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Osaka University
SPECTRAL PROPERTIES OF THE LAPLACE OPERATOR IN $L^p(R)$

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1. Introduction. One of the useful tools for analyzing a linear operator $T$ in a Banach space $X$, if available, is a functional calculus. In general, no reasonable functional calculus may exist. If it is known that $T$ is a closed operator then there is available a restricted functional calculus for $T$ based on functions which are holomorphic in a neighbourhood of the spectrum $\sigma(T)$, of $T$, and have a limit at infinity, [4; Ch. VII]. To admit a richer functional calculus it would be expected that $T$ should satisfy some additional properties.

For $0<\alpha<\pi$, define the open cone $S_\alpha = \{z \in \mathbb{C} \setminus \{0\} \mid \arg(z) < \alpha\}$. A closed operator $T$ in $X$ is said to be of type $\omega$ [12], where $0 \leq \omega < \pi$, if $\sigma(T)^{\bar{\omega}}$ (the bar denotes closure and, by definition, $\bar{S}_\omega = [0, \infty]$) and, for $0 < \varepsilon < (\pi - \omega)$ there is a positive constant $c_\varepsilon$ such that

$$||R(\lambda; T)|| \lesssim c_\varepsilon |\lambda|^{-1}, \quad \lambda \in \bar{S}_{\omega + \varepsilon}.$$  

Here $R(\lambda; T)$ denotes the resolvent operator of $T$ at $\lambda$. We remark that $-T$, for the case $0 \leq \omega \leq \pi/2$, is the infinitesimal generator of a holomorphic semigroup [12; Theorems 3.3.1 and 3.3.2].

In the case when $X$ is a Hilbert space and $T$ is of type $\omega$ there are results of A. Yagi [13] and more recently, of A. McIntosh [10], which give conditions equivalent to the existence of a functional calculus for $T$ based on the algebra $H^\omega(S_{\omega+\varepsilon})$, for every $0 < \varepsilon < (\pi - \omega)$. For example, this is the case if the purely imaginary powers $T^{iu}$, $u \in \mathbb{R}$, exist as bounded operators in $X$ or if $T$ satisfies certain square function estimates. However, these results are specific to Hilbert space. The situation in Banach spaces, even reflexive ones, is less clear and more complex; some positive results in this setting can be found in [2].

Perhaps one of the simplest examples to consider is the Laplace operator $L = -d^2/dx^2$ in $L^p(R)$ for $1 < p < \infty$. In this case, it turns out that $L$ is of type $\omega = 0$ and, as indicated in Section 2, $L$ has an $H^\omega(S_\varepsilon)$-functional calculus for every $\varepsilon > 0$. Another algebra of functions acting on $L$ is the space $BV(R^+)$ of functions on $[0, \infty)$ which are of bounded variation. We note that these

* This paper is dedicated to the late Professor N. Dunford.
algebras are distinct. Indeed, the function \( z \mapsto z^i \) belongs to \( H^\infty(S_\epsilon) \) for every \( 0 < \epsilon < \pi \) but its restriction to \( [0, \infty) \) is surely not of bounded variation. It is just as easy to exhibit elements of \( BV(R^+) \) which are not the restriction to \( [0, \infty) \) of any element of \( H^\infty(S_\epsilon) \) for any \( \epsilon > 0 \); the characteristic function \( \chi_J \) of any interval \( J \subseteq [0, \infty) \), other than \([0, \infty) \) itself, will do.

The most desirable functional calculus is one admitting the largest possible class of functions defined on \( \sigma(L) = [0, \infty) \). If \( p = 2 \), then \( L \) is self-adjoint and hence it is possible to form a continuous linear operator \( \psi(L) \) for every bounded Borel function \( \psi \) on \([0, \infty) \). The question arises of whether this is still the case for \( p = 2 \), that is, whether \( L \) is a scalar-type spectral operator in the sense of N. Dunford [5]? As noted above an operator \( \psi(L) \) exists whenever \( \psi = \chi_J \) for some interval \( J \subseteq [0, \infty) \). Since such sets generate the Borel subsets of \([0, \infty) \) one might be hopeful of a positive answer. Unfortunately, the main aim of this note is to show that \( L \) is not a scalar-type spectral operator in Dunford’s sense if \( p = 2 \); see Theorem 1 below.

2. Some functional calculi for \( L \). Unless stated otherwise it is assumed that \( p \in (1, \infty) \). Consider the closed operator \( L \) in \( L^p(R) \) given by \( L = -\frac{d^2}{dx^2} \). The domain of \( L \) is taken to be the dense subspace of \( L^p(R) \) specified by

\[
\mathcal{D}(L) = \{ f \in L^p(R); f' \in AC(R), f'' \in L^p(R) \}
\]

where \( AC(R) \) is the space of functions on \( R \) which are absolutely continuous on bounded intervals. Then \( \sigma(L) = [0, \infty) \) and \( -L \) is the infinitesimal generator of a strongly continuous \( C_0 \)-semigroup of contractions, namely the Gauss-Weierstrass semigroup given by

\[
(G_t f)(u) = \left( \frac{1}{2 \pi t} \right)^{\nu_2} \int_{-\infty}^{\infty} f(u - w) e^{-w^2/2t} dw, \quad f \in L^p(R),
\]

for each \( t > 0 \) [7; § 21.4]. It is known that

\[
(1) \quad ||R(\lambda; L)|| \leq 1 / |\lambda| \sin^2 \left( \frac{1}{2} \text{arg}(\lambda) \right), \quad \lambda \in \rho(L) = C \setminus [0, \infty),
\]

[8; IX § 1.8], from which it follows that \( L \) is of type \( \omega = 0 \). Let \( D = -id/dx \) denote the differentiation operator with domain

\[
\mathcal{D}(D) = \{ f \in L^p(R); f \in AC(R), f' \in L^p(R) \}.
\]

Then \( D \) is closed, densely defined and \( \sigma(D) = R \).

