} \end{tikzcd}$$

commutes, where the vertical arrows are the canonical morphisms. The restriction of $D^b\Gamma_{\mathrm{mot}}$ to the full image $D^b_{\mathrm{mot}\mathfrak{sh}}(\mathcal{V})_R$ of $K^b_{\mathrm{mot}\mathfrak{sh}}(\mathcal{V})_R$ is a triangulated pseudo-tensor functor.

It follows from Proposition 3.7 and the construction of the functor $D^b\Gamma_{\mathrm{mot}}$ that it has the desired properties. This completes the proof of Theorem 3.5. $\square$

Corollary 3.8 (Theorem 0.1)**.** *Let $\mathcal{C}$ be a weakly idempotent complete exact tensor category* (*cf. Example* 2.3) *whose unit object is projective and that admits arbitrary filtered direct limits* (*we do not assume exactness properties for the filtered direct limits.*)*. Let R be a commutative ring that is flat over $\mathbf{Z}$ such that the ring $\mathcal{C}(1,1)$ is an R-algebra. Then, all $\mathcal{C}$-valued geometric cohomology theories* $(\Gamma,\boxtimes,\mathrm{cl})$ *satisfy the assumptions of Theorem* 3.5*. In particular, there exists a triangulated functor*

$$\mathcal{R}_{\Gamma,\sharp}\colon \mathcal{DM}(\mathcal{V})_R\to D^+\mathcal{C}_\sharp$$

that is compatible with the cycle class maps cl *and exhibits the following two properties*:

(1) *The composite functor*

$$\mathcal{V}^{\mathrm{op}}\times\mathbf{Z}\to\mathcal{DM}(\mathcal{V})_R\xrightarrow{\mathcal{R}_{\Gamma,\sharp}} D^+\mathcal{C}_\sharp$$

is isomorphic to the functor Γ via the functor G (*defined in Definition* 3.1) *and the canonical functor $C^+\mathcal{C}\to D^+\mathcal{C}\to D^+\mathcal{C}_\sharp$.*

(2) *The restriction of $\mathcal{R}_{\Gamma,\sharp}$ to the triangulated tensor category $D^b_{\mathrm{mot}\mathfrak{sh}}(\mathcal{V})_R$ is a triangulated pseudo-tensor functor that is compatible with the external product $\boxtimes$ of Γ.*

Assume further that the triangulated category $D^+\mathcal{C}$ equips the countable self-direct sums for all objects of $D^+(\mathcal{C})$. Then, the composite

$$\mathcal{R}_{\Gamma}\colon \mathcal{DM}(\mathcal{V})_R\xrightarrow{\mathcal{R}_{\Gamma,\sharp}} D^+\mathcal{C}_\sharp\to D^+\mathcal{C}$$

of $\mathcal{R}_{\Gamma,\sharp}$ and a quasi-inverse of $D^+\mathcal{C}\to D^+\mathcal{C}_\sharp$ (*cf. Proposition* 1.35) *exhibits the same properties listed above.*

Proof. Because the unit object 1 of $\mathcal{C}$ is projective, the canonical morphisms

$$K^+\mathcal{C}(1,M)\to D^+\mathcal{C}(1,M)$$

are isomorphisms for all objects M of $K^+\mathcal{C}$. Hence, the assumption holds for any geometric cohomology theory $(\Gamma,\boxtimes,\mathrm{cl})$. Assertion (1) follows from Lemma 3.2. The other is trivial. $\square$

4. Examples of Realization

4.1. **Hodge realization.** (1) *The Hodge cohomology theory* (cf. [41, Chapter V, 2.3]). For a Noetherian subring R of $\mathbf{R}$, let $\mathcal{C}$ be the dg tensor category $C^+_{\mathcal{H}'_R}$ of bounded below enlarged R-mixed Hodge complexes (cf. [41, Chapter V, 2.3.1]). Furthermore, let $\mathcal{T}$ be its derived category $D^+_{\mathcal{H}'_R}$, and let F be the composite of the total complex functor $K^+C^+_{\mathcal{H}'_R}\to H^0C^+_{\mathcal{H}'_R}=: K^+_{\mathcal{H}'_R}$ and the canonical functor $K^+_{\mathcal{H}'_R}\to D^+_{\mathcal{H}'_R}$,

$$F\colon K^+C^+_{\mathcal{H}'_R}\to H^0C^+_{\mathcal{H}'_R}\to D^+_{\mathcal{H}'_R}.$$

Then, the triple $(\mathcal{C}, \mathcal{T}, F)$ forms a triangulated tensor system. We note that the triangulated tensor category $\mathcal{T}$ is canonically equivalent to the bounded below derived category of the abelian tensor category of polarizable R-mixed Hodge structures (cf. [41, Chapter V, 2.3.3]).

Let $\mathcal{V}$ be the category $\mathbf{Sm}_{\mathbf{C}}$ of smooth and quasi-projective schemes over $\mathbf{C}$, and let $\Gamma\colon \mathcal{V}^{\mathrm{op}} \times \mathbf{Z} \to C^+\mathcal{C}$ be the functor $R^{\mathrm{Hdg}}(-)$ constructed in [41, Chapter V, 2.3.9]. Then, combining the usual external product $\boxtimes$ and the usual cycle class maps $\mathrm{cl}^q_{W,X}$, the functor Γ forms a $(\mathcal{C}, \mathcal{T}, F)$-valued geometric cohomology on $\mathcal{V}$.

Levine constructed the object P^* in the category $\mathcal{T}$ which is isomorphic to the unit object such that the map of the represented functors

$$K^+\mathcal{C}(P^*, -) \to \mathcal{T}(P^*, -)$$

is an isomorphism (cf. [41, Chapter V, 2.3.10.1]). Hence, we have a realization functor associated with this geometric cohomology theory by means of Theorem 3.5 when $\mathfrak{d} \cong 1$ is $P^* \cong 1$.

(2) *The real Hodge cohomology theory* (cf. [41, Chapter V, 2.3.11]). For a Noetherian subring R of $\mathbf{R}$, let $\mathcal{C}$ be the dg tensor category $C^{+\infty}_{\mathcal{H}'_R}$ of bounded below enlarged real R-mixed Hodge complexes (cf. [41, Chapter V, 2.3.11]). Furthermore, $\mathcal{T}$ be its derived category $D^{+\infty}_{\mathcal{H}'_R}$, and let F be the composite of the total complex functor $K^bC^{+\infty}_{\mathcal{H}'_R} \to H^0C^{+\infty}_{\mathcal{H}'_R}$ and the canonical functor $H^0C^{+\infty}_{\mathcal{H}'_R} \to D^{+\infty}_{\mathcal{H}'_R}$. Then, the triple $(\mathcal{C}, \mathcal{T}, F)$ forms a triangulated tensor system.

Let $\mathcal{V}$ be the category $\mathbf{Sm}_{\mathbf{R}}$, and let $\Gamma\colon \mathcal{V}^{\mathrm{op}} \times \mathbf{Z} \to C^+\mathcal{C}$ be the functor $R^{\mathrm{Hdg}\infty}(-)$. Then, there also exist the usual external product $\boxtimes$ and the usual cycle class maps $\mathrm{cl}^q_{W,X}$ for Γ such that they form a $(\mathcal{C}, \mathcal{T}, F)$-valued geometric cohomology on $\mathcal{V}$. In a method similar to the above case, we have the realization functor associated with this geometric cohomology theory.

4.2. **Étale realization.** (1) *The l-adic étale cohomology theory* (cf. [41, Chapter V, 2.2]). The base scheme S is smooth and essentially of finite type over a ring R, where R is either an algebraically closed field, a global field, a local field, a finite field, or a ring of integers in a global field or in a local field (for simplicity). For a rational prime l which is invertible on S, let $\mathrm{Sh}^{\mathbf{Z}/l^*}_{\text{ét}}(S)$ denote the abelian tensor category of étale sheaves of $\mathbf{Z}/l^*$-modules on S. Let $\mathcal{C}$ be the full dg tensor subcategory of $C^+_{\mathrm{dg}}\mathrm{Sh}^{\mathbf{Z}/l^*}_{\text{ét}}(S)$ whose objects are all normalized $\mathbf{Z}/l^*$-modules. Furthermore, $\mathcal{T}$ is its derived category $D^+\lim\mathrm{Sh}^{\mathbf{Z}_l}_{\text{ét}}(S)$, and let F be the restriction of the composite functor

$$K^bC^+_{\mathrm{dg}}\mathrm{Sh}^{\mathbf{Z}/l^*}_{\text{ét}}(S) \xrightarrow{\mathrm{Tot}} K^+\mathrm{Sh}^{\mathbf{Z}/l^*}_{\text{ét}}(S) \to D^+\mathrm{Sh}^{\mathbf{Z}/l^*}_{\text{ét}}(S)$$

to the full subcategory $K^b\mathcal{C}$. Then, the triple $(\mathcal{C}, \mathcal{T}, F)$ forms a triangulated tensor system.

Let $\mathcal{V}$ be the category $\mathbf{Sm}_S$, and let $\Gamma\colon \mathcal{V}^{\mathrm{op}} \times \mathbf{Z} \to C^+\mathcal{C}$ be the functor which sends any object $(X \xrightarrow{p} S, j)$ in $\mathcal{V} \times \mathbf{Z}$ to $Rp_*\mathbf{Z}_{\acute{e}t,X,l}(j)$ (cf. [41, Chapter V, 2.2.8]). Then, combining the usual external product $\boxtimes$ and the usual cycle class maps $\mathrm{cl}^q_{W,X}$, the functor Γ forms a $(\mathcal{C}, \mathcal{T}, F)$-valued geometric cohomology on $\mathcal{V}$. By [41, Chapter V, 2.2.8.1], we can apply Theorem 3.5 to this geometric cohomology theory. We thus have the associated realization functor.

(2) *The étale cohomology theory with coefficients in $\mathbf{Z}/l^r$.* Composing the canonical functor from $\mathrm{Sh}^{\mathbf{Z}/l^*}_{\text{ét}}(S)$ to the category $\mathrm{Sh}^{\mathbf{Z}/l^r}_{\text{ét}}(S)$ of étale sheaves of $\mathbf{Z}/l^r$-modules on S, we have a geometric cohomology theory attached to the étale cohomology on S with coefficients in $\mathbf{Z}/l^r$ and the associated realization functor.

4.3. **Mixed absolute Hodge realization.** *The mixed absolute Hodge cohomology theory* (cf. [41, Chapter V, 2.4]). For a field k of finite type over $\mathbf{Q}$, and a Noetherian subring R of $\mathbf{R}$, let $\mathcal{C}$ be the dg tensor category $C^+_{MAH,k,R}$ of polarizable R-mixed absolute Hodge complexes over

k, and let $\mathcal{T}$ be its derived category $D^+_{MAH,k,R}$. We define the tensor triangulated functor F to be the composite

$$K^+\mathcal{C} \to H^0\mathcal{C} \to \mathcal{T}.$$

of the total complex functor and the canonical functor. Then, the triple $(\mathcal{C}, \mathcal{T}, F)$ forms a triangulated tensor system.

Let $\mathcal{V}$ be the category $\mathbf{Sm}_k$, and let $\Gamma\colon \mathcal{V}^{\mathrm{op}} \times \mathbf{Z} \to C^+\mathcal{C}$ be the functor which sends any object $(X \xrightarrow{p} S, j)$ in $\mathcal{V} \times \mathbf{Z}$ to the one $R_{MAH,k}(j)$ defined in [41, Chapter V, 2.4.8 and 2.4.9]. Then, combining the usual external product $\boxtimes$ and the usual cycle class maps $\mathrm{cl}^q_{W,X}$, the functor Γ forms a $(\mathcal{C}, \mathcal{T}, F)$-valued geometric cohomology on $\mathcal{V}$ (cf. [41, p. 291]). Furthermore, we can apply Theorem 3.5 to this geometric cohomology theory (cf. [loc. cit.]). We thus have the associated realization functor.

5. p-adic Hodge Cohomology

5.1. ***p*-adic Hodge geometric cohomology theory.** In this section, let p be a prime number, and let $\mathcal{O}$ be a complete discrete variation ring of mixed characteristic with a residue field k of characteristic p. Let K denote the quotient field of $\mathcal{O}$, let K_0 denote the maximal unramified extension of $\mathbf{Q}_p$ in K, and let σ denote the Frobenius automorphism of K_0. Let V_K denote the category of K-vector spaces, let V_0 denote that of K_0-vector spaces with a σ-linear endomorphism, and let V_{dR} denote that of K-vector spaces with a separated exhaustive decreasing filtration. For any field F and scheme X, we denote $X \times_{\mathrm{Spec}\,\mathbf{Z}} \mathrm{Spec}\, F$ by X_F.

To construct a *p-adic Hodge realization functor* from $\mathcal{DM}(\mathcal{O})$, for each object of $\mathbf{Sm}_{\mathcal{O}}$, we first present a functorial (at the level of complexes) construction of a triple of the following:

(1) A complex of the de Rham cohomology of the generic fiber with the Hodge filtration.
(2) A complex of the rigid cohomology of the special fiber with the Frobenius automorphism.
(3) A morphism of complexes from the former to the latter, called the specialization map.

In this construction, it is essential for us to treat a prime filter of $\mathcal{X}$ as a point of $\mathcal{X}$ for each rigid analytic space $\mathcal{X}$. This is because the usual points of $\mathcal{X}$ are not sufficient to use the Godement resolution as a canonical resolution of sheaves on $\mathcal{X}$, but the prime filters of $\mathcal{X}$ are sufficient. From this perspective, a functorial construction of a complex of the rigid cohomology is given by Besser [11], and one for a specialization map on complexes is given by Chiarellotto, Ciccioni and Mazzari [16]. Our construction of such a triple is based on these studies.

This section begins with the construction of the exact tensor category pHS such that such a triple belongs to the objects of C^+pHS; this category contains the category of filtered Frobenius modules (cf. [26]) as a full subcategory. Categories similar to C^+pHS have previously been studied by Bannai [4, Section 2] and Chiarellotto et al. [16, Section 2].

Definition 5.1. We define the exact categories pHS and pHS_0 as follows:

(1) The objects of pHS are systems of the form $M = ((M_0, \phi), M_K, (M_{\mathrm{dR}}, F), c_M, s_M)$, where (M_0, ϕ) is an object of V_0, M_K is a K-vector space, (M_{dR}, F) is an object of V_{dR}, and $c\colon M_0 \otimes K \to M_K$ and $s\colon M_{\mathrm{dR}} \to M_K$ are K-linear maps.
(2) For any two objects M and N of pHS, a morphism in pHS is given by a system $(f_0, f_K, f_{\mathrm{dR}})$, where $f_?\colon M_? \to N_?$ is a morphism in $V_?$ for each $? = 0, K$, or dR such that the diagram

$$\begin{tikzcd} M_0 \arrow[r, "c_M"] \arrow[d, "f_0"'] & M_K \arrow[d, "f_K"'] & M_{\mathrm{dR}} \arrow[l, "s_M"'] \arrow[d, "f_{\mathrm{dR}}"'] \\ N_0 \arrow[r, "c_N"'] & N_K & N_{\mathrm{dR}} \arrow[l, "s_N"] \end{tikzcd}$$

commutes.

(3) A kernel-cokernel pair $A \xrightarrow{f} B \xrightarrow{g} C$ in pHS (i.e., f and g are morphisms in pHS, $A = \mathrm{Ker} g$ and $C = \mathrm{Coker} f$) is a short exact sequence if the pair $A_? \xrightarrow{f_?} B_? \xrightarrow{g_?} C_?$ is a short exact sequence in $V_?$ for each $? = 0, K$, or dR.

The exact category pHS_0 is obtained by forgetting the σ-linear maps ϕ.

We will use the category pHS_0 in the next section.

Remark 5.2. (1) The exact categories pHS and pHS_0 satisfy the assumption regarding $\mathcal{C}$ in Example 2.3 with respect to natural tensor structures. Therefore, D^+pHS and D^+pHS_0 are triangulated tensor systems. The unit object in this exact category pHS_0 is projective; however, that in pHS is not.

(2) Let M be an object of C^+pHS such that $M_?$ is acyclic for each $? = 0, K$, or dR. Then, M is acyclic.

Lemma 5.3. *Let $f\colon M \to N$ be a morphism in C^+pHS.*

(1) *Suppose that M_{dR} and N_{dR} are strict and that the morphisms $H^i(f_{\mathrm{dR}})$ in V_{dR} are strict for all $i \in \mathbf{Z}$. Then, the cone of f_{dR} is also strict.*

(2) *Suppose further that $f_?\colon M_? \to N_?$ is a quasi-isomorphism of complexes of vector spaces for each $? = 0, K$, or* dR. *Then, f is a quasi-isomorphism in C^+pHS.*

Proof. Assertion (1) is proven in [6, 3.1], and the other follows from (1) and Remark 5.2(2). □

Next, we recall the Godement resolution of sheaves, which will be needed later (cf. [41, Part II, Chapter IV]). For a Grothendieck site $\mathscr{X}$ and a commutative ring R, we denote the category of sheaves of R-modules on $\mathscr{X}$ by $\mathrm{Sh}(\mathscr{X}, R)$. Let P be a conservative family of points of $\mathscr{X}$ (i.e., (1) the functors $\mathrm{Sh}(\mathscr{X}, R) \ni F \mapsto F_p \in \mathrm{Sh}(\mathrm{pt}, R)$ are exact for all points p of $\mathscr{X}$ in P and (2) $F_p = 0$ for all $p \in P$ implies $F = 0$ for an $F \in \mathrm{Sh}(\mathscr{X}, R)$). Then, let $G^*_{R,P}\colon \mathrm{Sh}(\mathscr{X}, R) \to C^+\mathrm{Sh}(\mathscr{X}, R)$ denote the Godement resolution associated with P, and let $\alpha_P\colon \mathrm{id} \to G^*_{R,P}$ denote the augmentation. We extend this functor $G^*_{R,P}$ to the functor $C^+\mathrm{Sh}(\mathscr{X}, R) \to C^+\mathrm{Sh}(\mathscr{X}, R)$ by composing the total complex functor. We again denote this extended functor by $G^*_{R,P}$. Then, we can show that the Godement resolution has the following properties.

Lemma 5.4. (1) *For each object F of $C^+\mathrm{Sh}(\mathscr{X}, R)$ and each object X of $\mathscr{X}$, $G^*_{R,P}F$ is acyclic with respect to the functor $\Gamma(X, -)$ and the morphism $F \to G^*_{R,P}F$ induced by the augmentation α_P is a quasi-isomorphism.*

(2) *Furthermore, let $u\colon \mathrm{Sh}(\mathscr{X}, R) \to \mathrm{Sh}(\mathscr{Y}, R)$ be a morphism of Grothendieck topoi, and let Q be a conservative family of points of $\mathscr{Y}$ such that the composite of p and f belongs to Q for each point p in P. Then, for a morphism $f\colon G \to u_*F$ of complexes of sheaves on $\mathscr{Y}$, where F and G are objects of $C^+\mathrm{Sh}(\mathscr{X}, R)$ and $C^+\mathrm{Sh}(\mathscr{Y}, R)$, respectively, there exists a natural morphism $g_f\colon G^*_{R,Q}G \to f_*G^*_{R,P}F$ such that $g_f \circ \alpha_Q = u_*(\alpha_P) \circ f$.*

Proof. Assertion (1) follows from [41, Part II, Chapter IV, 2.2]. See [16, 3.1.2] for a proof of the other. □

Lemma 5.5. *We assume that $\mathscr{X}$ has a finial object $\mathfrak{x}$ and that products over $\mathfrak{x}$ exist. Let F_1, F_2, and F_3 be objects of $C^+\mathrm{Sh}(\mathscr{X}, R)$, and let μ be a morphism $p_1^*F_1 \otimes p_2^*F_2 \to F_3 \circ \times$ of complexes of sheaves on $\mathscr{X} \times \mathscr{X}$, where the $p_i\colon \mathscr{X} \times \mathscr{X} \to \mathscr{X}$ are the projections to the i-th component. Then, there exists a morphism of complexes of sheaves of R-modules*

$$G^*_{R,P}\mu\colon p_1^*G^*_{R,P}F_1 \otimes p_2^*G^*_{R,P}F_2 \to G^*_{R,P}F_3 \circ \times$$

that is natural in μ and is compatible with μ via the augmentations

$$p_1^*\alpha_P \otimes_R p_2^*\alpha_P\colon p_1^*F_1 \otimes_R p_2^*F_2 \to p_1^*G^*_{R,P}F_1 \otimes_R p_2^*G^*_{R,P}F_2,$$

$$\alpha_P \circ \times \colon F_3 \circ \times \to G^*_{R,P} F_3 \circ \times.$$

If $F_1 = F_2 = F_3$ and if μ is associated and commutative, then $G^*_{R,P}\mu$ is also associated and commutative.

Furthermore, let us use the notations used in Lemma 5.4(2), and let us also assume that $\mathscr{Y}$ has a finial object $\mathfrak{y}$ and that products over $\mathfrak{y}$ exist. Let $f_i \colon G_i \to u_* F_i$ be a morphism in $C^+\mathrm{Sh}(\mathscr{Y}, R)$ for $i = 1, 2, 3$, and let ν be a morphism $q_1^* G_1 \otimes q_2^* G_2 \to G_3 \circ \times$, where the $q_i \colon \mathscr{Y} \times \mathscr{Y} \to \mathscr{Y}$ are the projections, such that ν is compatible with μ via the morphisms f_i. Then, $G^*_{R,Q}\nu$ is also compatible with $G^*_{R,P}\mu$ via the morphisms g_{f_i}.

