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SPECTRAL PROPERTIES OF THE LAPLACE OPERATOR IN $L^p(R)$

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1. Introduction. One of the useful tools for analyzing a linear operator T in a Banach space X, if available, is a functional calculus. In general, no reasonable functional calculus may exist. If it is known that T is a closed operator then there is available a restricted functional calculus for T based on functions which are holomorphic in a neighbourhood of the spectrum $\sigma(T)$, of T, and have a limit at infinity, [4; Ch. VII]. To admit a richer functional calculus it would be expected that T should satisfy some additional properties. For $0\leq\alpha<\pi$, define the open cone $S_\alpha=\{z\in \boldsymbol{C}\setminus\{0\};\ |\arg(z)|<\alpha\}$. A closed operator T in X is said to be of type ω [12], where $0\leq\omega<\pi$, if $\sigma(T)\subseteq\bar{S}_\omega$ (the bar denotes closure and, by definition, $\bar{S}_0=[0, \infty]$) and, for $0<\varepsilon<(\pi-\omega)$ there is a positive constant c_ε such that

$$||R(\lambda;\ T)||\leq c_\varepsilon|\lambda|^{-1}, \qquad \lambda\notin\bar{S}_{\omega+\varepsilon}\,.$$

Here $R(\lambda;\ T)$ denotes the resolvent operator of T at λ. We remark that $-T$, for the case $0\leq\omega\leq\pi/2$, is the infinitesimal generator of a holomorphic semigroup [12; Theorems 3.3.1 and 3.3.2].

In the case when X is a Hilbert space and T is of type ω there are results of A. Yagi [13] and more recently, of A. McIntosh [10], which give conditions equivalent to the existence of a functional calculus for T based on the algebra $H^\infty(S_{\omega+\varepsilon})$, for every $0<\varepsilon<(\pi-\omega)$. For example, this is the case if the purely imaginary powers T^{iu}, $u\in\boldsymbol{R}$, exist as bounded operators in X or if T satisfies certain square function estimates. However, these results are specific to Hilbert space. The situation in Banach spaces, even reflexive ones, is less clear and more complex; some positive results in this setting can be found in [2].

Perhaps one of the simplest examples to consider is the Laplace operator $L=-d^2/dx^2$ in $L^p(\boldsymbol{R})$ for $1<p<\infty$. In this case, it turns out that L is of type $\omega=0$ and, as indicated in Section 2, L has an $H^\infty(S_\varepsilon)$-functional calculus for every $\varepsilon>0$. Another algebra of functions acting on L is the space $BV(\boldsymbol{R}^+)$ of functions on $[0, \infty)$ which are of bounded variation. We note that these

* This paper is dedicated to the late Professor N. Dunford.

algebras are distinct. Indeed, the function $z \to z^i$ belongs to $H^\infty(S_\varepsilon)$ for every $0 < \varepsilon < \pi$ but its restriction to $[0, \infty)$ is surely not of bounded variation. It is just as easy to exhibit elements of $BV(\boldsymbol{R}^+)$ which are not the restriction to $[0, \infty)$ of any element of $H^\infty(S_\varepsilon)$ for any $\varepsilon > 0$; the characteristic function χ_J of any interval $J \subseteq [0, \infty)$, other than $[0, \infty)$ itself, will do.

The most desirable functional calculus is one admitting the largest possible class of functions defined on $\sigma(L) = [0, \infty)$. If $p = 2$, then L is self-adjoint and hence it is possible to form a continuous linear operator $\psi(L)$ for every bounded Borel function ψ on $[0, \infty)$. The question arises of whether this is still the case for $p \neq 2$, that is, whether L is a scalar-type spectral operator in the sense of N. Dunford [5]? As noted above an operator $\psi(L)$ exists whenever $\psi = \chi_J$ for some interval $J \subseteq [0, \infty)$. Since such sets generate the Borel subsets of $[0, \infty)$ one might be hopeful of a positive answer. Unfortunately, the main aim of this note is to show that L is not a scalar-type spectral operator in Dunford's sense if $p \neq 2$; see Theorem 1 below.

2. Some functional calculi for L. Unless stated otherwise it is assumed that $p \in (1, \infty)$. Consider the closed operator L in $L^p(\boldsymbol{R})$ given by $L = -d^2/dx^2$. The domain of L is taken to be the dense subspace of $L^p(\boldsymbol{R})$ specified by

$$\mathcal{D}(L) = \{f \in L^p(\boldsymbol{R}); f' \in AC(\boldsymbol{R}), f'' \in L^p(\boldsymbol{R})\}$$

where $AC(\boldsymbol{R})$ is the space of functions on $\boldsymbol{R}$ which are absolutely continuous on bounded intervals. Then $\sigma(L) = [0, \infty)$ and $-L$ is the infinitesimal generator of a strongly continuous C_0-semigroup of contractions, namely the Gauss-Weierstrass semigroup given by

$$(G_t f)(u) = \frac{1}{2}(\pi t)^{-1/2} \int_{-\infty}^{\infty} f(u-w) e^{-w^2/4t}\, dw\,, \qquad f \in L^p(\boldsymbol{R})\,,$$

for each $t > 0$ [7; §21.4]. It is known that

$$(1) \qquad \|R(\lambda; L)\| \leq 1/|\lambda| \sin^2\left(\frac{1}{2}\arg(\lambda)\right), \qquad \lambda \in \rho(L) = \boldsymbol{C} \setminus [0, \infty),$$

[8; IX § 1.8], from which it follows that L is of type $\omega = 0$. Let $D = -i d/dx$ denote the differentiation operator with domain

$$\mathcal{D}(D) = \{f \in L^p(\boldsymbol{R}); f \in AC(\boldsymbol{R}),\ f' \in L^p(\boldsymbol{R})\}\,.$$

Then D is closed, densely defined and $\sigma(D) = \boldsymbol{R}$.