For ease of presentation we now assume that \( p \in (1, 2) \). Then it is possible to reformulate the domains of \( L \) and \( D \) in terms of the Fourier transform mapping \( \hat{\cdot}: L^p(R) \rightarrow L^q(R) \) where \( q \) is the conjugate index to \( p \). Indeed,
\[ \mathcal{D}(L) = \{ f \in L^p(\mathbb{R}); \xi^2 \hat{f}(\xi) = \hat{g}(\xi) \text{ for some } g \in L^p(\mathbb{R}) \} \]

and, for each \( f \in \mathcal{D}(L) \), it turns out that \( Lf = g \) where \( g \in L^p(\mathbb{R}) \) satisfies \( \hat{g}(\xi) = \xi^2 \hat{f}(\xi) \) \cite{7; §21.4}. Similarly,

\[ \mathcal{D}(D) = \{ f \in L^p(\mathbb{R}); \xi \hat{f}(\xi) = \hat{g}(\xi) \text{ for some } g \in L^p(\mathbb{R}) \} \]

and, for each \( f \in \mathcal{D}(D) \), it is the case that \( Df = g \) where \( g \in L^p(\mathbb{R}) \) satisfies \( \hat{g}(\xi) = \xi \hat{f}(\xi) \).

Let the bounded measurable function \( m: \mathbb{R} \to \mathbb{C} \) be a \( p \)-multiplier \cite{11; IV §3}. Then there exists a bounded operator in \( L^p(\mathbb{R}) \), say \( T_m \), such that

\[ (T_m f) \hat{\gamma}(\xi) = m(\xi) \hat{f}(\xi), \quad f \in L^p(\mathbb{R}) \cap L^p(\mathbb{R}). \]

Observing that \( (Df) \hat{\gamma}(\xi) = \xi \hat{f}(\xi) \), for each \( f \in \mathcal{D}(D) \), it is natural to define \( m(D) \) to be the operator \( T_m \). If \( \gamma: \mathbb{C} \to \mathbb{C} \) is the function \( \gamma(z) = z^2 \), then \( \gamma(D) = D^2 = L \) where \( D^2 \) is defined in the usual way for positive integral powers of an unbounded operator. So, if \( m \) is a bounded measurable function on \([0, \infty)\) such that \( m \circ \gamma : \mathbb{R} \to \mathbb{C} \) is a \( p \)-multiplier, then we can define an operator \( m(L) \) by

\[ m(L) = (m \circ \gamma)(D). \]

Since the linear space of bounded measurable functions \( m: [0, \infty) \to \mathbb{C} \) such that \( m \circ \gamma : \mathbb{R} \to \mathbb{C} \) is a \( p \)-multiplier forms an algebra under pointwise multiplication it follows that the action of such functions \( m \) on \( L \) as specified by (2) is multiplicative. It is the formula (2) which will imply that \( H^\omega(S_\varepsilon) \) acts on \( L \) for each \( \varepsilon > 0 \).

The following result on multipliers will be needed. It is essentially Theorem 3 of \cite{11; p. 96}. An examination of its proof shows that the constant \( A_p \) specified there has the form of the right-hand-side of (3) for some universal constant \( \alpha_p \).

**Lemma 1.** Let \( 1 < p < \infty \). There exists a constant \( \alpha_p \) such that if \( m: \mathbb{R} \to \mathbb{C} \) is any \( C^1 \)-function in \( \mathbb{R} \setminus \{0\} \) for which both \( m \) and \( \xi \mapsto \xi m'(\xi) \), \( \xi \neq 0 \), are bounded, then \( m \) is a \( p \)-multiplier and the associated operator \( T_m \), considered in \( L^p(\mathbb{R}) \), satisfies

\[ ||T_m|| = ||m(D)|| \leq \alpha_p \max \{ ||m||_{\infty}, ||\xi m'(\xi)||_{\infty} \}. \]

Now, fix \( 0 < \varepsilon < \pi \) and let \( \psi \in H^\omega(S_\varepsilon) \). Then \( \psi \circ \gamma \in H^\omega(C_{\varepsilon \rho}) \) where, for any \( 0 \leq \rho < \pi/2 \), \( C_\rho \) is the open double cone \( S_\rho \cup (-S_\rho) \) and \( -S_\rho = \{-z; z \in S_\rho\} \). Furthermore, the norm \( ||\psi \circ \gamma||_{\infty} = \sup \{ |\psi(z^2)|; z \in C_{\varepsilon \rho}\} \) of \( \psi \circ \gamma \in H^\omega(C_{\varepsilon \rho}) \) coincides with the norm \( ||\psi||_{\infty} = \sup \{ |\psi(w)|; w \in S_{\varepsilon^2}\} \) of \( \psi \in H^\omega(S_\varepsilon) \). If \( \phi \) is any element of \( H^\omega(C_{\varepsilon \rho}) \), then it follows from the Cauchy integral formula that