Proof. The first half of this assertion is proven in [41, Part II, Chapter IV, 2.3.7], and the compatibility follows from the constructions of g_{f_i}, $G^*_{R,P}\mu$ and $G^*_{R,P}\nu$. □

To use the Godement resolution, following [16, Section 3 and 4], we prepare the conservative families of points of rigid analytic spaces with the rigid analytic site and of schemes with the Zariski site, respectively. For a rigid analytic space $\mathcal{X}$, let $Pt(\mathcal{X})$ be the set of points of $\mathcal{X}$ that consists of all prime filters (cf. [51, Section 2]) of $\mathcal{X}$. For an object X of $\mathbf{Sm}_K$, we let X^{an} denote the rigid analytic space associated with X. Furthermore, let $Pt(X)$ denote the set of points of X that consists of all Zariski points of X with discrete topology. We regard the set $Pt(X^{\mathrm{an}})$ as the set of points of X via the canonical morphism $X^{\mathrm{an}} \to X$ of sites.

Lemma 5.6. *Using the above notations, $Pt(\mathcal{X})$ (resp. $Pt(X)$) is a conservative family of points of $\mathcal{X}$ (resp. X). In particular, $Pt(X) \cup Pt(X^{\mathrm{an}})$ is a conservative family of points of X.*

Proof. See [51, Section 4] after the proof of Theorem 1. □

For simplicity, we will write $\mathrm{Gd}_{\mathrm{az}} = G^*_{R,Pt(X)\cup Pt(X^{\mathrm{an}})}$ and $\mathrm{Gd}_{\mathrm{a}} = G^*_{R,Pt(\mathcal{X})}$ when the sets $Pt(X) \cup Pt(X^{\mathrm{an}})$ and $Pt(\mathcal{X})$ and the ring R are clear from the context. Now, in this section, we will begin to construct the desired functor.

Step (1). We present a functorial construction of a complex of the de Rham cohomology. For details of this cohomology theory, we refer to [22].

Let X be an object of $\mathbf{Sm}_K$, and let $\Omega^\bullet_X$ denote the de Rham complex of X. For each compactification $\overline{X}$ of X with a normal crossing divisor $D := \overline{X} \setminus X$, let $\Omega^\bullet_{\overline{X}}\langle D\rangle$ denote the de Rham complex of $\overline{X}$ with logarithmic poles along the divisor D. Then, the pair

$$(\Gamma(\overline{X}, \mathrm{Gd}^2_{\mathrm{az}} \Omega^\bullet_{\overline{X}}\langle D\rangle), \Gamma(\overline{X}, \mathrm{Gd}^2_{\mathrm{az}} \Omega^{\bullet \geq n}_{\overline{X}}\langle D\rangle))$$

of complexes forms an object of C^+V_{dR} whose cohomology is the de Rham cohomology of X with the Hodge filtration. Then, we define the object $\Gamma_{\mathrm{dR}}(X)$ of C^+V_{dR} to be the direct limit of the pair

$$(\Gamma(\overline{X}, \mathrm{Gd}^2_{\mathrm{az}} \Omega^\bullet_{\overline{X}}\langle D\rangle), \Gamma(\overline{X}, \mathrm{Gd}^2_{\mathrm{az}} \Omega^{\bullet \geq n}_{\overline{X}}\langle D\rangle)$$

with respect to the direct system that consists of all normal crossing compactifications $\overline{X}$ of X.

Proposition 5.7 ([32, 7.1.2])**.** *The above correspondence $X \mapsto \Gamma_{\mathrm{dR}}(X)$ can be naturally extended to a functor $\Gamma_{\mathrm{dR}} \colon \mathbf{Sm}_K^{\mathrm{op}} \to C^+V_{\mathrm{dR}}$ that exhibits the following properties*:

- *Let X be an object of $\mathbf{Sm}_K$. Then, $\Gamma_{\mathrm{dR}}(X)$ is a strict complex. The cohomology groups $H^i(\Gamma_{\mathrm{dR}}(X))$ are of finite dimension for all $i \in \mathbf{Z}$ and are zero if $i \gg 0$.*
- *The $H^i(\Gamma_{\mathrm{dR}}(f))$ are strict morphisms for all morphisms f of $\mathbf{Sm}_K$ and all $i \in \mathbf{Z}$.*

Step (2). Next, we present a functorial construction of a complex of the rigid cohomology with the Frobenius automorphism. This construction is based on [11, Section 4] and is the same as that in [16, Section 4]. We refer to [8], [9] for the basic notations and properties in the theory of the rigid cohomology.

A system $(X, \overline{X}, \mathcal{P})$ is called a *rigid triple* if X is an object of $\mathbf{Sm}_k$, $\overline{X}$ is a compactification of X, and $\overline{X} \to \mathcal{P}$ is a closed immersion into a p-adic formal $\mathcal{O}$-scheme $\mathcal{P}$ that is smooth in a neighborhood of X. For each rigid triple $(X, \overline{X}, \mathcal{P})$, let $\mathcal{P}_K$ denote the generic fiber of $\mathcal{P}$, and let $]X[_{\mathcal{P}}$ be the tube of X in $\mathcal{P}_K$. For each strict neighborhood V of $]X[_{\mathcal{P}}$ in $]\overline{X}[_{\mathcal{P}}$, $\Gamma(V, \mathrm{Gd_a} j^\dagger \mathrm{Gd_a} \Omega_V^\bullet)$ is a complex of the rigid cohomology of X.

Besser provided a functorial construction of a complex of the rigid cohomology by taking the limit over the data $(\overline{X}, \mathcal{P})$ (cf. [11, p. 8–14]). Here, we recall this construction of such a complex.

Definition 5.8 ([11, 4.10, 4.11])**.** Let X be an object of $\mathbf{Sm}_k$.

(1) We let $\mathcal{PT}_X$ be a set of all pairs $(f, (\overline{Y}, \mathcal{P}))$, where $f\colon X \to Y$ is a morphism in $\mathbf{Sm}_k$ and $(Y, \overline{Y}, \mathcal{P})$ is a rigid triple.

(2) Let SET_X^0 be the category whose objects are all finite subsets $(f_a, (\overline{Y}_a, \mathcal{P}_a))_{a \in A}$ of PT_X such that f_a is the identity for some $a \in A$ and whose morphisms are inclusions.

Lemma 5.9 ([11, 4.12])**.** *For each object of $A = (f_a, (\overline{Y}_a, \mathcal{P}_a)_{a \in A})$ of SET_X^0, we let $\overline{X}_A$ be the closure of the image of the map $\coprod_{a \in A} f_a \colon X \to \coprod_{a \in A} \overline{Y}_a$, and we let $\mathcal{P}_A$ be the formal scheme $\coprod_{a \in A} \mathcal{P}_a$. Then, $(X, \overline{X}_A, \mathcal{P}_A)$ forms a rigid triple. Furthermore, we define*

$$\Gamma_{\mathrm{rig}}(X/K)_{\overline{X}_A, \mathcal{P}_A} = \varinjlim_{V} \Gamma(V, \mathrm{Gd_a} j^\dagger \mathrm{Gd_a} \Omega_V^\bullet),$$

where V runs over all strict neighborhoods of $]X[_{\mathcal{P}}$ in $]\overline{X}[_{\mathcal{P}}$. Then, for each subset B of A, the canonical projections $\overline{X}_A \to \overline{X}_B$ and $\mathcal{P}_A \to \mathcal{P}_B$ naturally induce a quasi-isomorphism

$$\Gamma_{\mathrm{rig}}(X/K)_{\overline{X}_B, \mathcal{P}_B} \to \Gamma_{\mathrm{rig}}(X/K)_{\overline{X}_A, \mathcal{P}_A}.$$

Proposition 5.10 ([11, 4.9, 4.22], [9, 3.1])**.** *The correspondence*

$$X \mapsto \Gamma_{\mathrm{rig}}(X/K) := \varinjlim_{A \in SET_X^0} \Gamma_{\mathrm{rig}}(X/K)_{\overline{X}_A, \mathcal{P}_A}$$

can be naturally extended to a functor $\Gamma_{\mathrm{rig},K} \colon \mathbf{Sm}_k^{\mathrm{op}} \to C^+V_K$ such that the cohomology groups $H^i(\Gamma_{\mathrm{rig},K}(X))$ are of finite dimension for all objects X of $\mathbf{Sm}_k$ and all $i \in \mathbf{Z}$. Moreover, if $K = K_0$, we can naturally extend $\Gamma_{\mathrm{rig},K_0}$ to a functor $\Gamma_0 \colon \mathbf{Sm}_k^{\mathrm{op}} \to C^+V_0$.

Step (3). Finally, we construct a natural transformation $sp\colon \Gamma_{\mathrm{dR}} \to \Gamma_{\mathrm{rig},K}$ that induces the specialization map on the cohomology theories. To this end, we present another functorial construction of a complex of the rigid cohomology of the special fiber for each object of $\mathbf{Sm}_{\mathcal{O}}$.

Let X be an object of $\mathbf{Sm}_{\mathcal{O}}$. For each compactification $\overline{X}$ of X, let $\hat{\overline{X}}$ denote the formal completion of $\overline{X}$ along its special fiber. Then, $(X_k, \overline{X}_k, \hat{\overline{X}})$ forms a rigid triple. Because $\overline{X}$ is proper over $\mathcal{O}$, $\hat{\overline{X}}_K$ is canonically isomorphic to $\overline{X}_K^{\mathrm{an}}$. Using this isomorphism, X_K^{an} can be regarded as a strict neighborhood of the tube $]X_k[_{\hat{\overline{X}}}$ in $\hat{\overline{X}}_K$. Hence, $\Gamma(X_K^{\mathrm{an}}, \mathrm{Gd_a} j^\dagger \mathrm{Gd_a} \Omega^\bullet_{X_K^{\mathrm{an}}})$ is a complex of the rigid cohomology of X_k. Then, we define the complex

$$\Gamma_{\mathrm{rig}}(X) = \varinjlim_{\overline{X}} \Gamma(X_K^{\mathrm{an}}, \mathrm{Gd_a} j^\dagger \mathrm{Gd_a} \Omega^\bullet_{X_K^{\mathrm{an}}}),$$

where $\overline{X}$ runs over all compactifications of X. In this case, we need not determine a method of taking a direct limit because there exists a canonical rigid triple $(X_k, \overline{X}_k, \hat{\overline{X}})$ and a canonical strict neighborhood X_K^{an} of $]X_k[_{\hat{\overline{X}}}$ in $\hat{\overline{X}}_K$ for each object X of $\mathbf{Sm}_{\mathcal{O}}$ and its compactification $\overline{X}$. By construction, there naturally exists a natural transformation on $\mathbf{Sm}_{\mathcal{O}}$:

$$\Gamma_{\mathrm{rig}}(-) \to \Gamma_{\mathrm{rig},K}((-)_k).$$

Let X be an object of $\mathbf{Sm}_{\mathcal{O}}$. For each normal crossing compactification $X_K \xrightarrow{i} \overline{X}_K$ of X_K, we obtain the canonical morphism

$$\Gamma(\overline{X}_K, \mathrm{Gd}^2_{\mathrm{az}} \Omega^\bullet_{\overline{X}_K} \langle D \rangle) \to \Gamma(X_K, \mathrm{Gd}^2_{\mathrm{az}} \Omega^\bullet_{X_K})$$

induced by the natural map $\Omega^\bullet_{\overline{X}_K}\langle D\rangle \to i_*\Omega^\bullet_{X_K}$ by Lemma 5.4(2). By taking the limit with respect to the direct system that consists of all normal crossing compactifications $\overline{X}_K$ of X_K, we obtain the morphism

$$\Gamma_{\mathrm{dR}}(X_K) \to \Gamma(X_K, \mathrm{Gd}^2_{\mathrm{az}}\Omega^\bullet_{X_K})$$

of complexes, which is functorial in X.

For each object X of $\mathbf{Sm}_{\mathcal{O}}$ and each compactification $\overline{X}$ of X, let ϵ denote the natural morphism $X_K^{\mathrm{an}} \to X_K$ of sites. Then, by Lemma 5.4(2), we obtain the composite morphism

$$\Gamma(X_K, \mathrm{Gd}^2_{\mathrm{az}}\Omega^\bullet_{X_K}) \to \Gamma(X_K^{\mathrm{an}}, \mathrm{Gd}^2_{\mathrm{a}}\Omega^\bullet_{X_K^{\mathrm{an}}}) \to \Gamma(X_K^{\mathrm{an}}, \mathrm{Gd}_{\mathrm{a}} j^\dagger \mathrm{Gd}_{\mathrm{a}}\Omega^\bullet_{\mathcal{X}^{\mathrm{an}}}),$$

where each of the arrows is induced by one of following the natural morphisms:

$$\Omega^\bullet_{X_K} \to \epsilon_*\Omega^\bullet_{X^{\mathrm{an}}},\ \mathrm{Gd}_{\mathrm{a}}\Omega^\bullet_{X_K^{\mathrm{an}}} \to j^\dagger\mathrm{Gd}_{\mathrm{a}}\Omega^\bullet_{\mathcal{X}^{\mathrm{an}}},$$

respectively. From the construction, the above composite morphism is functorial in $(X, \overline{X})$. Therefore, by taking the limit that consists of the direct systems of all compactifications $\overline{X}$ of X, we obtain the functorial morphism

$$\Gamma(X_K, \mathrm{Gd}^2_{\mathrm{az}}\Omega^\bullet_{X_K}) \to \Gamma_{\mathrm{rig}}(X).$$

We define the morphism $sp\colon \Gamma_{\mathrm{dR}}(X_K) \to \Gamma_{\mathrm{rig}}(X_k/K)$ as the composite of the above three morphisms:

$$\Gamma_{\mathrm{dR}}(X_K) \to \Gamma(X_K, \mathrm{Gd}^2_{\mathrm{az}}\Omega^\bullet_{X_K}) \to \Gamma_{\mathrm{rig}}(X) \to \Gamma_{\mathrm{rig}}(X_k/K).$$

Then, we obtain a functor with the desired properties that is formed by complexes of the triple consisting of the rigid cohomology, the de Rham cohomology, and the specialization map.

Proposition 5.11. *For each integer q, we define $\Gamma_0(X_k, q) = \Gamma_0(X_k)\otimes K(q)$ (resp. $\Gamma_{\mathrm{dR}}(X_k, q) = \Gamma_{\mathrm{dR}}(X_k)\otimes K(q)$), where $K(q)$ is the Tate object of degree q in V_0 (resp. V_{dR}) (cf. [16, 2.2.1(i)]). Then, we can naturally extend the correspondence*

$$(X, q) \mapsto (\Gamma_0(X_k, q), \Gamma_{\mathrm{rig}}(X_k/K), \Gamma_{\mathrm{dR}}(X_K, q), c, sp),$$

where c is the canonical quasi-isomorphism $\Gamma_{\mathrm{rig}}(X_k/K_0)\otimes_{K_0} K \to \Gamma_{\mathrm{rig}}(X_k/K)$ (cf. [11, 4.21]), to a functor $\Gamma_{pH}\colon \mathbf{Sm}_{\mathcal{O}}^{\mathrm{op}} \times \mathbf{Z} \to C^+pHS$ with the following properties:

- *Let X be an object of $\mathbf{Sm}_{\mathcal{O}}$. Then, the object $\Gamma_{\mathrm{dR}}(X_K)$ of C^+V_{dR} is strict. The all cohomology groups $H^i(\Gamma_{\mathrm{rig}}(X_k/K))$, $H^i(\Gamma_0(X_k/K_0))$, and $H^i(\Gamma_{\mathrm{dR}}(X_K/K))$ are of finite dimension for $i \in \mathbf{Z}$ and are zero if $i \gg 0$.*
- *The morphisms $H^i(\Gamma_{\mathrm{dR}}(f_K))$ of V_{dR} are strict for all morphisms f of $\mathbf{Sm}_{\mathcal{O}}$ and for $i \in \mathbf{Z}$.*

We remark that the specialization map $sp\colon \Gamma_{\mathrm{dR}}(X_K) \to \Gamma_{\mathrm{rig}}(X_k/K)$ is not a quasi-isomorphism in general, unlike the case of the ordinary Hodge cohomology, because for a proper closed subscheme Z of the special fiber of X, the generic fibers of X and $X \setminus Z$ are equal, but the special fibers are distinct.

Next, we will see that the Künneth maps on both cohomology theories can be extended to the commutative external product of the functor Γ_{pH}.

Proposition 5.12. *There exists a canonical commutative external product $\boxtimes$ of Γ_{pH} that induces the Künneth maps on both cohomology theories and satisfies axiom (5) of Definition 2.7 (that is, the axiom called the 'Künneth formula').*

Proof. Let us consider the diagram

$$\begin{tikzcd} \Gamma_{\mathrm{dR}}(X_K) \arrow[r] \arrow[rrr, bend right, "sp"'] & \Gamma(X_K, \mathrm{Gd}^2_{\mathrm{az}}\Omega^\bullet_{X_K}) \arrow[r] & \Gamma_{\mathrm{rig}}(X) \arrow[r] & \Gamma_{\mathrm{rig}}(X_k/K) & \Gamma_{\mathrm{rig}}(X_k/K_0)\otimes K \arrow[l, "c"'] \end{tikzcd}$$

of cohomology theories on $\mathbf{Sm}_{\mathcal{O}}^{\mathrm{op}}$ (i.e., functors from $\mathbf{Sm}_{\mathcal{O}}^{\mathrm{op}}$ to $C^+(pHS)$), where X is an object of $\mathbf{Sm}_{\mathcal{O}}$. By Lemma 5.5, each cohomology theory in this diagram admits the Künneth map (cf. [2, 4.1], [10, 3.1], [23, 8.1.24]). Moreover, these Künneth maps induce commutative external products of these cohomology theories. To demonstrate that these external products can be naturally extended to the external product $\boxtimes$ of Γ_{pH}, it is sufficient to show that these external products are compatible with the maps sp and c. This follows from the second half of Lemma 5.5. This external product $\boxtimes$ satisfies this axiom because the Künneth formula holds for each cohomology theory (cf. [2, 4.3], [10, 3.2], [23, 8.2.10]). ☐

Before we prove that the functor Γ_{pH} with the external product $\boxtimes$ can be extended to a pHS-valued geometric cohomology theory, we must consider the derived category $D^+(pHS)$ of the exact tensor category pHS.

Remark 5.13 ([16, p. 545, and 2.2.1(iii)]). For each of the objects M and N of C^+pHS, the complexes $\Gamma_0(M,N)$ and $\Gamma_1(M,N)$ are defined as

$$\Gamma_0(M,N) = C^+_{\mathrm{dg}}V_{K_0}(M_0,N_0) \oplus C^+_{\mathrm{dg}}V_K(M_K,N_K) \oplus C^+_{\mathrm{dg}}V_{\mathrm{dR}}(M_{\mathrm{dR}},N_{\mathrm{dR}}),$$

$$\Gamma_1(M,N) = C^+_{\mathrm{dg}}V_{K_0}((M_0)^\sigma,N_0) \oplus C^+_{\mathrm{dg}}V_{K_0}(M_0,N_K) \oplus C^+_{\mathrm{dg}}V_K(M_{\mathrm{dR}},N_K),$$

where $(M_0)^\sigma := M_0 \otimes_\sigma K_0$. We define the maps $\phi_1, \phi_2 \colon \Gamma_0(M,N) \to \Gamma_1(M.N)$ as

$$\phi_1(f_0,f_K,f_{\mathrm{dR}}) = (f_0 \circ \phi_M, c_N \circ f_0, f_0 \circ s_M),$$

$$\phi_2(f_0,f_K,f_{\mathrm{dR}}) = (\phi_N \circ f_0, f_K \circ c_M, s_N \circ f_{\mathrm{dR}})$$

for $(f_0,f_K,f_{\mathrm{dR}}) \in \Gamma_0(M,N)$. Furthermore, let $\psi_{M,N} = \phi_1 - \phi_2$, and let us define the complex $\Gamma(M,N)$ as $\mathrm{Cone}\,\psi_{M,N}[-1]$. Then, the following properties hold:

(1) The kernel of $\psi_{M,N}$ is equal to $C^+pHS_{\mathrm{dg}}(M,N)$ as a complex.

(2) When $M = 1$, the cone of the morphism $\psi_{M,N}$ is naturally quasi-isomorphic to the cone of the map

$$\psi_N \colon N_0 \oplus F^0N_{\mathrm{dR}} \to N_0 \oplus N_K$$

with $\psi_N(n_0,n_{\mathrm{dR}}) = (n_0 - \phi_N(n_0), c_N(n_0) - s_N(n_{\mathrm{dR}}))$, by the canonical isomorphisms

$$C^+_{\mathrm{dg}}V_{K_0}((M_0)^\sigma,N_0) \cong N_0,\ \ C^+_{\mathrm{dg}}V_{\mathrm{dR}}(M_{\mathrm{dR}},N_{\mathrm{dR}}) \cong F^0N_{\mathrm{dR}}.$$

(3) For any quasi-isomorphism $N \to L$ in C^+pHS, the canonical map $\Gamma(M,N) \to \Gamma(M,L)$ is a quasi-isomorphism because $\Gamma(M,-)$ is an exact functor.

Lemma 5.14 ([16, 2.2.2])**.** *For each $i \in \mathbf{Z}$, there exists an isomorphism $D^+pHS(M,N[i]) \to H^i(\Gamma(M,N))$ that is functorial in M and N, and the diagram*

$$\begin{array}{ccc} D^+pHS(M,N[i]) & \xrightarrow{\cong} & H^i(\Gamma(M,N)) \\ \uparrow & & \uparrow \\ K^+pHS(M,N[i]) & = & H^i(\mathrm{Ker}\,\psi_{M,N}) \end{array}$$

commutes, where the vertical arrows are the canonical maps.

