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Construction of higher dimensional field theory with dynamical boundary conditions

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Abstract

In field theory defined on multiply connected manifolds, the boundary conditions imposed on fields have the arbitrariness due to the gauge symmetry in Lagrangian density. In the present study of field theory on extra dimensions, these boundary conditions are given by hand. This is called the arbitrariness problem. In this thesis we construct the model including the boundary condition dynamics in order to approach to the arbitrariness problem. We determine the physically realized boundary conditions from the dynamics. As a result we get the nontrivial restriction for the boundary conditions. Especially in the $SU(5)$ case, the symmetry breaking to the standard model gauge symmetry $SU(3) \times SU(2) \times U(1)$ is naturally realized, and the fermion matter content of the standard model can be also realized with a pair of Higgs fields.

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Contents

1 Introduction

The standard model(SM) is extremely successful theory to describe particle physics. In recent years, the Higgs boson has been discovered by LHC experiment[1, 2]. By this pleasing news, all SM particle were observed in collider experiments. However the SM has several problems which we should solve in the further. One of them is so-called "Higgs mass hierarchy problem"[3, 4]. In the SM, Higgs potential is given by

$$V = m_H^2|H|^2 + \lambda_H|H|^4, \tag{1.1}$$

where H is complex scalar field. The SM requires $m_H^2 < 0$ and $\lambda_H > 0$ so that H acquires the vacuum expectation value which lead to the spontaneous electro-weak symmetry breaking. We know that $\langle H \rangle$ must be electro-weak scale experimentally: $\langle H \rangle \sim \mathcal{O}(100)$[GeV]. Then, the problem occurs at quantum corrections. m_H^2 receives huge quantum corrections form the other SM particles. For example, let us consider a quantum correction from a Dirac fermion ψ_f, and suppose the Higgs field couples to ψ_f by the term $-\lambda_f H\bar{\psi}_f\psi_f$ in Lagrangian. Then we have the mass correction δm_H^2 from ψ_f at one loop level

$$\delta m_H^2 \sim -\frac{|\lambda_f|^2}{8\pi^2}\Lambda_{\mathrm{UV}}^2 + \cdots , \tag{1.2}$$

where Λ_{UV} denotes an ultraviolet momentum cutoff to regulate the divergent loop integral. This momentum scale should be the energy scale where a new physics comes in and alters the physical behaviors. The largest correction in (1.2) is given by the top quark and in this case $|\lambda_f| \approx 1$. Then if Λ_{UV} is order Plank scale $M_P \approx 2.8 \times 10^{18}$ [GeV], we need strict fine-tuning at about 30 order of magnitude in order that we get the vacuum expectation value $\langle H \rangle$ at electro-weak scale. Therefore, it is needed that we investigate the beyond standard model to solve the above unnatural fine-tuning problem.

It is known that the gauge-Higgs unification(GHU) model is a candidate of beyond standard model. GHU is defined on higher dimensional spacetime than normal four dimensional Minkowski spacetime. The higher dimensional spacetime was firstly discussed by Kaluza and Klein[5] to unify gravity and gauge interactions. After that, the study of field theories on higher dimensional spacetimes gives us rich insight for particle physics and has been of interest for a long time. For example, the higher dimensional theory gives a new direction to understand large hierarchies in Yukawa coupling and flavor mixing of SM fermions. These hierarchies can be naturally realized by dynamically localizing SM fermions at slightly distinct points in a extra dimensional direction[6]-[9]. The extra dimension can also play a crucial role to give phenomenologically acceptable supersymmetry(SUSY) breaking scenario. For this SUSY breaking via extra dimension, there are two famous proposals: anomaly mediation[10]-[13] and gaugino mediation[14]-[17] scenarios. In these proposals, visible supersymmetric matter sector and hidden SUSY breaking sector are located at different branes. In anomaly mediation scenario, gravity mediates SUSY breaking effect from the hidden sector to the visible matter sector. In gaugino mediation scenario, the superpartnars of the SM gauge field mediate the SUSY breaking effect. However in order to formulate the field theory in higher dimensional spacetime, there arises the chiral fermion problem. Generally in five dimensional spacetime two component Weyl fermion is no longer representation in Lorentz group. Therefore, five dimensional Lagrangian forbids chiral fermion unlike the SM matter content. In order to solve this chiral fermion problem, the orbifold extra dimension has been introduced and can achieve chiral fermion due to the fact that the right- and left-handed fermions are expanded as the different forms in KK-expansion. Moreover S^1/Z_2 orbifold extra dimension can give the natural explanation for large mass hierarchies among three family of quark and lepton in Randall-Sundrum model[18]. Field theory on orbifolds have been extensively applied to the constructions of realistic Grand unified theory(GUT) and electro-weak symmetry breaking(EWSB) scenario. $SU(5)$ GUT on S^1/Z_2 orbifold has been investigate in Ref [19]-[25]. In this model, one can get the natural solution for the doublet-triplet mass splitting problem. $SO(10)$ GUT on 5D or 6D orbifold was also discussed in Ref [26]-[32]. Moreover, $SO(11)$ GUT has been proposed by Yamatsu and Hosotani[33, 34].

The Hosotani mechanism[35, 36] can give the powerful tool to investigate the field theory defined on multiply connected space. He has revealed the relation between boundary conditions and the dynamics of Wilson line phases on multiply connected manifolds, and classified boundary conditions to equivalence classes. For investigating the physics of field theory on multiply connected manifolds, the dynamics of Wilson line phases, which are regarded as physical degrees of freedom, play a crucial role, and leads to a new symmetry breaking mechanism as an alternative to the Higgs mechanism. By this dynamics of Wilson line phases, one can unify gauge fields and Higgs scalar fields on multiply connected manifolds and it is called Gauge-Higgs Unification. In GHU, the mass terms of Higgs fields, which are part of gauge field, are forbidden by gauge symmetry and these mass terms can arise by quantum correction. Hatanaka, Inami and Lim pointed out that the gauge hierarchy problem can be solved in GHU scenario thanks to gauge symmetry[37]. Therefore, GHU is extensively applied to the construction of EWSB model. 5D GHU on S^1/Z_2 have been investigated for $SU(3)$ gauge group in Ref [38]-[41] and for $SO(5) \times U(1)$ gauge group in Ref [42]-[47]. 6D GHU was also investigated in Ref [48]-[50]. The other interesting approach for GHU is given in Ref [51]-[53]. Matsumoto and Sakamura gave the general analysis of GHU on T^2/Z_N by group theoretical analysis[54]. Haba, Harada, Hosotani and Kawamura have investigated the physics of $SU(N)$ gauge theory on S^1/Z_2 orbifold in detail[55, 56]. They classified the boundary conditions to equivalence classes completely for $SU(N)$ gauge group, and analyzed the dynamics of Wilson line phases in each equivalence class especially, for $SU(5)$ gauge theory.

However, for field theory on extra dimensions we also have several problems to solve. One of them is that the boundary conditions for extra dimensional directions which we impose on each field have the arbitrariness due to the gauge symmetry imposed on Lagrangian density and in present study of the field theories on extra dimensions these boundary conditions are given by hand. We refer this subtlety as the arbitrariness problem[57]. Therefore, the purpose of our research is to solve this arbitrariness problem for the boundary conditions. It means that we try to determine the boundary conditions imposed on fields from the theoretical consequence. For this goal, we need the model including the dynamic of boundary conditions. In order to construct the model including the dynamics of boundary conditions and determine the physically realized boundary conditions, we make the following two assumptions:

- The boundary condition dynamics is realized by path integral formula where the integration is defined by invariant measure.
- The equivalence class which has the lowest energy density among each equivalence class is physically chosen.

In this paper, we construct the model including the dynamics of boundary conditions for $SU(N)$ gauge theory on S^1/Z_2 orbifold extra dimension. In the model, we regard the boundary conditions as dynamical variables. Then, we need to analyze systematically all possible configurations of boundary conditions in one framework. We achieve it by using the matrix model analysis. By investigating the property of invariant measure in detail, we have revealed only restricted equivalence classes practically contribute to the partition function of system in the $SU(N)$ case although all possible boundary conditions are included in the integration[58] and we discuss that in section 4. For the further analysis, we use the second assumption. Then, we determine the physically realized boundary conditions by comparing the minimal values of effective potential among sets of boundary conditions which practically contribute to partition function. Unfortunately, for comparing the minimal values of effective potential, there arises ambiguity: The difference between the minimal energy densities in two equivalence classes may be divergent due to the contributions of gauge and scalar fields to the effective potentials in the non-supersymmetric case[56]. In order to avoid this ambiguity, we put a restriction on the matter content of scalar fields, and there is no ambiguity coming from gauge field contribution to effective potential in our model. In supersymmetric case, the energy difference of effective potentials in two equivalence classes becomes always finite. We discuss that in section 6 in detail.

The content of this paper is as follows: In section 2 we give the brief review for the Hosotani mechanism in multiply connected manifolds. In section 3 we give the basic knowledge of field theory on S^1/Z_2 extra dimension and classify the boundary conditions into equivalence classes. In section 4 we construct the model including the dynamics of boundary conditions for $SU(N)$ gauge theory on S^1/Z_2 orbifold extra dimension and investigate the nature of invariant measure. In section 5 we calculate the effective potential for $SU(5)$ gauge theory by background method. In section 6 we determine the physically realized boundary conditions from the dynamics for non-supersymmetric and supersymmetric case in each matter content. Finally section 7 is devoted to the conclusion and discussion.

2 The Hosotani mechanism on multiply connected manifold

In this section we review the Hosotani mechanism on multiply connected manifolds, he has revealed that in gauge theory on multiply connected manifold the eigenvalues of Wilson line along non-contractable loop become the dynamical variables on system and the boundary conditions imposed on each field can be classified into equivalence classes by the dynamics of Wilson line.

The following discussion is based on Ref[36]

We consider gauge theory defined on multiply connected manifold M_0 and suppose M_0 is a homogeneous manifold, namely every point $x \in M_0$ is equivalent to others. The Lagrangian density is given by

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}i\not{D}\psi \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu] \\ D_\mu\psi &= (\partial_\mu + igT^\alpha A^\alpha_\mu)\psi, \end{aligned} \tag{2.1}$$

where T^α denotes appropriate representation matrix of gauge group, and α denotes the index of generator for gauge group. For simplicity, we consider only massless fermions for matter fields. Since M_0 is multiply connected manifolds, non-contractable loops appear. We suppose there are n generators of non-contractable loops and represent the transitions along these loops by h_a $(a = 1 \sim n)$.

In general principle Lagrangian density must be single valued on M_0. It mean that the Lagrangian density must be a periodic function under loop translations along h_a $(a = 1 \sim n)$:

$$\mathcal{L}[h_a(x)] = \mathcal{L}[x], \quad (a = 1 \sim n). \tag{2.2}$$

In order to satisfy this requirement, appropriate boundary conditions must be imposed on fields. Under the Lagrangian density (2.1) the boundary conditions are generally represented by

$$\begin{aligned} &(U_a,\ \beta_a) \quad (a = 1 \sim n) \\ A_\mu[h_a(x)] &= U_a A_\mu[x] U_a^\dagger \\ \psi[h_a(x)] &= e^{i\beta_a} T_\psi[U_a]\psi[x], \end{aligned} \tag{2.3}$$

where U_a can be a element of gauge group, and $T_\psi[U_a]$ denotes appropriate representation matrix of U_a for fermion. $(U_a,\ \beta_a)$ must be independent on spacetime coordinates. It is required by the homogeneity of space. Therefore, one must give both Lagrangian density and boundary conditions in order to define the field theory on multiply connected manifolds.

Next, we show that the gauge invariance on system dose not equal to the gauge symmetry in Lagrangian density. Let us consider a gauge transformation $\Omega(x)$:

$$\begin{aligned} A_\mu \to A'_\mu(x) &= \Omega(x) A_\mu \Omega^\dagger(x) - \frac{i}{g}\Omega(x)\partial_\mu\Omega^\dagger(x) \\ \psi(x) \to \psi'(x) &= T_\psi[\Omega(x)]\psi(x). \end{aligned} \tag{2.4}$$

For new fields A'_μ, ψ', the boundary conditions change, for example for A'_μ the relation (2.3) is transformed into

$$\begin{aligned} A'_\mu[h_a(x)] = U'_a A'_\mu U'^\dagger_a + \frac{i}{g}\Omega[h_a(x)]U_a(\partial_\mu \Omega^\dagger[x])U'^\dagger_a \\ - \frac{i}{g}\Omega[h_a(x)](\partial_\mu \Omega^\dagger[h_a(x)]), \end{aligned} \tag{2.5}$$

where

$$U'_a = \Omega[h_a x]U_a \Omega^\dagger[x]. \tag{2.6}$$

Therefore, general gauge transformation can not be regarded as gauge invariance on system since it spoils the relation (2.3).

If the gauge transformation $\Omega[x]$ satisfies the condition

$$\Omega[h_a(x)] = U_a \Omega[x] U_a^\dagger, \tag{2.7}$$

$\Omega[x]$ remains the boundary conditions invariant, so it is the residual gauge invariance on system. Ω is not single valued if $[\Omega[x], U_a] \neq 0$. Then the residual global gauge symmetry is given by

$$[\Omega^{\text{global}}, U_a] = 0 \qquad (a = 1 \sim n). \tag{2.8}$$

It means that the residual global gauge symmetry is generated by gauge subgroup whose generator commute with U_a. The global gauge transformation which satisfies the relation (2.8) is called the symmetry of boundary conditions. It is noted that the physical symmetry on system can be different from the symmetry of boundary conditions due to the dynamics of Wilson line phases.

For the gauge field on multiply connected manifold, the non-vanishing expectation value can arise even if the field strength has the vanishing value. This is caused by the fact gauge transformations which remain the system invariant are subject to the condition (2.7). The condition $\langle F_{\mu\nu}\rangle = 0$ leads to

$$\langle A_\mu \rangle = -\frac{i}{g} V^\dagger \partial_\mu V. \tag{2.9}$$

For the boundary condition (2.3), (2.9) implies

$$V^\dagger[h_a(x)]\partial_\mu V[h_a(x)]U_a = U_a V^\dagger[x]\partial_\mu V[x]. \tag{2.10}$$

If the condition $CV[h_a(x)] = U_a CV[x]U_a^\dagger$ is not satisfied for some constant unitary matrix C, there is no gauge transformation Ω which transforms $\langle A_\mu \rangle$ to 0 because Ω is restricted by the condition (2.7). In this case, we must regard $\langle A_\mu \rangle$ as physically different configuration from $\langle A_\mu \rangle = 0$ and non-zero expectation value leads to the spontaneous gauge symmetry breaking. The expectation value $\langle A_\mu \rangle$ is determined by dynamics on system, not arbitrary, under given boundary conditions. Since $CV(x)$ and $V(x)$ give the identical value $\langle A_\mu \rangle$, so $V(x)$ in (2.9) is not unique. Then $V(x)$ depends on the boundary conditions $(U_a,\ \beta_a)$ and matter content on system through the dynamics of Wilson line phases and can be determined up to a constant unitary matrix.

The different sets of boundary conditions (U_a, β_a) and (U'_a, β_a) can lead to the same physical content if the two sets are connected by the dynamics of Wilson line phase. Under the general gauge transformation (2.4) the boundary conditions (U_a, β_a) are changed to (2.5), but if gauge transformation Ω satisfies the condition

$$\Omega^\dagger[h_a(x)]\partial_\mu \Omega[h_a(x)]U_a = U_a \Omega^\dagger[x]\partial_\mu \Omega[x], \tag{2.11}$$

then gauge transformed fields A'_μ and ψ' again satisfy the boundary conditions (2.3) where the set (U_a, β_a) is replaced by the new boundary condition set (U'_a, β_a) defined by (2.6). The homogeneity of boundary condition, namely x-independence of U'_a is indeed preserved under the condition (2.11). This transformation is not the gauge invariance on system and transforms to the different system which is related to the original system by the dynamics of Wilson line phases. If the two sets of boundary conditions (U_a, β_a) and (U'_a, β_a) are related by gauge transformation which satisfies the condition (2.11) these two sets are called equivalent:

$$(U_a,\ \beta_a) \sim (U'_a\ \beta'_a), \tag{2.12}$$

and the boundary condition sets related by equivalence relation (2.12) lead to the same physical content: all physical observables are identical in $(U_a,\ \beta_a)$ and $(U'_a,\ \beta'_a)$. Then we can classify the boundary conditions into the equivalence classes by using the relation (2.12). This fact plays crucial role to analyzing the field theory defined on multiply connected manifolds.

Let us consider the physical symmetry on system by using the equivalence relation (2.12). In the given boundary conditions (U_a, β_a) the gauge field can develop non-zero vacuum expectation value in (2.9) for the case $\langle F_{\mu\nu} \rangle = 0$.