\[ |\phi'(x)| \leq ||\phi||_{\infty} |x| \sin(\varepsilon/2), \quad x \in \mathbb{R} \setminus \{0\}, \]
and hence
\[(\psi \circ \gamma)'(x)| \leq |\psi \circ \gamma|_w / |x| \sin(\varepsilon/2) = |\psi|_w / |x| \sin(\varepsilon/2),\]
for each $x \in R \setminus \{0\}$. Defining $(\psi \circ \gamma)(0)$ to be zero, say, it follows from Lemma 1 that the restriction to $R$ of $\psi \circ \gamma$, again denoted by $\psi \circ \gamma$, is a $p$-multiplier and hence the bounded operator $\psi(L) = (\psi \circ \gamma)(D)$ certainly exists. Noting that $1/\sin(\varepsilon/2) \geq 1$ it follows from (4) that
\[
\max \{||\psi \circ \gamma||_w, ||(\psi \circ \gamma)'(\xi)||_w\} = ||\psi \circ \gamma||_w / \sin(\varepsilon/2) = ||\psi||_w / \sin(\varepsilon/2)
\]
and hence, (3) implies the continuity of the mapping $\psi \mapsto \psi(L) = (\psi \circ \gamma)(D)$ from $H^n(S_\varepsilon)$ into the space of bounded linear operators on $L^p(R)$ equipped with the uniform operator topology. Accordingly, $L$ admits a $H^n(S_\varepsilon)$ functional calculus.

It is worth noting that this functional calculus includes the resolvent operators of $L$. Indeed, if $w \in C \setminus [0, \infty)$, then there exists $u \in C \setminus R$ such that $u^2 = w$. Of course, the other square root of $w$ is then $-u$. Let $R_u(x) = (x - w)^{-1}$ for $x \neq w$. Let $\varepsilon \in (0, \pi)$ be any number such that $R_u \in H^n(S_\varepsilon)$ in which case $R_u \circ \gamma \in H^n(C_{\varepsilon/2})$. It follows from the definition that $R_u(L) = (R_u \circ \gamma)(D)$ since $R_u(x^2) = (x^2 - w)^{-1}$, $x \in R$, is a $p$-multiplier. But, $R_u(x) = \psi_1(x) - \psi_2(x)$ for each $x \in R$, where $\psi_1(x) = [2u(x - u)]^{-1}$, $x \in R$, and $\psi_2(x) = [2u(x + u)]^{-1}$, $x \in R$. Lemma 1 implies that both $\psi_1$ and $\psi_2$ are $p$-multipliers and so $R_u(L) = (R_u \circ \gamma)(D)$ $= \psi_1(D) - \psi_2(D)$. But, noting that $u$ and $-u$ are in the resolvent set of $D$, it is easily checked from the definition of $D$ in terms of the Fourier transform that $\psi_1(D) = (2u)^{-1}(D - uI)^{-1}$ and $\psi_2(D) = (2u)^{-1}(D + uI)^{-1}$. Since $D$ is a closed operator it follows, for each $\lambda \in \rho(D)$, that the range of $D - \lambda I$ on $\Omega(D)$ is all of $L^p(R)$, [7; Theorem 2.16.3], and hence, that the operator $(D - \lambda I)^{-1}$ is everywhere defined. Accordingly, $(D - \lambda I)^{-1} = R(\lambda; D)$ and so the resolvent identities for $D$ imply that
\[
\psi_1(D) - \psi_2(D) = R(u; D)R(-u; D) = (D - u)^{-1}(D + u)^{-1}
\]
\[
= (D^2 - u^2)^{-1} = (L - w)^{-1}.
\]
But, $L$ is also a closed operator and hence $(L - w)^{-1} = R(w; L)$. It follows that $R_u(L) = (R_u \circ \gamma)(D) = R(w; L)$.

We remark that if $\psi(z) = f(z)/g(z)$ where $f$ and $g$ are polynomials such that $\deg(f) \leq \deg(g)$ and the zeros of $g$ are in the resolvent set $C \setminus [0, \infty)$ of $L$, then it is natural to define a bounded operator $\tilde{\psi}(L)$ by
\[
\tilde{\psi}(L) = \sum_{n=1}^k \sum_{j=0}^{m_n} a_{nj} R(w_n; L)^j = \sum_{n=1}^k \sum_{j=0}^{m_n} a_{nj} [(L - w_n)^{-1}]^j
\]
where $\psi(z) = \sum_{n=1}^k \sum_{j=0}^{m_n} a_{nj}(z - w_n)^{-j}$ is the partial fraction decomposition of $\tilde{\psi}$. Here
\{w_1, \ldots, w_k\} are the zeros of \(g\) and, for each \(1 \leq n \leq k\), the multiplicity of the zero \(w_n\) is \(m_n\). Now if \(\varepsilon \in (0, \pi)\) is any number such that \(\{w_n\}_{n=1}^k \cap \bar{S}_\varepsilon = \emptyset\), then \(\varphi \in H^\infty(S_\varepsilon)\) and hence there is also the operator \(\varphi(L)\) defined via (2). It is clear from the previous paragraph that the operators \(\varphi(L)\) and \(\psi(L)\) coincide.