Lemma 5.15 ([6, 1.11])**.** *Let α be an element of K_0. For any objects M and N of C^+pHS, we define the homomorphism $\cup_\alpha \colon \Gamma(1,M) \otimes \Gamma(1,N) \to \Gamma(1,M \otimes N)$ by the formulas*

$$(f^0,0) \cup_\alpha (g^0,0) = (f^0 \otimes g^0, 0),\ \ (0,f^1) \cup_\alpha (0,g^1) = (0,0),$$

$$(f^0,0) \cup_\alpha (0,g^1) = (0,(-1)^{\deg(f^0)}(\alpha\phi_1(f^0) + (1-\alpha)\phi_2(f^0)) \otimes g^1),$$

$$\text{and}\ \ (0,f^1) \cup_\alpha (g^0,0) = (0, f^1 \otimes (1-\alpha)\phi_1(g^0) + \alpha\phi_2(g^0))$$

for $(f^0, f^1) \in \Gamma(1, M) = \Gamma_0(1, M) \oplus \Gamma_1(1, M)[-1]$ *and* $(g^0, g^1) \in \Gamma(1, N)$*. Here,* ϕ_1 *and* ϕ_2 *are the maps defined in Remark 5.13. Then, the map* $\cup_\alpha$ *is functorial in* M *and* N*, is independent of the choice of* α *up to homotopy equivalence, and is associative if* $\alpha = 0$ *or* 1*. The diagram*

$$\begin{tikzcd}
\Gamma(1, M) \otimes \Gamma(1, N) \arrow[r, "\cup_\alpha"] \arrow[d] & \Gamma(1, M \otimes N) \arrow[d] \\
\Gamma_0(1, M) \otimes \Gamma_0(1, N) \arrow[r] & \Gamma_0(1, M \otimes N)
\end{tikzcd}$$

commutes, where arrows other than $\cup_\alpha$ *are canonical maps. Moreover, the diagram*

$$\begin{tikzcd}
D^+pHS(1, M) \otimes D^+pHS(1, N) \arrow[r] \arrow[d, "\cong"'] & D^+pHS(1, M \otimes N) \arrow[d, "\cong"'] \\
H^0(\Gamma(1, M)) \otimes H^0(\Gamma(1, N)) \arrow[r, "H^0(\cup_\alpha)"] & H^0(\Gamma(1, M \otimes N))
\end{tikzcd}$$

also commutes, where the top arrow is the canonical map.

Proof. It is trivial to prove all assertions except the last one. For quasi-isomorphisms $M \to M'$ and $N \to N'$, we consider a canonical hexahedron

$$\begin{tikzcd}
& K(1, M') \otimes K(1, M') \arrow[rr] \arrow[dl] \arrow[dd] & & K(1, M' \otimes N') \arrow[dl] \arrow[dd] \\
D(1, M) \otimes D(1, M) \arrow[rr, crossing over] \arrow[dd] & & D(1, M \otimes N) & \\
& H^0(\Gamma(1, M')) \otimes H^0(\Gamma(1, N')) \arrow[rr, "H^0(\cup_\alpha)"] & & H^0(\Gamma(1, M' \otimes N')) \\
H^0(\Gamma(1, M)) \otimes H^0(\Gamma(1, N)) \arrow[rr, "H^0(\cup_\alpha)"] \arrow[ur, "\cong"'] & & H^0(\Gamma(1, M \otimes N)) \arrow[ur, "\cong"'] \arrow[from=uu, crossing over] &
\end{tikzcd}$$

where we set $K(-,-) = K^+pHS(-,-)$ and $D(-,-) = D^+pHS(-,-)$. All subdiagrams except for the front square are clearly commutative. Hence, the remaining front square is also commutative. $\square$

At the end of this section, we prove that the functor Γ_{pH} with the external product $\boxtimes$ can be extended to a pHS-valued geometric cohomology theory on $\mathbf{Sm}_{\mathcal{O}}$ using fundamental results of the rigid cohomology and the de Rham cohomology.

Proposition 5.16 ([16, 1.6.2])**.** *There exist maps*

$$\mathrm{cl}^q_{W,X,pH} \colon \mathcal{Z}^q_W(X) \to D^+pHS(1, \Gamma_{pH,W}(X, q)[2q])$$

for all objects X *of* $\mathbf{Sm}_{\mathcal{O}}$*, as well as closed subschemes* W *of* X *of codimension* q*, such that the functor* Γ_{pH} *satisfies axiom* (2) *of Definition* 2.7 (*that is, the axiom labeled 'cycle classes'*) *with respect to the class of these maps,* cl_{pH}*, and this cycle class is compatible with that in the rigid cohomology and that in the de Rham cohomology via the map*

$$D^+pHS(1, \Gamma_{pH,W}(X, q)[2q]) \to H^{2q}_{\mathrm{rig},W_k}(X_k/K_0) \oplus F^q H^{2q}_{\mathrm{dR},W_K}(X_K)$$

induced by Remark 5.13(2) *and Lemma* 5.14*. Moreover, such a class of maps* cl_{pH} *is unique.*

Proof. Let M denote $\Gamma_{pH,W}(X, q)$. Then, by Remark 5.13(2) and Lemma 5.14, we have an exact sequence

$$H^{2q-1}(M_0) \oplus H^{2q-1}(M_K) \to D^+pHD(1, M[2q])$$
$$\to H^{2q}(M_0) \oplus F^0 H^{2q}(M_{\mathrm{dR}}) \xrightarrow{\psi_M} H^{2q}(M_0) \oplus H^{2q}(M_K).$$

It follows from [16, 1.6.2] that there exists a map

$$\eta\colon \mathcal{Z}^q_W(X) \to H^{2q}(M_0) \oplus F^0H^{2q}(M_{\mathrm{dR}})$$

such that the following statements hold:

(1) For each cycle C in $\mathcal{Z}^q_W(X)$, let $\eta_0(C_k)$ be the cycle class of C_k in $H^{2q}(M_0)$ and let $\eta_{\mathrm{dR}}(C_K)$ be that of C_K in $F^0H^{2q}(M_{\mathrm{dR}})$. Then, $\eta(C){=}(\eta_0(C_k), \eta_{\mathrm{dR}}(C_K))$.
(2) $\psi_M \circ \eta = 0$.

Because the semi-purity in the rigid cohomology [9, 5.7] implies that $H^{2q-1}(M_0)$ is zero, the map η uniquely lifts to a morphism

$$\mathcal{Z}^q_W(X) \to D^+pHS(1, M[2q]).$$

Now, we define the map $\mathrm{cl}^q_{X,W,pH}$ for Γ_{pH} to be precisely this lifted morphism. These maps $\mathrm{cl}_{\mathrm{pH}}$ satisfy axiom (2)(a) of Definition 2.7 by [16, 1.6.3]. Because the Künneth maps and the Poincaré duality pairings are compatible for the rigid and de Rham cohomology theories (cf. [2, 4.3], [10, 3.1]), the above cycle class maps η_0 and η_{dR} satisfy axiom (2)(b) of Definition 2.7. Hence, by Lemma 5.15, this axiom holds for the class of maps cl_{pH} because the specialization maps are compatible with the Poincaré duality pairings (cf. [2, 6.9]). □

Here, we note that the range $D^+pHS(1, \Gamma_{pH,W}(X,q)[2q])$ of the cycle class map above is the rigid syntomic cohomology defined by Besser [11, 6.1], which is a p-adic analog of the Deligne cohomology.

Proposition 5.17. *For an object X of $\mathbf{Sm}_{\mathcal{O}}$ and $i, q \in \mathbf{Z}$, there exists a canonical isomorphism*

$$D^+pHS(1, \Gamma_{pH}(X,q)[i]) \cong H^i_{\mathrm{syn}}(X, K(q)),$$

where the right-hand side is the rigid syntomic cohomology.

Proof. This proposition is true if we replace the triangulated category D^+pHS with the one pHD defined in [16, 2.1.1(ii)] (cf. [16, 5.3.4]). It is clear that the category pHD is a full triangulated subcategory of D^+pHS. □

Theorem 5.18. *The triple $(\Gamma_{pH}, \boxtimes, \mathrm{cl}_{pH})$ is a pHS-valued geometric cohomology theory on $\mathbf{Sm}_{\mathcal{O}}$.*

Proof. First, the existence of $\Gamma_{pH,s}$ (defined in 2.5) follows from the fact that $F^n\Gamma_{\mathrm{dR}}(X) = 0$ for all X in $\mathbf{Sm}_K$ and $n > \dim X$.

Next, we recall that both the rigid cohomology and the de Rham cohomology satisfy axioms (3), (4), (7), and (8) of Definition 2.7, namely, semi-purity, homotopy equivalence, excision, and the unit property (cf. [2], [8], [9], [31]).
(a) The Künneth formula in Γ_{pH} is proven in Proposition 5.12.
(b) Semi-purity in Γ_{pH} follows from the semi-purity in both cohomologies.
(c) The direct sum property in Γ_{pH} is evident.
(d) Excision in Γ_{pH} follows from the fact that the excision isomorphisms in both of the cohomologies commute with the specialization maps *sp*. Similarly, because the Gysin maps in each cohomology commute with the specialization maps *sp*, homotopy equivalence holds for Γ_{pH}.
(e) The cycle class property in Γ_{pH} is proven in Proposition 5.16.
(f) The unit property in Γ_{pH} follows from the construction of the cycle class map cl_{pH}.
(g) Gysin isomorphism in Γ_{pH} follows from the fact that under the assumption of Definition 2.7(6), the diagram

$$
\begin{tikzcd}
\Gamma(X) \arrow[r, "\mathrm{id}\otimes\mathrm{cl}(W)"] & \Gamma(X)\otimes\Gamma_W(X,q)[2q] \arrow[r, "\boxtimes"] & \Gamma_{X\times_S W}(X\times_S X,q)[2q] \arrow[r, "\Delta^*"] & \Gamma_W(X,q)[2q] \\
\Gamma(W) \arrow[u, "p^*"] \arrow[d, "\mathrm{id}\otimes\mathrm{cl}(S)"'] \arrow[r, "\mathrm{id}\otimes\mathrm{cl}(W)"] & \Gamma(W)\otimes\Gamma_W(X,q)[2q] \arrow[u, "p^*\otimes\mathrm{id}"'] \arrow[r, "\boxtimes"] & \Gamma_{W\times_S W}(W\times_S X,q)[2q] \arrow[u, "(p\times\mathrm{id})^*"'] \arrow[r, "\Delta^*"] & \Gamma_W(X,q)[2q] \arrow[u, "="'] \\
\Gamma(W)\otimes\Gamma(S) \arrow[r, "\mathrm{id}\otimes\pi^*"] \arrow[dr, "="] & \Gamma(W)\otimes\Gamma(W) \arrow[u, "\mathrm{id}\otimes G_{X,W}"'] \arrow[r, "\boxtimes"] & \Gamma(W\times_S W) \arrow[u, "G_{W\times X,W\times W}"'] \arrow[r, "\Delta^*"] & \Gamma(W) \arrow[u, "G_{X,W}"'] \\
& \Gamma(W)\otimes\Gamma(S) \arrow[u, "\mathrm{id}\otimes\pi^*"'] \arrow[r, "\boxtimes"] & \Gamma(W\times_S S) \arrow[u, "(\mathrm{id}\times\pi)^*"'] \arrow[r, "\Delta^*"] & \Gamma(W) \arrow[u, "="']
\end{tikzcd}
$$

commutes and the bottom maps are isomorphisms in D^+pHS. In this diagram, Δ is the diagonal morphism, π is the structural morphism of W over S, and $G_{X,W}\colon \Gamma(W)\to\Gamma_W(X,q)[2q]$ is the Gysin isomorphism.

From the above, we see that the triple $(\Gamma_{pH},\boxtimes,\mathrm{cl}_{pH})$ is a pHS-valued geometric cohomology theory on $\mathbf{Sm}_{\mathcal{O}}$. ☐

5.2. **Realization associated with p-adic Hodge cohomology.** In this section, we will prove Theorem 0.2. By applying the forgetful functor $pHS\to pHS_0$ (i.e., by forgetting the Frobenius endomorphisms, cf. Definition 5.1) to the pHS-valued geometric cohomology theory constructed above, we obtain a pHS_0-valued cohomology theory $(\Gamma_{pH_0},\boxtimes,\mathrm{cl}_{pH_0})$. Because the unit object of pHS_0 is projective, this geometric cohomology theory can be extended to a realization functor by Corollary 3.8. Furthermore, because D^+pHS_0 equips the countable self-direct sums for all objects, we can obtain a realization functor as follows.

Theorem 5.19 (Theorem 0.2)**.** *Let R be a commutative ring that is flat over $\mathbf{Z}$ such that the ring $K_0=pHS_0(1,1)$ is an R-algebra. Then, there exists a triangulated functor*

$$\mathcal{R}_{pHS_0}\colon \mathcal{DM}(\mathcal{O})_R\to D^+pHS_0$$

that is compatible with the cycle class maps cl_{pH_0} and has the following properties:

(1) *The composite*

$$\mathbf{Sm}_{\mathcal{O}}^{\mathrm{op}}\times\mathbf{Z}\to\mathcal{DM}(\mathcal{O})_R\xrightarrow{\mathcal{R}_{pHS_0}} D^+pHS_0$$

is isomorphic to Γ_{pH} via the functor G that is defined in Definition 3.1 *and the forgetful functor $pHS\to pHS_0$.*

(2) *The restriction of $\mathcal{R}_{pHS_0}$ to the triangulated tensor subcategory $D^b_{\mathrm{motsh}}(\mathbf{Sm}_{\mathcal{O}})_R$ is a triangulated pseudo-tensor functor that is compatible with the external product $\boxtimes$.*

(3) *Let D be an object of $\mathcal{DM}(\mathcal{O})_R$.*

- *For each $?=0$, K, or dR and $i\in\mathbf{Z}$, the vector space $H^i(\mathcal{R}_{pHS_0}(D)_?)$ is of finite dimension and is zero if $i\gg 0$.*
- *The map $c\colon H^i(\mathcal{R}_{pHS_0}(D)_0)\otimes K\to H^i(\mathcal{R}_{pHS_0}(D)_K)$ is an isomorphism for each $i\in\mathbf{Z}$.*

Proof. Assertions (1) and (2) follow from Corollary 3.8 and the construction of $\mathcal{R}_{pHS_0}$. The last assertion follows from the same corollary, Proposition 5.11, and the following lemma in the case that $\mathcal{T}$ is the strictly full subcategory generated by the image of Γ_{pH}. We note that the category $\mathcal{DM}(\mathcal{O})$ is the pseudo-abelian hull of the full subcategory generated (as a triangulated category) by the objects $\mathbf{Z}_X(q)[i]$ for all X in $\mathbf{Sm}_{\mathcal{O}}$ and for $i,q\in\mathbf{Z}$. ☐

Lemma 5.20. *For a strictly full subcategory $\mathcal{T}$ of D^+pHS_0, let $(D^+pHS_0)_{\mathcal{T}}$ denote the full triangulated subcategory of D^+pHS_0 generated by all objects of $\mathcal{T}$.*

(1) *If $\mathcal{T}$ is closed under the operation of taking cones, then $(D^+pHS_0)_{\mathcal{T}}$ is equal to $\mathcal{T}$.*

(2) *If $\mathcal{T}$ is closed under the operation of taking direct summands, then the restriction of $D^+pHS_0 \to (D^+pHS_0)_\sharp$ to $(D^+pHS_0)_{\mathcal{T}}$ is an equivalence $(D^+pHS_0)_{\mathcal{T}} \to (D^+pHS_0)_{\mathcal{T},\sharp}$ of triangulated categories.*

Proof. Assertions (1) is evident. A quasi-inverse of $D^+pHS_0 \to (D^+pHS_0)_\sharp$ sends an object (M, p) of $(D^+pHS_0)_\sharp$ to the object $M(p)$ of D^+pHS_0 that satisfies the properties below (cf. [41, Part II, Chapter II, 2.4.8.1]):

(1) $M(p)$ is a direct summand of M.
(2) The projector $p \colon M \to M$ is equal to the composite $M \to M(p) \to M$ of the projection and the inclusion.

These statements prove assertion (2). □

Remark 5.21. For an object D (resp. a morphism f) of $\mathcal{DM}(\mathcal{O})_R$, it is not simple, in general, to determine whether the object $\mathcal{R}_{pHS_0}(D)_{\mathrm{dR}}$ of C^+V_{dR} (resp. the morphism $H^i(\mathcal{R}_{pHS_0}(f)_{\mathrm{dR}})$ of V_{dR} for $i \in \mathbf{Z}$) is strict. However, if D is associated with a simplicial object of $\mathbf{Sm}_{\mathcal{O}}$, then we can show that $\mathcal{R}_{pHS_0}(D)_{\mathrm{dR}}$ is strict by Lemma 5.3 and Proposition 5.11.

Part 2. **Chern Class Map**

6. Chern Class Map and Chern Character

We begin with this section to define an *abstract cohomology theory* for which we will construct a Chern class map and a Chern character.

6.1. **Definition and examples of abstract cohomology theory.**

Definition 6.1. Let $\mathcal{V}$ be a full subcategory of $\mathbf{Sch}_S$ that satisfies the following conditions

- The base scheme S and the empty scheme is in $\mathcal{V}$.
- The classifying scheme $BGL_{N,S}$ of $GL_{N,S}$ (cf. [41, Appendex B, 1.1.2]) is a simplicial scheme in $\mathcal{V}$ for each $N \in \mathbf{Z}_{\geq 0}$.
- For a scheme X in $\mathcal{V}$, all open subschemes of X are in $\mathcal{V}$.

(1) A 4-tuple $(\mathcal{C}, \Gamma, \mathcal{T}, F)$, where $\mathcal{C}$ is a dg category, $\Gamma = \{\Gamma(j)\}_{j \in \mathbf{Z}}$ is a $\mathbf{Z}$-graded functor from $\mathcal{V}^{\mathrm{op}}$ to $Z^0\mathcal{C}$, $\mathcal{T}$ is a triangulated category, and F is a triangulated functor from $K^b\mathcal{C}$ to $\mathcal{T}$ is called an *abstract cohomology theory on* $\mathcal{V}$ if the following conditions hold for $j \in \mathbf{Z}$:

- $\Gamma(j)$ preserves arbitrary direct sums.
- The canonical map $F\Gamma(j, X) \to F\Gamma(j, U)^*$ in $\mathcal{T}$ is an isomorphism for each simplicial scheme U attached to a finite affine open covering of a scheme X in $\mathcal{V}$.

(2) For an abstract cohomology theory $\mathcal{H} = (\mathcal{C}, \Gamma, \mathcal{T}, F)$ on $\mathcal{V}$, we define the *$\mathcal{H}$-cohomology group* $H^i_{\mathcal{H}}(X, j)$ of a simplicial scheme X in $\mathcal{V}$ to be

$$H^i_{\mathcal{H}}(X, j) = \varprojlim_n \mathcal{T}(F\Gamma(0, S), F\Gamma(j, X)^{*\leq n}[i])$$

for $i, j \in \mathbf{Z}$. Here, $\Gamma(j, X)^{*\leq n}$ is the (stupid) truncation of $\Gamma(j, X)^*$ to degree $\leq n$ for $n \geq 0$ (cf. [41, p. 288]).

Remark 6.2. For a $(\mathcal{C}, \mathcal{T}, F)$-valued geometric cohomology theory $(\Gamma, \boxtimes, \mathrm{cl})$ on $\mathcal{V}$, the 4-tuple $(C^+\mathcal{C}, \Gamma, \mathcal{T}, F)$ forms an abstract cohomology theory on $\mathcal{V}$. For such an abstract cohomology theory, the unit object of the tensor category $\mathcal{T}$ is isomorphic to $F\Gamma(0, S)$ via the cycle class map (cf. Definition 2.7). In this thesis, we use this identification.

Remark 6.3. Let $(\mathcal{C}, \Gamma, \mathcal{T}, F)$ be a geometric cohomology theory. Suppose that $\mathcal{C}$ is the dg category $C^+_{\mathrm{dg}}\mathcal{A}$ of bounded below complexes on an abelian category $\mathcal{A}$, $\mathcal{T}$ is its derived category

$D^+\mathcal{A}$, and F is the composite

$$K^b C^+_{\mathrm{dg}}\mathcal{A} \xrightarrow{\mathrm{Tot}} K^+\mathcal{A} \to D^+\mathcal{A}$$

of the total complex functor (cf. [41, Part II, Chapter II, 1.2.9]) and the canonical one. Then, via the total complex functor

$$\mathrm{Tot}\colon C^+C^+\mathcal{A} \to C^+\mathcal{A} = Z^0\mathcal{C},$$

the functor Γ is canonically extended to a contravariant one from the category $s.\mathcal{V}$ of simplicial objects in $\mathcal{V}$ to $Z^0\mathcal{C}$ without taking truncations,

$$s.\mathcal{V}^{\mathrm{op}} \xrightarrow{\Gamma^*} C^+C^+\mathcal{A} \xrightarrow{\mathrm{Tot}} C^+\mathcal{A}.$$

Suppose further that $\mathcal{A}$ is the category $\mathbf{Mod}_R$ of R-modules for a ring R, and that the unite object R is isomorphic to $F\Gamma(S)$ in $\mathcal{T}$. Then, under fixing such an isomorphism, the group $H^i_{\mathcal{H}}(X,j)$ is canonically isomorphic to $H^i(F\Gamma(j,X)^*)$,

$$H^i_{\mathcal{H}}(X,j) \cong D^+\mathcal{A}(R, F\Gamma(j,X)[i]) \cong K^+\mathcal{A}(R, F\Gamma(j,X)[i]) = H^i(F\Gamma(j,X)^*).$$

This identification is often used in the rest of this thesis.