Then under the gauge transformation $\Omega[x] = V[x]$ the boundary conditions change to

$$U_a \to U'_a = V[h_a(x)]U_a V^\dagger[x] \equiv U_a^{\rm sym}. \tag{2.13}$$

In boundary conditions (U'_a, β_a) in (2.13) $\langle A'_\mu \rangle = 0$ and we can find that $U_a^{\rm sym}$ is independent on x due to (2.10). Hence we get the equivalence relation

$$\big(U_a,\ \beta_a; \langle A_\mu \rangle = -\frac{i}{g}V^\dagger \partial_\mu V\big) \sim (U_a^{\rm sym},\ \beta_a; \langle A_\mu \rangle = 0). \tag{2.14}$$

$U_a^{\rm sym}$ is not unique since $V(x)$ has the arbitrariness for constant unitary matrix, but the eigenvalues of $U_a^{\rm sym}$ can be uniquely determined. These eigenvalues are important variable for parameterizing the physics on system. In the boundary conditions $(U_a^{\rm sym}, \beta_a)$ the physical global symmetry equals to the symmetry of boundary conditions since $\langle A'_\mu \rangle = 0$. Therefore, the physical global gauge symmetry is given by

$$[\Omega^{\rm global},\ U_a^{\rm sym}] = 0 \quad (a = 1 \sim n) \tag{2.15}$$

from (2.8). For the realistic model we consider the case that M_0 is factorized into the product of four dimensional Minkowski spacetime M^4 and a multiply connected space N. We assume that the size of N is much smaller than the energy scale which is experimentally observed and denote the spacetime coordinates by $x \in M^4$ and $y \in N$ respectively. In this case, the physical gauge invariance is given by gauge transformation $\Omega[y, x]$ which satisfies the condition $\Omega[h_a(y), x] = U^{\rm sym}\Omega[y, x](U^{\rm sym})^\dagger$. In low energy limit the gauge potential Ω is independent on $y \in N$ and the physical symmetry is given by

$$\Omega(x)U_a^{\rm sym} = U_a^{\rm sym}\Omega(x). \tag{2.16}$$

It means that the physical symmetry at low energy effective theory corresponds to the gauge subgroup whose generator commutes with $U_a^{\rm sym}$ just as the global gauge symmetry.

Next we will show that the eigenvalues of $U_a^{\rm sym}$ can be related to Wilson line along non-contractable loop with boundary conditions U_a in (2.12). The Wilson line phases are defined by the phases of $W_a[x, C]$, where

$$W_a[x; C] = W_a^{(0)}[x; C]U_a = \mathcal{P}\exp\left\{ -ig\int_{h_a(x)}^{x} A_\mu dx^\mu \right\} U_a. \tag{2.17}$$

x again denotes the coordinate of general multiply connected manifold M_0 and $W_a^{(0)}[x; C]$ is the Wilson line along non-contractable loop C. Then,

$W_a[x, C]$ transforms in covariant form under gauge transformation due to the condition (2.7). So the Wilson line phases are gauge invariant quantities. For the field configuration (2.9), $W_a[x; C]$ is given by

$$W_a[x; C] = V^\dagger[x] V[h_a(x)] U_a. \tag{2.18}$$

It follows from the fact that $W_a[x; C]$ remains invariant under a continuous deformation of the path C due to $F_{\mu\nu} = 0$. Under the boundary changing gauge transformation $\Omega(x) = V(x)$, The quantity $W_a[x; C]$ is replaced with $W'_a[x; C] = U_a^{\rm sym}$ and

$$U_a^{\rm sym} = V[h_a(x)] U_a V^\dagger[x] = V[x] W_a[x; C] V^\dagger[x]. \tag{2.19}$$

That is, $U_a^{\rm sym}$ has the identical eigenvalues with $W_a[x; C]$. $U_a^{\rm sym}$ is independent on x, so the eigenvalues of $W_a[x; C]$ are also x-independent. In general, these phase factors must be regarded as dynamical degree of freedom for field theory defined on multiply connected manifold, because these can not be gauged away once boundary conditions are given. The Wilson line phases are determined by calculating effective potential, and given by field configuration which minimizes the value of effective potential.

3 The field theory on S^1/Z_2 orbifold extra dimension

In this section, we give the basic knowledges of field theory on $M^4 \times S^1/Z_2$. Suppose x and y denote the coordinates of M^4 and S^1/Z_2 respectively, and the radius of S^1 is R. The S^1/Z_2 orbifold is obtained by identifying two points on S^1 by Z_2 symmetry. In other words, we identify the coordinates $(x, y + 2\pi R)$ and $(x, -y)$ with (x, y) by S^1 compactification and orbifold condition:$(x, y) \sim (x, y + 2\pi R) \sim (x, -y)$. The discussion follows Ref[55, 56]

3.1 The boundary conditions on S^1/Z_2 orbifold

As general principle, Lagrangian density has to single-valued on $M^4 \times S^1/Z_2$ as I mentioned in section 2. Then Lagrangian density must satisfy the conditions

$$\begin{aligned} \mathcal{L}(x, y + 2\pi R) &= \mathcal{L}(x, y) \\ \mathcal{L}(x, -y) &= \mathcal{L}(x, y), \end{aligned} \tag{3.1}$$

the second condition is by orbifold Z_2 symmetry. Therefore, we must impose the boundary conditions on each field on system in order to satisfy the conditions (3.1). It is known that in S^1/Z_2 case, these boundary conditions are realized by two parity transformations around $y = 0$ and $y = \pi R$ respectively. We denote the boundary conditions for these parity transformations by P_0 and P_1. For gauge field A_M ($M = 0 \sim 3, 5$) they are given by

$$\begin{pmatrix} A_\mu(x,-y) \\ A_y(x,-y) \end{pmatrix} = P_0 \begin{pmatrix} A_\mu(x,y) \\ -A_y(x,y) \end{pmatrix} P_0^\dagger \tag{3.2}$$

and

$$\begin{pmatrix} A_\mu(x,\pi R-y) \\ A_y(x,\pi R-y) \end{pmatrix} = P_1 \begin{pmatrix} A_\mu(x,\pi R+y) \\ -A_y(x,\pi R+y) \end{pmatrix} P_1^\dagger. \tag{3.3}$$

We should note that for A_y component, the opposite sign relative to A_μ is required since the derivative ∂_y gives extra minus sign under these transformations. The repeated parity transformations must equal to the identical transformation, so the condition

$$P_0^2 = P_1^2 = 1 \tag{3.4}$$

must be imposed. Note that the boundary conditions P_0 and P_1 have arbitrariness due to the gauge symmetry in Lagrangian density. In general, P_0 and P_1 can be the elements of gauge group as long as the condition (3.4) is satisfied. For $SU(N)$ case, P_0 and P_1 are the element of $U(N)$. The arbitrariness of boundary conditions corresponds to the possible choices to construct the different model, leading to the different symmetry breaking patterns. The boundary condition for S^1 loop transition can be represented by P_0 and P_1. For gauge field, it is given by

$$A_M(x, y + 2\pi R) = U A_M(x,y) U^\dagger, \tag{3.5}$$

where $U = P_1 P_0$. U is called S^1 boundary condition.

For scalar field, the boundary conditions are specified by

$$\begin{aligned} \phi(x,-y) &= \pm T_\phi[P_0]\phi(x,y) \\ \phi(x,\pi R-y) &= \pm e^{i\pi\beta_\phi} T_\phi[P_1]\phi(x,\pi R+y) \\ \phi(x,y+2\pi R) &= e^{i\pi\beta_\phi} T_\phi[U]\phi(x,y), \end{aligned} \tag{3.6}$$

where $T_\phi[U]$ denotes an appropriate representation matrix for gauge group and the relation $T_\phi[U] = T_\phi[P_0]T_\phi[P_1]$ is satisfied. The arbitrariness of sign in (3.6) is allowed as long as the interaction terms in Lagrangian remain invariant. β_ϕ must be 0 or 1 due to Z_2 parity.

For Dirac fermion, the boundary conditions are given by

$$\begin{aligned} \psi(x,-y) &= \pm T_\psi[P_0]\gamma^5\psi(x,y) \\ \psi(x,\pi R-y) &= \pm e^{i\pi\beta_\psi}T_\psi[P_1]\gamma^5\psi(x,\pi R+y) \\ \psi(x,y+2\pi R) &= e^{i\pi\beta_\psi}T_\psi[U]\psi(x,y). \end{aligned} \tag{3.7}$$

The phase β_ψ must be 0 or 1 just as for scalar field and $(\gamma^5)^2 = 1$ in our convention.

Gauge transformation changes the given boundary conditions P_0 and P_1 as well as each field. Under a gauge transformation $\Omega(x,y)$ each field transforms to

$$\begin{aligned} A_M(x,y) \to A'_M(x,y) = \Omega(x,y)A_M(x,y)\Omega^\dagger(x,y) - \frac{i}{g}\Omega(x,y)\partial_M\Omega^\dagger(x,y), \\ \phi(x,y) \to \phi'(x,y) = T_\phi[\Omega(x,y)]\phi, \qquad \psi(x,y) \to \psi'(x,y) = T_\psi[\Omega(x,y)]\psi. \end{aligned} \tag{3.8}$$

For this gauge transformation, the boundary conditions (3.2), (3.3) and (3.5) also change to

$$\begin{aligned} A'_M(x,y+2\pi R) &= U'A'_M(x,y)U'^\dagger - \frac{i}{g}U'\partial_M U'^\dagger \\ \begin{pmatrix} A'_\mu(x,-y) \\ A'_y(x,-y) \end{pmatrix} &= P'_0 \begin{pmatrix} A'_\mu(x,y) \\ -A'_y(x,y) \end{pmatrix} P'^\dagger_0 - \frac{i}{g}P'_0 \begin{pmatrix} \partial_\mu \\ -\partial_y \end{pmatrix} P'^\dagger_0 \\ \begin{pmatrix} A'_\mu(x,\pi R-y) \\ A'_y(x,\pi R-y) \end{pmatrix} &= P'_1 \begin{pmatrix} A'_\mu(x,\pi R+y) \\ -A'_y(x,\pi R+y) \end{pmatrix} P'^\dagger_1 - \frac{i}{g}P'_1 \begin{pmatrix} \partial_\mu \\ -\partial_y \end{pmatrix} P'^\dagger_1 \end{aligned} \tag{3.9}$$

where,

$$\begin{aligned} U' &= \Omega(x,y+2\pi R)U\Omega^\dagger(x,y) \\ P'_0 &= \Omega(x,-y)P_0\Omega^\dagger(x,y) \\ P'_1 &= \Omega(x,\pi R-y)P_1\Omega^\dagger(x,\pi R+y), \end{aligned} \tag{3.10}$$

for new field A'_M. Scalar and fermion fields ϕ' and ψ' satisfy the similar relations to (3.5) and (3.6) with $(P'_0,\ P'_1,\ U')$.

In general, the gauge symmetry in Lagrangian density can not be regarded as the symmetry on system because gauge transformation may deform the form of boundary conditions. The gauge transformations which preserve the boundary conditions give the residual gauge invariance on system. It is given by the gauge transformations satisfying the relations

$$\begin{aligned} \Omega(x, y+2\pi R)U &= U\Omega(x,y) \\ \Omega(x,-y)P_0 &= P_0\Omega(x,y) \\ \Omega(x,\pi R-y)P_1 &= P_1\Omega(x,\pi R+y), \end{aligned} \tag{3.11}$$

We call the gauge transformation satisfying the relations (3.11) the symmetry of boundary conditions. Typically, we are interested in the case that the size of extra dimension is much smaller than the experimental scale. In this case, the gauge potential at low energy becomes independent on y: $\Omega = \Omega(x)$ and the condition (3.11) reduces to

$$\Omega(x)U = U\Omega(x), \quad \Omega(x)P_0 = P_0\Omega(x), \quad \Omega(x)P_1 = P_1\Omega(x). \tag{3.12}$$

It implies that the symmetry of boundary conditions is generated by generators in gauge group which commute with U, P_0 and P_1. We should separate the symmetry of boundary conditions from the physical symmetry on system since the dynamics of Wilson line phase come in.

If the gauge transformed boundary conditions (3.9) satisfy the conditions

$$\partial_M P_0' = 0, \quad \partial_M P_1' = 0, \quad \partial_M U' = 0, \tag{3.13}$$

then the two sets are equivalent :

$$(P_0', P_1', U') \sim (P_0, P_1, U). \tag{3.14}$$

From the conditions (3.13) we can find that $(P_0')^\dagger = P_0'$ and $(P_1')^\dagger = P_1'$, and $(P_0',\ P_1',\ U')$ also satisfy (3.4) and $U' = P_1' P_0'$. The equivalence relation (3.14) is ensured by the Hosotani mechanism, leading the same physical content, as we reviewed in section 2. Therefore, we can classify the boundary conditions into equivalence classes by using the equivalence relation (3.13), and this fact play a crucial role in order to approach to the arbitrariness problem for boundary conditions.

Let us determine the physical symmetry on system by the Hosotani mechanism. The Hosotani mechanism allows us to determine the physical symmetry on system once the boundary conditions and matter content are given. We assume that a constant $\langle A_y \rangle$ minimize the effective potential V_{eff} and $\exp(i2\pi g R\langle A_y\rangle) \neq I$. Under the gauge transformation

$\Omega(x,y) = \exp\{ig(y+\alpha)\langle A_y\rangle\}$ the expectation value $\langle A_y\rangle$ transforms to $\langle A'_y\rangle = 0$. Boundary conditions also change to

$$(P_0^{\rm sym}, P_1^{\rm sym}, U^{\rm sym}, \beta) \equiv (e^{2ig\alpha\langle A_y\rangle}P_0,\ e^{2ig(\alpha+\pi R)\langle A_y\rangle}P_1,\ WU,\ \beta). \quad (3.15)$$

WU denote the Wilson line given by (2.14) along S^1/Z_2 direction. We should note that only gauge fields (A_y^a, $\frac{1}{2}\lambda^a \in \mathcal{H}_W$) where,

$$\mathcal{H}_W = \left\{\frac{\lambda^a}{2};\ \ \{\lambda^a, P_0\} = \{\lambda^a, P_1\} = 0\right\}, \quad (3.16)$$

can acquire the vacuum expectation values, because only these field have the zero-mode in KK-expansions, namely the constant mode for y, and from the requirement that spacetime is homogeneous the other fields can not have the vacuum expectation value. Hence the boundary conditions (3.15) indeed satisfy the conditions (3.4) and $U^{\rm sym} = P_1^{\rm sym}P_0^{\rm sym}$. In this gauge, the physical symmetry on system agrees with the symmetry of boundary conditions as $\langle A_y\rangle = 0$. Then the physical symmetry is generated by gauge subgroup whose generators are given by

$$H^{\rm sym} = \left\{\frac{\lambda^a}{2};\ \ [\lambda^a, P_0^{\rm sym}] = [\lambda^a, P_1^{\rm sym}] = 0\right\}. \quad (3.17)$$

3.2 The classification of equivalence class for $SU(N)$ gauge theory

In this subsection, we consider the classification of boundary conditions into the equivalence classes for $SU(N)$ case. First, we note that the eigenvalues of P_0 and P_1 must be $+1$ or -1 due to the parity condition (3.4). The diagonal P_0 and P_1 can be specified by three non-negative integers (p,q,r) as follows:

$$\begin{aligned} \text{diag } P_0 &= \overbrace{(+1,\cdots,+1,+1,\cdots,+1,-1,\cdots,-1,-1,\cdots,-1)}^{N} \\ \text{diag } P_1 &= (\underbrace{+1,\cdots,+1}_{p},\underbrace{-1,\cdots,-1}_{q},\underbrace{+1,\cdots,+1}_{r},\underbrace{-1,\cdots,-1}_{s=N-p-q-r}), \end{aligned} \quad (3.18)$$

where $N \geq p,q,r,s \geq 0$. We denote the boundary conditions indicated by (p,q,r) as $[p;q,r;s]$.

Let us find the equivalence relations for diagonal P_0 and P_1. First, we consider $SU(2)$ case with the boundary conditions $(P_0,\ P_1,\ U) = (\tau_3,\ \tau_3,\ I)$.

τ_i denotes the Pauli matrix. In this case, under the boundary changing gauge transformation $\Omega = \exp\left\{i\left(\frac{\alpha y}{2\pi R}\right)\tau_2\right\}$ we get the equivalence relation

$$(\tau_3, \tau_3, I) \sim (\tau_3, e^{i\alpha\tau_2}\tau_3, e^{i\alpha\tau_2}), \tag{3.19}$$

and for $\alpha = \pi$ we get

$$(\tau_3, \tau_3, I) \sim (\tau_3, -\tau_3, -I). \tag{3.20}$$

In $SU(N)$ case, we can obtain the following equivalence relations by using (3.20) as $SU(2)$ subgroup gauge transformation.

$$\begin{aligned} [p, q, r, s] &\sim [p-1; q+1, r+1; s-1] \quad \text{for} \;\; p, s \geq 1 \\ &\sim [p+1; q-1, r-1; s+1] \quad \text{for} \;\; q, r \geq 1. \end{aligned} \tag{3.21}$$

Next, we consider the non-diagonal P_0 and P_1. In this case, we can always diagonalize one of them by global gauge transformation, and suppose P_0 is diagonalized. In general, P_1 is not diagonal, but by the appropriate boundary changing gauge transformation we can always relate non-diagonal P_0 and P_1 to the both diagonal one by the equivalence relation. This calculation is summarized in Appendix A. It implies that each equivalence class includes at least one both diagonal representation of P_0 and P_1.