We now outline, briefly, the action of \(BV(\mathbb{R}^+)\) on \(L\). If \(f: \mathbb{R} \to \mathbb{C}\) is any function, then \(V(f)\) denotes the total variation of \(f\). The linear space \(BV(\mathbb{R})\) consists of all \(\mathbb{C}\)-valued functions on \(\mathbb{R}\) which have finite total variation. It is a Banach algebra with respect to pointwise multiplication and norm defined by

\[
\|f\|_{BV} = \|f\|_\infty + V(f), \quad f \in BV(\mathbb{R}).
\]

Fix \(1 < p < \infty\). Then each \(m \in BV(\mathbb{R})\) is a \(p\)-multiplier and the mapping \(m \mapsto m(D), m \in BV(\mathbb{R})\), is a continuous algebra homomorphism for the uniform operator topology [1; pp. 208–209]. Define \(BV(\mathbb{R}^+)\) to be the closed subalgebra of \(BV(\mathbb{R})\) consisting of those functions \(f\) such that \(f \equiv 0\) in \((-\infty, 0)\). Then, for each \(f \in BV(\mathbb{R}^+)\), the function \(f \circ \gamma: x \mapsto f(x^2), x \in \mathbb{R}\), belongs to \(BV(\mathbb{R})\) and \(V(f \circ \gamma) \leq 2V(f)\). Accordingly, the map

\[
m \mapsto m(L) = (m \circ \gamma)(D), \quad m \in BV(\mathbb{R}^+),
\]

is a functional calculus for \(L\). We remark that if \(w \in \rho(L) = C \setminus [0, \infty)\), then the restriction to \([0, \infty)\) of \(R_w(z) = (z - w)^{-1}, z \neq w\), belongs to \(BV(\mathbb{R}^+)\) since its derivative is an element of \(L^1([0, \infty))\). As noted previously, the operator \(R_w(L)\), defined to be \((R_w \circ \gamma)(D)\), agrees with the resolvent operator \(R(w; L) = (L - wI)^{-1}\).

3. The non-spectrality of \(L\). At this stage it is natural to inquire whether \(L\) admits a functional calculus based on some richer family of functions. Indeed, this is the case for \(p = 2\). Suppose that \(J \subseteq [0, \infty)\) is an interval. Then \(\chi_J \circ \gamma \in BV(\mathbb{R}^+)\) is the characteristic function of the set \(\{t^{1/2}; t \in J\} \cup \{-t^{1/2}; t \in J\}\) which, with obvious notation, is the union of the two intervals \(J^{1/2}\) and \(-J^{1/2}\). Accordingly, \(\chi_J \circ \gamma = \chi_{J^{1/2}} + \chi_{-J^{1/2}}\). \(\chi_J(0)\chi_{[0]}\), and so the operator \(\chi_J(L)\) defined via (2) is just \(\chi_J(\gamma)(D) + \chi_{-J^{1/2}}(D); \gamma\) is a projection commuting with \(L\). Furthermore, the family of projections \(\{\chi_J(L); J\ \text{an interval in } [0, \infty)\}\) is uniformly bounded in \(L^p(\mathbb{R})\), [11; p. 100]. For the case \(p = 2\) this family of projections can be extended so that a projection is assigned to each Borel subset of \([0, \infty)\) and the so extended family forms the resolution of the identity for the self-adjoint operator \(L\). However, if \(p \neq 2\), then the state of affairs is quite different as seen by the following

**Lemma 2.** Let \(\mathcal{R}^+\) denote the algebra of subsets of \((0, \infty)\) generated by all intervals in \([0, \infty)\), in which case the additive set function \(J \mapsto \chi_J(L)\) has a unique extension from the semi-algebra of all intervals in \([0, \infty)\) to \(\mathcal{R}^+\). If \(p \in (1, \infty)\), but \(p \neq 2\), then the family of projections \(\{\chi_E(L); E \in \mathcal{R}^+\}\) is not uniformly bounded in \(L^p(\mathbb{R})\).
Proof. We proceed by contradiction. Suppose then that

\[
\sup \{ \|\chi_E(L)\|_p; E \in \mathcal{R}^+ \} < \infty
\]

where \(\|\cdot\|_p\) denotes the operator norm considered with respect to the Banach space \(L^p(\mathbb{R})\). Let \(\mathcal{R}\) denote the algebra of subsets of \(\mathbb{R}\) generated by the intervals in \(\mathbb{R}\) and let \(\mathcal{R}_0 = \{F \subset \mathbb{R}; F = -F\}\). If \(F \in \mathcal{R}_0\), then it is clear that \(F^2 = \{t^2; t \in F\}\) is an element of \(\mathcal{R}^+\). The discussion prior to Lemma 2 together with the finite additivity of \(E \rightarrow \chi_E(D), E \in \mathcal{R}\) and \(E \rightarrow \chi_E(L), E \in \mathcal{R}^+\) implies that \(\chi_E(L) = \chi_E(D)\). It follows from (5) that

\[
\sup \{ \|\chi_E(D)\|_p; F \in \mathcal{R}_0 \} < \infty.
\]

Let \(F \in \mathcal{R}\). Then \(F_- = F \cap (-\infty, 0)\) is a finite disjoint union of intervals in \((-\infty, 0)\) and \(F_+ = F \cap [0, \infty)\) is a finite disjoint union of intervals in \([0, \infty)\). Define \(F(1) = F_- \cup (-F_-)\) and \(F(2) = F_+ \cup (-F_+)\). Since both \(F(1)\) and \(F(2)\) are elements of \(\mathcal{R}_0\), it follows from (6), the identities \(\chi_{F_-} = \chi_{F(1)}\chi_{(-\infty, 0)}\), \(\chi_{F_+} = \chi_{F(2)}\chi_{[0, \infty)}\) and the finite additivity of \(\chi_E(D)\) that

\[
\sup \{ \|\chi_E(D)\|_p; F \in \mathcal{R} \} < \infty.
\]

That this is not the case is well known.