Example 6.4. *Levine's motivic abstract cohomology theory.* Let $\mathcal{V}$ be a category which satisfies the conditions $(a),(b)$ and (c) in Definition 1.32. Then, we have an abstract cohomology theory $\mathcal{H}_{\mathrm{Le}} = (\mathcal{C}_{\mathrm{Le}}, \Gamma_{\mathrm{Le}}, \mathcal{T}_{\mathrm{Le}}, F_{\mathrm{Le}})$ on $\mathcal{V}$ associated with Levine's motivic categories with coefficients in a ring R as follows:

- The dg category $\mathcal{C}_{\mathrm{Le}}$ is $\mathcal{A}_{\mathrm{mot}}(\mathcal{V})_R$ (cf. Definition 1.25).
- The functor Γ_{Le} is $\mathcal{V}^{\mathrm{op}} \times \mathbf{Z} \to Z^0\mathcal{A}_{\mathrm{mot}}(\mathcal{V})_R$ which sends (X,j) to Levine's motive $\mathbf{Z}_X(j)$ (cf. Definition 1.7).
- The triangulated category $\mathcal{T}_{\mathrm{Le}}$ is $D^b_{\mathrm{mot}}(\mathcal{V})_R$ (cf. Definition 1.32).
- The triangulated functor F_{Le} is the canonical one $K^b\mathcal{A}_{\mathrm{mot}}(\mathcal{V})_R \to D^b_{\mathrm{mot}}(\mathcal{V})_R$.

We note that the $\mathcal{H}_{\mathrm{Le}}$-cohomology $H^i_{\mathcal{H}_{\mathrm{Le}}}(X,j)$ is Levine's motivic cohomology

$$H^i_{\mathrm{Le}}(X, R(j)) := D^b_{\mathrm{mot}}(\mathcal{V})_R(\mathbf{Z}_S(0), \mathbf{Z}_X(j)[i])$$

with coefficients in R (cf. [41, Part I, Chapter I, 2.2.7]).

6.2. **Construction of Chern class map and Chern character.** Following Huber's method in [32, Section 18], we next see that, for an abstract cohomology theory $\mathcal{H}$ on $\mathcal{V}$, an element c of $\prod_j \varprojlim_N H^{2j}_{\mathcal{H}}(BGL_{N,S}, j)$ gives us a Chern class map

$$c^{j,2j-i}_{\mathcal{H},c}\colon K_i(X) \to H^{2j-i}_{\mathcal{H}}(X,j)$$

to the $\mathcal{H}$-cohomology group.

We first construct a Hurewicz map. To do this, we use the following lemmas about the K-group and the homotopy limit.

Lemma 6.5. *We set $GL = \varinjlim_N GL_N$. Then, for a scheme X, there exists a canonical map*

$$K_i(X) \to \varinjlim_U \pi_i(\mathbf{Z} \times \operatorname{holim} BGL(\mathcal{O}_X(U))^+),$$

where U runs over all simplicial schemes attached to a finite affine open covering of X. This map is an isomorphism if X is regular.

Proof. See [41, Appendix B, 2.1.1.2, 2.1.2]. $\square$

Lemma 6.6. *For a simplicial-cosimplicial abelian group A, there exists a canonical isomorphism*

$$\pi_i(\operatorname{holim} A) \cong H^{-i}(\operatorname{Tot} NA_*^*)$$

for $i \in \mathbf{Z}_{\geq 0}$.

Proof. See [41, Appendex B, 2.2.1]. □

We define a Hurewicz map

$$K_i(X) \to \varinjlim_{U,N} H^{-i}(\operatorname{Tot} N\mathbf{Z}\mathcal{V}(U, BGL_{N,S})_*^*) \tag{1}$$

to be the composite

$$\begin{aligned} K_i(X) &\xrightarrow{6.5} \varinjlim_{U} \pi_i(\operatorname{holim} BGL(\mathcal{O}_X(U))^+) \\ &\to \varinjlim_{U} \pi_i(\operatorname{holim} \mathbf{Z}BGL(\mathcal{O}_X(U))^+) \\ &\xrightarrow{6.6} \varinjlim_{U} H^{-i}(\operatorname{Tot}(N\mathbf{Z}BGL(\mathcal{O}_X(U))^+)_*^*) \\ &\xleftarrow{\cong} \varinjlim_{U} H^{-i}(\operatorname{Tot} N\mathbf{Z}BGL(\mathcal{O}_X(U))_*^*) \\ &= \varinjlim_{U,N} H^{-i}(\operatorname{Tot} N\mathbf{Z}\mathcal{V}(U, BGL_{N,S})_*^*) \end{aligned}$$

where the second map is induced by the map $BGL \to \mathbf{Z}BGL$ of simplicial presheaves on X and the fourth is induced by the weak homotopy equivalence $BGL \to BGL^+$.

We next construct a map

$$K_i(X) \to \varinjlim_{N,n} \mathcal{T}(F\Gamma(j, BGL_{N,S})^{*\leq n}, F\Gamma(j, X)[i])$$

for $j \in \mathbf{Z}$. Let X and Y be truncated simplicial objects in $\mathcal{V}$, i.e., X and Y are simplicial objects in $\mathcal{V}$ which satisfies $X_i = Y_i = \emptyset$ for $i \gg 0$. Then, the functor Γ induces a map

$$\mathbf{Z}\mathcal{V}(X, Y) \to \mathcal{C}(\Gamma(j, Y), \Gamma(j, X))$$

of simplicial-cosimplicial complexes of abelian groups. Taking the associated chain-cochain complexes and normalizing them, we have a map

$$N\mathbf{Z}\mathcal{V}(X, Y)_*^* \to N\mathcal{C}(\Gamma(j, Y), \Gamma(j, X))_*^* \tag{2}$$

of triple cochain complexes. Taking the total complex twice and using the lemma below give us a map

$$\operatorname{Tot} N\mathbf{Z}\mathcal{V}(X, Y)_*^* \to C^b_{\mathrm{dg}}\mathcal{C}(\Gamma(j, Y)^*, \Gamma(j, X)^*), \tag{3}$$

which is natural in X and Y.

Lemma 6.7. *Let $A^\bullet$ and $B^\bullet$ be bounded complexes in a dg category $\mathcal{D}$. Then, there exists a canonical isomorphism*

$$\operatorname{Tot} \circ \operatorname{Tot}_i (N\mathcal{D}(A^i, B^j)_{i,j\in\mathbf{Z}}) \cong C^b_{\mathrm{dg}}\mathcal{D}(A^\bullet, B^\bullet).$$

Here, Tot_i *means taking the total complex under fixing the index j.*

Proof. It follows from [13, Section 2, Proposition 3], because $C^b_{\mathrm{dg}}\mathcal{D}$ is the minimal strictly full dg subcategory of the dg category Pre-Tr($\mathcal{D}$) of twisted complexes in $\mathcal{C}$ which is closed under the operation of taking cones (cf. [41, Part II, Chapter II, 1.2.7]). □

We compose the Hurewicz map (1)

$$K_i(X) \to \varinjlim_{U,N} H^{-i}(\mathrm{Tot}\, N\mathbf{Z}\mathcal{V}(U, BGL_{N,S})^*_*)$$

with the colimit (with respect to n, U and N) of the composite map

$$\begin{aligned} H^i(\mathrm{Tot}\, N\mathbf{Z}\mathcal{V}(U, BGL_{N,S})^*_{*\le n}) &\xrightarrow{(3)} H^i(C^b_{\mathrm{dg}}\mathcal{C}(\Gamma(j, BGL_{N,S}))^{*\le n}, \Gamma(j,U)^*) \\ &= K^b\mathcal{C}(\Gamma(j, BGL_{N,S})^{*\le n}, \Gamma(j,U)^*[i]) \\ &\xrightarrow{F} \mathcal{T}(F\Gamma(j, BGL_{N,S})^{*\le n}, F\Gamma(j,U)^*[i]) \quad (4) \\ &\xleftarrow{\cong} \mathcal{T}(F\Gamma(j, BGL_{N,S})^{*\le n}, F\Gamma(j,X)[i]), \quad (5) \end{aligned}$$

where the bottom map is induced by the canonical isomorphism $F\Gamma(j,X) \to F\Gamma(j,U)^*$. We thus have a functorial map

$$K_i(X) \otimes \varprojlim_N H^j_{\mathcal{H}}(BGL_{N,S}, k) \to H^{j-i}_{\mathcal{H}}(X, k) \tag{6}$$

for $j, k \in \mathbf{Z}$. We here name an element of $\prod_j \varprojlim_N H^{2j}_{\mathcal{H}}(BGL_{N,S}, j)$.

Definition 6.8. A *universal Chern class for an abstract cohomology theory* $\mathcal{H}$ is an element $c = (c^j)_j \in \prod_j \varprojlim_N H^{2j}_{\mathcal{H}}(BGL_{N,S}, j)$.

Hence, a universal Chern class c for the abstract cohomology theory $\mathcal{H}$ gives us the Chern class map

$$c^{j,2j-i}_{\mathcal{H},c} = \gamma(- \otimes c^j)\colon K_i(X) \to H^{2j-i}_{\mathcal{H}}(X, j).$$

In the case where $i = j = 0$, we add the rank function

$$K_0(X) \to \mathbf{Z} \to H^0_{\mathcal{H}}(X, 0), \tag{7}$$

where the second map sends 1 to the map $\Gamma(0,S) \xrightarrow{\Gamma(X \to S)} \Gamma(0,X)$.

We here define the Chern character for an abstract cohomology theory with a universal Chern class.

Definition 6.9. The Chern character

$$ch^{i,j}_{\mathcal{H},c}\colon K_i(X) \to H^{2j-i}_{\mathcal{H}}(X, j) \otimes \mathbf{Q}$$

for an abstract cohomology theory $\mathcal{H}$ with a universal Chern class c is defined as follows:

$$ch^{i,j}_{\mathcal{H},c} = \frac{(-1)^{j-1}}{(j-1)!}\, c^{j,2j-i}_{\mathcal{H},c} \quad \text{if } j > 0.$$

$ch^{0,0}_{\mathcal{H},c}$ is the rank function (7), $K_0(X) \to \mathbf{Z} \to H^0_{\mathcal{H}}(X,0) \otimes \mathbf{Q}$. Otherwise, $ch^{i,j}_{\mathcal{H},c} = 0$.

Remark 6.10. We note that the definition of Huber's Chern character [32, 18.3.12] is not correct. The map is not multiplicative (see [32, 18.3.11]).

$$ch = \delta_{0i} e + \sum_{j \ge 1} \frac{(-1)^{j-1}}{(j-1)!} c^i_j$$

is the correct definition.

7. Naturalities of Chern Class Map

7.1. **Naturalities.** We note that naturalities of our Chern class map.

Proposition 7.1. *Let $(\mathcal{H}, c)$ be an abstract cohomology theory with a universal Chern class. Furthermore, let G be a dg functor $\mathcal{C} \to \mathcal{C}'$ and let G' be a triangulated functor $\mathcal{T} \to \mathcal{T}'$ such that the diagram*

$$\begin{array}{ccc} K^b\mathcal{C} & \xrightarrow{K^bG} & K^b\mathcal{C}' \\ {\scriptstyle F}\big\downarrow & & \big\downarrow{\scriptstyle F'} \\ \mathcal{T} & \xrightarrow{G'} & \mathcal{T}' \end{array}$$

commutes up to an isomorphism. Then, we have an abstract cohomology theory $\mathcal{H}' = (\mathcal{C}', G\Gamma, \mathcal{T}', F')$ with a universal Chern class $G'c$. The Chern class map for $(\mathcal{H}', G'c)$ coincides with the composite

$$K_i(X) \xrightarrow{c^{j,2j-i}_{\mathcal{H},c}} H^{2j-i}_{\mathcal{H}}(X, j) \xrightarrow{G'} H^{2j-i}_{\mathcal{H}'}(X, j).$$

Proposition 7.2. *For a map $p\colon S \to T$ of schemes and a full subcategory $\mathcal{W}$ of Sch_T that satisfies the conditions on $\mathcal{V}$ in Definition 6.1. Assume that the image of $\mathcal{W}$ under the base change map $p^*\colon Sch_T \to Sch_S$ contains the category $\mathcal{V}$. Then, an abstract cohomology theory $(\mathcal{H}, c)$ on $\mathcal{V}$ with a universal Chern class gives a one $(p^*\mathcal{H}, p^*c)$ on $\mathcal{W}$ via p^* and its Chern class map $c^{j,2j-i}_{p^*\mathcal{H},p^*c}$ coincides with the composite*

$$K_i(X) \to K_i(p^*X) \xrightarrow{c^{j,2j-i}_{\mathcal{H},c}} H^{2j-i}_{\mathcal{H}}(p^*X, j) = H^{2j-i}_{p^*\mathcal{H}}(X, j).$$

Proposition 7.3. *Let $(\mathcal{H}, c)$ and $(\mathcal{H}', c')$ be abstract cohomology theories on $\mathcal{V}$ with a universal Chern class such that $\mathcal{C} = \mathcal{C}'$, $\mathcal{T} = \mathcal{T}'$ and $F = F'$. Furthermore, let ϕ be a map*

$$\varprojlim_n K^b\mathcal{C}(\Gamma(0, S), \Gamma(-,-)^{*\leq n}[-]) \to \varprojlim_n K^b\mathcal{C}(\Gamma'(0, S), \Gamma'(-,-)^{*\leq n}[-])$$

of functors from $s.\mathcal{V}^{\mathrm{op}} \times \mathbf{Z} \times \mathbf{Z}$ to $\mathbf{Ab}$ such that $F\phi$ sends c to c'. Then, the Chern class map for $(\mathcal{H}', c')$ coincides with the composite

$$K_i(X) \xrightarrow{c^{j,2j-i}_{\mathcal{H},c}} H^{2j-i}_{\mathcal{H}}(X, j) \xrightarrow{F\phi} H^{2j-i}_{\mathcal{H}'}(X, j).$$

Proof. The map (3) induces a one

$$\begin{aligned} \delta_{\mathcal{H}}\colon H^i(\mathrm{Tot}\, N\mathbf{Z}\mathcal{V}(U, BGL_{N,S})^*_{*\leq n}) \times H^j C^b_{\mathrm{dg}}\mathcal{C}(\Gamma(0, S), \Gamma(j, BGL_{N,S})^{*\leq n}) \\ \to H^{i+j} C^b_{\mathrm{dg}}(\Gamma(0, S), \Gamma(j, U)^*). \end{aligned}$$

It is enough to show that the diagram

$$\begin{array}{ccc} H^j C^b_{\mathrm{dg}}\mathcal{C}(\Gamma(0, S), \Gamma(j, BGL_{N,S})^{*\leq n}) & \xrightarrow{\delta_{\mathcal{H}}(f,-)} & H^{i+j} C^b_{\mathrm{dg}}\mathcal{C}(\Gamma(0, S), \Gamma(j, U)^*) \\ \big\downarrow & & \big\downarrow \\ H^j C^b_{\mathrm{dg}}\mathcal{C}(\Gamma'(0, S), \Gamma'(j, BGL_{N,S})^{*\leq n}) & \xrightarrow{\delta_{\mathcal{H}'}(f,-)} & H^{i+j} C^b_{\mathrm{dg}}\mathcal{C}(F\Gamma'(0, S), F\Gamma'(j, U)^*) \end{array}$$

commutes for each element f in $H^i(\mathrm{Tot}\, N\mathbf{Z}\mathcal{V}(U, BGL_{N,S})^*_{*\leq n})$, where the vertical maps are induced by ϕ. By Lemma 6.7, this commutativity is equivalent to the one of a corresponding diagram

$$\begin{array}{ccc} H^j \mathrm{Tot}(\mathcal{C}(\Gamma(0, S), \Gamma(j, BGL_{N,S}))^{*\leq n}) & \longrightarrow & H^{i+j} \mathrm{Tot}\,(\mathcal{C}(\Gamma(0, S), \Gamma(j, U))^*) \\ \big\downarrow & & \big\downarrow \\ H^j \mathrm{Tot}\,(\mathcal{C}(\Gamma'(0, S), \Gamma'(j, BGL_{N,S}))^{*\leq n}) & \longrightarrow & H^{i+j} \mathrm{Tot}\,(\mathcal{C}(\Gamma'(0, S), \Gamma'(j, U))^*). \end{array}$$

It follows from that the spectral sequence whose $E_1^{j,k-i}$ is the commutative diagram

$$\begin{array}{ccc} H^j\mathcal{C}(\Gamma(0,S),\Gamma(j,B_{k-i}GL_{N,S})) & \xrightarrow{\Gamma(f_k)\circ-} & H^{i+j}\mathcal{C}(\Gamma(0,S),\Gamma(j,U_k)) \\ \phi\downarrow & & \downarrow\phi \\ H^j\mathcal{C}(\Gamma'(0,S),\Gamma'(j,B_{k-i}GL_{N,S})) & \xrightarrow{\Gamma'(f_k)\circ-} & H^{i+j}\mathcal{C}(\Gamma'(0,S),\Gamma'(j,U_k)) \end{array}$$

converges to the above one, where $(f_k)_k$ is an element in $\oplus_k \mathbf{Z}\mathcal{V}(U_k, B_{k-i}GL_{N,S})$ which represents f. □

By this proposition, we see that the following properties hold.

Remark 7.4. (1) Our Chern class map is uniquely determined up to homotopy.

(2) Suppose that the maps

$$K^b\mathcal{C}(\Gamma(0,S),\Gamma(-)) \xrightarrow{F} \mathcal{T}(F\Gamma(0,S),F\Gamma(-,-)),$$
$$K^b\mathcal{C}(\Gamma'(0,S),\Gamma(-)) \xrightarrow{F} \mathcal{T}(F\Gamma'(0,S)C,F\Gamma(-,-))$$

of functors from $s.\mathcal{V}^{\mathrm{op}} \times \mathbf{Z}$ to $\mathbf{Ab}$ are isomorphisms. Then, the ϕ is a map

$$H^-_{\mathcal{H}}(-,-) \to H^-_{\mathcal{H}'}(-,-)$$

of $\mathcal{H}$-cohomology groups on $s.\mathcal{V} \times \mathbf{Z} \times \mathbf{Z}$, and the Chern class map for $(\mathcal{H}',\phi(c))$ coincides with the composite of the one for $(\mathcal{H},c)$ and the map ϕ.

7.2. **Relation to realization.** In this section, we will define a *realization functor* for an abstract cohomology theory $\mathcal{H}$ and prove that Levine's motivic Chern class map is compatible with the one for $\mathcal{H}$ via its realization functor.

As mentioned in Section 1, Levine constructed the dg tensor category $\mathcal{A}^{\mathfrak{sh}}_{\mathrm{mot}}(\mathcal{V})$ and the dg tensor functor $\mathcal{A}^{\mathfrak{sh}}_{\mathrm{mot}}(\mathcal{V}) \to \mathcal{A}_{\mathrm{mot}}(\mathcal{V})$. $D^b_{\mathrm{mot}\mathfrak{sh}}(\mathcal{V})_R$ is the full image of the composite

$$K^b(\mathcal{A}^{\mathfrak{sh}}_{\mathrm{mot}}(\mathcal{V})_R) \to K^b(\mathcal{A}_{\mathrm{mot}}(\mathcal{V})_R) \to D^b_{\mathrm{mot}}(\mathcal{V})_R.$$

We note that the triangulated tensor functor

$$D^b_{\mathrm{mot}\mathfrak{sh}}(\mathcal{V})_R \to D^b_{\mathrm{mot}}(\mathcal{V})_R$$

is an equivalences of triangulated categories, since the functor

$$K^b(\mathcal{A}^{\mathfrak{sh}}_{\mathrm{mot}}(\mathcal{V})_R) \to K^b(\mathcal{A}_{\mathrm{mot}}(\mathcal{V})_R)$$

is already an equivalence (see Proposition 1.30).

Definition 7.5. Let $\mathcal{H}$ be an abstract cohomology theory on $\mathcal{V}$ with a universal Chern class c. Then, we say that the abstract cohomology theory $\mathcal{H}$ on $\mathcal{V}$ is *extended to a (an abstract) realization functor* if there exist a dg functor

$$\Gamma_{\mathrm{mot}} \colon \mathcal{A}^{\mathfrak{sh}}_{\mathrm{mot}}(\mathcal{V})_R \to \mathcal{C}$$

and a triangulated functor

$$R_{\mathcal{H}} \colon D^b_{\mathrm{mot}\mathfrak{sh}}(\mathcal{V})_R \to \mathcal{T}$$

for some ring R that is flat over $\mathbf{Z}$ such that the diagram

$$\begin{array}{ccccc} \mathcal{V}^{\mathrm{op}} & \xrightarrow{R(j)^{\mathfrak{sh}}} & Z^0\mathcal{A}^{\mathfrak{sh}}_{\mathrm{mot}}(\mathcal{V})_R & \longrightarrow & D^b_{\mathrm{mot}\mathfrak{sh}}(\mathcal{V})_R \\ \| & & \Gamma_{\mathrm{mot}}\downarrow & & R_{\mathcal{H}}\downarrow \\ \mathcal{V}^{\mathrm{op}} & \xrightarrow{\Gamma(j)} & Z^0\mathcal{C} & \xrightarrow{F} & \mathcal{T} \end{array}$$

commutes up to isomorphisms for each $j \in \mathbf{Z}$.