From these information we can count the number of equivalence class for $SU(N)$ case on $M^4 \times S^1/Z_2$. The discussion reduces to counting number of both diagonal P_0 and P_1 representations in (3.18) subtracted by the number of equivalence relations in (3.21). We denote the former number by n_1 and the later number by n_2 and write $[p; q, r; s] = [N-k; q, r; k-j]$, where $j = q + r$. The number k runs from 0 to N, and j runs from 0 to k. For fixed (k, j), there are $(j+1)$ configurations for (q, r). Then the number n_1 is given by

$$\sum_{k=0}^{N}\sum_{j=0}^{k}(j+1) = \frac{1}{6}(N+1)(N+2)(N+3). \tag{3.22}$$

The equivalence relation (3.21) can be written as $[N-k; q, r; k-j] = [N-k-1; q+1, r+1; k-j-1]$ and k and j run from 0 to $N-1$ and from 0 to $k-1$ respectively. Hence

$$\sum_{k=0}^{N-1}\sum_{j=0}^{k-1}(j+1) = \frac{1}{6}(N-1)N(N+1). \tag{3.23}$$

From (3.22) and (3.23) we can fine that the number of equivalence classes equals to $(N+1)^2$ (Strictly speaking, we can find that the number of equivalence classes is at most $(N+1)^2$ from above discussion. Then we suppose that no more equivalence relation exists).

3.3 The classification of mode expansions for $SU(2)$ gauge theory

In this subsection, we summarize the mode expansions in general boundary conditions for $SU(2)$ case. These mode expansions are useful for calculating the effective potential in general $SU(N)$ case. The general boundary conditions for $SU(2)$ case are classified into three cases for P_0 and P_1.

The first case is P_0 and $P_1 = I$ or $-I$. In this case, $U = I$ or $-I$ and the residual gauge invariance on system is given by

$$\Omega(x, y + 2\pi R) = \Omega(x, y), \quad \Omega(x, -y) = \Omega(x, y), \tag{3.24}$$

from (3.11). Therefore, a gauge potential is written as

$$\begin{aligned} \Omega(x, y) &= \exp\left\{ i \sum_{a=1}^{3} \omega_a(x, y)\tau_a \right\} \\ \omega_a(x, y) &= \frac{1}{\sqrt{\pi R}}\omega_{a,0}(x) + \sqrt{\frac{2}{\pi R}} \sum_{n=1}^{\infty} \omega_{a,n} \cos\frac{ny}{R}. \end{aligned} \tag{3.25}$$

The y-independent mode corresponds to the residual gauge invariance at low energy. Components of field is either even or odd under parity transformations for this case. The mode expansions correspond to the following four types:

$$\begin{aligned} \phi^{(++)}(x, y) &= \frac{1}{\sqrt{\pi R}}\phi_0(x) + \sqrt{\frac{2}{\pi R}} \sum_{n=1}^{\infty} \phi_n(x) \cos\frac{ny}{R} \\ \phi^{(--)}(x, y) &= \sqrt{\frac{2}{\pi R}} \sum_{n=1}^{\infty} \phi_n(x) \sin\frac{ny}{R} \\ \phi^{(+-)}(x, y) &= \sqrt{\frac{2}{\pi R}} \sum_{n=0}^{\infty} \phi_n(x) \cos\frac{(n+\frac{1}{2})y}{R} \\ \phi^{(-+)}(x, y) &= \sqrt{\frac{2}{\pi R}} \sum_{n=0}^{\infty} \phi_n(x) \sin\frac{(n+\frac{1}{2})y}{R}. \end{aligned} \tag{3.26}$$

The notation $\phi^{(\pm\pm)}$ represents even or odd under P_0, P_1 parity transformations respectively.

The second case is $P_0 = I$ or $-I$ and $P_1 = \tau_3$. If $P_0 \propto I$ we can diagonalize P_1 by global gauge transformation. Hence if P_1 is not proportional to I we can always take $P_1 = \tau_3$. In this case, the residual gauge invariance is given by

$$\Omega(x, y + 2\pi R)\tau_3 = \tau_3\Omega(x, y), \quad \Omega(x, -y) = \Omega(x, y), \tag{3.27}$$

and gauge potential is written as

$$\begin{aligned} \begin{pmatrix} \omega_1(x,y) \\ \omega_2(x,y) \end{pmatrix} &= \sqrt{\frac{2}{\pi R}} \sum_{n=0}^{\infty} \begin{pmatrix} \omega_{1,n}(x) \\ \omega_{2,n}(x) \end{pmatrix} \cos\frac{(n+\frac{1}{2})y}{R} \\ \omega_3(x,y) &= \frac{1}{\sqrt{\pi R}}\omega_{3,0}(x) + \sqrt{\frac{2}{\pi R}} \sum_{n=1}^{\infty} \omega_{3,n}(x) \cos\frac{ny}{R}. \end{aligned} \tag{3.28}$$

Therefore the low energy residual gauge invariance corresponds to $U(1)$. The mode expansions for field components are included in (3.26)

The third case is $P_0 = \tau_3$ and $P_1 = \tau_3 e^{2\pi i(\alpha_1\tau_1 + \alpha_2\tau_2)}$. We can set $\alpha_1 = 0$ and $\alpha_2 = \alpha$ without loss of generality. Then, $U = e^{-2\pi i\alpha\tau_2}$ and the residual gauge invariance is given by

$$\begin{aligned} \Omega(x, y + 2\pi R) &= e^{-2\pi i\alpha\tau_2}\Omega(x,y)e^{2\pi i\alpha\tau_2} \\ \Omega(x, -y) &= \tau_3\Omega(x,y)\tau_3, \end{aligned} \tag{3.29}$$

and (3.29) leads to

$$\begin{aligned} \omega_2(x,y) &= \sqrt{\frac{2}{\pi R}} \sum_{n=1}^{\infty} \omega_{2,n}(x) \sin\frac{ny}{R} \\ \begin{pmatrix} \omega_1(x,y) \\ \omega_3(x,y) \end{pmatrix} &= \sqrt{\frac{1}{\pi R}} \sum_{n=-\infty}^{\infty} v_n(x) \begin{pmatrix} \sin\frac{(n+2\alpha)y}{R} \\ \cos\frac{(n+2\alpha)y}{R} \end{pmatrix}. \end{aligned} \tag{3.30}$$

In this case, whole $SU(2)$ symmetry is broken, and the low energy $U(1)$ residual gauge invariance appears at $\alpha = 0, \pm\frac{1}{2}, \pm 1, \cdots$. The mode expansion depends on the representation in $SU(2)$. As a example, we consider a scalar field in fundamental representation. The mode expansion is given by

$$\begin{pmatrix} \phi_1(x,y) \\ \phi_2(x,y) \end{pmatrix} = \sqrt{\frac{1}{\pi R}} \sum_{n=-\infty}^{\infty} \phi_n(x) \begin{pmatrix} \cos\frac{(n+\alpha)y}{R} \\ \sin\frac{(n+\alpha)y}{R} \end{pmatrix}. \tag{3.31}$$

For scalar field in adjoint representation, the mode expansion is the same as (3.30).

4 The formulation of gauge theory on $M^4 \times S^1/Z_2$ with the dynamics of boundary conditions

4.1 The definition of model

In previous section, we reviewed the field theory on S^1/Z_2 orbifold and found there is the arbitrariness for boundary conditions imposed on fields. From now on, we try to determine physically realized boundary conditions from the dynamics. For this purpose, we need the model including the dynamics of boundary conditions, therefore in this section we give the definition of model with dynamical boundary conditions. This section is based on our original work[58]. We focus on $SU(N)$ gauge theory on $M^4 \times S^1/Z_2$, and define the partition function on system by

$$\mathrm{Z} = \int_C dP_0 \int_C dP_1 \int \mathcal{D}A_M \mathcal{D}\phi \mathcal{D}\psi \Big|_{P_0,P_1} e^{iS(A_M,\phi,\psi,P_0,P_1)}, \tag{4.1}$$

where,

$$C = \{P_j \in U(N), \quad P_j^2 = 1\} \quad j = 0, 1 \tag{4.2}$$

and $S(A_M, \phi, \psi, P_0, P_1)$ is the action depending on gauge, fermion, scalar fields and boundary condition variables. The symbol $|_{P_0,P_1}$ means we restrict the functional integral regions for fields to preserve the boundary conditions (P_0, P_1). dP_0 and dP_1 are defined by $U(N)$ invariant measures. We suppose the action $S(A_M, \phi, \psi, P_0, P_1)$ is $SU(N)$ gauge invariant.

4.2 Analysis of integral on dP_0 and dP_1

In this subsection, we discuss the general properties of integration over boundary conditions $\int_C dP_0 \int_C dP_1$. First, under the following transformation for integral variable

$$P_0 = U^\dagger P_0' U, \quad U \in U(N) \tag{4.3}$$

we can find

$$\int_C dP_0 = \int_C dP_0', \tag{4.4}$$

from the property $d[U^\dagger P_0' U] = dP_0'$ for invariant measure, and $(P_0')^2 = 1$ for any $P_0' \in UCU^\dagger$. The same discussion can be applied to P_1.

Next, we will give the method to splits the integrations P_0 and P_1 between diagonal variables and off-diagonal variables[59, 60]. We start with a integration

$$\mathrm{F} = \int_C dP_0 \int_C dP_1 f(P_0, P_1), \tag{4.5}$$

where $f(P_0, P_1)$ is a function depending on P_0 and P_1, and we assume $f(P_0, P_1)$ is invariant under transformations $P_0 \to UP_0U^\dagger, \quad P_1 \to UP_1U^\dagger$ for $U \in U(N)$:

$$f(UP_0U^\dagger, UP_1U^\dagger) = f(P_0, P_1). \tag{4.6}$$

Before the analysis, we need to regularize the integral region C by infinitesimal regularization parameter μ as follows:

$$C \to \hat{C} \equiv \{P_j \in U(N),\ \rho_i = \pm e^{i\mu_i},\ 0 \le \mu_i \le \mu \ll 1\} \quad \mu : \text{real} \tag{4.7}$$

for reason that we will mention later. ρ_i $(1 \le i \le N)$ denote eigenvalues of P_j $(j = 0, 1)$ and in the limit $\mu \to 0$ we restore the original definition.

Then, we define the function

$$\begin{aligned} \Delta^{-1}(P_0) &= \int dU \prod_{1 \le i < j \le N} \delta^{(2)}[(UP_0U^\dagger)_{ij}] \\ \delta^{(2)}[(UP_0U^\dagger)_{ij}] &= \delta[\Re(UP_0U^\dagger)_{ij}]\delta[\Im(UP_0U^\dagger)_{ij}], \end{aligned} \tag{4.8}$$

where dU denotes the invariant measure for $U(N)$ and the symbol $\int dU$ means we integrate over whole region on $U(N)$. Substituting the function (4.8) to (4.5) we find

$$\mathrm{F} = \int_{\hat{C}} dP_0 \int_{\hat{C}} dP_1 \Delta(P_0) \int dU \prod_{1 \le i < j \le N} \delta^{(2)}[(UP_0U^\dagger)_{ij}] f(P_0, P_1). \tag{4.9}$$

We change the variables as $P_0 = U^\dagger P_0' U$ and $P_1 = U^\dagger P_1' U$, then, using (4.4), (4.6) and the fact that function (4.8) is invariant under the unitary transformation of variable, (4.9) is transformed to

$$\mathrm{F} = \int dU \int_{\hat{C}} dP_0' \int_{\hat{C}} dP_1' \Delta(P_0') \prod_{1 \le i < j \le N} \delta^{(2)}[(P_0')_{ij}] f(P_0', P_1'). \tag{4.10}$$

Then, we normalize $\int dU = 1$ and carry out integration of P_0' with delta function. (4.10) becomes

$$\mathrm{F} = \int d\Lambda_0 \int_{\hat{C}} dP_1' \Delta(\Lambda_0) f(\Lambda_0, P_1'), \tag{4.11}$$

where

$$\Delta^{-1}(\Lambda_0) = \frac{(2\pi)^N}{\prod\limits_{1\le i<j\le N} |\epsilon_i - \epsilon_j e^{i\mu_{ij}}|^2}, \quad \mu_{ij} = \mu_j - \mu_i. \tag{4.12}$$

ϵ_i, ϵ_j are $+1$ or -1. The symbol $\int d\Lambda_0$ denote the integration over only diagonal elements in $\hat{C}$, and it is given by

$$\int d\Lambda_0 = \sum_{\pm 1} \int_0^{\mu} \prod_{1\le n\le N} d\mu_n. \tag{4.13}$$

$\sum\limits_{\pm 1}$ means the summation over all combinations we assign $+1$ or -1 to ϵ_i $(1 \le i \le N)$ in (4.11) and (4.12).

We can apply the same calculation from (4.8) to (4.12) to P_1 part and we have

$$\mathrm{F} = \int d\Lambda_0 \int d\Lambda_1 \Delta(\Lambda_0)\Delta(\Lambda_1) \int dU f(\Lambda_0, U^\dagger \Lambda_1 U), \tag{4.14}$$

where,

$$\Delta^{-1}(\Lambda_1) = \frac{(2\pi)^N}{\prod\limits_{1\le p<q\le N} |\epsilon_p' - \epsilon_q' e^{i\mu_{pq}'}|^2}, \quad \mu_{pq}' = \mu_q' - \mu_p', \tag{4.15}$$

$$\int d\Lambda_1 = \sum_{\pm 1} \int_0^{\mu'} \prod_{1\le m\le N} d\mu_m'. \tag{4.16}$$

$\mu' \ll 1$ is the regularization parameter, and ϵ_p', ϵ_q' are $+1$ or -1. For the variables P_0, $P_1 \in U(N)$, $N \ge 3$, taking limit $\mu, \mu' \to 0$ leads to $F \to 0$ for any $f(P_0, P_1)$. It means that the integral regions for boundary conditions correspond to measure zero regions in $U(N)$ invariant measure. Therefore we must redefine the partition function (4.1) to make it well-defined.

4.3 Integration of partition function over boundary conditions

In this subsection, we apply the analysis discussed in section 4.2 to our model defined in section 4.1. As a result, we will find that only restricted sets of boundary conditions practically contribute to the partition function. First, we need to redefine our partition function in order to make it well-defined as I mentioned in previous section. The modified partition function is given by

$$\mathrm{Z} = V^{-1} \int_{\hat{C}} dP_0 \int_{\hat{C}} dP_1 \int \mathcal{D}A_M \mathcal{D}\phi \mathcal{D}\psi \Big|_{P_0,P_1} e^{iS(A_M,\phi,\psi,P_0,P_1)}, \tag{4.17}$$

where

$$V \equiv \int_{\hat{C}} dP_0 \int_{\hat{C}} dP_1 = \int d\Lambda_0 \int d\Lambda_1 \Delta(\Lambda_0)\Delta(\Lambda_1). \tag{4.18}$$

We regularized the integral regions for boundary conditions $C \to \hat{C}$ with the regularization parameters μ and μ' just as section 4.2.

The field values are not defined in this regularization since $P_0^2, P_1^2 \neq 1$. Therefore, we also redefine the parity transformation matrices (P_0, P_1) as

$$\hat{P}_0 \equiv (P_0^{-2})^{\frac{1}{2}} P_0, \quad \hat{P}_1 \equiv (P_1^{-2})^{\frac{1}{2}} P_1, \tag{4.19}$$

where

$$A^{\frac{1}{2}} = U\Lambda^{\frac{1}{2}}U^\dagger, \;\; \Lambda^{\frac{1}{2}} = \begin{pmatrix} \sqrt{a_1} & & \\ & \sqrt{a_2} & \\ & & \ddots \end{pmatrix} \quad A \in U(N). \tag{4.20}$$

$a_i \;\; (i = 1, 2, \cdots)$ are the eigenvalues of A, and we choose the positive square root of the eigenvalues as the convention. In this prescription, we find the eigenvalues of $\hat{P}_0, \hat{P}_1$ are $+1$ or -1 and $\hat{P}_0^2 = \hat{P}_1^2 = 1$. We can restore these to the original definition in the limit $\mu, \mu' \to 0$ and the field values are well-defined. From now on, the symbol $|_{P_0,P_1}$ implies that we restrict the functional integral regions for fields A_M, ϕ, ψ to preserve the boundary condition $\hat{P}_0, \hat{P}_1$.