Lemma 2 implies that the family of projections \(\{\chi_E(L); E \in \mathcal{R}^+\}\) cannot be enlarged to form a spectral measure in \(L^p(\mathbb{R})\), [5; XVII Lemma 3.3 and Corollary 3.10]. This point suggests that \(L\) ought not to be a scalar-type spectral operator. However, to make a precise argument along these lines would require showing that if there were some spectral measure in \(L^p(\mathbb{R})\), say \(P\), a priori having no connection whatsoever with the projectors \(\chi_E(L)\), for which \(L = \int_0^\infty \lambda dP(\lambda)\), then necessarily \(P\) arises by extension of the set function \(J \mapsto \chi_J(L)\), with domain all intervals \(J\) in \([0, \infty)\), to the collection of all Borel sets in \([0, \infty)\). That is, it would have to be established that \(P(J) = \chi_J(L)\) for each such interval \(J\). Rather than pursuing this approach directly we prefer a slightly different argument to establish the following result.

**Theorem 1.** If \(1 < p < \infty\) and \(p \neq 2\), then \(L\) is not a scalar-type spectral operator in \(L^p(\mathbb{R})\).

Before indicating a proof we recall more precisely the notion of a scalar-type spectral operator, briefly, a scalar operator. So, let \(X\) be a Banach space and \(L(X)\) be the space of all continuous linear operators from \(X\) into itself. By a spectral measure in \(X\) is meant a set function \(P: \Sigma \rightarrow L(X)\), where \(\Sigma\) is a \(\sigma\)-algebra of subsets of some set \(\Omega\), such that \(P\) is multiplicative (i.e. \(P(E \cap F) = P(E)P(F)\) for every \(E \in \Sigma\) and \(F \in \Sigma\), \(P(\Omega)\) is the identity operator \(I\) in \(X\) and \(P\) is countably additive for the strong operator topology in \(L(X)\). Given a
$C$-valued, $\Sigma$-measurable function on $\Omega$, say $\psi$, it is possible to define a closed, densely defined operator $P(\psi)$ in $X$ as follows: the domain $\mathcal{D}(P(\psi))$ of $P(\psi)$ consists of those elements $x \in X$ such that $\psi$ is integrable with respect to the $X$-valued measure $P(\cdot): E \mapsto P(E) x, E \in \Sigma$ (in the usual sense [9]), in which case $P(\psi)x$ is defined to be the element $\int_\Omega \psi(w)dP(w)x$, denoted briefly by $\int_\Omega \psi dP x$. It turns out that $P(\psi) \in L(X)$ if and only if $\psi$ is $P$-essentially bounded on $\Omega$. A linear operator $T$ in $X$ is said to be a scalar operator if there exists a spectral measure $P: \Sigma \to L(X)$ and a $\Sigma$-measurable function $\psi$ such that $T = P(\psi)$. This is the case if and only if there exists a spectral measure $Q$ in $X$ defined on the Borel sets $\mathcal{B}(\sigma(T))$ of $\sigma(T)$ such that $T = Q(\lambda)$. Here $\lambda$ denotes the identity function in $C$. All of the above definitions and statements concerning scalar operators can be found in [3] and [5].

The idea of the proof of Theorem 1 is as follows. Since $iD$ is the infinitesimal generator of the translation group in $L^p(R)$ given by $T_t f = f(t + \cdot)$, $t \in R$, that is, $T_t = e^{itp}$, $t \in R$, it follows from [6; Theorem 2] and [5; XVIII Theorem 2.17] that $iD$ and hence, also $D$, is not a scalar operator if $p \neq 2$. Now, if $L$ were a scalar-operator, then it ought to follow from $L = D^2$ that $D = L^{1/2}$ and hence, $D$ would also be a scalar operator [5; XVIII Theorem 2.17] which is a contradiction. Although this is not quite correct (if it were, then $\sigma(D) = \sigma(L^{1/2})$ would be $[0, \infty)$!) it is the spirit in which the proof will proceed. The difficulty is that $D$ is "not quite" a function of $L$ (see (7)). So, it is necessary to identify the positive square root $L^{1/2}$ of $L$, more precisely.

Suppose again that $p \in (1, 2)$. Let $H \in L(L^p(R))$ denote the Hilbert transform. That is, $H$ is the operator corresponding to the $p$-multiplier $\xi \mapsto \text{sgn}(\xi)$, $\xi \in R$. Then $H^2 = I$ and so $\sigma(H) = \{-1, 1\}$. Define a closed operator $S$ in $L^p(R)$ with dense domain

$$\mathcal{D}(S) = \{f \in L^p(R); |\xi| \hat{f}(\xi) = \hat{g}(\xi) \text{ for some } g \in L^p(R)\}$$

by $Sf = g$, $f \in \mathcal{D}(S)$, where $g \in L^p(R)$ satisfies $\hat{g}(\xi) = |\xi| \hat{f}(\xi)$. To see that $S$ is actually closed and densely defined we observe that $-S$ is the infinitesimal generator of a strongly continuous $C_0$-semigroup, namely the Poisson semigroup given by

$$(P_t f)(w) = t^{-1} \int_{-\infty}^{\infty} f(w-u)(t^2+u^2)^{-1} du, \quad f \in L^p(R),$$

for each $t > 0$; see [7; § 21.4], for example. It is clear from the definition of $L$ in terms of the Fourier transform that $S$ is the natural candidate to be called the positive square root of $L$. Indeed, $S^2 = L$ and, in addition, $\sigma(S) = [0, \infty)$. To see this, we note that if $f \in \mathcal{D}(S)$, then
\[
((S - \lambda I)f)(\xi) = (|\xi| - \lambda)f(\xi), \quad \lambda \in \mathbb{C}.
\]