Remark 7.6. By Theorem 3.5, a geometric cohomology theory which satisfies the assumptions in this theorem is extended to an abstract realization functor.

Remark 7.7. An abstract cohomology theory $\mathcal{H}$ on $\mathcal{V}$ that is extended to an abstract realization functor has a canonical universal Chern class. This is the image of Levine's motivic Chern class

$$c_{\mathrm{Le}} \otimes 1_R \in \prod_j \varprojlim_N H^{2j}_{\mathrm{Le}}(BGL_{N,S}, R(j))$$

under the map

$$H^{2j}_{\mathrm{Le}}(BGL_{N,S}, R(j)) \to H^{2j}_{\mathcal{H}}(BGL_{N,S}, j)$$

induced by the abstract realization functor $R_{\mathcal{H}}$. Here, we fix isomorphisms which make the diagram in Definition 7.5 commutative.

Remark 7.8. It is natural to define an abstract realization functor $(\Gamma_{\mathrm{mot}}, R_{\mathcal{H}})$ whose domains are the motivic categories with the index $\mathfrak{sh}$. In fact, the flow of Levine's (and thus ours) construction of a realization functor $\mathfrak{R}$ on $\mathcal{DM}_R$ is as follows (cf. the proof of Theorem 3.5, or [41, Chapter V]):

(1) Construct the dg tensor functor Γ_{mot} on the category $\mathcal{A}^{\mathfrak{sh}}_{\mathrm{mot}}(\mathcal{V})$ for a certain cohomology theory (cf. Proposition 3.7).
(2) Extends to the triangulated tensor functor on $K^b_{\mathrm{mot}\mathfrak{sh}}(\mathcal{V})_R$, and composing a quasi-inverse of $K^b_{\mathrm{mot}\mathfrak{sh}}(\mathcal{V})_R \to K^b_{\mathrm{mot}}(\mathcal{V})_R$ (cf. Proposition 1.30).
(3) Extends to the triangulated tensor functor $R_{\mathcal{H}}$ on $D^b_{\mathrm{mot}\mathfrak{sh}}(\mathcal{V})_R$, and taking the pseudo-abelian hull.

Proposition 7.9. *Let $\mathcal{H}$ be an abstract cohomology theory on $\mathcal{V}$ that is extended to an abstract realization functor*

$$R_{\mathcal{H}} \colon D^b_{\mathrm{mot}\mathfrak{sh}}(\mathcal{V})_R \to \mathcal{T}$$

in the sense of Definition 7.5 *for some ring R that is flat over $\mathbf{Z}$. Then, the composite*

$$K_i(X) \xrightarrow{c^{j,2j-i}_{\mathcal{H}_{\mathrm{Le}},c_{\mathrm{Le}}} \otimes 1_R} H^{2j-i}_{\mathrm{Le}}(X, R(j)) \xrightarrow{R_{\mathcal{H}}} H^{2j-i}_{\mathcal{H}}(X, j)$$

coincides with our Chern class map $c^{j,2j-i}_{\mathcal{H},c}$ for $(\mathcal{H}, c)$. Here, $(\mathcal{H}_{\mathrm{Le}}, c_{\mathrm{Le}})$ is the Levine's motivic abstract cohomology theory with the universal Chern class (see Example 6.4 *and Remark* 7.7*), and c is the universal Chern class for $\mathcal{H}$ induced by the abstract realization functor (cf. Remark* 7.7*). Especially, after tensoring with $\mathbf{Q}$, the same property holds for the Chern characters.*

Proof. The 4-tuple consisting of the two categories $(\mathcal{A}^{\mathfrak{sh}}_{\mathrm{mot}}(\mathcal{V}), D^b_{\mathrm{mot}\mathfrak{sh}}(\mathcal{V}))$ and the two functors

$$\mathbf{Z} \colon \mathcal{V}^{\mathrm{op}} \times \mathbf{Z} \to Z^0\mathcal{A}^{\mathfrak{sh}}_{\mathrm{mot}}(\mathcal{V}) \quad \text{and} \quad K^b\mathcal{A}^{\mathfrak{sh}}_{\mathrm{mot}}(\mathcal{V}) \to D^b_{\mathrm{mot}\mathfrak{sh}}(\mathcal{V})$$

forms an abstract cohomology theory. The $\mathcal{H}$-cohomology of this abstract cohomology theory equals to the Levine's motivic cohomology via the equivalence (cf. Proposition 1.30). Hence, Proposition 7.1 tells us that the Chern class map for this abstract cohomology theory with the universal Chern class c_{Le} coincides with the one for $(\mathcal{H}_{\mathrm{Le}}, c_{\mathrm{Le}})$. Applying Proposition 7.1 to the abstract realization functor $(\Gamma_{\mathrm{mot}}, R_{\mathcal{H}})$ implies this theorem. ☐

8. Levine's Motivic Chern Class Map

As constructed in Section 6, we have a Chern class map

$$c^{j,2j-i}_{\mathcal{H}_{\mathrm{Le}},c_{\mathrm{Le}}} \colon K_i(X) \to H^{2j-i}_{\mathcal{H}_{\mathrm{Le}}}(X, j) = H^{2j-i}_{\mathrm{Le}}(X, \mathbf{Z}(j))$$

to Levine's motivic cohomology theory. On the other hand, Levine already constructed a Chern class map

$$c^{j,2j-i}_{\mathrm{Le}} K_i(X) \to H^{2j-i}_{\mathrm{Le}}(X, \mathbf{Z}(j))$$

to his motivic cohomology (cf. [41, Part I, Chapter III, 1.4]), and studied this map and the associated Chern character (cf. [41, Part I, Chapter III, 3.3.6.1]). For example, he proved his own Chern character induces an isomorphism from the K-theoretical motivic cohomology if the base scheme S is the spectrum of a field.

Proposition 8.1. *Suppose that the base scheme S is the spectrum of a field k. Then, Levine's Chern character*

$$ch^{i,j}_{\mathrm{Le}}\colon K_i(X)^{(j)} \to H^{2j-i}_{\mathrm{Le}}(X, \mathbf{Q}(j))$$

is an isomorphism for a scheme X in $\mathcal{V}$. Here, $K_i(X)^{(j)}$ is the weight j eigenspace of the rational K-group $K_i(X) \otimes \mathbf{Q}$ of the Adams operators (*cf.* [41, Part I, Chapter III, 3.6.3]).

Proof. See [41, Part I, Chapter III, 3.6.12]. ☐

The aim of this section is to prove the proposition below that says these two Chern class maps coincide.

Proposition 8.2. *Our Chern class map $c^{j,2j-i}_{\mathcal{H}_{Le},c_{Le}}$ for the abstract cohomology theory $\mathcal{H}_{Le}$ with the universal Chern class c_{Le} coincides with Levine's motivic Chern class map $c^{j,2j-i}_{\mathrm{Le}}$.*

Remark 8.3. By the above proposition, Proposition 7.9 holds if we replace our Chern class map (resp. Chern character) in this proposition with Levine's.

In this section, the term *sheaf* will refer to a Zariski sheaf. If the category $\mathcal{V}$ is clear from the context, we often omit $\mathcal{V}$ from Levine's motivic categories. Via the embedding $\mathbf{Z}(j)$, we often regard $\mathcal{V}^{\mathrm{op}}$ as the subcategory of $Z^0\mathcal{A}_{\mathrm{mot}}(\mathcal{V})_R$ for $j \in \mathbf{Z}$. We here note that the canonical map

$$H^i_{\mathrm{Le}}(X, \mathbf{Z}(j)) \otimes R \to H^i_{\mathrm{Le}}(X, R(j))$$

is an isomorphism.

8.1. **Construction.** We begin to recall the construction of Levine's motivic Chern class map. For details, see [41, Part I, Chapter III, 1.4]. For a scheme X in $\mathcal{V}$, let $\mathrm{Zar}(X)$ denote the full subcategory of open subschemes of X. Then, for $j \in \mathbf{Z}$, let $\mathcal{A}_{\mathrm{mot}}(X, j)$ denote the additive subcategory of $\mathcal{A}_{\mathrm{mot}}(\mathcal{V})$ generated by the image of the functors

$$\mathbf{Z}(j) \otimes \mathfrak{e}^{\otimes -}\colon \mathrm{Zar}(X)^{\mathrm{op}} \times \mathbf{Z}_{\geq 0} \to \mathcal{A}_{\mathrm{mot}}(\mathcal{V}),$$

where $\mathfrak{e}$ is the object of $\mathcal{A}_{\mathrm{mot}}(\mathcal{V})$ (cf. [41, Part I, Chapter I, 1.4.6]). We note that $\mathfrak{e}$ is canonically isomorphic to the unit object $\mathbf{Z}_S(0)$ in the category $D^b_{\mathrm{mot}}(\mathcal{V})_R$.

Remark 8.4. The additive subcategory $\mathcal{A}_{\mathrm{mot}}(\mathrm{Zar}(X, \mathrm{id}))$ of $\mathcal{A}_{\mathrm{mot}}(\mathcal{V})$ constructed in [41, Part I, Chapter II, 1.5.2] is canonically equivalent to the additive category $\coprod_j \mathcal{A}_{\mathrm{mot}}(X, j)$.

Lemma 8.5. (1) *$\mathcal{A}_{\mathrm{mot}}(X, j)$ is isomorphic to the free additive category on $\mathrm{Zar}(X)_c \times \mathbf{Z}_{\geq 0}$, where $\mathrm{Zar}(X)_c$ is the full subcategory of connected open subschemes of X.*

(2) *A Zariski presheaf P on X that takes values in an additive category $\mathcal{A}$ and that sends disjoint unions to direct sums is canonically extended to an additive functor $P_{\mathrm{mot}}\colon \mathcal{A}_{\mathrm{mot}}(X, j) \to \mathcal{A}$. Especially, an additive functor $Q\colon \mathcal{A}_{\mathrm{mot}}(X, j) \to \mathcal{A}$ is canonically isomorphic to $(Q \circ \mathbf{Z}(j))_{\mathrm{mot}}$.*

Proof. See [41, Part I, Chapter II, 1.5.2]. The assertion (2) follows from (1). ☐

Let Z be an object of the category $s.\mathcal{V}$ of simplicial objects in $\mathcal{V}$. Then, using Lemma above and taking the dg category C^b_{dg} of bounded complexes, the map (2)

$$N\mathbf{Z}\mathcal{V}(-, Z)_{*\leq n} \to N\mathcal{A}_{\mathrm{mot}}(\mathbf{Z}_Z(j), \mathbf{Z}(j))_{*\leq n}$$

of presheaves on X (for $\mathcal{H} = \mathcal{H}_{\mathrm{Le}}$) that takes values in $C^bC^+\mathbf{Ab}$ is canonically extended to a map

$$C^b_{\mathrm{dg}}((N\mathbf{Z}\mathcal{V}(-, Z)_{*\leq n})_{\mathrm{mot}}) \to C^b_{\mathrm{dg}}(N\mathcal{A}_{\mathrm{mot}}(\mathbf{Z}_Z(j), -)_{*\leq n})$$

of functors from $C^b_{\mathrm{dg}}\mathcal{A}_{\mathrm{mot}}(X,j)$ to $C^b_{\mathrm{dg}}C^bC^+\mathbf{Ab}$. Composing the total complex functor twice

$$C^b_{\mathrm{dg}}C^bC^+\mathbf{Ab} \xrightarrow{C^b_{\mathrm{dg}}\mathrm{Tot}} C^b_{\mathrm{dg}}C^+\mathbf{Ab} \xrightarrow{\mathrm{Tot}} C^+_{\mathrm{dg}}\mathbf{Ab}$$

and using Lemma 6.7,

$$\mathrm{Tot} \circ \mathrm{Tot}_i\,(NC^b_{\mathrm{mot}}(\mathbf{Z}_Z(j)^i, A^j)_{i\le n, j\in\mathbf{Z}}) \cong C^b_{\mathrm{dg}}C^b_{\mathrm{mot}}(\mathbf{Z}_Z(j)^*, A^\bullet),$$

we have a map

$$\mathrm{Tot}\, C^b_{\mathrm{dg}}(N\mathbf{Z}\mathcal{V}(-,Z)^*_{*\le n})_{\mathrm{mot}} \to C^b_{\mathrm{dg}}C^b_{\mathrm{mot}}(\mathbf{Z}_Z(j)^{*\le n}, -))$$

of functors from $C^b_{\mathrm{mot}}(X,j)$ to $C^+_{\mathrm{dg}}\mathbf{Ab}$. Here, we set $C^b_{\mathrm{mot}} = C^b_{\mathrm{dg}}\mathcal{A}_{\mathrm{mot}}$ and $C^b_{\mathrm{mot}}(X,j) = C^b_{\mathrm{dg}}\mathcal{A}_{\mathrm{mot}}(X,j)$. The above functor $\mathrm{Tot}\, C^b_{\mathrm{dg}}(N\mathbf{Z}\mathcal{V}(-,Z)^*_{*\le n})_{\mathrm{mot}}$ was constructed in [41, p. 120] and was denoted by $C^b(C^{*\ge -n}_X(Z;\mathbf{Z}))$. Composing the total complex functor $C^b_{\mathrm{dg}}C^b_{\mathrm{mot}} \to C^b_{\mathrm{mot}}$ gives us the map

$$\xi_n(Z,\mathrm{id})\colon C^b(C^{*\ge -n}_X(Z;\mathbf{Z})) \to C^b_{\mathrm{mot}}(\mathbf{Z}_Z(j)^{*\le n}, -)$$

of dg functors from $C^b_{\mathrm{mot}}(X,j)$ to $C^+_{\mathrm{dg}}\mathbf{Ab}$ which was constructed in [41, p. 120]. Here, we set $C^b_{\mathrm{mot}}(X,j) = C^b_{\mathrm{dg}}\mathcal{A}_{\mathrm{mot}}(X,j)$. From construction, we see the following lemma.

Lemma 8.6. *As a functor from $s.\mathcal{V}^{\mathrm{op}}$ to $C^b\mathbf{Ab}$,*

$$C^b(C^{*\ge -n}_X(Z;\mathbf{Z}))(\mathbf{Z}(j)^*) = \mathrm{Tot}\, N\mathbf{Z}\mathcal{V}(-,Z)^*_{*\le n},$$

and the natural transformation $\xi_n(Z,\mathrm{id}) \circ \mathbf{Z}(j)$ is equal to the map (2)

$$\mathrm{Tot}\, N\mathbf{Z}\mathcal{V}(-,Z)^*_{*\le n} \to C^b_{\mathrm{mot}}(\mathbf{Z}_Z(j)^{*\le n}, \mathbf{Z}(j)^*).$$

We denote $HR_{\mathbf{Z}_X(j)}$ the category of hyper-resolutions of $\mathbf{Z}_X(j)$ constructed by Levine. This is a subcategory of $C^b_{\mathrm{mot}}(X,j)$ and exhibits the following properties (cf. [41, Part I, Chapter II, 1.4, 1.5]).

Lemma 8.7. (1) *For any object A of $HR_{\mathbf{Z}_X(j)}$, there exists a canonical morphism $\epsilon\colon \mathbf{Z}_X(j) \to A$ in $C^b_{\mathrm{mot}}(X,j)$, called the augmentation, such that ϵ is an isomorphism in $D^b_{\mathrm{mot}}(\mathcal{V})$.*

(2) *Let U be a simplicial object in $\mathcal{V}$ associated with a finite open covering of X. Then, the object $\mathbf{Z}_U(j)^*$ of $C^b_{\mathrm{mot}}(X,j)$ belongs to $HR_{\mathbf{Z}_X(j)}$ and the canonical map $U \to X$ induces the augmentation $\mathbf{Z}_X(j) \to \mathbf{Z}_U(j)^*$.*

Proof. See [41, Part I, Chapter II, 1.4.1, 1.4.2]. □

Lemma 8.8. *The image of the category $HR_{\mathbf{Z}_X(j)}$ in the homotopy category $K^b\mathcal{A}_{\mathrm{mot}}(\mathcal{V})$ is right-filtering. Furthermore, for an object X of $\mathcal{V}$ and an object Z of $s.\mathcal{V}$, let $S(Z)_n$ denote the sheafification of the presheaf $N\mathbf{Z}\mathcal{V}(-,Z)_{*\le n}$ of cochain complexes on X. Then, there exists a canonical isomorphism*

$$\varinjlim_{A\in HR_{\mathbf{Z}_X(j)}} H^i(C^b(C^{*\ge -n}_X(Z;\mathbf{Z}))(A)) \cong \mathbf{H}^i(X, S(Z)_n),$$

where the right group is the i-th hypercohomology of $S(Z)_n$.

Proof. See [41, Part I, Chapter II, 1.4.3, 1.5.3]. □

For a hyper-resolution A of $\mathbf{Z}_X(j)$ with the augmentation ϵ, we have a functorial map

$$H^i(C^b(C^{*\ge -n}_X(Z;\mathbf{Z}))(A)) \xrightarrow{\epsilon^{-1}\circ\xi_n(Z)(A)} D^b_{\mathrm{mot}}(\mathbf{Z}_Z(j)^{*\le n}, \mathbf{Z}_X(j)[i]).$$

Taking the colimit over $HR_{\mathbf{Z}_X(j)}$ and using the above lemma, we have a functorial map

$$\mathbf{H}^i(X, S(Z)_n) \to D^b_{\mathrm{mot}}(\mathbf{Z}_Z(j)^{*\le n}, \mathbf{Z}_X(j)[i]).$$

This gives us a functorial map

$$\varinjlim_n \mathbf{H}^i(X, S(Z)_n) \otimes \varprojlim_n D^b_{\mathrm{mot}}(1, \mathbf{Z}_Z(j)^{*\le n}[k]) \to D^b_{\mathrm{mot}}(1, \mathbf{Z}_X(j)[i+k]).$$

We now let $S(Z)$ denote the sheafification of the presheaf $\mathrm{Tot}\, N\mathbf{Z}\mathcal{V}(-, Z)_*$ on X. Then, because the cohomological dimension of X is finite, the canonical map

$$\varinjlim_n \mathbf{H}^i(X, S(Z)_n) \to \mathbf{H}^i(X, S(Z))$$

is an isomorphism. We thus have a functorial map

$$\mathbf{H}^i(X, S(Z)) \otimes H^k_{\mathrm{Le}}(Z, \mathbf{Z}(j)) \to H^{i+k}_{\mathrm{Le}}(X, \mathbf{Z}(j)).$$

The Hurewicz map

$$K_i(X) \to \varinjlim_N \mathbf{H}^{-i}(X, S(BGL_{N,S})) \tag{8}$$

constructed in [41, Appendix B, 2.2.1.3] is the composite

$$K_i(X) \to \varinjlim_{U,N} H^{-i}(\mathrm{Tot}\, N\mathbf{Z}\mathcal{V}(U, BGL_{N,S})^*_*) \to \varinjlim_N \mathbf{H}^{-i}(X, S(BGL_{N,S})),$$

where the first map is the Hurewicz map (1) constructed in the previous section and the other is the canonical map. Composing this, we have a functorial map

$$\gamma_{\mathrm{Le}}\colon K_i(X) \otimes \varprojlim_N H^k_{\mathrm{Le}}(BGL_{N,S}, \mathbf{Z}(j)) \to H^{k-i}_{\mathrm{Le}}(X, \mathbf{Z}(j)).$$

Finally, the universal Chern class

$$c_{\mathrm{Le}} \in \prod_j \varprojlim_N H^{2j}_{\mathrm{Le}}(BGL_{N,S}, \mathbf{Z}(j))$$

for Levine's motivic cohomology theory (cf. [41, Part I, Chapter III, 1.3.4]) gives us Levine's motivic Chern class map

$$c^{j,2j-i}_{\mathrm{Le}} = \gamma_{\mathrm{Le}}(- \otimes c_{\mathrm{Le}})\colon K_i(X) \to H^{2j-i}_{\mathrm{Le}}(X, \mathbf{Z}(j)).$$

8.2. **Proof of Theorem 8.2.** To prove Theorem 8.2, we will use the two lemmas below.

Lemma 8.9. *For a simplicial object Z in $\mathcal{V}$, the canonical map*

$$\varinjlim_U H^i(\mathrm{Tot}\, N\mathbf{Z}\mathcal{V}(U, Z)^*_{*\le n}) \to \varinjlim_{A \in HR_{\mathbf{Z}_X(j)}} H^i(C^b(C^{*\ge -n}_X(Z; \mathbf{Z}))(A))$$

induced by Lemma 8.6 *and* 8.7(2) *is equal to the composite*

$$\begin{aligned}\varinjlim_U H^i(\mathrm{Tot}\, N\mathbf{Z}\mathcal{V}(U, Z)^*_{*\le n}) &\to \mathbf{H}^{-i}(X, S(Z)_n)\\ &\xrightarrow{8.8} \varinjlim_{A \in HR_{\mathbf{Z}_X(j)}} H^i(C^b(C^{*\ge -n}_X(Z; \mathbf{Z}))(A))\end{aligned}$$

of the canonical maps.