The next step is to split the integral variables for boundary conditions into diagonal components and off-diagonal components according to section 4.2. Using the function (4.8), the partition function becomes

$$\mathrm{Z} = V^{-1} \int_{\hat{C}} dP_0 \int_{\hat{C}} dP_1 \int \mathcal{D}A_M \mathcal{D}\phi \mathcal{D}\psi \Big|_{P_0,P_1} \Delta(P_0) \int dU \delta^{(2)}(UP_0U^\dagger) e^{iS(A_M,\phi,\psi,P_0,P_1)}, \tag{4.21}$$

where,

$$\delta^{(2)}(UP_0U^\dagger) \equiv \prod_{1\le i<j\le N} \delta^{(2)}[(UP_0U^\dagger)_{ij}]. \tag{4.22}$$

Then, we change the variable $P_0 \to P_0' = UP_0U^\dagger$ and use (4.4) we get

$$\begin{aligned} Z = V^{-1} \int dU \int_{\hat{C}} dP_0' \int_{\hat{C}} dP_1 \int \mathcal{D}A_M \mathcal{D}\phi\mathcal{D}\psi \Big|_{U^\dagger P_0' U, P_1} &\Delta(P_0')\delta^{(2)}(P_0') \\ &\times e^{iS(A_M,\phi,\psi,U^\dagger P_0' U,P_1)}. \end{aligned} \tag{4.23}$$

and integrate out $\int_{\hat{C}} dP_0$ with the delta function $\delta^{(2)}(P_0')$, it becomes

$$\begin{aligned} Z = V^{-1} \int dU \int d\Lambda_0 \int_{\hat{C}} dP_1 \Delta(\Lambda_0) \int \mathcal{D}A_M \mathcal{D}\phi\mathcal{D}\psi \Big|_{U^\dagger \Lambda_0 U, P_1} \\ \times e^{iS(A_M,\phi,\psi,U^\dagger \Lambda_0 U,P_1)}. \end{aligned} \tag{4.24}$$

$U^\dagger \Lambda_0 U$ is a unitary transformation for $U \in U(N)$, but we can regard it as a unitary transformation for $U' \in SU(N)$ because $U^\dagger \Lambda_0 U$ remains invariant under $U \to U' = \Lambda U$ for a diagonal $\Lambda \in U(N)$, and it is written as $U'^\dagger \Lambda_0 U' = U^\dagger \Lambda_0 U$ for $U' = \Lambda U \in SU(N)$ by suitable Λ. Then, we can rewrite (4.24) as

$$\begin{aligned} Z = V^{-1} \int dU \int d\Lambda_0 \int_{\hat{C}} dP_1 \Delta(\Lambda_0) \int \mathcal{D}A_M \mathcal{D}\phi\mathcal{D}\psi \Big|_{U'^\dagger \Lambda_0 U', P_1} \\ \times e^{iS(A_M,\phi,\psi,U'^\dagger \Lambda_0 U',P_1)} \\ U' \in SU(N). \end{aligned} \tag{4.25}$$

Change the variable $P_1 \to P_1' = U' P_1 U'^\dagger$ and use (4.4), we have

$$\begin{aligned} Z = V^{-1} \int dU \int d\Lambda_0 \int_{\hat{C}} dP_1' \Delta(\Lambda_0) \int \mathcal{D}A_M \mathcal{D}\phi\mathcal{D}\psi \Big|_{U'^\dagger \Lambda_0 U', U'^\dagger P_1' U'} \\ \times e^{iS(A_M,\phi,\psi,U'^\dagger \Lambda_0 U',U'^\dagger P_1' U')}. \end{aligned} \tag{4.26}$$

The integrand in (4.26) equals to the original system with the boundary conditions $(\hat{\Lambda}_0, \hat{P}_1')$ up to the global gauge transformation U'. The system should be independent on global gauge. Then, (4.26) becomes

$$\begin{aligned} Z = V^{-1} \int dU \int d\Lambda_0 \int_{\hat{C}} dP_1' \Delta(\Lambda_0) \int \mathcal{D}A_M \mathcal{D}\phi\mathcal{D}\psi \Big|_{\Lambda_0, P_1'} \\ \times e^{iS(A_M,\phi,\psi,\Lambda_0,P_1')}. \end{aligned} \tag{4.27}$$

Normalize $\int dU = 1$, and apply the same procedure to P'_1. Then, equation (4.27) becomes

$$\mathrm{Z} = \frac{\sum_{\pm 1} \int_0^{\mu} \prod_{1\le n\le N} d\mu_n \int_0^{\mu'} \prod_{1\le m\le N} d\mu'_m \Delta(\Lambda_0)\Delta(\Lambda_1) I(A_M,\phi,\psi,\Lambda_0,\Lambda_1)}{\sum_{\pm 1} \int_0^{\mu} \prod_{1\le n'\le N} d\mu_{n'} \int_0^{\mu'} \prod_{1\le m'\le N} d\mu'_{m'} \Delta(\Lambda_0)\Delta(\Lambda_1)} \tag{4.28}$$

where

$$I(A_M,\psi,\Lambda_0,\Lambda_1) \equiv \int dU \int \mathcal{D}A_M \mathcal{D}\phi\mathcal{D}\psi \Big|_{\Lambda_0, U^\dagger \Lambda_1 U} e^{iS(A_M,\phi,\psi,\Lambda_0,U^\dagger\Lambda_1 U)}. \tag{4.29}$$

We suppose $I(A_M,\phi,\psi,\Lambda_0,\Lambda_1)$ is almost constant value on the integral variables μ_n and μ'_m, compared with $\Delta(\Lambda_0), \Delta(\Lambda_1)$. Then, we can replace the function $\int_0^{\mu} \prod_{1\le n\le N} d\mu_n \int_0^{\mu'} \prod_{1\le m\le N} d\mu'_m \Delta(\Lambda_0)\Delta(\Lambda_1)$ with the integrand on particular values $\mu_n, \mu'_m \quad (0 < \mu_n, \mu'_m < \mu)$ times the integral regions by mean-value theorem. The conditions $\{\mu_{ij} \neq 0, \mu'_{pq} \neq 0 \quad (1 \le i,p < j,q \le N)\}$ is required for any finite μ if ϵ_i and ϵ_j or ϵ'_p and ϵ'_q have the same sign, since these values correspond to the maximum or minimum of the integrand. After this replacement, the integral regions of $d\mu_n, d\mu'_m$ between the denominator and numerator in (4.28) cancel out. As a result, we have

$$\mathrm{Z} = \frac{\sum_{\pm 1} \prod_{1\le i,p<j,q\le N} |\epsilon_i - \epsilon_j e^{i\mu_{ij}}|^2 |\epsilon'_p - \epsilon'_q e^{i\mu'_{pq}}|^2 I(A_M,\phi,\psi,\Lambda_0,\Lambda_1)}{\sum_{\pm 1} \prod_{1\le k,v<l,w\le N} |\epsilon_k - \epsilon_l e^{i\mu_{kl}}|^2 |\epsilon'_v - \epsilon'_w e^{i\mu'_{vw}}|^2}. \tag{4.30}$$

In the summation $\sum_{\pm 1}$, the factors $|1-e^{i\mu_{ij}}|^2$ and $|1-e^{i\mu'_{pq}}|$ give 0 to each term in (4.30) in the limit $\mu, \mu' \to 0$. We suppose "a" is the lowest number of the factors ,such as $|1-e^{i\mu_{ij}}|$, each term has in (4.30). The lowest number of the factors such as $|1-e^{i\mu'_{pq}}|$ is also a. Then, we multiply the denominator and numerator in (4.30) by $|1-e^{i\mu}|^{-2a}$. As taking the limit $\mu \to 0$, we can see

$$\frac{|1-e^{i\mu_{ij}}|}{|1-e^{i\mu}|} = \left|\frac{\mu_{ij}}{\mu}\right| \to C_{ij} > 0. \tag{4.31}$$

C_{ij} must be finite value in order to be consistent with mean-value theorem. There is at least one term which has finite value in denominator and numerator of (4.30) in this limit. Such finite terms correspond to the combination which has the highest number of pairs such as $\{+1, -1\}$ in ϵ_i $(1 \leq i \leq N)$. The other terms go to 0 in the limit $\mu \to 0$ and after the limit, the set $\{\epsilon_i,\ (1 \leq i \leq N)\}$ corresponds to the eigenvalues of P_0. Therefore we conclude that only the eigenvalue set of P_0 which have the largest number of $\{+1, -1\}$ pairs contribute to the partition function. We have the same conclusion for P_1 integral.

Next, let us consider the case that boundary conditions (Λ_0', Λ_1') are related to a diagonal boundary conditions (Λ_0, Λ_1) by the permutation of the eigenvalue sets. we show $I(A_M, \phi, \psi, \Lambda_0', \Lambda_1') = I(A_M, \phi, \psi, \Lambda_0, \Lambda_1)$. Note that the factor $\displaystyle\prod_{1\leq i<j\leq N} |\epsilon_i - \epsilon_j e^{i\mu_{ij}}|^2$, $\displaystyle\prod_{1\leq p<q\leq N} |\epsilon_p' - \epsilon_q' e^{i\mu_{pq}'}|^2$ gives the identical contribution to $I(A_M, \phi, \psi, \Lambda_0', \Lambda_1')$ and $I(A_M, \phi, \psi, \Lambda_0, \Lambda_1)$ since these are transformed into the identical form by relabels of integral variables μ_n and μ_m'. Since (Λ_0', Λ_1') is the permutation of eigenvalues sets in (Λ_0, Λ_1), they satisfy the relations

$$\begin{aligned} \Lambda_0' = V_0^\dagger \Lambda_0 V_0 \qquad \Lambda_1' &= V_1^\dagger \Lambda_1 V_1 \\ V_0, V_1 &\in SU(N), \end{aligned} \tag{4.32}$$

then, for the boundary condition (Λ_0, Λ_1), we have

$$\begin{aligned} &I(A_M, \phi, \psi, \Lambda_0', \Lambda_1') \\ &= \int dU \int \mathcal{D}A_M \mathcal{D}\phi \mathcal{D}\psi \Big|_{\Lambda_0', U^\dagger \Lambda_1' U} e^{iS(A_M, \phi, \psi, \Lambda_0', U^\dagger \Lambda_1' U)} \\ &= \int dU \int \mathcal{D}A_M \mathcal{D}\phi \mathcal{D}\psi \Big|_{V_0^\dagger \Lambda_0 V_0, U^\dagger V_1^\dagger \Lambda_1 V_1 U} e^{iS(A_M, \phi, \psi, V_0^\dagger \Lambda_0 V_0, U^\dagger V_1^\dagger \Lambda_1 V_1 U)}. \end{aligned} \tag{4.33}$$

Under global gauge transformation $\Lambda_0' \to V_0 \Lambda_0' V_0^\dagger, \quad U^\dagger \Lambda_1' U \to V_0 U^\dagger \Lambda_1' U V_0^\dagger$, we find

$$\begin{aligned} &I(A_M, \phi, \psi, \Lambda_0', \Lambda_1') \\ &= \int dU \int \mathcal{D}A_M \mathcal{D}\phi \mathcal{D}\psi \Big|_{\Lambda_0, V_0 U^\dagger V_1^\dagger \Lambda_1 V_1 U V_0^\dagger} e^{iS(A_M, \phi, \psi, \Lambda_0, V_0 U^\dagger V_1^\dagger \Lambda_1 V_1 U V_0^\dagger)}. \end{aligned} \tag{4.34}$$

Using the property of $\int dU$ invariant measure, we get

$$I(A_M, \phi, \psi, \Lambda_0', \Lambda_1') = I(A_M, \phi, \psi, \Lambda_0, \Lambda_1). \tag{4.35}$$

Then, according to the discussion in section 3.2, we find that the boundary conditions (Λ_0, Λ_1) and (Λ_0', Λ_1') are in the same equivalence class. Hence (4.35) is consistent with the Hosotani mechanism. Moreover, from the discussion in Appendix A we can see that there is at least one both diagonal representation of (P_0, P_1) in each equivalence class. Then, on the process that arbitrary set of boundary conditions transforms to both diagonal representations by global and local gauge transformations, there is no transformation which changes the eigenvalues set of the boundary conditions. Therefore, arbitrary boundary conditions (P_0, P_1) and its eigenvalue set (Λ_0, Λ_1) belong to the same equivalence class. Since a permutation (Λ_0', Λ_1') of diagonal representations (Λ_0, Λ_1) belong to the equivalence class with (Λ_0, Λ_1), we conclude equivalence classes for $SU(N)$ gauge theory on S^1/Z_2 are completely classified by eigenvalues sets for boundary conditions. Therefore, on the process that we compute some physical observables, the integrand on $\int dU$ in (4.29) is independent of the variable U, so it is sufficient to compute only about the both diagonal representations $(P_0, P_1) = (\Lambda_0, \Lambda_1)$ if we want to know some physical observables.

4.4 Application to several examples

In previous section, we gave the general discussion for dynamics of boundary conditions in $SU(N)$ gauge theory on S^1/Z_2 and got the result that only restricted sets of boundary conditions practically contribute to the partition function. Let us apply our formulation to $SU(2)$, $SU(3)$ and $SU(5)$ cases. Especially in $SU(5)$ case, we will show that SM gauge symmetry $SU(3) \times SU(2) \times U(1)$ naturally appears in our formulation.

First, we consider $SU(2)$ case as the simplest example. In this case, only one equivalence class gives non-vanishing contribution to partition function and it is characterized by the eigenvalue set

$$\left\{ \begin{array}{l} P_0 = \{+1, -1\} \\ P_1 = \{+1, -1\} \end{array} \right\} . \tag{4.36}$$

Non-vanishing contribution is generated by unitary transformations of (4.36). The diagonal representation (4.36) leads to $SU(2) \to U(1)$ symmetry of boundary conditions.

Then, for $SU(3)$ case the four equivalence classes contribute to partition function, and these equivalence classes are characterized by the following eigenvalue sets

$$\begin{array}{ll} (1) & (2) \\ \left\{\begin{array}{l} P_0=\{+1,+1,-1\} \\ P_1=\{+1,+1,-1\} \end{array}\right\} & \left\{\begin{array}{l} P_0=\{+1,+1,-1\} \\ P_1=\{-1,-1,+1\} \end{array}\right\} \\ (3) & (4) \\ \left\{\begin{array}{l} P_0=\{-1,-1,+1\} \\ P_1=\{+1,+1,-1\} \end{array}\right\} & \left\{\begin{array}{l} P_0=\{-1,-1,+1\} \\ P_1=\{-1,-1,+1\} \end{array}\right\}. \end{array} \tag{4.37}$$

The symmetry of boundary conditions correspond to $SU(2)\times U(1)$ for $(1)\sim(4)$. The physical symmetry can be different among $(1)\sim(4)$ depending on matter content on the system. The partition function (4.28) is written as

$$\mathrm{Z}=C_1\mathrm{I}_{(1)}+C_2\mathrm{I}_{(2)}+C_3\mathrm{I}_{(3)}+C_4\mathrm{I}_{(4)}. \tag{4.38}$$

Here, $I_{(i)}$ $i=1\sim4$ indicate the $I(A_M,\psi,\Lambda_0,\Lambda_1)$ in (4.29) with the boundary conditions (i), $i=1\sim4$. Since the factor $\prod_{1\le i<j\le N}|\epsilon_i-\epsilon'_j e^{i\mu_{ij}}|^2$, $\prod_{1\le p<q\le N}|\epsilon'_p-\epsilon'_q e^{i\mu'_{pq}}|^2$ gives the identical contribution to (4.38) we dropped this factor as overall constant. C_i denote the coefficients corresponding to all permutation in the eigenvalue sets (i). In $SU(3)$ case, these constants are

$$C_i=({}_3C_1)^2 \qquad i=1\sim4. \tag{4.39}$$

So, we can see all coefficients are the same, and drop this coefficients as overall constants.

Finally, we consider $SU(5)$ case. Just as in the $SU(3)$ example, four equivalence classes contribute to the partition function. These equivalence classes are characterized by

$$\begin{array}{ll} (1) & (2) \\ \left\{\begin{array}{l} P_0=\{+1,+1,+1,-1,-1\} \\ P_1=\{+1,+1,+1,-1,-1\} \end{array}\right\} & \left\{\begin{array}{l} P_0=\{+1,+1,+1,-1,-1\} \\ P_1=\{-1,-1,-1,+1,+1\} \end{array}\right\} \\ (3) & (4) \\ \left\{\begin{array}{l} P_0=\{-1,-1,-1,+1,+1\} \\ P_1=\{+1,+1,+1,-1,-1\} \end{array}\right\} & \left\{\begin{array}{l} P_0=\{-1,-1,-1,+1,+1\} \\ P_1=\{-1,-1,-1,+1,+1\} \end{array}\right\}. \end{array} \tag{4.40}$$

In (1) $\sim$ (4), the symmetry of boundary conditions corresponds to SM gauge symmetry $SU(3) \times SU(2) \times U(1)$. Therefore, we can get $SU(3) \times SU(2) \times U(1)$ physical symmetry if the expectation value of gauge field $\langle A_y \rangle$ does not arise, in other words Wilson line phase have the trivial configuration. We give the further analysis for $SU(5)$ case in the following sections. The partition function is written as (4.38) and the factor $\prod_{1\le i<j\le N} |\epsilon_i - \epsilon'_j e^{i\mu_{ij}}|^2$, $\prod_{1\le p<q\le N} |\epsilon'_p - \epsilon'_q e^{i\mu'_{pq}}|^2$ and C_i give the identical contribution for boundary conditions (1) $\sim$ (4) and can be dropped as overall constant just as in $SU(3)$ case.

5 Calculation of the effective potential for $SU(5)$ gauge theory on S^1/Z_2

In previous section, we showed that only restricted sets of boundary conditions contribute to the partition function. Especially, for $SU(5)$ case these sets are given by (4.40) and their equivalence classes. Then, we try to determine the physically realized boundary condition by further analysis: we suppose that the equivalence class which has the lowest energy density is physically selected as I mentioned in the introduction. For this purpose, we need to evaluate the effective potential on system. In this section, we review the method to calculate the effective potential for SU(5) case with the diagonal boundary conditions (4.40) at one loop level. This section is based on Ref [55]. First, we note the boundary conditions (4.40) identically contribute to gauge field since gauge field is in adjoint representation. Hence, the degree of freedom of Wilson line also takes place identically. The difference for these boundary conditions arise in matter content on system.