Since \(|\xi| \to (|\xi| - \lambda)^{-1}, \xi \in \mathbb{R},\) is a \(p\)-multiplier whenever \(\lambda \in [0, \infty)\) (cf. Lemma 1), it is clear that the corresponding operator is the resolvent operator of \(S\) at \(\lambda\). This shows that \(\sigma(S) \subseteq [0, \infty)\) and it is not difficult to show equality. If \(f\) is a "nice function", then a direct computation shows that

\[
(Df)(\xi) = \xi f(\xi) = |\xi| f(\xi) \operatorname{sgn}(\xi) = (SHf)(\xi) = (HSf)(\xi),
\]
a formula which suggests the known equality \(D = SH = HS\) [7; §22.5], written more suggestively as

\[
(7) \quad D = HL^{1/2} = L^{1/2}H.
\]

It is this identity, the correct version of "\(D = L^{1/2}\)", which will lead to a proof of Theorem 1.

So, suppose that \(L\) is a scalar operator. The first aim is to show that \(S\) is then also a scalar operator for which the following result is needed. The proof is immediate from the fact that \(\sigma(L) = [0, \infty)\) and the estimates (1).

**Lemma 3.** If \(A = -L\), then \(R(\lambda; A)\) exists for \(\Re(\lambda) > 0\) and

\[
\sup \{|\Re(\lambda)| \cdot |R(\lambda; A)|; \Re(\lambda) > 0\} < \infty.
\]

It follows from Lemma 3 that

\[
(8) \quad -\pi^{-1} \sin(\alpha \pi) \int_0^\infty \lambda^{\alpha-1}(\lambda I + L)^{-1}Lf d\lambda, \quad f \in \mathcal{D}(L),
\]

is defined for each \(0 < \alpha < 1\) [14; Ch. IX, §11 Theorem 3]. In the notation of §11 of Chapter IX in [14] with \(A = -L\), if \(\hat{A}_\alpha\) is the infinitesimal generator of the holomorphic semigroup \(\hat{T}_\alpha = \hat{\mathcal{T}}\) defined there, then for each \(f \in \mathcal{D}(A) = \mathcal{D}(L)\) the value \(\hat{A}_\alpha f\) is equal to (8); see [14; (3) and (4), p. 260]. Noting that \(\hat{A}_{1/2}\) is precisely the generator of the Poisson semigroup [14; p. 268], that is, \(\hat{A}_{1/2} = -S\), it follows from (8) with \(\alpha = 1/2\) that

\[
(9) \quad Sf = -(-Sf) = \pi^{-1} \int_0^\infty \lambda^{-1/2}(\lambda I + L)^{-1}Lf d\lambda, \quad f \in \mathcal{D}(L).
\]

In particular, \(\mathcal{D}(L) \subseteq \mathcal{D}(S)\).

Now, by assumption, \(L = \int_0^\infty \mu dV(\mu) = V(\mu)\) for some spectral measure \(V: \mathcal{B}([0, \infty)) \to L(L^p(\mathbb{R}))\). Accordingly, if \(f \in \mathcal{D}(L)\), then the functional calculus for scalar operators implies that

\[
(\lambda I + L)^{-1}Lf = \int_0^\infty \mu (\lambda + \mu)^{-1}dV(\mu)f, \quad \lambda > 0.
\]
SPECTRAL PROPERTIES OF THE LAPLACE OPERATOR

Substituting this expression into (9) and using Fubini’s theorem gives

\[ Sf = \pi^{-1} \int_0^\infty \int_0^\infty \lambda^{-1/2} \mu^{-1/2} d\lambda d\mu f = \pi^{-1} \int_0^\infty \mu^{-1/2} \int_0^\infty \lambda^{-1/2} (\mu + \lambda)^{-1} d\lambda d\mu f, \]

for each \( f \in \mathcal{D}(L) \). To justify the use of Fubini’s theorem it must be established that the function \( \mu \mapsto \mu^{1/2}, \mu \geq 0, \) is \( V(\cdot)f \)-integrable whenever \( f \in \mathcal{D}(L) \). But, if \( f \in \mathcal{D}(L)=\mathcal{D}(V(\mu)) \), then by definition of the operator \( V(\mu) \) the identity function \( \mu \) on \([0, \infty)\) is \( V(\cdot)f \)-integrable and hence, so is \( \mu \mapsto \mu^{1/2} \chi_{(0,\infty)}(\mu), \mu \geq 0; \) see [9; Ch. II, § 3 Theorem 1]. Since \( \mu \mapsto \mu^{1/2} \chi_{(0,\infty)}(\mu), \mu \geq 0, \) is bounded on \([0, \infty)\) it is also \( V(\cdot)f \)-integrable [9; Ch. II § 3 Lemma 1] and the desired conclusion follows.