Proof. It follows from that the diagram

$$\begin{array}{ccc} H^i(\mathrm{Tot}\, N\mathbf{Z}\mathcal{V}(U, Z)^*_{*\le n}) & \longrightarrow & \mathbf{H}^{-i}(X, S(Z)_n) \\ \downarrow & & \| \\ H^i(C^b(C^{*\ge -n}_X(Z; \mathbf{Z}))(\mathbf{Z}_U(j)^*)) & \longrightarrow & \mathbf{H}^{-i}(X, S(Z)_n) \end{array}$$

commutes for the simplicial scheme U attached to a finite affine open covering of X. $\square$

Lemma 8.10. *For a simplicial object Z in $\mathcal{V}$, the diagram*

$$\begin{tikzcd}
\varinjlim_{U} H^i(\mathrm{Tot}\, N\mathbf{Z}\mathcal{V}(U,Z)^*_{*\le n}) \arrow[r] \arrow[d, "(3)"'] & \varinjlim_{A\in HR_{\mathbf{Z}_X(j)}} H^i(C^b(C_X^{*\ge -n}(Z;\mathbf{Z}))(A)) \arrow[d, "\varinjlim \xi_n(A)"'] \\
\varinjlim_{U} H^i(C^b_{\mathrm{mot}}(\mathbf{Z}_Z(j)^{*\le n}, \mathbf{Z}_U(j)^*)) \arrow[r, "8.7(2)"] \arrow[d, "(4)+(5)"'] & \varinjlim_{A\in HR_{\mathbf{Z}_X(j)}} H^i(C^b_{\mathrm{mot}}(\mathbf{Z}_Z(j)^{*\le n}, A)) \arrow[d, "\varinjlim \epsilon_A^{-1}"'] \\
D^b_{\mathrm{mot}}(\mathbf{Z}_Z(j)^{*\le n}, \mathbf{Z}_X(j)[i]) \arrow[r, equal] & D^b_{\mathrm{mot}}(\mathbf{Z}_Z(j)^{*\le n}, \mathbf{Z}_X(j)[i])
\end{tikzcd}$$

commutes, where the top arrow is the map in Lemma above.

Proof. It follows from Lemma 8.6 and 8.7 (2). $\square$

Proof of Theorem 8.2. Lemma 8.9 shows the two Hurewicz maps coincide, i.e., the diagram

$$\begin{tikzcd}
K_i(X) \arrow[r, "(1)"] \arrow[d, "(8)"'] & \varinjlim_{U,N} H^{-i}(\mathrm{Tot}\, N\mathbf{Z}\mathcal{V}(U, BGL_{N,S})^*_{*\le n}) \arrow[d, "8.8"'] \\
\varinjlim_{N} \mathbf{H}^{-i}(X, S(BGL_{N,S})) \arrow[r, "8.9"] & \varinjlim_{N,A\in \mathrm{HR}_{\mathbf{Z}_X(j)}} H^{-i}(C^b(C_X^{*\ge -n}(BGL_{N,S};\mathbf{Z}))(A))
\end{tikzcd}$$

commutes. Hence, Lemma 8.10 shows that the map (6)

$$K_i(X)\otimes \varprojlim_{N} H^k_{\mathrm{Le}}(BGL_{N,S}, \mathbf{Z}(j)) \to H^{k-i}_{\mathrm{Le}}(X, \mathbf{Z}(j))$$

for $\mathcal{H}_{\mathrm{Le}}$ coincides with the γ_{Le}. Then, the motivic universal Chern class c_{Le} defines the same Chern class maps. $\square$

9. Beilinson's Regulator

In this section, we first compare Beilinson's regulator with our Chern character for the Hodge cohomology theory (in Section 9.1). After that, we study the Chern character for the real Hodge cohomology theory (in Section 9.2) and for the Hodge cohomology theory over a subfield field of $\mathbf{C}$ (in Section 9.3). We close this section with proving Theorem 0.3 (Corollary 9.13).

9.1. **Over C.** We will show that our Chern character for the Hodge cohomology theory $\mathcal{H}_{HS}$ in Example 4.1(1) with some universal Chern class c_{HS} is canonically isomorphic to Beilinson's regulator (cf. [5]). To state the claim precisely, we recall the Deligne-Beilinson cohomology theory which is the range of Beilinson's regulator. We here note that the $\mathcal{H}_{HS}$-cohomology is the absolute Hodge cohomology (cf. [6, Section 5]).

Let $C^+_{\mathcal{H}_R}$ denote the dg category of bounded below R-mixed Hodge complexes (cf. [6, Section 3], [41, Chapter V, 2.3.1]). Then, there exists a canonical dg functor $C^+_{\mathcal{H}'_R} \to C^+_{\mathcal{H}_R}$ that induced an equivalence $D^+_{\mathcal{H}'_R} \to D^+_{\mathcal{H}_R}$ between their derived categories.

Let $\widetilde{\Gamma}_{\mathcal{H}}$ denote the additive functor $Z^0C^+_{\mathcal{H}_R} \to C^+\mathbf{Mod}_R$ which preserves arbitrary homotopies and was constructed in [6, 1.2] (see also [32, 4.2]). We define the functor Γ_{DB} to be the composite

$$\mathcal{V}^{\mathrm{op}} \xrightarrow{\Gamma_{HS}} Z^0C^+_{\mathcal{H}'_R} \to Z^0C^+_{\mathcal{H}_R} \xrightarrow{\widetilde{\Gamma}_{\mathcal{H}}} C^+\mathbf{Mod}_R.$$

The associated cohomology theory (cf. Remark 6.3) is called the Deligne-Beilinson cohomology theory and is denoted by $\mathcal{H}_{\mathrm{DB}}$.

Remark 9.1. In the derived category, the functor Γ_{DB} is isomorphic to the global sections of an injective resolution of the Deligne-Beilinson complex $\oplus_{j\in\mathbf{Z}} R_{\mathcal{D},Zar}(j)$ in [25, 5.5]. Hence, the $\mathcal{H}_{DB}$-cohomology is equal to the Deligne-Beilinson cohomology,

$$H^i_{\mathcal{H}_{DB}}(X,j) = H^i_{DB}(X,R(j)),$$

by Remark 6.3.

The functor $\widetilde{\Gamma}_{\mathcal{H}}$ can be canonically extended to a triangulated functor

$$D^+_{\mathcal{H}_R} \to D^+\mathbf{Mod}_R$$

and there exists a canonical quasi-isomorphism $R \to \widetilde{\Gamma}_{\mathcal{H}}1$ in $C^+\mathbf{Mod}_R$. Furthermore, the following lemma holds.

Lemma 9.2. *The map*

$$D^+_{\mathcal{H}_R}(1,-) \to D^+\mathbf{Mod}_R(\widetilde{\Gamma}_{\mathcal{H}}1, \widetilde{\Gamma}_{\mathcal{H}}-)$$

of functors on $D^+\mathbf{Mod}_R$ induced by $\widetilde{\Gamma}_{\mathcal{H}}$ is an isomorphism.

Proof. By construction, the map in this lemma coincides with the composite of the isomorphism

$$D^+_{\mathcal{H}_R}(1,-) \cong D^+\mathbf{Mod}_R(R, \widetilde{\Gamma}_{\mathcal{H}}-)$$

(cf. [32, 4.2.3]) and the one

$$D^+\mathbf{Mod}_R(R, \widetilde{\Gamma}_{\mathcal{H}}-) \cong D^+\mathbf{Mod}_R(\widetilde{\Gamma}_{\mathcal{H}}1, \widetilde{\Gamma}_{\mathcal{H}}-)$$

induced by the $R \cong \widetilde{\Gamma}_{\mathcal{H}}1$. □

From this, we identify the absolute Hodge cohomology $H^i_{\mathcal{H}_{\mathcal{HS}}}(X,j)$ with the Deligne-Beilinson cohomology $H^i_{DB}(X,R(j))$.

Because the Hodge cohomology theory is extended to a realization functor, this equips a canonical universal Chern class c_{HS} (see Remark 7.7). On the other hand, the universal Chern class

$$c_{DB} \in \prod_j \varprojlim_N H^{2j}_{DB}(BGL_{N,\mathbf{C}}, R(j))$$

in the Deligne-Beilinson cohomology was constructed (cf. [48, p. 20]). We here note that they coincide up to sign.

Lemma 9.3. *In the Deligne-Beilinson cohomology*

$$\varprojlim_N H^{2j}_{DB}(BGL_{N,\mathbf{C}}, R(j)),$$

we have the equality $c^j_{HS} = (-1)^j c^j_{DB}$.

Proof. Both universal Chern classes are constructed by the projective bundle formula. Comparing their definitions implies this lemma. □

In this section, we will prove the following proposition.

Proposition 9.4. *Assume that the ring R contains $\mathbf{Q}$. Then, our Chern character*

$$ch^{i,j}_{\mathcal{H}_{HS},c_{HS}} \colon K_i(X) \to H^{2j-i}_{DB}(X, R(j))$$

for the Hodge cohomology theory $\mathcal{H}_{HS}$ with the universal Chern class c_{HS} coincides with Beilinson's regulator multiplied by $(-1)^j$.

To show this proposition, we recall the construction of Beilinson's regulator following [48, Section 4].

Lemma 9.5. *Let $\mathscr{F}$ be a complex of Zariski sheaves on $\mathcal{V}$. Then, for a simplicial scheme X in $\mathcal{V}$, the map*

$$\mathcal{V}(-,X) \to C\mathbf{Ab}(\mathscr{F}(X),\mathscr{F}(-))$$

of cosimplicial sheaves on $\mathcal{V}$ induces an isomorphism

$$H^i(X,\mathscr{F}) \cong D(\mathcal{V})(N\mathbf{Z}\mathcal{V}(-,X)_*,\mathscr{F}[i]),$$

where $D(\mathcal{V})$ is the derived category of abelian Zariski shaves on $\mathcal{V}$.

Proof. See [48, p. 15, Lemma]. □

Applying this lemma to the case where $\mathscr{F}$ is the Godement resolution of the real Deligne-Beilinson complex of weight j (see Remark 9.1) and $X = BGL_{N,\mathbf{C}}$, we have an isomorphism

$$H^{2j}(BGL_{N\mathbf{C}},\mathscr{F}) \cong D(\mathcal{V})(N\mathbf{Z}\mathcal{V}(-,BGL_{N,\mathbf{C}})_*,\mathscr{F}[2j]).$$

For an affine scheme $Y = \operatorname{Spec} A$ in $\mathcal{V}$, because

$$H_i(GL_N(A),\mathbf{Z}) = H^{-i}(N\mathbf{Z}\mathcal{V}(Y,BGL_{N,\mathbf{C}})_*)$$

(cf. [41, Part I, Appendix B, 1.1]), we have a map

$$H^{2j}(BGL_{N,\mathbf{C}},\mathscr{F}) \to \mathbf{Ab}(H_i(GL_N(A),\mathbf{Z}),H^{2j-i}(\mathscr{F}(Y))).$$

Taking the limit with respect to N and composing the Hurewicz map (1), we have a map

$$\varprojlim_N H^{2j}(BGL_{N,\mathbf{C}},\mathscr{F}) \to \mathbf{Ab}(K_i(Y),H^{2j-i}(\mathscr{F}(Y))). \tag{9}$$

Finally, the j-th universal Chern class

$$c^j_{DB} \in \varprojlim_N H^{2j}(BGL_{N,\mathbf{C}},\mathscr{F})$$

(cf. [48, p. 20]) gives us Beilinson's Chern class map

$$c^{i,2j-i}_{Be}\colon K_i(Y) \to H^{2j-i}(\mathscr{F}(Y)) = H^{2j-i}_{DB}(Y,R(j)).$$

In the case $i = j = 0$, we add the rank function

$$K_0(Y) \to \mathbf{Z} \xrightarrow{\text{rank}} H^0_{DB}(Y,R(0)).$$

Furthermore, we define Beilinson's Chern class map

$$c^{i,2j-i}_{Be}\colon K_i(X) \to H^{2j-i}_{DB}(X,R(j)).$$

for a scheme X in $\mathcal{V}$ as follows (cf. [48, p. 17]): Jouanolou's lemma [38, 1.5] tells us that there exists a torsor Y on X for a vector bundle that is an affine scheme. Then, the map $Y \to X$ induces isomorphisms

$$K_i(X) \xrightarrow{\cong} K_i(Y) \quad \text{and} \quad H^{2j-i}_{DB}(X,R(j)) \xrightarrow{\cong} H^{2j-i}_{DB}(Y,R(j))$$

because of the homotopy invariance. Beilinson's Chern class map for X is defined to be the composite map

$$K_i(X) \xrightarrow{\cong} K_i(Y) \xrightarrow{c^{i,2j-i}_{Be}} H^{2j-i}_{DB}(Y,R(j)) \xleftarrow{\cong} H^{2j-i}_{DB}(X,R(j))$$

that is independent on the choice of such a Y.

The Beilinson regulator is defined to be the associated Chern character (cf. [48, p. 28]).

Lemma 9.6. *Our Chern class map for the Deligne-Beilinson cohomology theory $\mathcal{H}_{DB}$ with the universal Chern class c_{DB} coincides with Beilinson's Chern class map.*

Proof. Let Y be an affine scheme in $\mathcal{V}$. Then, the map (9) canonically induces a map

$$K_i(Y) \otimes \varprojlim_N H^{2j}_{DB}(BGL_{N,S}, R(j)) \to H^{2j-i}_{DB}(Y, R(j))$$

which coincides with the map (6) for the cohomology theory associated with the global sections of $\mathscr{F}$. This follows from the comparison between their constructions and Lemma 9.5. Proposition 7.3 and Remark 9.1 tell us that the above map equals to the map (6) for the Deligne-Beilinson cohomology theory. Hence, the universal Chern class c_{DB} gives the same Chern class maps. By functoriality, these maps coincide for each scheme in $\mathcal{V}$. □

Proof of Proposition 9.4. By Proposition 7.1, we have a commutative diagram

$$\begin{array}{ccc} K_i(Y) & \longrightarrow & H^{2j-i}_{\mathcal{H}_{HS}}(Y, j) \\ {\scriptstyle \times(-1)^j}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{\Gamma}_{\mathcal{H}}} \\ K_i(Y) & \longrightarrow & H^{2j-i}_{\mathcal{H}_{DB}}(Y, j), \end{array}$$

where the horizontal arrows are our Chern class maps for $(\mathcal{H}_{HS}, c_{HS})$ and $(\mathcal{H}_{DB}, c_{DB})$. This proposition thus follows from Lemma 9.6. □

The following theorem is the terminus of this section.

Theorem 9.7. *Assume that the ring R contains $\mathbf{Q}$. Let $R_{\mathcal{H}_{HS}}$ denote the realization functor*

$$D^b_{\mathrm{mot}\mathfrak{sh}}(\mathbf{Sm}_{\mathbf{C}})_R \to D^+_{\mathcal{H}_R}$$

for the Hodge cohomology theory $\mathcal{H}_{HS}$. Then, the diagram

$$\begin{array}{ccc} K_i(X) & \xrightarrow{(-1)^j ch^{i,j}_{\mathrm{Le}} \otimes 1_R} & H^{2j-i}_{\mathrm{Le}}(X, R(j)) \\ {\scriptstyle ch^{i,j}_{Be}}\big\downarrow & & \big\downarrow{\scriptstyle R_{\mathcal{H}_{HS}}} \\ H^{2j-i}_{DB}(X, R(j)) & = & H^{2j-i}_{\mathcal{H}_{HS}}(X, j) \end{array}$$

commutes for a scheme X in $\mathcal{V}$.

Proof. It follows from Remark 8.3 and Proposition 9.4. □

9.2. **Over R.** We let $\mathcal{H}^\infty_{HS}$ denote the real Hodge cohomology theory in Example 4.1(2). Because this cohomology theory is extended to a realization functor, it equips a canonical universal Chern class c^∞_{HS}. In this section, we prove that Beilinson's regulator coincides with the Chern character for $(\mathcal{H}^\infty_{HS}, c^\infty_{HS})$ up to unit. We here note that the $\mathcal{H}^\infty_{HS}$-cohomology is the real part of the absolute Hodge cohomology if $1/2 \in R$ (see Lemma 9.9).

By construction, forgetting the action of $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$ gives us a commutative diagram

$$\begin{array}{ccccc} \mathbf{Sm}_{\mathbf{R}} & \xrightarrow{\Gamma_{\mathcal{H}^\infty_{HS}}} & Z^0 C^{+\infty}_{\mathcal{H}'_R} & \longrightarrow & D^{+\infty}_{\mathcal{H}'_R} \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathbf{Sm}_{\mathbf{C}} & \xrightarrow{\Gamma_{\mathcal{H}_{HS}}} & Z^0 C^{+}_{\mathcal{H}'_R} & \longrightarrow & D^{+}_{\mathcal{H}'_R} \end{array}$$

of categories, where the left vertical arrow is the base change map and the other verticals are the forgetful functor (see [41, Chapter V, 2.3]). Proposition 7.1 and 7.2 thus tell us the commutativity of the diagram below.

Proposition 9.8. *For a scheme X in $\mathbf{Sm_R}$, the diagram*

$$\begin{array}{ccc} K_i(X) & \xrightarrow{c^{2j-i,j}_{\mathcal{H}^\infty_{HS},c^\infty_{HS}}} & H^{2j-i}_{\mathcal{H}^\infty_{HS}}(X,j) \\ \downarrow & & \downarrow \\ K_i(X_{\mathbf{C}}) & \xrightarrow{c^{2j-i,j}_{\mathcal{H}_{HS},c_{HS}}} & H^{2j-i}_{\mathcal{H}_{HS}}(X_{\mathbf{C}},j) \end{array}$$

commutes, where the left vertical arrow is the base change map and the right one is induced by the forgetful functor $D^{+\infty}_{\mathcal{H}'_R} \to D^{+}_{\mathcal{H}'_R}$.

The real absolute Hodge cohomology $H^{2j-i}_{\mathcal{H}^\infty_{HS}}(X,j)$ is equals to the real part of the absolute cohomology, if the ring R contains $1/2$.

Lemma 9.9. *Assume that the ring R contains $1/2$. Then, for a scheme X in $\mathbf{Sm_R}$, the map*

$$H^i_{\mathcal{H}^\infty_{HS}}(X,j) \to H^i_{\mathrm{DB}}(X_{\mathbf{C}},R(j))$$

induced by the forgetful functor is injective, and its image is the invariant part under the action of $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$. Here, we set $X_{\mathbf{C}} = X \otimes_{\mathbf{R}} \mathbf{C}$.

Proof. This was proved in [6, Section 7]. □

The corollary below is the goal of this section.

Corollary 9.10. *Let $R_{\mathcal{H}^\infty_{HS}}$ denote the realization functor*

$$D^b_{\mathrm{mots}\mathfrak{h}}(\mathbf{Sm_R})_R \to D^{+\infty}_{\mathcal{H}_R}$$

for the real Hodge cohomology theory $\mathcal{H}_{HS}$. Then, if the ring R contains $\mathbf{Q}$, the diagram

$$\begin{array}{ccccc} K_i(X) & \xrightarrow{(-1)^j ch^{i,j}_{\mathrm{Le}} \otimes 1_R} & H^{2j-i}_{\mathrm{Le}}(X,R(j)) & \xrightarrow{R_{\mathcal{H}^\infty_{HS}}} & H^{2j-i}_{\mathcal{H}^\infty_{HS}}(X,j) \\ \downarrow & & & & \cong\downarrow \\ K_i(X_{\mathbf{C}}) & \xrightarrow{ch^{i,j}_{Be}} & H^{2j-i}_{DB}(X_{\mathbf{C}},R(j))^\infty & = & H^{2j-i}_{\mathcal{H}_{HS}}(X_{\mathbf{C}},j)^\infty \end{array}$$

commutes for a scheme X in $\mathbf{Sm_R}$, where the left vertical arrow is the base change map. Here, we denote the invariant submodule of a $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$-module M by M^∞.

Proof. By Remark 8.3, our Chern class map for the real Hodge cohomology theory is compatible with Levine's. On the other hand, Proposition 9.4 tells that our Chern class map for Hodge cohomology coincides with Beilinson's. Thus, combing Proposition 9.8, we have the commutativity of this diagram. □

9.3. **Over a subfield of C.** We let $\mathcal{H}_{MAH}$ denote the mixed absolute Hodge cohomology theory in Example 4.3. We define the category $C^+_{\mathcal{H}_{k,R}}$ of real R-mixed Hodge complexes on k to be the full dg subcategory of $C^+_{MAH,k,R}$ which consists of the objects whose data other than the real R-mixed Hodge complex on k are 0, i.e., the data (i), (ii), (v), (vi), (vii) and (x) in [41, Chapter V,2.4.1] are 0. Then, we have a forgetful functor $C^+_{MAH,k,R} \to C^+_{\mathcal{H}_{k,R}}$. We define $s(\sigma)$ to be

$$s(\sigma) = \begin{cases} \infty & \text{if } \sigma(k) \subset \mathbf{R} \\ \emptyset & \text{otherwise} \end{cases}$$

for an embedding σ of k in $\mathbf{C}$. Then, picking up the (real) R-mixed Hodge complex on an embedding $\sigma\colon k \to \mathbf{C}$ gives us a forgetful functor $C^+_{\mathcal{H}_{k,R}} \to C^{+s(\sigma)}_{\mathcal{H}_R}$. They are extended to triangulated functors

$$D^+_{MAH,k,R} \to D^+_{\mathcal{H}_{k,R}} \to D^{+s(\sigma)}_{\mathcal{H}_R}$$

between their derived categories. Furthermore, the diagram

$$\begin{array}{ccccc}
\mathbf{Sm}_k & \xrightarrow{\Gamma_{MAH}} & Z^0C^+_{MAH,k,R} & \longrightarrow & D^+_{MAH,k,R} \\
\| & & \downarrow & & \downarrow \\
\mathbf{Sm}_k & \longrightarrow & Z^0C^+_{\mathcal{H}_{k,R}} & \longrightarrow & D^+_{\mathcal{H}_{k,R}} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{Sm}_{\mathbf{C}^{s(\sigma)}} & \xrightarrow{\Gamma_{\mathcal{H}^{s(\sigma)}_{HS}}} & Z^0C^{+s(\sigma)}_{\mathcal{H}'_R} & \longrightarrow & D^{+s(\sigma)}_{\mathcal{H}'_R}
\end{array}$$

commutes by their constructions, where the left vertical arrow is the base change map via σ. We thus have a map

$$\rho_\sigma\colon H^i_{\mathcal{H}_{MAH}}(X,j) \to H^i_{\mathcal{H}^{s(\sigma)}_{HS}}(X_\sigma, j)$$

of the $\mathcal{H}$-cohomology groups for a scheme X in $\mathbf{Sm}_k$, where we set $X_\sigma = X \otimes_{k,\sigma} \mathbf{C}$ (cf. Section 9.1 and 9.2).