Then we calculate the effective potential by background field method at one loop level. The Lagrangian density on D dimensional spacetime is given by

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{matter}} \\
\mathcal{L}_{\text{gauge}} &= -\frac{1}{2}\text{Tr}F_{MN}F^{MN} - \frac{1}{\alpha}\text{Tr}F[A]^2 - \text{Tr}\left(\bar{\eta}\frac{\delta F[A]}{\delta A_M}D^M\eta\right) \qquad (5.1) \\
\mathcal{L}_{\text{matter}} &= \bar{\psi}i\gamma_M D^M\psi + |D_M\phi|^2 - V[\phi,\psi],
\end{aligned}$$

where the gauge fixing term with a gauge parameter α and ghost term correspond to the second and third terms respectively, and we choose $\alpha = 1$. ψ and ϕ denote Dirac and complex scalar fields and $V[\phi, \psi]$ is any interaction term. For simplicity, we consider only massless case for fermion and scalar fields. The covariant derivative is given by $D_M = \partial_M + igT^a A_M$, where T^a is an appropriate representation matrix of gauge group.

Then, we split the gauge field A_M into the classical part A^0_M, which corresponds to background field and the solution of classical equation of motion, and the quantum part A^q_M, which corresponds to integral variable for functional integral. Under the gauge transformation Ω, each field transforms to

$$\begin{aligned} A^0_M &\to A'^0_M = \Omega A^0_M \Omega^\dagger - \frac{i}{g}\Omega \partial_M \Omega^\dagger \\ A^q_M &\to A'^q_M = \Omega A^q_M \Omega^\dagger \\ \psi &\to \psi' = T_\psi[\Omega]\psi, \quad \phi \to \phi' = T_\phi[\Omega]\phi. \end{aligned} \tag{5.2}$$

Then, we impose the following gauge fixing condition

$$F[A] = D_M(A^0)A^{qM} = \partial_M A^{qM} + ig[A^0_M, A^{qM}] = 0, \tag{5.3}$$

and suppose the background field also satisfies $F[A^0] = 0$.

The effective potential V_{eff} depends on not only A^0_M buy also the boundary conditions $(P_0, P_1, U, e^{i\pi\beta})$ so we denote V_{eff} as $V_{eff}[A_0; P_0, P_1, U, \beta]$. Let us consider the boundary changing gauge transformation Ω. We suppose the boundary conditions $(P_0, P_1, U, e^{i\pi\beta})$ transform to the new boundary condition set $(P'_0, P'_1, U', e^{i\pi\beta})$ which satisfies the condition (3.13) under this gauge transformation. Then, the action is invariant expect the gauge fixing term. If the gauge transformation satisfies the additional condition

$$D^{0M}(\partial_M \Omega^\dagger \Omega) + ig[A^{0M}, \partial_M \Omega \Omega^\dagger] = 0, \tag{5.4}$$

the entire action remains invariant. Here, we denoted $D_M(A^0)$ by D^0_M for short. Therefore, if the gauge transformation Ω satisfies the conditions (3.13) and (5.4), the effective potentials satisfy the relation

$$V_{eff}[A^0; P_0, P_1, U, \beta] = V_{eff}[A'^0; P'_0, P'_1, U', \beta]. \tag{5.5}$$

This relation is required by the Hosotani mechanism since the entire action is invariant. From (5.5) we find the effective potential is a invariant function under the gauge transformation which remains the boundary conditions invariant. Moreover, if we consider the case that the boundary conditions $(P_0, P_1, U, e^{i\pi\beta})$ change to $(P_0^{\rm sym}, P_1^{\rm sym}, U^{\rm sym}, e^{i\pi\beta})$ in (3.15) under the gauge transformation, then the gauge transformation satisfies the condition (5.4) for $A^0 = \langle A \rangle$ in $(P_0, P_1, U, e^{i\pi\beta})$. Hence we get the relation

$$V_{eff}[\langle A^0 \rangle; P_0, P_1, U, \beta] = V_{eff}[\langle A'^0 \rangle = 0; P_0^{sym}, P_1^{sym}, U', \beta]. \tag{5.6}$$

By using these information, we can calculate the effective potential with the boundary conditions (4.40). We note the degree of freedom of Wilson line appears in only the component A_y^a whose generator is in $\mathcal{H}_W$ given by (3.16). Therefore, the vacuum expectation value A^0 for the boundary conditions (4.40) can be parameterized as

$$A_y^0 = \frac{1}{2gR} \begin{pmatrix} 0 & \Theta \\ \Theta^\dagger & 0 \end{pmatrix}. \tag{5.7}$$

Θ denotes a 3×2 matrix. Then, the boundary conditions (4.40) have $G_{SM} = SU(3) \times SU(2) \times U(1)$ as the symmetry of boundary conditions. Therefore some of components in Θ are unphysical parameters. Under a appropriate global gauge transformation in G_{SM}, Θ brings in the form

$$\Theta = \begin{pmatrix} \alpha & \gamma \\ 0 & \beta \\ 0 & 0 \end{pmatrix}, \tag{5.8}$$

where α and γ are complex parameters and β is a real parameter. The effective potential must be a invariant function of these α, β and γ under G_{SM} gauge transformation. Only possible combinations of these parameters appear as the eigenvalues of $\Theta\Theta^\dagger$ and $\Theta^\dagger\Theta$ where

$$\Theta\Theta^\dagger = \begin{pmatrix} |\alpha|^2 + |\gamma|^2 & \beta\gamma & 0 \\ \beta\gamma^* & \beta^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Theta^\dagger\Theta = \begin{pmatrix} |\alpha|^2 & \gamma\alpha^* \\ \gamma^*\alpha & \beta^2 + |\gamma|^2 \end{pmatrix}. \tag{5.9}$$

The eigenvalues of $\Theta\Theta^\dagger$ are given by λ_+, λ_- and 0, and those of $\Theta^\dagger\Theta$ are given by λ_+ and λ_-, where

$$\lambda_\pm = \frac{1}{2}\big(\beta^2 + |\alpha|^2 + |\gamma|^2 \pm \sqrt{(\beta^2 + |\alpha|^2 + |\gamma|^2)^2 - 4|\alpha|^2\beta^2}\big). \tag{5.10}$$

Therefore the effective potential is a function of λ_+ and λ_-. It means that one can simplify the form of Θ without loss of generality, then we can set

$\alpha = a$, $\beta = b$ and $\gamma = 0$ where a and b are real values. In this case, $\lambda_\pm$ correspond to a^2 and b^2. Hence we evaluate V_{eff} for

$$\Theta = \begin{pmatrix} a & 0 \\ 0 & b \\ 0 & 0 \end{pmatrix}. \tag{5.11}$$

Then, the resultant $V_{eff}(a,b)$ should be interpreted as $V_{eff}(\sqrt{\lambda_+}, \sqrt{\lambda_-})$.

The one-loop effective potential for A^0 is obtained by integrating out the fields A^q_M, η, ψ and ϕ up to quadratic terms of these fields. The contributions from each field are given by

$$\begin{aligned} V_{eff}[A^0] &= V_{eff}[A^0]^{g+gh} + V_{eff}[A^0]^{\text{fermion}} + V_{eff}[A^0]^{\text{scalar}} \\ V_{eff}[A^0]^{g+gh} &= -(D-2)\frac{i}{2}\text{Tr}\ln D^0_L D^{0L} \\ V_{eff}[A^0]^{\text{fermion}} &= f(D)\frac{i}{2}\text{Tr}\ln D^0_L D^{0L}, \quad f(D) = 2^{[D/2]} \\ V_{eff}[A^0]^{\text{scalar}} &= -2\frac{i}{2}\text{Tr}\ln D^0_L D^{0L}, \end{aligned} \tag{5.12}$$

where we supposed $F^0_{MN} = 0$. Then the expectation values in D^0_M arise in the components in (5.7) and (5.11).

6 Determination of equivalence class from the dynamics of boundary conditions

6.1 Non-supersymmetric $SU(5)$

In this section, we determine the physically realized equivalence class from the dynamics of boundary conditions. As we mentioned in the previous section, we assume that the equivalence class which has the lowest vacuum energy density is physically selected, so we determine the equivalence class by evaluating the effective potential on system. We focus our attention to the $SU(5)$ case, but the discussion is easily generalized to $SU(N)$ case.

In section 4 we found that only restricted equivalence classes contribute to the partition function and for $SU(5)$ case those are specified by the following diagonal boundary condition sets:

$$\left\{\begin{array}{l} P_0 = \{-1,-1,-1,+1,+1\} \\ P_1 = \{-1,-1,-1,+1,+1\} \end{array}\right\} \quad \left\{\begin{array}{l} P_0 = \{+1,+1,+1,-1,-1\} \\ P_1 = \{+1,+1,+1,-1,-1\} \end{array}\right\} \tag{6.1}$$

$$\begin{Bmatrix} P_0 = \{-1,-1,-1,+1,+1\} \\ P_1 = \{+1,+1,+1,-1,-1\} \end{Bmatrix} \quad \begin{Bmatrix} P_0 = \{+1,+1,+1,-1,-1\} \\ P_1 = \{-1,-1,-1,+1,+1\} \end{Bmatrix}. \tag{6.2}$$

The equivalence classes which practically contribute to the partition function are generated by the separate unitary transformations of P_0 and P_1 in (6.1) and (6.2). In following calculation, we will use the above diagonal representations in (6.1) and (6.2) for evaluating the effective potential. Note the two sets in (6.1) (or (6.2)) are related by the transformations $P_0 \to -P_0$ and $P_1 \to -P_1$ and the transformation $P_1 \to -P_1$ changes the two sets in (6.1) to the two sets in (6.2). Therefore, these four boundary condition sets make no difference for gauge fields because gauge field belongs to the adjoint representation of $SU(5)$. It is important that the symmetry of boundary conditions corresponds to $SU(5) \to SU(3) \times SU(2) \times U(1)$ in four boundary condition sets in (6.1) and (6.2), therefore the physical symmetry on system has SM symmetry only if Wilson line phase has trivial configuration, namely all gauge fields have vanishing vacuum expectation values. The differences of these boundary condition sets appear in matter content of the system (ψ, ϕ), and these differences affect shape of the effective potential for gauge fields.

In general, the differences of minimal values of effective potentials among these four equivalence classes become divergent due to the contributions of scalar fields. Then if we suppose scalar fields exist in the pair like

$$\begin{aligned} \begin{pmatrix} \phi \\ \phi' \end{pmatrix}(x,-y) &= \pm P_0 \begin{pmatrix} \phi \\ -\phi' \end{pmatrix}(x,y) \\ \begin{pmatrix} \phi \\ \phi' \end{pmatrix}(x,\pi R - y) &= \pm e^{i\pi\beta_\phi} P_1 \begin{pmatrix} \phi \\ -\phi' \end{pmatrix}(x,\pi R + y), \end{aligned} \tag{6.3}$$

where $\beta_\phi = 0$ or 1. It is guaranteed that the differences of effective potential among the four equivalence classes become finite. For the purpose that we compare the minimal values of effective potentials, we put the rule that scalar fields exist in the pair (6.3). In supersymmetric case, such pairs naturally appear in hypermultiplets. Under this rule, the effective potentials for the two boundary condition sets in (6.1) or (6.2) are completely degenerate at one loop level. Then we can not compare the vacuum energy densities between the two sets in (6.1) or (6.2). This degenerates may be lifted at the higher loop correction. For the moment, we compare the energy differences between in (6.1) and (6.2). We compare the minimal values of effective potential between the boundary condition sets on the left sides in (6.1) and (6.2) and suppose the degenerates in (6.1), (6.2) are lifted by higher loop corrections.

We call the boundary condition set on the left side in (6.1) Type (I) and the boundary condition set on the left side in (6.2) Type (II). We introduce the notation

$$P \equiv \{-1, -1, -1, +1, +1\}. \tag{6.4}$$

For Type (I), the boundary conditions of fermions in **5** representation are written as

$$\begin{aligned} \psi^5(x, -y) &= \pm P\gamma_5\psi(x, y) \\ \psi^5(x, \pi R - y) &= \pm e^{-i\pi\beta_\psi^5} P\gamma_5\psi(x, \pi R + y) \\ \psi^5(x, y + 2\pi R) &= e^{-i\pi\beta_\psi^5}\psi(x, y), \end{aligned} \tag{6.5}$$

where $\beta_\psi^5 = 0$ or 1. For scalar fields, the boundary condition set is given by (6.3), where P_0 and P_1 are replaced by P's. The interchanging overall constant $+$ and $-$ corresponds to interchanging the mode expansions between right- and left-handed for fermions or ϕ and ϕ' for scalar pairs, therefore give the identical contributions to the effective potential. Therefore we do not distinguish $\pm$ signs for overall constants from now on. We suppose there are n_f and m_f fermion fields with $\beta_\psi^5 = 0, 1$ in **5** representation respectively, and n_s and m_s scalar fields with $\beta_\phi = 0, 1$ in (6.3) respectively. For Type (II), the boundary condition sets of fermion in **5** representation and scalar fields are added by extra minus sign for $(x, \pi R - y) \to (x, \pi R + y)$ parity transformation compared with the boundary condition sets for Type (I). It means that we take the replacement $n_f \leftrightarrow m_f$ and $n_s \leftrightarrow m_s$ in order to change from Type (I) to Type (II) boundary condition sets. For fermion fields in **10** representation, the boundary condition set is represented by

$$\begin{aligned} \psi_{ij}^{10}(x, -y) &= \pm(P)_i^{i'}(P)_j^{j'}\gamma_5\psi_{i'j'}(x, y) \\ \psi_{ij}^{10}(x, \pi R - y) &= \pm e^{-i\pi\beta_\psi^{10}}(P)_i^{i'}(P)_j^{j'}\gamma_5\psi_{i'j'}(x, \pi R + y) \\ \psi_{ij}^{10}(x, y + 2\pi R) &= e^{-i\pi\beta_\psi^{10}}\psi_{ij}(x, y), \end{aligned} \tag{6.6}$$

where $\beta_\psi^{10} = 0$ or 1. Since changing from P to $-P$ preserve the boundary conditions (6.6), there is no difference for changing from Type (I) to (II). We suppose there are p_f, q_f fermion fields in **10** representation with $\beta_\psi^{10} = 0, 1$ respectively in Type (I) and (II) case.

We can calculate the contributions to effective potential from gauge and matter fields for the boundary conditions (6.1) and (6.2). For simplicity, we suppose all fermion and scalar fields are massless, then the Lagrangian density is given by (5.1). The detail of calculation is summarized in Appendix

B and C. For gauge and ghost fields, the contribution to effective potential is given by

$$V_{eff}^{g+gh}(a,b) = -3C\big\{f_5(a)+f_5(b)+f_5(a+b)+f_5(a-b)+\frac{1}{2}f_5(2a)+\frac{1}{2}f_5(2b)\big\}, \tag{6.7}$$

where

$$C = \frac{3}{64\pi^7 R^5}, \qquad f_5(x) = \sum_{n=1}^{\infty} \frac{\cos(n\pi x)-1}{n^5}. \tag{6.8}$$

There are the other a, b independent contributions. Since these term will not contribute to our analysis, we dropped these terms. For a fermion field in **5** representation, the contribution to effective potential is given by

$$V_{eff}^{5\ fermion}(a,b) = C\big\{f_5(\beta_\psi^5)+2f_5(a-\beta_\psi^5)+2f_5(b-\beta_\psi^5)+5\int \frac{d^4p}{(2\pi)^4}\ln\Big[p^2+(\frac{n}{R})^2\Big]\big\} \tag{6.9}$$

For scalar field pairs with the boundary condition set given by (6.3) which is replaced $P_0, P_1 \to P$'s, the contribution to effective potential is given by (6.9) with opposite sign for overall constant and replacement $\beta_\psi^5 \to \beta_\phi$. For fermion fields in **10** representation, the contribution to effective potential is

$$\begin{aligned} V_{eff}^{10\ fermion}(a,b) &= C\big\{2f_5(\beta_\psi^{10}) + 2f_5(a-\beta_\psi^{10}) + 2f_5(b-\beta_\psi^{10}) \\ &+2f_5(a+b-\beta_\psi^{10}) + 2f_5(a-b-\beta_\psi^{10}) + 10\int \frac{d^4p}{(2\pi)^4}\ln\Big[p^2+(\frac{n}{R})^2\Big]\big\} \end{aligned} \tag{6.10}$$

Then, we find that the effective potential $V_{eff}(a,b)$ is symmetric under $a \leftrightarrow b$, $a \to -a$ and $b \to -b$ and periodic in a and b: $V_{eff}(a+2,b) = V_{eff}(a,b+2) = V_{eff}(a,b)$.

There may be another contribution coming from the action $S(P_0,P_1)$ depending on P_0, P_1 variables. However these variables P_0, P_1 are independent on the spacetime coordinates, therefore $S(P_0,P_1)$ does not contribute to effective potential unless the coefficients of each term in $S(P_0,P_1)$ are proportional to the spacetime volume. Then we assume the coefficients in $S(P_0,P_1)$ are not proportional to the spacetime volume, and ignore these contribution to effective potential.