Now, define a set function \( P: \mathcal{B}(\mathbb{R}) \to L(L(\mathbb{R})) \) by \( P(I) = \int_{\mu>0} \mu^{1/2} \chi_{[0, \infty)}(\mu) d\mu I \) for each Borel set \( I \subseteq [0, \infty) \). Then \( P \) is a spectral measure and \( \mathcal{S} = P(\lambda) = \int_0^\infty \lambda dP(\lambda) \) is a scalar operator such that

\[ \mathcal{S} = \int_0^\infty \mathcal{S} f = \int_0^\infty \mu^{1/2} dV(\mu) f, \quad f \in \mathcal{D}(P(\lambda)) = \mathcal{D}(V(\mu^{1/2})); \]

see [5; XVIII Theorem 2.17]. In particular, \( \sigma(\mathcal{S}) = [0, \infty) \), [5; XVIII Lemma 2.25]. The argument used above to justify the use of Fubini’s theorem in (10) shows that \( \mathcal{D}(L) \subseteq \mathcal{D}(\mathcal{S}). \)

The claim is that \( \mathcal{S} = \mathcal{S} \). The formulae (10) and (11) show that

\[ (\mathcal{S} - \lambda I) f = (\mathcal{S} - \lambda I) f, \quad f \in \mathcal{D}(L). \]

Since \( \sigma(\mathcal{S}) = [0, \infty) = \sigma(\mathcal{S}), \) the resolvent sets \( \rho(\mathcal{S}) \) and \( \rho(\mathcal{S}) \) also coincide. If \( \lambda \) belongs to this common resolvent set, then it follows from (12) that

\[ (\mathcal{S} - \lambda I) f = (\mathcal{S} - \lambda I) f, \quad f \in \mathcal{D}(L). \]

Operate on the left with the bounded resolvent operator \( R(\lambda; \mathcal{S}) \) gives

\[ R(\lambda; \mathcal{S}) (\mathcal{S} - \lambda I) f = f, \quad f \in \mathcal{D}(L). \]

But, \( f = R(\lambda; \mathcal{S}) (\mathcal{S} - \lambda I) f \) whenever \( f \in \mathcal{D}(L) \subseteq \mathcal{D}(\mathcal{S}) \) and it follows that \( R(\lambda; \mathcal{S}) g = R(\lambda; \mathcal{S}) g \) for all \( g \) in the range of the operator \( (\mathcal{S} - \lambda I) \) restricted to \( \mathcal{D}(L) \). Assume for the moment that the space of all such functions \( g \) is dense in \( L^2(\mathbb{R}) \) whenever \( \lambda < 0 \). Then \( R(\lambda; \mathcal{S}) = R(\lambda; \mathcal{S}) \) for all \( \lambda < 0 \). Both \( \mathcal{S} \) and \( \mathcal{S} \) are closed operators and so \( R(\lambda; \mathcal{S}) = (\mathcal{S} - \lambda I)^{-1} \) and \( R(\lambda; \mathcal{S}) = (\mathcal{S} - \lambda I)^{-1} \) for each \( \lambda \in \rho(\mathcal{S}) \). Accordingly, the equality \( R(\lambda; \mathcal{S}) = R(\lambda; \mathcal{S}) \), valid for each \( \lambda < 0 \), implies that

\[ \mathcal{D}(\mathcal{S}) = \text{Range}(\mathcal{S} - \lambda I)^{-1} = \text{Range}(\mathcal{S} - \lambda I)^{-1} = \mathcal{D}(\mathcal{S}). \]
Fix \( \lambda < 0 \). If \( f \in \mathcal{D}(S) = \mathcal{D}(\bar{S}) \), then

\[
(S - \lambda I)^{-1}(S - \lambda I)f = f = (\bar{S} - \lambda I)^{-1}(\bar{S} - \lambda I)f = (S - \lambda I)^{-1}(\bar{S} - \lambda I)f
\]

from which \( Sf = \bar{S}f \) follows by injectivity of \( (S - \lambda I)^{-1} \). Accordingly, \( S = \bar{S} \).

So, it remains to establish the following

**Lemma 4.** Let \( \lambda < 0 \). Then the space of functions \( \{ (S - \lambda I)f; f \in \mathcal{D}(L) \} \) is dense in \( L^p(\mathbb{R}) \).

**Proof.** The aim is to show that the stated space of functions contains the set \( \mathcal{D}(S - \lambda I) = \mathcal{D}(S) \) and hence, it will be dense in \( L^p(\mathbb{R}) \). So, if \( h \in \mathcal{D}(S - \lambda I) \), then it is to be shown that \( h = (S - \lambda I)f \) for some \( f \in \mathcal{D}(L) \).

By definition of \( \mathcal{D}(S - \lambda I) \) there is \( g \in L^p(\mathbb{R}) \) such that \( (|\xi| - \lambda)\hat{h}(\xi) = \hat{g}(\xi) \) and hence, \( \hat{h}(\xi) = (|\xi| - \lambda)^{-1}\hat{g}(\xi) = (|\xi| - \lambda)(|\xi| - \lambda)^{-2}\hat{g}(\xi) \). Since \( \xi \mapsto (|\xi| - \lambda)^{-2} \) is a \( p \)-multiplier (cf. Lemma 1) there is \( \xi \mapsto (|\xi| - \lambda)^{-2}\hat{g}(\xi) = \hat{f}(\xi) \).

In particular, \( \hat{h}(\xi) = (|\xi| - \lambda)\hat{f}(\xi) \) and so it remains to show that \( f \in \mathcal{D}(L) \). But, \( \xi^p\hat{f}(\xi) = \xi^p(|\xi| - \lambda)^{-2}\hat{g}(\xi) \). Since \( \xi \mapsto \xi^p(|\xi| - \lambda)^{-2} \) is also a \( p \)-multiplier (by Lemma 1 again) there is \( \varphi \in L^p(\mathbb{R}) \) such that \( \xi^p(|\xi| - \lambda)^{-2}\hat{g}(\xi) = \varphi(\xi) \) and hence \( \xi^p\hat{f}(\xi) = \varphi(\xi) \). This shows that \( f \in \mathcal{D}(L) \) and completes the proof of the lemma.