Proposition 7.1 and 7.2 tells us the following one.

Proposition 9.11. *Let $c^{j,2j-i}_{MAH}$ denote our Chern class map*

$$K_i(X) \to H^{2j-i}_{\mathcal{H}_{MAH}}(X,j)$$

induced by the realization functor for the mixed absolute Hodge cohomology theory (cf. Remark 7.7). For a scheme X in $\mathbf{Sm}_k$ and an embedding σ of k in $\mathbf{C}$, the diagram

$$\begin{array}{ccc}
K_i(X) & \xrightarrow{c^{j,2j-i}_{MAH}} & H^{2j-i}_{\mathcal{H}_{MAH}}(X,j) \\
\downarrow & & \downarrow{\scriptstyle \rho_\sigma} \\
K_i(X \otimes_{k,\sigma} \mathbf{C}^{s(\sigma)}) & \xrightarrow{c^{2j-i,j}_{\mathcal{H}^\infty_{HS},c^\infty_{HS}}} & H^{2j-i}_{\mathcal{H}^\infty_{HS}}(X \otimes_{k,\sigma} \mathbf{C}^{s(\sigma)}, j)
\end{array}$$

commutes, where the left vertical map is the base change map.

Proof. By their constructions, these universal Chern classes coincide via the map ρ_σ. Hence, we can apply Proposition 7.1 to this case. $\square$

Lemma 9.12. *Assume that k is algebraic over $\mathbf{Q}$. Then, the diagram*

$$\begin{array}{ccc}
K_i(X \otimes_{\mathbf{Q}} \mathbf{C}) & \xrightarrow{c^{j,2j-i}_{Be}} & H^{2j-i}_{DB}(X \otimes_{\mathbf{Q}} \mathbf{C}, R(j)) \\
\cong\downarrow & & \downarrow\cong \\
\oplus_\sigma K_i(X_\sigma) & \xrightarrow{\oplus_\sigma c^{j,2j-i}_{Be}} & \oplus_\sigma H^{2j-i}_{DB}(X_\sigma, R(j))
\end{array}$$

of $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$*-modules commutes, where the direct sums runs through over all embeddings σ of k in $\mathbf{C}$ and the vertical arrows induced by the map $X_\sigma \to X \otimes_{\mathbf{Q}} \mathbf{C}$. Here, the action of* $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$ *on $\oplus_\sigma K_i(X_\sigma)$ is induced by the actions on the set of embeddings of k and on the group $K_i(X_\sigma)$ for $\sigma\colon k \to \mathbf{R} \to \mathbf{C}$, and* $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$ *acts on $\oplus_\sigma H^{2j-i}_{DB}(X_\sigma, R(j))$ similarly.*

Proof. The maps $X_\sigma \to X \otimes_{\mathbf{Q}} \mathbf{C}$ give us the isomorphism $X \otimes_{\mathbf{Q}} \mathbf{C} \cong \coprod_\sigma X_\sigma$ of $\mathbf{C}$-schemes with an action of $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$. We thus have the commutativity of this diagram. □

Corollary 9.13. *Let k be a finite extension field over $\mathbf{Q}$ and let R_{MAH} denote the realization functor*

$$D^b_{\mathrm{mot}\mathfrak{sh}}(\mathbf{Sm}_k)_R \to D^+_{MAH,R}$$

for the mixed absolute cohomology theory $\mathcal{H}_{MAH}$. Assume that the ring R contains $\mathbf{Q}$. Then, the diagram

$$\begin{array}{ccccc} K_i(X) & \xrightarrow{(-1)^j ch^{i,j}_{\mathrm{Le}} \otimes 1_R} & H^{2j-i}_{\mathrm{Le}}(X, R(j)) & \xrightarrow{R_{MAH}} & H^{2j-i}_{\mathcal{H}_{MAH}}(X, j) \\ \downarrow & & & & \downarrow \oplus_\sigma \rho_\sigma \\ K_i(X_{\mathbf{C}})^\infty & \xrightarrow{ch^{i,j}_{Be}} & H^{2j-i}_{DB}(X_{\mathbf{C}}, R(j))^\infty & \xrightarrow[9.12]{\cong} & \oplus_\sigma H^{2j-i}_{DB}(X_\sigma, R(j))^{s(\sigma)} \end{array}$$

commutes for a scheme X in $\mathbf{Sm}_k$, where the direct sums runs through all infinite places of k and the left vertical arrow is the base change map. Here, we set $X_{\mathbf{C}} = X \otimes_{\mathbf{Q}} \mathbf{C}$.

Proof. Combining the diagrams in Proposition 9.11 and Lemma 9.12 via Proposition 9.8, we have a commutative diagram

$$\begin{array}{ccc} K_i(X) & \xrightarrow{(-1)^j c^{j,2j-i}_{MAH}} & H^{2j-i}_{\mathcal{H}_{MAH}}(X, j) \\ \downarrow & & \downarrow \oplus_\sigma \rho_\sigma \\ \oplus_\sigma K_i(X \otimes_{k,\sigma} \mathbf{C}^{s(\sigma)}) & \xrightarrow{\oplus_\sigma c^{j,2j-i}_{Be}} & \oplus_\sigma H^{2j-i}_{DB}(X_\sigma, R(j))^{s(\sigma)} \\ \downarrow & & \downarrow \\ K_i(X_{\mathbf{C}}) & \xrightarrow{c^{j,2j-i}_{Be}} & H^{2j-i}_{DB}(X_{\mathbf{C}}, R(j)). \end{array}$$

Remark 8.3 thus tells us the desired commutativity. □

10. Soulé's Chern Class Map

In this section, we study on an l-adic analog of the previous section and prove Theorem 0.4. As already mentioned, we take Soulé's Chern class as an l-adic analog of Beilinson's.

Let $\mathcal{H}_{ét}$ denote the l-adic étale cohomology theory in Example 4.2. Then, the $\mathcal{H}_{ét}$-cohomology group is the continuous étale cohomology,

$$H^i_{\mathcal{H}_{ét}}(X, j) = H^i_{\mathrm{cont}}(X, \mathbf{Z}_l(j)).$$

As we mentioned, there exists the abstract realization functor for this abstract cohomology theory $\mathcal{H}_{ét}$. This realization functor canonically gives us a universal Chern class $c_{ét}$ (cf. Remark 7.7).

On the other hand, Soulé constructed a Chern class map

$$\bar{c}^{j,2j-1}_{\mathrm{So}} K_i(A, \mathbf{Z}/l^r) \to H^{2j-i}_{ét}(\mathrm{Spec}\, A, \mu^{\otimes j}_{l^r})$$

with coefficients in $\mathbf{Z}/l^r$ for a ring A (which is a Dedekind domain when $i = 1$) (cf. [49, II.2.3]). Taking the limit with respect to r (cf. [49, IV.3.1]), we have Soulé's Chern class map

$$c^{j,2j-1}_{\mathrm{So}} \colon K_i(A, \mathbf{Z}_l) \to H^{2j-i}_{ét}(\mathrm{Spec}\, A, \mathbf{Z}_l(j))$$

with coefficients in $\mathbf{Z}_l$ (cf. [50, Section 1]). We denote the associated Chern character by $ch^{i,j}_{\mathrm{So}}$. The goal of this section is to prove the following proposition.

Proposition 10.1. *Let $c_{ét}^{j,2j-i}$ denote the Chern class map for the l-adic étale cohomology theory $\mathcal{H}_{ét}$ with $c_{ét}$. Let X be an affine scheme $\operatorname{Spec} A$ in $\mathcal{V}$ and i be a non-negative integer. If $i = 1$, we assume that A is a Dedekind domain. Then, the diagram*

$$\begin{tikzcd} K_i(X) \arrow[r, "(-1)^j c_{ét}^{j,2j-i}"] \arrow[d] & H^{2j-i}_{\mathrm{cont}}(X, \mathbf{Z}_l(j)) \arrow[d] \\ K_i(X, \mathbf{Z}_l) \arrow[r, "c_{\mathrm{So}}^{j,2j-i}"] & H^{2j-i}_{ét}(X, \mathbf{Z}_l(j)) \end{tikzcd}$$

commutes, where the vertical arrows are the canonical maps.

Theorem 10.2. *With notations as above, Soulé's Chern class map coincides with the composite*

$$K_i(X) \xrightarrow{(-1)^j c_{\mathrm{Le}}^{j,2j-i}} H^{2j-i}_{\mathrm{Le}}(X, \mathbf{Z}(j)) \to H^{2j-i}_{\mathrm{cont}}(X, \mathbf{Z}_l(j)) \to H^{2j-i}_{ét}(X, \mathbf{Z}_l(j)),$$

where the middle map is induced by the realization functor $R_{ét}$ for the l-adic étale cohomology theory. After tensoring with $\mathbf{Q}$, the same property holds for the Chern characters.

Proof. It follows from the above proposition and Remark 8.3. $\square$

Let $\mathcal{H}_{ét,l^r}$ denote the étale cohomology theory with coefficients in $\mathbf{Z}/l^r$ in Example 4.2(2). Then, the diagram

$$\begin{tikzcd} \mathcal{V}^{\mathrm{op}} \arrow[r, "\Gamma_{ét}"] \arrow[d, equal] & C^+\mathrm{Sh}_{\text{ét}}^{\mathbf{Z}/l^*}(S) \arrow[r, "F_{ét}"] \arrow[d] & D^+\mathrm{Sh}_{\text{ét}}^{\mathbf{Z}/l^*}(S) \arrow[d] \\ \mathcal{V}^{\mathrm{op}} \arrow[r, "\Gamma_{ét,l^r}"] & C^+\mathrm{Sh}_{\text{ét}}^{\mathbf{Z}/l^r}(S) \arrow[r, "F_{ét,l^r}"] & D^+\mathrm{Sh}_{\text{ét}}^{\mathbf{Z}/l^r}(S) \end{tikzcd}$$

commutes. Hence, the universal Chern class $c_{ét}$ for the abstract cohomology theory $\mathcal{H}_{ét}$ gives us a one $c_{ét,l^r}$ for $\mathcal{H}_{ét,l^r}$. Proposition 10.1 can be deduced from the one below.

Proposition 10.3. *With notations as above, let $c_{ét,l^r}^{j,2j-i}$ denote the Chern class map for the abstract cohomology theory $\mathcal{H}_{ét,l^r}$ with the universal Chern class $c_{ét,l^r}$. Then, the diagram*

$$\begin{tikzcd} K_i(X) \arrow[r, "(-1)^j c_{ét,l^r}^{j,2j-i}"] \arrow[d] & H^{2j-i}_{ét}(X, \mu_{l^r}^{\otimes j}) \arrow[d, equal] \\ K_i(X, \mathbf{Z}/l^r) \arrow[r, "\bar{c}_{\mathrm{So}}^{j,2j-1}"] & H^{2j-i}_{ét}(X, \mu_{l^r}^{\otimes j}) \end{tikzcd}$$

commutes, the left vertical arrow is the canonical map.

Lemma 10.4. *Proposition* 10.1 *is deduced from Proposition* 10.3.

Proof. By Proposition 7.1, the diagram

$$\begin{tikzcd} K_i(X) \arrow[r, "c_{ét}^{j,2j-i}"] \arrow[d, equal] & H^{2j-i}_{\mathrm{cont}}(X, \mathbf{Z}_l(j)) \arrow[d] \\ K_i(X) \arrow[r, "\varprojlim_r c_{ét,l^r}^{j,2j-i}"] & H^{2j-i}_{ét}(X, \mathbf{Z}_l(j)) \end{tikzcd}$$

commutes. On the other hand, taking the limit of the diagram in Proposition 10.3, we have a commutative one

$$\begin{array}{ccc} K_i(X) & \xrightarrow{(-1)^j \varprojlim_r c^{j,2j-i}_{\acute{e}t,l^r}} & H^{2j-i}_{\acute{e}t}(X, \mathbf{Z}_l(j)) \\ \downarrow & & \| \\ K_i(X, \mathbf{Z}_l) & \xrightarrow{c^{j,2j-1}_{\mathrm{So}}} & H^{2j-i}_{\acute{e}t}(X, \mathbf{Z}_l(j)). \end{array}$$

Combining them tells us the desired commutativity. □

To prove Proposition 10.3, we first recall the construction of Soulé's Chern class map $\bar{c}^{j,2j-1}_{\mathrm{So}}$ with coefficients in $\mathbf{Z}/l^n$. For details, see [49, Section 2].

For a ring A, let G denote the general linear group scheme $GL_{N,A}$ of rank N over A. Then, for the identity map id as a representation of G, there exists the associated Chern class $c_j(\mathrm{id})$ in the equivariant étale cohomology $H^{2j}_{\acute{e}t}(\operatorname{Spec} A, G(A), \mu_{l^r}^{\otimes j})$ for $j \geq 0$ (cf. [29, (2.3)]), where G acts A trivially. For $i \in \mathbf{Z}$, using the map

$$\Phi\colon H^k_{\acute{e}t}(\operatorname{Spec} A, G(A), \mu_{l^r}^{\otimes j}) \to \mathbf{Ab}(H_i(G(A), \mathbf{Z}/l^r), H^{k-i}_{\acute{e}t}(\operatorname{Spec} A, \mu_{l^r}^{\otimes j})) \tag{10}$$

constructed in [49, Lemma 1], the Chern class $c_j(\mathrm{id})$ gives us a map

$$H_i(G(A), \mathbf{Z}/l^r) \to H^{2j-i}_{\acute{e}t}(\operatorname{Spec} A, \mu_{l^r}^{\otimes j}).$$

For $i \neq 1$, Soulé's Chern class map

$$\bar{c}^{j,2j-i}_{\mathrm{So}}\colon K_i(A, \mathbf{Z}/l^r) \to H^{2j-i}_{\acute{e}t}(\operatorname{Spec} A, \mu_{l^r}^{\otimes j})$$

with coefficients in $\mathbf{Z}/l^r$ is defined as follows:

We set $GL = \varinjlim_N GL_{N,A}$. If $i \geq 2$, the map is the composite with the Hurewicz map

$$\pi_i(BGL(A)^+, \mathbf{Z}/l^r) \to H_i(BGL(A)^+, \mathbf{Z}/l^r) = H_i(GL(A), \mathbf{Z}/l^r)$$

with coefficients in $\mathbf{Z}/l^r$ (cf. [49, II.2.2]).

$\bar{c}^{j,2j}_{\mathrm{So}}$ is the map which makes the diagram

$$\begin{array}{ccccc} K_0(A) & \longrightarrow & H_0(GL(A), \mathbf{Z}) & \longrightarrow & H_0(GL(A), \mathbf{Z}/l^r) \\ \downarrow & & & & \downarrow \Phi(c_j(\mathrm{id})) \\ K_0(A)/l^r K_0(A) & = & K_0(A, \mathbf{Z}/l^r) & \xrightarrow{\bar{c}^{j,2j}_{\mathrm{So}}} & H^{2j}_{\acute{e}t}(\operatorname{Spec} A, \mu_{l^r}^{\otimes j}) \end{array}$$

commutative.

If $i = 1$, for only a Dedekind ring A, the Chern class map

$$\bar{c}^{j,2j-1}_{\mathrm{So}}\colon K_1(A, \mathbf{Z}/l^r) \to H^{2j-1}_{\acute{e}t}(\operatorname{Spec} A, \mu_{l^r}^{\otimes j})$$

with coefficients in $\mathbf{Z}/l^r$ was constructed ad hoc (cf. [49, IV.1.3]).

To compare Soulé's Chern class map $\bar{c}^{j,2j-i}_{\mathrm{So}}$ with ours, we will use the following lemmas.

Lemma 10.5. *With notations as above, the diagram*

$$\begin{array}{ccccc} K_i(\operatorname{Spec} A) & \xrightarrow{(1)} & H_i(GL(A), \mathbf{Z}) & \longrightarrow & H_i(GL(A), \mathbf{Z}/l^r) \\ \| & & & & \downarrow \Phi(c_j(\mathrm{id})) \\ K_i(A) & \longrightarrow & K_i(A, \mathbf{Z}/l^r) & \xrightarrow{\bar{c}^{j,2j-i}_{\mathrm{So}}} & H^{2j-i}_{\acute{e}t}(\operatorname{Spec} A, \mu_{l^r}^{\otimes j}) \end{array}$$

commutes.

Proof. It follows from their constructions. □

Lemma 10.6. *The canonical map $H^{2j}_{\acute{e}t}(BG,\mu_{l^r}^{\otimes j}) \to H^{2j}_{\acute{e}t}(\operatorname{Spec} A, G(A), \mu_{l^r}^{\otimes j})$ constructed in* [27, p. 221] *sends the j-th universal Chern class $c^j_{\acute{e}t}$ to the Chern class $(-1)^j c_j(\mathrm{id})$ above.*

Proof. Comparing the constructions of these universal Chern classes implies this lemma. They were constructed by the projective bundle formulas. ☐

We have the isomorphism

$$H^k_{\acute{e}t}(BG,\mu_{l^r}^{\otimes j}) \cong D(\mathcal{V})(N\mathbf{Z}\mathcal{V}(-,BG)_*, R\epsilon_*\mu_{l^r}^{\otimes j}[k])$$

in Lemma 9.5, where $R\epsilon_*$ is the right derived functor associated with the push-forward to Zariski sheaves. Via the identification

$$H_i(G(A),\mathbf{Z}) = H^{-i}(N\mathbf{Z}\mathcal{V}(\operatorname{Spec} A, BG)_*),$$

taking the cohomology groups gives us a map

$$\Psi\colon H^k_{\acute{e}t}(BG,\mu_{l^r}^{\otimes j}) \to \mathbf{Ab}(H_i(G(A),\mathbf{Z}), H^{k-i}_{\acute{e}t}(\operatorname{Spec} A,\mu_{l^r}^{\otimes j})).$$

Lemma 10.7. *For $i \in \mathbf{Z}$, the diagram*

$$\begin{tikzcd} H^k_{\acute{e}t}(BG,\mu_{l^r}^{\otimes j}) \arrow[r, "\Psi"] \arrow[d, "10.6"'] & \mathbf{Ab}(H_i(G(A),\mathbf{Z}), H^{k-i}_{\acute{e}t}(\operatorname{Spec} A,\mu_{l^r}^{\otimes j})) \\ H^k_{\acute{e}t}(\operatorname{Spec} A, G(A),\mu_{l^r}^{\otimes j}) \arrow[r, "\Phi"] & \mathbf{Ab}(H_i(G(A),\mathbf{Z}/l^r), H^{k-i}_{\acute{e}t}(\operatorname{Spec} A,\mu_{l^r}^{\otimes j})) \arrow[u] \end{tikzcd}$$

commutes, where the right vertical arrow is the canonical map and Φ is the map (10)*.*

Proof. Comparing the construction of the map Φ with that of Ψ tells us this lemma. We note that, for a ring R,

$$H_i(G(A),R) = H^{-i}(N\mathbf{Z}\mathcal{V}(\operatorname{Spec} A, BG)_* \otimes R)$$

(cf. [41, Part I, Appendix B, 1.1]). ☐

Proof of Proposition 10.3. By Lemma 10.7, the maps Φ and Ψ give us a commutative diagram

$$\begin{tikzcd} H^k_{\acute{e}t}(BG,\mu_{l^r}^{\otimes j}) \otimes H_i(G(A),\mathbf{Z}) \arrow[r, "\Psi"] \arrow[d] & H^{k-i}_{\acute{e}t}(\operatorname{Spec} A,\mu_{l^r}^{\otimes j}) \arrow[d, equal] \\ H^k_{\acute{e}t}(\operatorname{Spec} A, G(A),\mu_{l^r}^{\otimes j}) \otimes H_i(G(A),\mathbf{Z}/l^r) \arrow[r, "\Phi"] & H^{k-i}_{\acute{e}t}(\operatorname{Spec} A,\mu_{l^r}^{\otimes j}). \end{tikzcd}$$

that is natural in the rank N of the group $G = GL_{N,A}$. Taking the colimit with respect to N and using Lemma 10.6, we have a commutative diagram

$$\begin{tikzcd} H_i(GL(A),\mathbf{Z}) \arrow[r, "\Psi(c^j_{\acute{e}t}\otimes -)"] \arrow[d] & H^{j-i}_{\acute{e}t}(\operatorname{Spec} A,\mu_{l^r}^{\otimes j}) \arrow[d, equal] \\ H_i(GL(A),\mathbf{Z}/l^r) \arrow[r, "(-1)^j\Phi(c_j(\mathrm{id})\otimes -)"] & H^{j-i}_{\acute{e}t}(\operatorname{Spec} A,\mu_{l^r}^{\otimes j}). \end{tikzcd}$$

Composing the Hurewicz map (1), $K_i(A) \to H_i(GL(A),\mathbf{Z})$, makes the top arrow in this diagram our Chern class map $c^{j,2j-i}_{\text{ét},l^r}$. Hence, by Lemma 10.5, we have the desired commutativity. ☐

We close this section with an application to the category of mixed Tate motives. For a finite extension field K over $\mathbf{Q}$, let $MT(K)$ denote the category of mixed Tate motives over K (cf. [40, Section 4]). This is a full admissible abelian tensor subcategory of $D^b_{\mathrm{mot}\mathfrak{sh}}(\mathbf{Sm}_K)_{\mathbf{Q}}$. Hence, restricting the rational l-adic realization functor

$$R_{\acute{e}t,\mathbf{Q}}\colon D^b_{\mathrm{mot}\mathfrak{sh}}(\mathbf{Sm}_K)_{\mathbf{Q}} \to D^+\mathrm{Sh}^{\mathbf{Z}/l^*}_{\text{ét}}(\operatorname{Spec} K)_{\mathbf{Q}}$$

to the subcategory $MT(K)$, we have an exact tensor functor

$$R_{\acute{e}t,\mathbf{Q}}\colon MT(K)_{\mathbf{Q}} \to \mathrm{Sh}_{\text{ét}}^{\mathbf{Z}/l^*}(\operatorname{Spec} K)_{\mathbf{Q}}.$$

We note that the domain $\mathrm{Sh}_{\text{ét}}^{\mathbf{Z}/l^*}(\operatorname{Spec} K)$ is canonically equivalent to the abelian tensor category of continuous $\mathbf{Z}_l[\mathrm{Gal}(\overline{K}/K)]$-modules (cf. [32, 9.1]).