Let us evaluate the minimum of effective potential for various matter contents. The effective potential is minimized at $(a,b) = (0,0)$, $(1,1)$ or $(1,0)$ for enough small $n_f, m_f, n_s, m_s, p_f, q_f$, where the effective potential is symmetric under $a \leftrightarrow b$, so the effective potential at $(0,1)$ is the equivalent to the value at $(1,0)$. However for large $n_f, m_f, n_s, m_s, p_f, q_f$, this condition is not valid. The reason is that, for example $p_f = q_f = 1$ case, the contribution of fermions to effective potential is given by the sum of (6.10) with $\beta_\psi^{10} = 0$ and 1, and the summation of $a,\ b$ depending part of fermion contributions are almost canceled, but there still leaves small contribution with the delicate dependence on a and b. For $n_f = m_f = 1$ case, the similar effect takes place but is smaller than the effect for **10** representation. Then, if $n_f,\ m_f, p_f,\ q_f$ become large, this effect also be large and changes the minimal point slightly different from $(a,b) = (0,0),\ (1,1)$ or $(1,0)$. In fact, in the case $n_f = m_f = 10, n_s = m_s = 0, p_f = q_f = 8$ the minimal point of effective potential becomes slightly different from $(a,b) = (0,0)$. From now on, we evaluate the effective potential for small $n_f, m_f, n_s, m_s, p_f, q_f$ so that the effective potential is minimized at $(a,b) = (0,0),\ (1,1)$ or $(1,0)$. We denote the effective potential with Type (I) boundary condition as $V_{eff}^{I}(a,b)$ and $V_{eff}^{II}(a,b)$ for Type (II) boundary condition, and introduce the notations

$$\begin{aligned} N_1 &\equiv (n_f - m_f) - (n_s - m_s) \\ N_2 &\equiv p_f - q_f. \end{aligned} \tag{6.11}$$

The minimal points of effective potential and the difference between $V_{eff}^{I}(a,b)$ and $V_{eff}^{II}(a,b)$ depend on only these combinations $N_1,\ N_2$, so we evaluate $V_{eff}^{I}(a,b),\ V_{eff}^{II}(a,b)$ for each $N_1,\ N_2$ as input parameters. The values of $N_1,\ N_2$ are evaluated in the point of view for Type (I) boundary condition, so we substitute $-N_1,\ N_2$ for matter content on system in order to evaluate $V_{eff}^{II}(a,b)$.

For Type (I) case, when $N_1,\ N_2$ satisfy

$$N_1 + N_2 < \frac{3}{2} \quad \text{and} \quad N_1 + 3N_3 < \frac{9}{2}\ , \tag{6.12}$$

$V_{eff}^{I}(a,b)$ is minimized at $(a,b) = (0,0)$. The physical symmetry on system is $SU(3) \times SU(2) \times U(1)$. When $N_1,\ N_2$ satisfy

$$N_1 + N_2 > \frac{3}{2} \quad \text{and} \quad -N_1 + N_2 < \frac{3}{2}\ , \tag{6.13}$$

$V_{eff}^{I}(a,b)$ is minimized at $(a,b) = (1,1)$, and the physical symmetry is $[SU(2)]^2 \times [U(1)]^2$. When $N_1,\ N_2$ satisfy

$$N_1 + 3N_2 > \frac{9}{2} \quad \text{and} \quad -N_1 + N_2 > \frac{3}{2}\ , \tag{6.14}$$

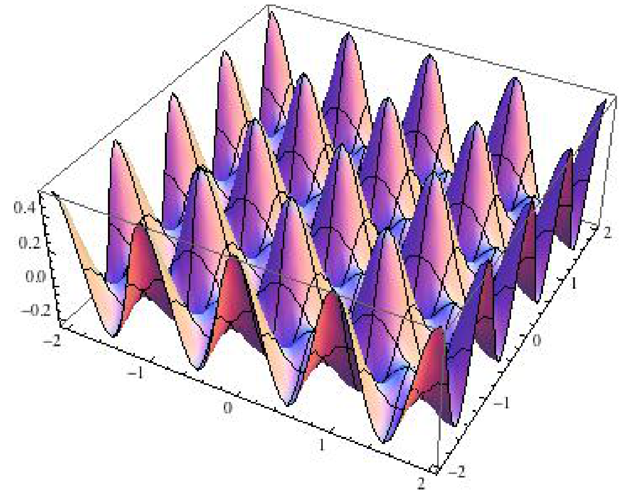

Figure 1: The $a,\ b$ depending part of summation of (6.10) with $\beta_\psi^{10} = 0$ and 1. The difference between maximum and minimum is much smaller than that of contribution from gauge part, but the getting number of matter larger the becoming this effect dominant.

$V_{eff}^I(a, b)$ is minimized at $(a, b) = (1, 0)$, and the physical symmetry is $SU(2) \times [U(1)]^3$.

In order to evaluate the minimum point of $V^{II}(a, b)$, we just take the replacement $N_1 \to -N_1$ in (6.12) - (6.14).

we compare the minimal value of $V_{eff}^I(a, b)$ to that of $V_{eff}^{II}(a, b)$ for each $N_1,\ N_2$ in (6.11). The results are summarized as the tables in Appendix D.

We evaluate KK-mode of fermion and scalar fields, and their zero modes correspond to the matter content of system at low energy scale. We introduce the notation $(\pm,\ \pm)$ whose signs represent even or odd under parity transformations $P_0,\ P_1$ for each component of field multiplets. We write down the parity assignment for " + " overall constant in (6.3), (6.5) and (6.6). For " − " overall constant, parity assignments in components are obtained by replacement $+ \leftrightarrow -$ for both parity transformations. For fermion case, only the components with $(+, +)$ or $(-, -)$ parity assignment have zero mode and may be regarded as SM particles. For fermion fields in **5** representation belonging to the boundary condition set (6.5),

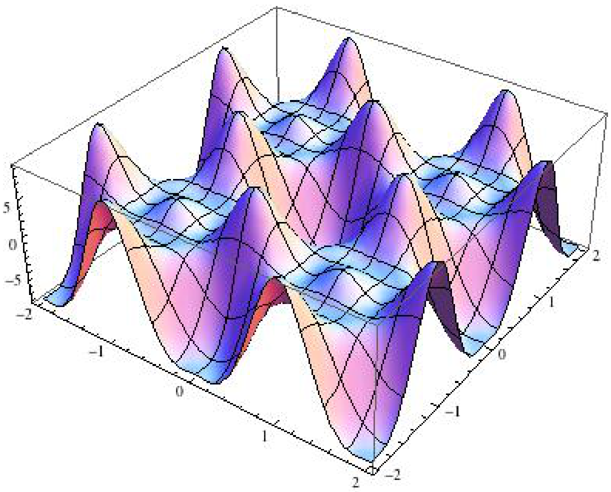

Figure 2: The effective potential $V_{eff}(a,b)/C$ for $n_f = m_f = 10,\ n_s = m_s = 0,\ p_f = q_f = 8$ case. The global minimum is located at $(a,b) = (0.1581,\ 0)$.

$$\psi^5 = \begin{pmatrix} (-,-) \\ (-,-) \\ (-,-) \\ (+,+) \\ (+,+) \end{pmatrix} \quad \text{for } \beta^5_\psi = 0, \tag{6.15}$$

and for $\beta^5_\psi = 1$ the parity assignments are given by taking the replacement $+ \leftrightarrow -$ for second parity in (6.15). For fermions in **10** representation belonging to boundary condition set (6.6),

$$\psi^{10} = \begin{pmatrix} & (+,+) & (+,+) & (-,-) & (-,-) \\ & & (+,+) & (-,-) & (-,-) \\ & & & (-,-) & (-,-) \\ & & & & (+,+) \\ & & & & \end{pmatrix} \quad \text{for } \beta^{10}_\psi = 0, \tag{6.16}$$

for $\beta^{10}_\psi = 1$ parity assignments are also obtained by the replacement $+ \leftrightarrow -$ for second parity in (6.16). For scalar field pairs, only components with $(+,+)$ parity assignment has zero modes, and for boundary condition set

(6.3) which is replaced $P_0,\ P_1$ with P's,

$$\phi = \begin{pmatrix} (-,-) \\ (-,-) \\ (-,-) \\ (+,+) \\ (+,+) \end{pmatrix} \quad \text{plus} \quad \phi' = \begin{pmatrix} (+,+) \\ (+,+) \\ (+,+) \\ (-,-) \\ (-,-) \end{pmatrix} \quad \text{for } \beta_\psi = 0 \qquad (6.17)$$

the replacement $+ \leftrightarrow -$ for second parity leads to the parity assignment for $\beta_\psi = 1$.

We investigate the two types of matter content on system with the boundary conditions which has the lower vacuum energy density: The one is that only Higgs field exists in bulk five dimensional spacetime and fermion fields are located on one fixed point. The other is that Higgs and fermion fields of SM particles live in the bulk, and in this case we need three fermions in **5** and **10** representations in parity assignments (6.15), (6.16) and one Higgs doublet. The former case corresponds to, for example $n_s = 1,\ n_f = m_f = m_s = p_f = q_f = 0$ (The Higgs field must arise in pair due to the condition (6.3)). In this case, the physical symmetry is the SM symmetry. The later case corresponds to, for example $n_f = 3,\ m_f \geq 3,\ p_f = 3,\ q_f \geq 2,\ n_s = 1,\ m_s = 0$, where $m_f,\ q_f$ fermions are needed for vacuum expectation values to be stable at the point which leads to SM gauge symmetry, and these fields do not be observed at low energy since they have no zero mode for any component. In both case, Type (I) boundary condition have the lower vacuum energy density and should be physically realized.

6.2 Supersymmetric $SU(5)$

Let us consider the supersymmetric $SU(5)$ case. In this case, the energy difference of effective potential for different boundary condition sets is always finite. Therefore our formulation become more natural in supersymmetric case. However if the system has exact supersymmetry, the contributions from bosonic and fermionic degree of freedom to effective potential are completely canceled at one loop level, so we require some supersymmetry-breaking mechanism in order to evaluate non-trivial effective potentials. We introduce soft SUSY breaking terms due to the Scherk-Schwarz SUSY breaking mechanism[61] since it gets the evaluation of effective potentials easier. The Scherk-Schwarz SUSY breaking on orbifold was investigated in Ref [62]-

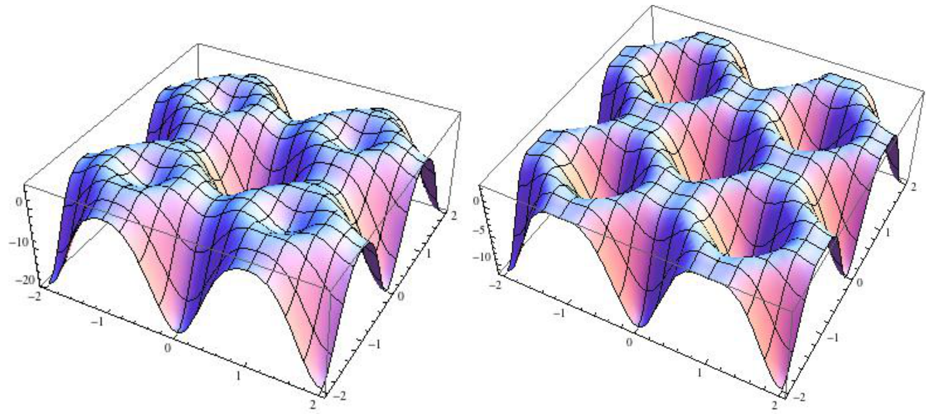

Figure 3: The evaluations of $V^{I}_{eff}(a,b)/C$ and $V^{II}_{eff}(a,b)/C$ in the matter content $n_s = 1$, $n_f = m_f = m_s = p_f = q_f = 0$. The left and right figures indicate $V^{I}_{eff}(a,b)/C$ and $V^{II}_{eff}(a,b)/C$ respectively, and in both case the global minimums are located at $(0,0)$. Type (I) boundary condition have the lower vacuum energy density.

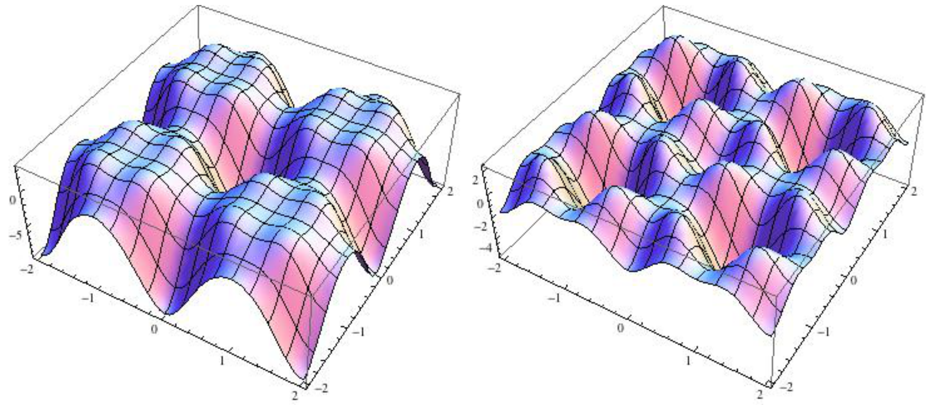

Figure 4: The evaluations of $V^{I}_{eff}(a,b)/C$ and $V^{II}_{eff}(a,b)/C$ in the matter content $n_f = m_f = p_f = 3$, $q_f = 2$, $n_s = 1$, $m_s = 0$. The left and right figures indicate $V^{I}_{eff}(a,b)/C$ and $V^{II}_{eff}(a,b)/C$ respectively. The global minimums are located at $(0,0)$ for $V^{I}_{eff}(a,b)$, and at $(1,1)$ for $V^{II}_{eff}(a,b)$. Type (I) boundary condition have the lower energy density.

[67]. The Hosotani mechanism with Scherk-Schwarz SUSY breaking was also analyzed on $M^3 \times S^1$ in Ref [68].

Five dimensional $N = 1$ SUSY corresponds to four dimensional $N = 2$ SUSY after dimensional reduction. A five dimensional gauge multiplet is given by

$$\mathcal{V} = (A^M,\ \lambda,\ \lambda',\ \sigma), \tag{6.18}$$

and it is decomposed into a vector superfield and chiral superfield

$$V = (A^\mu,\ \lambda), \quad \Sigma = (\sigma + iA^y,\ \lambda') \tag{6.19}$$

in four dimensional point of view. Five dimensional matter fields come in a hypermultiplet

$$\mathcal{H} = (h,\ h^{c\dagger},\ \tilde{h},\ \tilde{h}^{c\dagger}), \tag{6.20}$$

and it represented by two chiral super fields

$$H = (h,\ \tilde{h}), \quad H^c = (h^c,\ \tilde{h}^c). \tag{6.21}$$

H and H^c have conjugate transformation under the gauge group.

Under the requirement that Lagrangian density must be single valued and satisfy orbifold condition, the boundary conditions are written as, for gauge multiplet

$$\begin{aligned}
\begin{pmatrix} V \\ \Sigma \end{pmatrix}(x,-y) &= P_0 \begin{pmatrix} V \\ -\Sigma \end{pmatrix}(x,y)P_0^\dagger, \\
\begin{pmatrix} A_\mu \\ A_y \end{pmatrix}(x,\pi R - y) &= P_1 \begin{pmatrix} A_\mu \\ -A_y \end{pmatrix}(x,\pi R + y)P_1^\dagger, \\
\begin{pmatrix} \lambda \\ \lambda' \end{pmatrix}(x,\pi R - y) &= e^{-2\pi i\alpha\tau_2} P_1 \begin{pmatrix} \lambda \\ -\lambda' \end{pmatrix}(x,\pi R + y)P_1^\dagger, \\
\sigma(x,\pi R - y) &= -P_1 \sigma(x,\pi R + y)P_1^\dagger, \\
A_M(x,y+2\pi R) &= U A_M(x,y)U^\dagger, \\
\begin{pmatrix} \lambda \\ \lambda' \end{pmatrix}(x,y+2\pi R) &= e^{-2\pi i\alpha\tau_2} U \begin{pmatrix} \lambda \\ -\lambda' \end{pmatrix}(x,y)U^\dagger, \\
\sigma(x,y+2\pi R) &= U\sigma(x,y)U^\dagger,
\end{aligned} \tag{6.22}$$

and for hypermultiplet $\mathcal{H}$

$$\begin{aligned}
\begin{pmatrix} h \\ h^{c\dagger} \end{pmatrix}(x,-y) &= \pm T_{\mathcal{H}}[P_0]\begin{pmatrix} h \\ -h^{c\dagger} \end{pmatrix}(x,y), \\
\begin{pmatrix} h \\ h^{c\dagger} \end{pmatrix}(x,\pi R-y) &= \pm e^{-2\pi i(\alpha+\frac{1}{2}\beta)\tau_2} T_{\mathcal{H}}[P_1]\begin{pmatrix} h \\ -h^{c\dagger} \end{pmatrix}(x,\pi R+y), \\
\begin{pmatrix} h \\ h^{c\dagger} \end{pmatrix}(x,y+2\pi R) &= e^{-2\pi i(\alpha+\frac{1}{2}\beta)\tau_2} T_{\mathcal{H}}[U]\begin{pmatrix} h \\ h^{c\dagger} \end{pmatrix}(x,y),
\end{aligned}$$

$$\begin{aligned}
\begin{pmatrix} \tilde{h} \\ \tilde{h}^{c\dagger} \end{pmatrix}(x,-y) &= \pm T_{\mathcal{H}}[P_0]\begin{pmatrix} \tilde{h} \\ -\tilde{h}^{c\dagger} \end{pmatrix}(x,y), \\
\begin{pmatrix} \tilde{h} \\ \tilde{h}^{c\dagger} \end{pmatrix}(x,\pi R-y) &= \pm e^{-i\pi\beta} T_{\mathcal{H}}[P_1]\begin{pmatrix} \tilde{h} \\ -\tilde{h}^{c\dagger} \end{pmatrix}(x,\pi R+y), \\
\begin{pmatrix} \tilde{h} \\ \tilde{h}^{c\dagger} \end{pmatrix}(x,y+2\pi R) &= e^{-i\pi\beta} T_{\mathcal{H}}[U]\begin{pmatrix} \tilde{h} \\ \tilde{h}^{c\dagger} \end{pmatrix}(x,y),
\end{aligned} \tag{6.23}$$

where $\beta = 0$ or 1. For $(\lambda,\ \lambda')$ and $(h,\ h^{c\dagger})$ pairs, non-trivial twist on $SU(2)_R$ space is allowed for loop translation on S^1, introducing soft SUSY breaking terms parameterized by real-valued parameter α. $T_{\mathcal{H}}$ denotes the appropriate representation matrices and we consider only **5** and **10** representations just as in the non-SUSY case. In order to get the effective potential for Type (I), (II) boundary conditions, we need to evaluate the contributions to effective potential from each field where $P_0,\ P_1$ are replaced by $P = \{-1,-1,-1,+1,+1\}$ in (6.22), (6.23). All matter contributions for Type (I), (II) are obtained by the combination of overall constant $\pm$ and $\beta_{\mathcal{H}} = 0,\ 1$ and the contribution depend on $\beta_{\mathcal{H}}$ but not overall constant $\pm$. We suppose there are n_5 and m_5 hypermultiplet in **5** representation with $\beta_{\mathcal{H}} = 0$ and 1 respectively, and n_{10} and m_{10} hypermultiplet in **10** representation with $\beta_{\mathcal{H}} = 0$ and 1 in Type (I) boundary conditions. Changing from Type (I) to Type (II) corresponds to replacement $n_5 \leftrightarrow m_5$.