So, we are at the stage of having established that \( S = \bar{S} = \int_0^\infty \lambda dP(\lambda) \) is a scalar operator if \( L \) is a scalar operator.

Now, the Hilbert transform \( H \) is equal to \( Q_1 - Q_2 \) where \( Q_1 \) is the projection corresponding to the \( p \)-multiplier \( \chi_{(0, \infty)} \) and \( Q_2 \) is the projection corresponding to the \( p \)-multiplier \( \chi_{(-\infty, 0)} \). In particular, \( Q_1Q_2 = 0 = Q_2Q_1 \) and \( Q_1 + Q_2 = I \). If we define \( Q(E) = \chi_E(1)Q_1 + \chi_E(-1)Q_2 \) for each \( E \in \mathcal{B}(\mathbb{C}) \), then \( Q \) is a spectral measure in \( L^p(\mathbb{R}) \) such that \( H = \int_\Omega dQ(\mu) \). Since \( H \) and \( S \) commute, it follows that \( H^p(E) = P(E)H \) for each \( E \in \mathcal{B}([0, \infty)) \), [5; XVIII Corollary 2.4]. But, \( H \) is also a scalar operator, with \( Q \) its resolution of the identity, and hence \( Q_jP(E) = P(E)Q_j \) for each \( j \in \{1, 2\} \) and \( E \in \mathcal{B}([0, \infty)) \), [5; XV Corollary 3.7].

Let \( \Omega = [0, \infty) \times \{-1, 1\} \) and let \( \Sigma \) denote the Borel subsets of \( \Omega \). Define a set function \( \Lambda: \Sigma \to L(\mathcal{L}(\mathbb{R})) \) by

\[
\Lambda(U) = Q_1P\{ t \geq 0; (t, 1) \in U \} + Q_2P\{ t \geq 0; (t, -1) \in U \}, \quad U \in \Sigma.
\]

Then it is routine to check that \( \Lambda \) is a spectral measure which may be considered as being defined on all of \( \mathcal{B}(\mathbb{C}) \) with \( \Omega \) as its support. Let \( \psi: \Omega \to \mathbb{C} \) be the \( \Sigma \)-measurable function defined by \( (\lambda, \mu) \mapsto \lambda \mu \) for each \( (\lambda, \mu) \in \Omega \). The corresponding scalar operator \( \Lambda(\psi) \) that is so induced has domain given by

\[
\mathcal{D}(\Lambda(\psi)) = \{ f \in L^p(\mathbb{R}); \psi \text{ is } \Lambda(\cdot) \text{-integrable} \}.
\]

Since, for each \( U \in \Sigma \), we have the identity
Whenever \( f \in L^p(\mathbb{R}) \), it is clear that \( \psi \) is \( \Lambda(\cdot) \)-integrable if and only if the identity function \( \lambda \), on \([0, \infty)\), is \( P(\cdot) \)-integrable. Accordingly,

\[
\mathcal{D}(\Lambda(\psi)) = \{ f \in L^p(\mathbb{R}) ; \lambda \text{ is } P(\cdot) \text{-integrable} \} = \mathcal{D}(S),
\]

where we have used the fact that \( S = \mathcal{P}(\lambda) \). But, (7) implies that \( \mathcal{D}(S) = \mathcal{D}(D) \). Hence, if \( f \in \mathcal{D}(\Lambda(\psi)) = \mathcal{D}(D) \), then

\[
\Lambda(\psi)f = \int_\Omega \lambda_d\lambda(\lambda, \mu)f = Q_1 \int_0^\infty \lambda d\mathcal{P}(\lambda)f - Q_2 \int_0^\infty \lambda d\mathcal{P}(\lambda)f =
\]

\[
= Q_1 Sf - Q_2 Sf = HSf = Df
\]

which shows that \( D = \Lambda(\psi) \). Accordingly, \( D \) is a scalar operator. This is the desired contradiction and completes the proof of Theorem 1 for the case when \( 1 < p < 2 \).

For \( 2 < p < \infty \) we proceed via duality. Indeed, noting that the dual operator \( L^* \), of \( L \) (when \( L \) is considered in \( L^q(\mathbb{R}) \)), is just \( L \) in \( L^p(\mathbb{R}) \), it suffices to establish the fact that in a reflexive Banach space \( X \) the dual operator \( T^* \) of a scalar operator \( T \) is a scalar operator in \( X^* \). But, if \( T = \mathcal{P}(\psi) \) where \( T : \Sigma \rightarrow L(X) \) is a spectral measure and \( \psi \) is a \( \Sigma \)-measurable function, then it is an easy consequence of the reflexivity of \( X \) and the Orlicz-Pettis lemma that the set function \( \mathcal{P}^* : \Sigma \rightarrow L(X^*) \) defined by \( \mathcal{P}^*(E) = \mathcal{P}(E)^* \), \( E \in \Sigma \), is a spectral measure and hence \( \mathcal{P}^*(\psi) \) is a scalar operator in \( X^* \). It remains only to verify the identity \( T^* = \mathcal{P}^*(\psi) \). But, this follows from the reflexivity of \( X \) and [5; XVIII Theorem 2.11 (i)]. The proof of Theorem 1 is thereby complete.

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References


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