Corollary 10.8. *Let l be an odd prime and K be a finite extension field of $\mathbf{Q}$. Then, the canonical extension of the rational l-adic realization functor*

$$R_{\acute{e}t,\mathbf{Q}}\colon MT(K)_{\mathbf{Q}} \to \mathrm{Sh}_{\acute{e}t}^{\mathbf{Z}/l^*}(\operatorname{Spec} K)_{\mathbf{Q}}$$

to the $\mathbf{Q}_l$-linear category $MT(K)_{\mathbf{Q}_l}$ of mixed Tate motives is fully faithful.

Proof. The proof is similar to the Hodge case (see [24, 2.14]). We need that the injectivity of l-adic realization on $\mathrm{Ext}^1_{MT(K)_{\mathbf{Q}_l}}(\mathbf{Z}(0), \mathbf{Z}(j))$. When $j = 1$, this follows from [21, 2.4]. Otherwise, the proposition below and Theorem 10.2 imply this via the identification

$$K_{2j-1}(K)^{(j)} \cong H^1(\operatorname{Spec} K, \mathbf{Q}(j)) = \mathrm{Ext}^1_{MT(K)_{\mathbf{Q}}}(\mathbf{Z}(0), \mathbf{Z}(j))$$

induced by Levine's Chern class map $ch^{1,j}_{\mathrm{Le}}$. □

Proposition 10.9. *With notations as above, Soulé's Chern class map*

$$c^{j,2j-i}_{\mathrm{So}}\colon K_i(K) \otimes \mathbf{Q}_l \to H^{2j-i}_{\acute{e}t}(\operatorname{Spec} K, \mathbf{Q}_l(j))$$

is an isomorphism if $2j - i = 1$ or 2, and $i > 1$.

Proof. We have a commutative diagram

$$\begin{array}{ccc} K_{2j-i}(\mathcal{O}_K) \otimes \mathbf{Q}_l & \longrightarrow & H^i_{\acute{e}t}(\operatorname{Spec} \mathcal{O}_K[1/l], \mathbf{Q}_l(j)) \\ \downarrow & & \downarrow \\ K_{2j-i}(K) \otimes \mathbf{Q}_l & \longrightarrow & H^i_{\acute{e}t}(\operatorname{Spec} K, \mathbf{Q}_l(j)), \end{array}$$

where the horizontal arrows are Soulé's Chern class maps. The vertical arrows in this diagram are isomorphisms by the localization sequences. Moreover, Soulé proved that the top Chern class map is an isomorphism (see [50, Theorem 1]). Hence, the remaining map is also an isomorphism. □

11. Huber's Chern Character and Besser's Regulator

Huber [32] constructed a Chern character to the absolute cohomology of the cohomology theory which takes values in the category of mixed realizations. Using the method of her construction, Besser [11] constructed the rigid syntomic regulator as a Chern character which is a p-adic analog of Beilinson's regulator. In this section, we compare these maps with our Chern characters. We first define an abstract cohomology theory associated with Huber's which takes values in the category of mixed realizations.

For a field k of finite type over $\mathbf{Q}$, we define the *mixed realization cohomology theory* $\mathcal{H}_{\mathcal{MR}}$ on the category $\mathbf{Sm}_k$ of smooth and quasi-projective schemes as follows:

- $(\mathcal{C}_{\mathcal{MR}}, \mathcal{T}_{\mathcal{MR}}, F_{\mathcal{MR}})$ is associated with the abelian category $\mathcal{MR}$ defined in [32, 11.1.1] (cf. Remark 6.3).
- $\Gamma_{\mathcal{MR}}$ is the functor constructed in [32, 11.2].

We note that the difference between $C^+_{MAH,k,\mathbf{Q}}$ and $C^+_{\mathrm{dg}}\mathcal{MR}$ is a bit (cf. [32, 11.1], [41, p. 285]).

Proposition 11.1. *Let $c_{\mathcal{MR}}$ denote the universal Chern class for the mixed realization cohomology theory $\mathcal{H}_{\mathcal{MR}}$ constructed in* [32, Section 17]*. Then, our Chern class map $c^{j,2j-i}_{\mathcal{MR}}$ for the pair $(\mathcal{H}_{\mathcal{MR}}, c_{\mathcal{MR}})$ coincides with Huber's Chern class map $c^{j,2j-i}_{Hu}$ defined in* [32, 18.2.6] *up to sign. Especially, the associated Chern characters coincides up to sign after tensoring with* $\mathbf{Q}$.

Proof. As already mentioned, the construction of our Chern class map is a generalization of Huber's. The difference is only the construction of a Hurewicz map

$$K_i(X) \to \varinjlim_U H^{-i}(\mathrm{Tot}\, N\mathbf{Z}\mathcal{V}(U, BGL)^*_*)$$

(see [32, 18.2.4]) and a dg structure on the category $C\mathcal{A}$ of complexes in an additive category $\mathcal{A}$ (see [32, 2.2.7]). We remark that their total complex functors are the same (compare [32, 2.2.3] and [41, Part II, Chapter II, 1.2.9]). To construct a Hurewicz map which she used, the Bousfield-Kan completion $\mathbf{Z}_\infty$ (cf. [14, Chapter 1, Section 4]) was used instead of the +-construction. However, the simplicial presheaf BGL^+ is canonically weak homotopy equivalent to $\mathbf{Z}_\infty BGL$ via $\mathbf{Z}_\infty BGL^+$ (cf. [28, 2.16]). The difference The dg structure becomes equivalent to the other by multiplying $(-1)^{\sum_{n=0}^{i-1} n}$ on the group of maps of degree i, □

In a manner similar to Huber's construction, Besser constructed a Chern class map to the rigid syntomic cohomology and defined the rigid syntomic regulator as the associated Chern character (cf. [11, Section 7]). We note that the $\mathcal{H}$-cohomology of p-adic Hodge cohomology theory is Besser's rigid syntomic cohomology (see Proposition 5.17).

Remark 11.2. We note that the definition of the associated Chern character in [11, 7.6] is incorrect, because such a map is not multiplicative (cf. Remark 6.10). The correct definition is

$$ch = \sum_{j \geq 1} \frac{(-1)^{j-1}}{(j-1)!} c^i_j \quad (+\ \mathrm{Rank}\ \text{ if } i = j = 0).$$

Proposition 11.3. *Besser's rigid syntomic regulator whose range is the rigid syntomic cohomology group $H^i_{\mathrm{syn}}(X, K(j))$ coincides with our Chern character for the p-adic Hodge cohomology theory in Theorem* 5.18 *with the universal Chern class c_{Be} constructed in* [11, 7.4] *up to sign.*

Proof. Besser's regulator is a Chern character that is constructed following the method of Huber. On the same reason as the proposition above, these maps coincide up to sign. □

12. Gillet's and Asakura-Sato's Chern character

We close this paper with comparing our Chern characters with Gillet's [27] and with Asakura-Sato's [1].

For a $\mathbf{Z}$-graded bounded below complex $\mathit{\Gamma} = \{\mathit{\Gamma}(j)\}_{j \in \mathbf{Z}}$ of abelian Zariski sheaves on $\mathcal{V}$, we define an associated abstract cohomology theory $\mathcal{H}_\mathit{\Gamma}$ as follows:

- The triple $(\mathcal{C}, \mathcal{T}, F)$ is associated with the abelian category $\mathbf{Ab}$ of abelian groups (cf. Remark 6.3).
- The functor $\Gamma(j) \colon \mathcal{V}^{\mathrm{op}} \to C^+\mathbf{Ab}$ is the composite of the global sections of the Godement resolution of $\mathit{\Gamma}(j)$ with the total complex functor,

$$\mathcal{V}^{\mathrm{op}} \xrightarrow{\mathit{\Gamma}(j)} C^+\mathrm{Sh}^{\mathbf{Z}}_{\mathrm{Zar}}(X) \to C^+C^+\mathrm{Sh}^{\mathbf{Z}}_{\mathrm{Zar}}(X) \xrightarrow{H^0} C^+C^+\mathbf{Ab} \xrightarrow{\mathrm{Tot}} C^+\mathbf{Ab}.$$

We assume that this graded complex $\mathit{\Gamma}$ of sheaves satisfies one of the following conditions:

- $\mathit{\Gamma}$ is extended to a twisted duality theory on $\mathcal{V}$ with $d = 2$ [27, 1.2].
- $\mathit{\Gamma}$ is extended to an admissible cohomology theory on $\mathcal{V}$ [1, 2.5].

Then, the assumption gives us the universal Chern classes c_Γ in $\prod_j \varprojlim_N H^{2j}(BGL_{N,S}, \Gamma(j))$ (see [27, 2.8] or [1, 4.1]) and the Chern character

$$ch_\Gamma^{i,j}\colon K_i(X) \to H^{2j-i}(X, \Gamma(j)) \otimes \mathbf{Q}$$

(see [27, Section 2] or [1, Section 6]).

Remark 12.1. (1) Suppose that Γ equips structures of a twisted duality theory and an admissible cohomology theory simultaneously. Then, the associated universal Chern classes c_Γ and the Chern characters $ch_\Gamma^{i,j}$ coincide, because the method of Asakura and Sato's construction is the same to Gillet's.

(2) A twisted duality theory is almost an admissible cohomology theory. The bit difference is the axiom about the existence of a first Chern class map (compare [27, 1.2(xi)] and [1, 2.5(1)], see also [1, 2.6]).

Proposition 12.2. *With notations as above, the Chern character $ch_\Gamma^{i,j}$ coincides with our Chern character for $(\mathcal{H}_\Gamma, c_\Gamma)$.*

Proof. For a scheme X in $\mathcal{V}$, they constructed a map

$$H^{2j}(BGL_{N,X}, \Gamma(j)) \to \mathbf{Ab}(K_i(X), H^{2j-i}(X, \Gamma(j)))$$

using the notion of the homotopy category of simplicial sheaves (see [27, Section 2], [1, Section 5]). The pull-back of the universal Chern class c_Γ thus gives us the Chern class map $K_i(X) \to H^{2j-i}(X, \Gamma(j))$. $ch_\Gamma^{i,j}$ is the associated Chern character (cf. [27, 2.34], [1, 5.6]). Hence, it is enough to show that the diagram

$$\begin{array}{ccc} H^{2j}(BGL_{N,X}, \Gamma(j)) & \longrightarrow & \mathbf{Ab}(K_i(X), H^{2j-i}(X, \Gamma(j))) \\ \uparrow & & \| \\ H^{2j}(BGL_{N,S}, \Gamma(j)) & \longrightarrow & \mathbf{Ab}(K_i(X), H^{2j-i}_{\mathcal{H}_\Gamma}(X, j)) \end{array}$$

commutes. This follows from comparing their constructions. ☐

By Theorem 3.5, the geometric cohomology theory $\mathcal{H}_\Gamma$ associated with a twisted duality theory Γ on $\mathcal{V}$ is extended to the realization functor $R_\Gamma\colon D^b_{\mathrm{mot}\mathfrak{sh}}(\mathcal{V}) \to D^+\mathbf{Ab}$. Hence, Proposition 7.9 implies the proposition below.

Proposition 12.3. *The composite*

$$K_i(X) \xrightarrow{ch_{\mathrm{Le}}^{i,j}\otimes 1} H^{2j-i}_{\mathrm{Le}}(X, \mathbf{Q}(j)) \xrightarrow{R_\Gamma} H^{2j-i}_{\mathcal{H}_\Gamma}(X, j) \otimes \mathbf{Q} = H^{2j-i}(X, \Gamma(j)) \otimes \mathbf{Q}$$

coincides with Gillet's Chern character $ch_\Gamma^{i,j}$ multiplied by $(-1)^j$.

Proof. Let c_{R_Γ} denote the universal Chern class associated with the realization functor R_Γ (cf. Remark 7.7). Then, we have an equality

$$(-1)^j c^j_{R_\Gamma} = c^j_\Gamma \quad \text{in} \quad \varprojlim_N H^{2j}(BGL_{N,S}, \Gamma(j))$$

by construction. Hence, it follows from the above proposition and Theorem 8.3. ☐

13. A motivic interpretation of Besser's regulator

We now do not have the realization functor associated with the p-adic Hodge cohomology theory that is constructed in Section 5. However, we can construct another p-adic Hodge realization functor on a full subcategory of mixed Tate motives by composing the p-adic étale realization functor with the functor D_{cris} that sends a crystalline representation to a weakly admissible filtered ϕ-module. In this section, we give a motivic interpretation of Besser's regulator using this p-adic Hodge realization functor. We refer to [26] for the basic notations and properties in the theory of p-adic representation.

Let K be a finite extension of $\mathbf{Q}$ and let v be a place of K on the prime number p. Let K_v denote the completion of K with respect to v and let G_{K_v} denote the absolute Galois group of K_v. Furthermore, let $\mathcal{O}$ denote the ring of integers of K. In Section 10, we constructed the rational p-adic étale realization functor

$$R_{\acute{e}t,\mathbf{Q}} \colon MT(K)_{\mathbf{Q}} \to \mathrm{Sh}_{\text{ét}}^{\mathbf{Z}/l^*}(\operatorname{Spec} K)_{\mathbf{Q}}$$

on the $\mathbf{Q}$-linear category $MT(K)_{\mathbf{Q}}$ of mixed Tate motives over K (see the sentence before Corollary 10.8). Taking the associated Galois representation, we have a functor

$$MT(\mathcal{O}_{(v)})_{\mathbf{Q}} \to \mathrm{Rep}_{\mathrm{cris}}(G_{K_v})$$

to the category $\mathrm{Rep}_{\mathrm{cris}}(G_{K_v})$ of crystalline representations of G_{K_v}, where $MT(\mathcal{O}_{(v)})_{\mathbf{Q}}$ is the full subcategory of mixed Tate motives which are unramified at v (cf. [53, 4.2]). Composing the equivalence

$$D_{\mathrm{cris}} \colon \mathrm{Rep}_{\mathrm{cris}}(G_{K_v}) \to MF^f_{K_v}(\phi)$$

to the category $MF^f_{K_v}(\phi)$ of weakly admissible filtered ϕ-modules over K_v, we have a p-adic Hodge realization functor

$$R_{pH} \colon MT(\mathcal{O}_{(v)})_{\mathbf{Q}} \xrightarrow{R_{\acute{e}t,\mathbf{Q}}} \mathrm{Rep}_{\mathrm{cris}}(G_{K_v}) \xrightarrow{D_{\mathrm{cris}}} MF^f_{K_v}(\phi)$$

on $MT(\mathcal{O}_{(v)})_{\mathbf{Q}}$.

Using this p-adic Hodge realization functor R_{pH}, we can give a motivic interpretation of Besser's regulator as follows:

Theorem 13.1. *Let X be a smooth and projective scheme over the local ring $\mathcal{O}_{(v)}$ such that the $\mathbf{Z}_{X_K}(k)$ belongs to the category $MT(\mathcal{O}_{(v)})_{\mathbf{Q}}$ for some $k \in \mathbf{Z}$. Then, for $i \geq 0$ and $j \in \mathbf{Z}$, the map*

$$K_i(X) \otimes \mathbf{Q} \to K_i(X_{K_v}) \otimes \mathbf{Q} \to H^{2j-i}_{\mathrm{syn}}(X \otimes \mathcal{O}_v, K(j))$$

that is induced by the base change map and by the functor R_{pH} becomes equivalent to Besser's regulator $r^{i,j}_{\mathrm{Bes}}$ by multiplying $(-1)^j$, where $\mathcal{O}_v$ is the completion of $\mathcal{O}$ with respect to the place v.

Proof. Composing the representable functor $C^+_{\mathrm{dg}}\mathrm{Sh}_{\text{ét}}^{\mathbf{Z}/l^*}(\operatorname{Spec} K_v)(1,-)$ with the p-adic étale cohomology theory Γ in Example 4.2 (1), we have a twisted duality theory $\Gamma_{\text{ét}}$ on $\mathbf{Sm}_{K_v}$ in the sense of Gillet. Then, the corollary 9.10 of [11] means that the diagram

$$\begin{array}{ccc} K_i(X_{K_v}) \otimes \mathbf{Q} & \xrightarrow{ch^{i,j}_{\Gamma_{\text{ét}}}} & H^{2j-i}_{\mathrm{cont}}(X_{K_v}, \mathbf{Q}_p(j)) \\ \uparrow & & \downarrow \\ K_i(X \otimes \mathcal{O}_v) \otimes \mathbf{Q} & \xrightarrow{r^{i,j}_{\mathrm{Bes}}} & H^{2j-i}_{\mathrm{syn}}(X \otimes \mathcal{O}_v, K_v(j)) \end{array}$$

commutes, where the top horizontal arrow is Gillet's Chern character and the right vertical arrow is the canonical map. Since the $\mathcal{H}$-cohomology groups of Γ and $\Gamma_{\text{ét}}$ are coincide (cf. [41, Section V, 2.2.8]), our Chern character for them are also by Proposition 7.1. Hence, it follows from [11, Proposition 9.11] and Corollary 12.3. □

We close this thesis with giving an application of this Theorem. Yamashita [53] studied the p-adic Hodge realization of Huber-Wildeshaus' motivic polylogarithm class (that is constructed in [35]) via the realization functor

$$R_{pH} \colon MT(\mathcal{O}_{(v)})_{\mathbf{Q}} \to MF^f_K(\phi).$$

Combining his result with this theorem, we have an another proof of the corollary below which says that Besser's regulator sends Huber-Wildeshaus' polylogarithm class to a special value of Coleman's p-adic polylogarithm up to non-zero rational number.

Corollary 13.2. *Let N be a positive integer that is prime to the prime number p and v be a place of the N-th cyclotomic field $\mathbf{Q}(\mu_N)$ on p. Let $\mathcal{O}_{(v)}$ denote the ring of v-integers in $\mathbf{Q}(\mu_N)$ and $\mathcal{O}_v$ denote its completion. Furthermore, let K_v denote the field of fractions of $\mathcal{O}_v$. Then, for $j \geq 2$, the composite*

$$K_{2j-1}(\mathcal{O}_{(v)}) \otimes \mathbf{Q} \to K_{2j-1}(\mathcal{O}_v) \otimes \mathbf{Q} \xrightarrow{r^{1,j}_{\mathrm{Bes}}} H^1(\operatorname{Spec} \mathcal{O}_v, K_v(j)) \xrightarrow{\cong} K_v,$$

where the third arrow is the canonical isomorphism (*cf.* [53, p. 718])*, sends Huber-Wildeshaus' motivic polylogarithm class pol_ζ* (*see* [35, 9.4]) *to $j!\,\ell^{(p)}_j(\zeta)$ for an N-th root of unity ζ. Here, $\ell^{(p)}_j(\zeta)$ is the special value of Coleman's p-adic polylogarithm* (*see* [19, VI])

$$\ell^{(p)}_j(t) = \sum_{n \geq 1, (n,p)=1} \frac{t^n}{n^j}$$

at ζ.

Proof. It follows from [53, 4.10 and 4.11] and the theorem above. We remark that the coefficient

$$\left(1 - \frac{1}{p^n}\right)$$

in [53, 4.11] is incorrect. The correct coefficient is

$$\left(1 - \frac{\sigma}{p^n}\right),$$

where σ is the Frobenius automorphism. $\square$

Remark 13.3. By calculating the regulator explicitly, Besser and De Jeu already proved a result similar to this corollary that says Besser's regulator sends the element $[\zeta]_j$ in $K_{2j-1}(\mathcal{O}_{(v)}) \otimes \mathbf{Q}$ constructed by De Jeu [20] to a special value of p-adic polylogarithm up to some explicit constant (see [12, Theorem 1.12]). We can prove this corollary using their result. We remark that they used another identification of $H^1_{\mathrm{syn}}(\operatorname{Spec} \mathcal{O}_v, K_v(j))$ with K_v in their paper (see [12, 4.6]). Composing the automorphism

$$\frac{\sigma}{p^j} - \mathrm{id}$$

of K_v, their identification becomes equivalent to ours (see [53, 4.13]), where σ is the Frobenius automorphism.

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