The effective potential at one loop level can be calculated just as non-

SUSY case. The contribution from gauge multiplet is given by

$$\begin{aligned}
V^g_{eff}(a,b) = -2C\{&2f_5(a) - f_5(a+2\alpha) - f_5(a-2\alpha) + 2f_5(b) - f_5(b+2\alpha) \\
&-f_5(b-2\alpha) + 2f_5(a+b) - f_5(a+b+2\alpha) - f_5(a+b-2\alpha) + 2f_5(a-b) \\
&-f_5(a-b+2\alpha) - f_5(a-b-2\alpha) + f_5(2a) - \frac{1}{2}f_5(2a+2\alpha) - \frac{1}{2}f_5(2a-2\alpha) \\
&+f_5(2b) - \frac{1}{2}f_5(2b+2\alpha) - \frac{1}{2}f_5(2b-2\alpha)\} \\
\approx -8\pi^2 C\alpha^2\{&g_3(a) + g_3(b) + g_3(a+b) + g_3(a-b) + \frac{1}{2}g_3(2a) + \frac{1}{2}g_3(2b)\} \\
&\text{for } \alpha \ll 1,
\end{aligned} \tag{6.24}$$

where we dropped $a,\ b$ independent part, and use the notation

$$g_3(x) \equiv \sum_{n=1}^{\infty} \frac{\cos(n\pi x)}{n^3}. \tag{6.25}$$

The contribution from hypermultiplet in **5** representation is

$$\begin{aligned}
V^5_{eff}(a,b) = 2C\{&\frac{1}{2}f_5(\beta^5) - \frac{1}{2}f_5(2\alpha-\beta^5) + f_5(a-\beta^5) - \frac{1}{2}f_5(a+2\alpha-\beta^5) \\
&-\frac{1}{2}f_5(a-2\alpha-\beta^5) + f_5(b-\beta^5) - \frac{1}{2}f_5(b+2\alpha-\beta^5) - \frac{1}{2}f_5(b-2\alpha-\beta^5)\} \\
\approx 4\pi^2 C\alpha^2\{&\frac{1}{2}g_3(\beta^5) + g_3(a-\beta^5) + g_3(b-\beta^5)\} \qquad \text{for } \alpha \ll 1,
\end{aligned} \tag{6.26}$$

and for hypermultiplet in **10** representation,

$$\begin{aligned}
V^{10}_{eff}(a,b) = 2C\{&f_5(\beta^{10}) - f_5(2\alpha-\beta^{10}) + f_5(a-\beta^{10}) - \frac{1}{2}f_5(a+2\alpha-\beta^{10}) \\
&-\frac{1}{2}f_5(a-2\alpha-\beta^{10}) + f_5(b-\beta^{10}) - \frac{1}{2}f_5(b+2\alpha-\beta^{10}) - \frac{1}{2}f_5(b-2\alpha-\beta^{10}) \\
&+f_5(a+b-\beta^{10}) - \frac{1}{2}f_5(a+b+2\alpha-\beta^{10}) - \frac{1}{2}f_5(a+b-2\alpha-\beta^{10}) \\
&+f_5(a-b-\beta^{10}) - \frac{1}{2}f_5(a-b+2\alpha-\beta^{10}) - \frac{1}{2}f_5(a-b-2\alpha-\beta^{10})\} \\
\approx 4\pi^2 C\alpha^2\{&g_3(\beta^5) + g_3(a-\beta^5) + g_3(b-\beta^5) + g_3(a+b-\beta^5) + g_3(a-b-\beta^5)\} \\
&\text{for } \alpha \ll 1.
\end{aligned} \tag{6.27}$$

The logarithmic divergent part is canceled between fermion and bosonic degree of freedom. We can determine the physically realized boundary condition by these formula (6.24)-(6.27) for each $n_5,\ m_5,\ n_{10},\ m_{10}$. The minimal point of effective potential highly depends on the matter content $n_5,\ m_5,\ n_{10},\ m_{10}$ and the SUSY breaking parameter α. Therefore, the systematic evaluation of effective potential as we used in the non-SUSY case is difficult. Then, we investigate only two interest case. The first case is that there are only gauge and Higgs fields in bulk region and fermion fields are located on a fixed point. We suppose two Higgs hypermultiplets are introduced in **5** representation, given by the boundary conditions,

$$\begin{aligned} \begin{pmatrix} h_1 & h_2 \\ h_1^{c\dagger} & h_2^{c\dagger} \end{pmatrix}(x,-y) &= P_0 \begin{pmatrix} h_1 & -h_2 \\ -h_1^{c\dagger} & h_2^{c\dagger} \end{pmatrix}(x,y), \\ \begin{pmatrix} h_1 & h_2 \\ h_1^{c\dagger} & h_2^{c\dagger} \end{pmatrix}(x,\pi R-y) &= e^{-2\pi i\alpha\tau_2} P_1 \begin{pmatrix} h_1 & -h_2 \\ -h_1^{c\dagger} & h_2^{c\dagger} \end{pmatrix}(x,\pi R+y), \\ \begin{pmatrix} h_1 & h_2 \\ h_1^{c\dagger} & h_2^{c\dagger} \end{pmatrix}(x,y+2\pi R) &= e^{-2\pi i\alpha\tau_2} U \begin{pmatrix} h_1 & h_2 \\ h_1^{c\dagger} & h_2^{c\dagger} \end{pmatrix}(x,y). \end{aligned} \tag{6.28}$$

This model has been investigated in Ref [62]. Under the boundary conditions (6.28) these hypermultiplets are expanded as

$$\begin{aligned} \begin{pmatrix} h_{1,j} & h_{2,j} \\ h_{1,j}^{c\dagger} & h_{2,j}^{c\dagger} \end{pmatrix}(x,y) &= \frac{1}{\sqrt{\pi R}} \sum_{n=-\infty}^{\infty} \begin{pmatrix} h_{1,j}(x)\sin\frac{n-\alpha}{R}y & h_{2,j}(x)\cos\frac{n+\alpha}{R}y \\ h_{1,j}(x)\cos\frac{n-\alpha}{R}y & h_{2,j}(x)\sin\frac{n+\alpha}{R}y \end{pmatrix} \\ \begin{pmatrix} h_{1,i} & h_{2,i} \\ h_{1,i}^{c\dagger} & h_{2,i}^{c\dagger} \end{pmatrix}(x,y) &= \frac{1}{\sqrt{\pi R}} \sum_{n=-\infty}^{\infty} \begin{pmatrix} h_{1,i}(x)\cos\frac{n+\alpha}{R}y & h_{2,i}(x)\sin\frac{n-\alpha}{R}y \\ h_{1,i}(x)\sin\frac{n+\alpha}{R}y & h_{2,i}(x)\cos\frac{n-\alpha}{R}y \end{pmatrix}, \end{aligned} \tag{6.29}$$

where $j = 1,2,3$, $i = 4,5$ and these indices denote gauge indices. And these fields lead to soft SUSY breaking mass terms

$$\begin{aligned} &\sum_{n=-\infty}^{\infty} \left(\frac{n-\alpha}{R}\right)^2, \quad \text{for} \ \ h_{1,j},\ h_{2,i} \\ &\sum_{n=-\infty}^{\infty} \left(\frac{n+\alpha}{R}\right)^2, \quad \text{for} \ \ h_{1,i},\ h_{2,j}, \end{aligned} \tag{6.30}$$

and suppose SUSY breaking scale is much smaller than scale $\mathcal{O}(\frac{1}{R})$, namely $\alpha \ll 1$. This Higgs matter content is realized by, for example $n_5 = 2,\ m_5 =$

3 case, preserving SM symmetry unbroken. $m_5 = 3$ field are needed for preserving SM symmetry and boundary condition structure, and for these fields scalar components acquire SUSY breaking mass terms

$$\sum_{n=-\infty}^{\infty}\left(\frac{n-\alpha-\frac{1}{2}}{R}\right)^2 \quad \text{or} \quad \sum_{n=-\infty}^{\infty}\left(\frac{n+\alpha+\frac{1}{2}}{R}\right)^2, \tag{6.31}$$

and fermion superpertners have no zero mode, so these fields do not the low energy physics for $\alpha \ll 1$.

The second case is that all SM particles live in bulk region. In this case if Type (I) boundary condition is physically realized, we need $n_5 = 5,\ m_5 = 3$ fields for Higgs and fermion fields. However these matters make the point $(a, b) = (0, 0)$ unstable because the contributions of effective potential for $n_5,\ n_{10}$ fields have the minimum points at $(a, b) = (1, 1),\ (1, 0)$ respectively. Hence we need to put $m_5,\ m_{10}$ fields in system since the contributions of these fields have the minimum point at $(a, b) = (0, 0)$. Then we must evaluate whether the point $(a, b) = (0, 0)$ can be stable, in other words evaluate

$$\begin{aligned}\frac{\partial^2}{\partial a^2}V_{eff}^I(0,0) = \frac{\partial^2}{\partial b^2}V_{eff}^I(0,0) &= 2\pi^2 C\Big\{(10-n_5-3n_{10})\sum_{n=1}^{\infty}\frac{1}{n^3}(1-\cos[n\pi 2\alpha]) \\ &\qquad -(m_5+3m_{10})\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}(1-\cos[n\pi 2\alpha])\Big\} \\ &\approx \pi^4 C\Big\{(10-n_5-3n_{10})(-\log 2\alpha+\frac{3}{2})+(m_5+3m_{10})\log 2\Big\}(2\alpha)^2,\end{aligned} \tag{6.32}$$

for small α and we used the formula

$$\sum_{n=1}^{\infty}\frac{1-\cos(n\pi 2\alpha)}{n^3} \approx (-\frac{1}{2}\log 2\alpha+\frac{3}{4})(2\alpha)^2. \tag{6.33}$$

In the first case, the first and second terms in (6.32) are positive, so the point $(a, b) = (0, 0)$ is indeed stable. In the second case, the first term is the negative value and its coefficient become large for small α compared with the coefficient for second term. As a result, we need huge number of $m_5,\ m_{10}$ fields in order to make the point $(a, b) = (0, 0)$ stable. It means that it is difficult to realize SM particles in SM gauge symmetry.

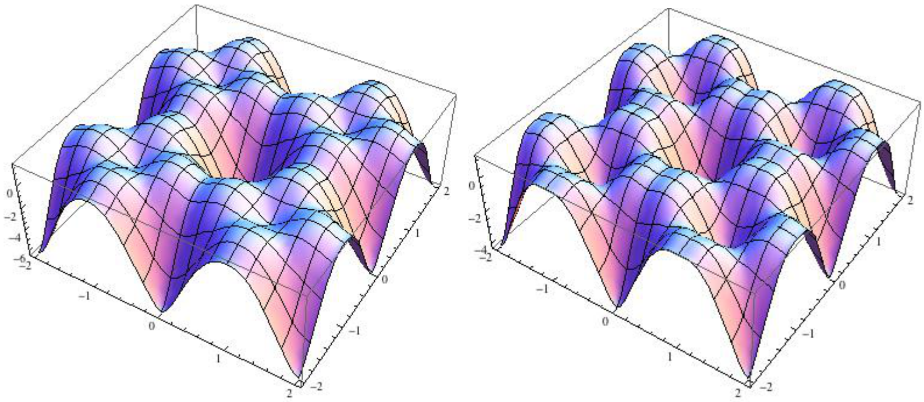

Figure 5: The evaluation of $V^{I}_{eff}(a,b)/8\pi^2 C\alpha^2$ and $V^{II}_{eff}(a,b)/8\pi^2 C\alpha^2$ in $n_5 = 2$, $m_5 = 3$, $n_{10} = m_{10} = 0$. The left and right figures indicate $V^{I}_{eff}(a,b)/8\pi^2 C\alpha^2$ and $V^{II}_{eff}(a,b)/8\pi^2 C\alpha^2$ respectively, and in both case the global minimums are located at $(a,b) = (0,0)$. In this matter content, Type (I) boundary condition has the lower energy density.

7 Conclusion and discussion

In present study of field theory defined on extra dimension, the boundary conditions imposed on fields are given by hand even though these boundary conditions play major role to determine the physical properties on system. Therefore, we constructed the higher dimensional field theory including the dynamics of boundary conditions in order to determine the physically realized boundary conditions from theoretical consequence. The process of determining the physically realized boundary conditions consists of two part: (i) The restriction of equivalence classes from the property of invariant measure. (ii) The selection of physically realized equivalence class by comparing vacuum energy densities among equivalence classes.

The first part is discussed in section 4. In section 4, we gave the formulation of model with the dynamics of boundary conditions for $SU(N)$ gauge theory. This dynamics is defined by path integral formula, namely integrate over all possible configurations of boundary conditions with some weighting action. The integration is defined by invariant measure for $U(N)$ group. By investigating the property of invariant measure, we have revealed that only restricted class of boundary conditions practically contributes to the partition function of system. Especially, for $SU(5)$ case we found that the symmetry breaking to SM gauge symmetry $SU(5) \to SU(3) \times SU(2) \times U(1)$ is naturally included in our model.

The second part is discussed in section 6. In section 6 we determined the physically realized equivalence class in the $SU(5)$ case by comparing minimal energy densities among the equivalence classes which practically contribute to partition function on system. Then we investigated non-supersymmetric and supersymmetric cases. In non-supersymmetric case, we encounter the problem that the difference of minimums for effective potentials is divergent between equivalence classes. We solved this problem by putting scalar fields in pairs. We can produce the SM fermion matter content with a pair of Higgs fields in the case $n_f = 3,\ m_f \geq 3,\ p_f = 3,\ q_f \geq 2,\ n_s = 1,\ m_s = 0$. There leaves a problem that color triplet components in Higgs field must be heavy compared with doublet components. In supersymmetric case, we can avoid the divergence problem for effective potential, therefore our formulation becomes more natural. We considered two cases: Only gauge and Higgs fields live in the bulk region, and all SM particle live in the bulk. The former case can be realized in the case $n_5 = 2,\ m_5 = 3$ but the later case can not, because the second derivative of effective potential become negative in any matter content for small SUSY breaking parameter α so we can not realize SM gauge symmetry in this case.

In our research, we assume that the boundary condition are governed by some dynamics. Then under this assumption we give one criterion to determine the physically realized boundary conditions. Unfortunately, at present we do not know whether or not the boundary conditions should be treated as dynamical values. To discuss that, we need to understand the more fundamental theory, like string theory, and get the further knowledges about the spacetime dynamics. From the spacetime dynamics we should discuss the boundary conditions are determine from some dynamics or the other principle. It may be possible that our model is generalized to the other classical groups, like $SO(N)$. It is the nontrivial work because our model for the $SU(N)$ case is strongly governed by the structure of manifold for $U(N)$ group, namely global structure of $U(N)$ group. Therefore, for example $SO(N)$ case, the result depends on the structure of manifold for $O(N)$ and this structure is more complicate than $U(N)$ case. Then we expect the different result from $SU(N)$ case.

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A Transformation of boundary conditions P_0, P_1 into diagonal representations

In this appendix, we give the proof that every equivalence class has both diagonal P_0 and P_1 representation for $SU(N)$ gauge theory on S^1/Z_2 as I mentioned in subsection 3.2. The calculation follows Ref [56].

P_0 and P_1 are unitary and Hermitian matrices. Then, Under appropriate global $SU(N)$ gauge transformation, P_0, P_1 matrices transform to

$$P_0 = \begin{pmatrix} I_m & \\ & -I_n \end{pmatrix}, \quad P_1 = \begin{pmatrix} A & C^\dagger \\ C & B \end{pmatrix}. \tag{A.1}$$

I_m denote unit matrix and $m + n = N$. Moreover, we can diagonalize A and B matrices since P_0 has still global gauge invariance under $SU(m) \times SU(n)$ subgroup. Hence we get

$$P_1 = \begin{pmatrix} a_1 & & & & \vec{c}_1^\dagger & \\ & \ddots & & & \vdots & \\ & & a_m & & \vec{c}_m^\dagger & \\ & & & b_1 & & \\ \vec{c}_1 & \cdots & \vec{c}_m & & \ddots & \\ & & & & & b_n \end{pmatrix} = \begin{pmatrix} a_1 & & & & & \\ & \ddots & & \vec{d}_1 & \cdots & \vec{d}_n \\ & & a_m & & & \\ & \vec{d}_1^\dagger & & b_1 & & \\ & \vdots & & & \ddots & \\ & \vec{d}_n^\dagger & & & & b_n \end{pmatrix}. \tag{A.2}$$

The variables in (A.2) must satisfy the conditions

$$\begin{array}{c} a_j^2 + \vec{c}_j^\dagger \vec{c}_j = 1, \quad b_j^2 + \vec{d}_j^\dagger \vec{d}_j = 1 \\ \vec{c}_j^\dagger \vec{c}_k = \vec{d}_j^\dagger \vec{d}_k = 0 \quad \text{if} j \neq k, \end{array} \tag{A.3}$$

by $P_1^2 = 1$. Let us suppose the rank of C is r. It means that only r vectors for $\vec{c}_j$ or $\vec{d}_j$ can be linearly independent, and from the condition (A.3) the other $(m-r)$ vectors for $\vec{c}_j$ or $(n-r)$ vectors for $\vec{d}_j$ become zero vectors. Therefore after a appropriate rearrangement of rows and columns the matrix C has the form

$$C = \left(\begin{array}{c|c} \tilde{C} & 0 \\ \hline 0 & 0 \end{array}\right), \tag{A.4}$$

where $\tilde{C}$ denote rank r, $r \times r$ matrix. Then, $\tilde{C}$ consist of r linearly independent vectors $\vec{e}_1, \cdots, \vec{e}_r$ and these vectors are orthogonal each other. By the rearrangement of rows and columns we can take these vectors into the form that the first component of $\vec{e}_1$ is non-zero, the second component of $\vec{e}_2$ is non-zero, and so on. Through the further rearrangement, we can take P_1 in the form

$$P_1 = \begin{pmatrix} \tilde{P}_1 & \\ & \hat{I}_{N-2r} \end{pmatrix}, \tag{A.5}$$

where

$$\tilde{P}_0 = \begin{pmatrix} a_1 & & & & & \\ & \ddots & & & \tilde{C}'^{\dagger} & \\ & & a_r & & & \\ & & & -a_1 & & \\ & \tilde{C}' & & & \ddots & \\ & & & & & -a_r \end{pmatrix}, \tag{A.6}$$

where $\tilde{C}'$ denotes the transformed $\tilde{C}$ so that the first component of $\vec{e}_1$ is non-zero, the second component of $\vec{e}_2$ is non-zero and so on. $\hat{I}_{N-2r}$ is a diagonal matrix with diagonal elements $+1$ or -1. We can find (A.6) because from the condition $(\tilde{P}_1)^2 = 1$ it follows that

$$(a_j + b_k)\tilde{C}'_{jk} = 0 \quad (1 \le j, k \le r) \tag{A.7}$$

Then, we can find $a_j = -b_j$ since $\tilde{C}'_{jj} \neq 0$. Therefore $\tilde{P}_0$ must have the form of (A.6). We note that $\tilde{C}_{jk} = 0$ if $a_j \neq -b_k$. By collecting the same values a_l in $\{a_j,\ j = 1 \sim r\}$ and the corresponding $-a_l$, $\tilde{P}_1$ can be transformed into the following block diagonal form:

$$\tilde{P}_1 = \begin{pmatrix} \tilde{P}_1^{(1)} & & \\ & \ddots & \\ & & \tilde{P}_1^{(t)} \end{pmatrix}, \quad \tilde{P}_1^{(l)} = \begin{pmatrix} a_l I_{s_l} & C_l^{\dagger} \\ C_l & -a_l I_{s_l} \end{pmatrix}, \tag{A.8}$$

where $(s_1 + s_2 + \cdots + s_l = r)$. Under the unitary transformation, we can bring $\tilde{P}_1^{(l)}$ into

$$\begin{pmatrix} I_{s_l} & \\ & U^{(l)} \end{pmatrix} \tilde{P}_1^{(l)} \begin{pmatrix} I_{s_l} & \\ & U^{(l)\dagger} \end{pmatrix} = \begin{pmatrix} a_l I_{s_l} & \sqrt{1-a_l^2} I_{s_l} \\ \sqrt{1-a_l^2} I_{s_l} & -a_l I_{s_l} \end{pmatrix}, \quad \text{(A.9)}$$

where $U^{(l)}$ is given by the unitary matrix $\sqrt{1-a_l^2} C_l^\dagger$. Under this transformation we can see P_0 indeed remains invariant. Hence we get the block diagonal form

$$\begin{pmatrix} \hat{P}_1^{(l)} & & \\ & \hat{P}_1^{(l)} & \\ & & \ddots \end{pmatrix} = \begin{pmatrix} \cos\theta_l & \sin\theta_l & & & \\ \sin\theta_l & -\cos\theta_l & & & \\ & & \cos\theta_l & \sin\theta_l & \\ & & \sin\theta_l & -\cos\theta_l & \\ & & & & \ddots \end{pmatrix} \quad \text{(A.10)}$$

for each $\tilde{P}_1^{(l)}$. For each $\hat{P}_1^{(l)}$, the corresponding P_0 part is τ_3. Therefore, the local $SU(2)$ transformation $\Omega(y) = e^{i(\theta_l y/2\pi R)\tau_2}$ in this subspace transforms $\hat{P}_1^{(l)}$ into τ_3, remaining P_0 form invariant. This completes the poof.

B Calculation of the effective potential for SU(5)

In this appendix, we calculate the effective potential V_{eff} for the boundary conditions (6.1) and (6.2). We show only the fermion field in **5** representation case, but the calculation can be applied to the case for the field in adjoint and **10** representations. The calculation is based on Ref [55].

The boundary conditions for fermion field in **5** representation are given by

$$\begin{aligned} \psi^5(x,-y) &= \pm P\gamma_5\psi(x,y) \\ \psi^5(x,\pi R - y) &= \pm e^{-i\pi\beta_\psi^5} P\gamma_5\psi(x,\pi R + y) \\ \psi^5(x, y + 2\pi R) &= e^{-i\pi\beta_\psi^5}\psi(x,y), \end{aligned} \quad \text{(B.1)}$$

where $\beta_\psi^5 = 0$ or 1. The left- and right-handed components are defined by $\psi_L(x,y) = \frac{1}{2}(1-\gamma_5)\psi(x,y)$ and $\psi_R(x,y) = \frac{1}{2}(1+\gamma_5)\psi(x,y)$. We consider the "+" sign in (B.1), the case for "-" sign corresponds to the interchanging mode expansions between left- and right-handed fermions in "+"

sign case. The contribution of effective potential from fermion field corresponds to $-f(D)\frac{i}{2}\mathrm{Tr}\ln D_L^0 D^{0L}$. Then, we need to evaluate the eigenvalues of $D_L^0 D^{0L} = \partial_\mu \partial^\mu - D_y^2$. Since the expectation values of gauge field are parameterized in (5.7) and (5.11), $-D_y^2$ is given by

$$-D_y^2 = \begin{pmatrix} -\partial_y\partial_y + \left(\frac{a}{2R}\right)^2 & 0 & 0 & i\frac{a}{2R} & 0 \\ 0 & -\partial_y\partial_y + \left(\frac{b}{2R}\right)^2 & 0 & 0 & i\frac{b}{2R} \\ 0 & 0 & -\partial_y\partial_y & 0 & 0 \\ i\frac{a}{2R} & 0 & 0 & -\partial_y\partial_y + \left(\frac{a}{2R}\right)^2 & 0 \\ 0 & i\frac{b}{2R} & 0 & 0 & -\partial_y\partial_y + \left(\frac{b}{2R}\right)^2 \end{pmatrix}. \tag{B.2}$$

Then in the gauge components of fermion, the pairs (ψ_1, ψ_4) and (ψ_2, ψ_5) are mixed by $-D_y^0 D^{0y}$. For the pair (ψ_1, ψ_4) we expand as

$$\begin{pmatrix} \psi_1(x,y) \\ i\psi_4(x,y) \end{pmatrix} = \frac{1}{\sqrt{\pi R}} \sum_{n=-\infty}^{\infty} \left\{ \psi_L^{(n)}(x) \begin{pmatrix} \cos(n+\frac{1}{2}\beta_\psi^5)\frac{y}{R} \\ \sin(n+\frac{1}{2}\beta_\psi^5)\frac{y}{R} \end{pmatrix} + \psi_R^{(n)}(x) \begin{pmatrix} \sin(n-\frac{1}{2}\beta_\psi^5)\frac{y}{R} \\ \cos(n-\frac{1}{2}\beta_\psi^5)\frac{y}{R} \end{pmatrix} \right\}. \tag{B.3}$$

The expansion indeed satisfies the boundary condition (B.2). By substituting the expansion (B.3) to (B.2) and performing the integration of $\int dy$, we obtain the mass spectrum for (ψ_1, ψ_4) pair:

$$\sum_{n=-\infty}^{\infty} \frac{(n - \frac{1}{2}(a - \beta_\psi^5))^2}{R^2}, \quad \text{for } \psi_L^{(n)} \tag{B.4}$$

and

$$\sum_{n=-\infty}^{\infty} \frac{(n + \frac{1}{2}(a - \beta_\psi^5))^2}{R^2}, \quad \text{for } \psi_R^{(n)}. \tag{B.5}$$

The similar calculation can be applied to (ψ_2, ψ_5) pair. The mass spectrums are given by (B.4) and (B.5) where a is replaced by b. The contribution from ψ_3 is independent on a and b. The mass spectrum is obtained by $(n+\frac{1}{2}\beta)^2/R^2$ by combining contributions from the left- and right-handed components. Therefore, we summarize the contribution from fermion field in

5 representation $\mathrm{Tr}\ln D_M^0 D^{0M}$ as

$$\sum_{n=-\infty}^{\infty}\ln\left\{-p^2+\left(\frac{n+\frac{1}{2}\beta_\psi^5}{R}\right)^2\right\}+2\sum_{n=-\infty}^{\infty}\ln\left\{-p^2+\left(\frac{n-\frac{1}{2}(a-\beta_\psi^5)}{R}\right)^2\right.$$
$$\left.+2\sum_{n=-\infty}^{\infty}\ln\left\{-p^2+\left(\frac{n-\frac{1}{2}(b-\beta_\psi^5)}{R}\right)^2\right\},\tag{B.6}$$

where p^μ denotes four dimensional momentum and (B.6) is integrated on p^μ.

C Zeta regularization

In this appendix, we review the method for calculating the logarithmic function given in (B.6). It is called the zeta regularization. The calculation in this appendix follows Ref [69].

We focus on massless field case so consider the following function

$$V(\theta)=\frac{1}{2}\sum_{n=-\infty}^{\infty}\int\frac{d^{d-1}p}{(2\pi)^{d-1}}\ln\left[p^2+\frac{1}{R^2}\left(n+\frac{\theta}{2\pi}\right)^2\right].\tag{C.1}$$

We define the generalized zeta function $\zeta(s)$ by

$$\zeta(s)=\frac{1}{\Gamma(s)}\sum_{n=-\infty}^{\infty}\int_0^\infty dt t^{s-1}\int\frac{d^{d-1}p}{(2\pi)^{d-1}}\exp\left\{-t\left[p^2+\frac{1}{R^2}\left(n+\frac{\theta}{2\pi}\right)^2\right]\right\},\tag{C.2}$$

where $\Gamma(s)$ denotes the Gamma function and we can find $V(\theta)=-\zeta'(0)/2$. After integrating over p variable we find

$$\zeta(s)=\frac{\pi^{\frac{d-1}{2}}}{(2\pi)^{d-1}\Gamma(s)}\frac{2\pi R}{\sqrt{4\pi}}\sum_{n=-\infty}^{\infty}e^{in\theta}\int_0^\infty dt t^{s-\frac{d}{2}-1}\exp\left[-\frac{(2\pi R)^2n^2}{4t}\right],\tag{C.3}$$

where we used the poisson's summation formula

$$\sum_{n=-\infty}^{\infty}\exp\left[-t\left(\frac{2\pi n+\theta}{2\pi R}\right)^2\right]=\frac{2\pi R}{\sqrt{4\pi t}}\sum_{l=-\infty}^{\infty}\exp\left(-\frac{(2\pi R)^2l^2}{4t}+il\theta\right).\tag{C.4}$$

We find the term corresponding to $n=0$ is divergent, but since this term is independent on θ we can eliminate it by subtracting the constant $V(0)$.

In the $s \to 0$ limit Gamma function behaves as $\Gamma(s) \sim 1/s$. Then, $\zeta'(0)$ is obtained by

$$\zeta'(0) = \frac{\pi^{\frac{d-1}{2}} 2\pi R}{(2\pi)^{d-1}\sqrt{4\pi}} \sum_{n=-\infty}^{\infty} (e^{in\theta} - 1) \int_0^\infty dt t^{-\frac{d}{2}-1} \exp\left[-\frac{(2\pi R)^2 n^2}{4t} \right]. \quad \text{(C.5)}$$

Since the integral part in (C.5) is represented by

$$\int_0^\infty dt t^{-\frac{d}{2}-1} \exp\left[-\frac{(2\pi R)^2 n^2}{4t} \right] = \frac{2^d}{(2\pi R)^d n^d} \Gamma\left(\frac{d}{2}\right), \quad \text{(C.6)}$$

eventually we get the form

$$V(\theta) = -\frac{\Gamma(\frac{d}{2})}{\pi^{\frac{d}{2}} (2\pi R)^{d-1}} \sum_{n=1}^{\infty} \frac{\cos(n\theta) - 1}{n^d}. \quad \text{(C.7)}$$

D Comparison of energy densities between equivalence classes

In this appendix, we compare the vacuum energy densities between in Type (I) and Type (II) boundary conditions in (6.1) and (6.2) for non-SUSY case. We suppose that there are n_s, m_s scalar field pairs in the boundary conditions (6.3) with $\beta_\phi = 0$, 1 respectively, n_f, m_f fermion fields in (6.5) with $\beta_\psi^5 = 0$, 1 and p_f, q_f fermion fields in (6.6) with $\beta_\psi^{10} = 0$, 1 in the point of view of Type (I). We can get the matter content in Type (II) by the replacement $n_f \leftrightarrow m_f$ and $n_s \leftrightarrow m_s$ from the matter content in Type (I). We introduce the notations $N_1 = (n_f - m_f) - (n_s - m_s)$ and $N_2 = p_f - q_f$. Since the effective potentials $V_{eff}^I(a,b)$ and $V_{eff}^{II}(a,b)$ depend on only these combinations N_1, N_2, we evaluate $V_{eff}^I(a,b)$ and $V_{eff}^{II}(a,b)$ for each N_1, N_2 case. The result is summarized in Table 1. In the third rows, we give the equivalence class which has the lowest energy density between in Type (I) and Type (II), and the corresponding minimal point is given in the fourth rows.

The values of N_1, N_2 are evaluated in the point of view of Type (I) boundary condition. We substitute $-N_1$, N_2 in order to calculate $V_{eff}^{II}(a,b)$. We show only $N_1 > 0$ case, and $N_1 < 0$ case is identical to the positive N_1 but the results in the third rows are obtained by the replacement (I) $\leftrightarrow$ (II). $N_1 = 0$ case corresponds to completely degenerating between Type (I) and (II).

The vacuum expectation values $(a, b) = (0, 0),\ (1, 1)$ and $(1, 0)$ correspond to $SU(3) \times SU(2) \times U(1),\quad [SU(2)]^2 \times [U(1)]^2$ and $SU(2) \times [U(1)]^3$ physical symmetry on system respectively.

$N_1 = 1$				
N_2	$\cdots$ 0	1	2	3$\cdots$
BCs	(II)	(II)	(I),(II)	(II)
(a,b)	(0,0)	(0,0)	(1,0) or (1,1)	(1,0)

$N_1 = 2$						
N_2	$\cdots$ -1	0	1	2	3	4$\cdots$
BCs	(II)	(II)	(II)	(I),(II)	(II)	(II)
(a,b)	(0,0)	(0,0)	(0,0)	(0,0) or (1,1)	(1,0)	(1,0)

$N_1 = 3$								
N_2	$\cdots$ -2	-1	0	1	2	3	4	5$\cdots$
BCs	(II)	(II)	(II)	(II)	(II)	(I),(II)	(II)	(II)
(a,b)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0) or (1,1)	(1,0)	(1,0)

$N_1 = 4$										
N_2	$\cdots$ -3	-2	-1	0	1	2	3	4	5	6$\cdots$
BCs	(II)	(II)	(II)	(II)	(II)	(II)	(I)	(II)	(II)	(II)
(a,b)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,1)	(1,0)	(1,0)	(1,0)

$N_1 = 5$									
N_2	$\cdots$-2	-1	0	1	2	3	4	5	6$\cdots$
BCs	(II)	(II)	(II)	(II)	(II)	(I)	(I),(II)	(II)	(II)
(a,b)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,1)	(1,1) or (1,0)	(1,0)	(1,0)

Table 1: The determination of equivalence class from comparison of vacuum energy density between Type (I) and (II). The equivalence class with the lowest energy density and the corresponding minimum point are shown in the third and fourth rows respectively. The case that two minimal points of $(a,\ b)$ are put in a slot indicates the minimum energy densities degenerate between Type (I) and (II).

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