For ease of presentation we now assume that $p \in (1,2)$. Then it is possible to reformulate the domains of L and D in terms of the Fourier transform mapping $\hat{\cdot}: L^p(\boldsymbol{R}) \to L^q(\boldsymbol{R})$ where q is the conjugate index to p. Indeed,

$$\mathscr{D}(L) = \{f \in L^p(\boldsymbol{R});\ \xi^2\hat{f}(\xi) = \hat{g}(\xi) \text{ for some } g \in L^p(\boldsymbol{R})\}$$

and, for each $f \in \mathscr{D}(L)$, it turns out that $Lf=g$ where $g \in L^p(\boldsymbol{R})$ satisfies $\hat{g}(\xi)=\xi^2\hat{f}(\xi)$ [7; § 21.4]. Similarly,

$$\mathscr{D}(D) = \{f \in L^p(\boldsymbol{R});\ \xi\hat{f}(\xi) = \hat{g}(\xi) \text{ for some } g \in L^p(\boldsymbol{R})\}$$

and, for each $f \in \mathscr{D}(D)$, it is the case that $Df=g$ where $g \in L^p(\boldsymbol{R})$ satisfies $\hat{g}(\xi)=\xi\hat{f}(\xi)$.

Let the bounded measurable function m: $\boldsymbol{R} \to \boldsymbol{C}$ be a p-multiplier [11; IV § 3]. Then there exists a bounded operator in $L^p(\boldsymbol{R})$, say T_m, such that

$$(T_m f)^\wedge(\xi) = m(\xi)\hat{f}(\xi)\,, \qquad f \in L^p(\boldsymbol{R}) \cap L^2(\boldsymbol{R})\,.$$

Observing that $(Df)^\wedge(\xi)=\xi\hat{f}(\xi)$, for each $f \in \mathscr{D}(D)$, it is natural to define $m(D)$ to be the operator T_m. If γ: $\boldsymbol{C} \to \boldsymbol{C}$ is the function $\gamma(z)=z^2$, then $\gamma(D)=D^2=L$ where D^2 is defined in the usual way for positive integral powers of an unbounded operator. So, if m is a bounded measurable function on $[0, \infty)$ such that $m\circ\gamma$: $\boldsymbol{R} \to \boldsymbol{C}$ is a p-multiplier, then we can define an operator $m(L)$ by

$$m(L) = (m\circ\gamma)(D)\,. \tag{2}$$

Since the linear space of bounded measurable functions m: $[0, \infty) \to \boldsymbol{C}$ such that $m\circ\gamma$: $\boldsymbol{R} \to \boldsymbol{C}$ is a p-multiplier forms an algebra under pointwise multiplication it follows that the action of such functions m on L as specified by (2) is multiplicative. It is the formula (2) which will imply that $H^\infty(S_\varepsilon)$ acts on L for each $\varepsilon>0$.

The following result on multipliers will be needed. It is essentially Theorem 3 of [11; p. 96]. An examination of its proof shows that the constant A_p specified there has the form of the right-hand-side of (3) for some universal constant α_p.

Lemma 1. *Let $1<p<\infty$. There exists a constant α_p such that if m: $\boldsymbol{R} \to \boldsymbol{C}$ is any C^1-function in $\boldsymbol{R}\setminus\{0\}$ for which both m and $\xi \mapsto \xi m'(\xi)$, $\xi \neq 0$, are bounded, then m is a p-multiplier and the associated operator T_m, considered in $L^p(\boldsymbol{R})$, satisfies*

$$\|T_m\| = \|m(D)\| \leqslant \alpha_p \max\{\|m\|_\infty,\ \|\xi m'(\xi)\|_\infty\}\,. \tag{3}$$

Now, fix $0<\varepsilon<\pi$ and let $\psi \in H^\infty(S_\varepsilon)$. Then $\psi\circ\gamma \in H^\infty(C_{\varepsilon/2})$ where, for any $0 \leq \rho < \pi/2$, C_ρ is the open double cone $S_\rho \cup (-S_\rho)$ and $-S_\rho=\{-z;\ z \in S_\rho\}$. Furthermore, the norm $\|\psi\circ\gamma\|_\infty=\sup\{|\psi(z^2)|;\ z \in C_{\varepsilon/2}\}$ of $\psi\circ\gamma \in H^\infty(C_{\varepsilon/2})$ coincides with the norm $\|\psi\|_\infty=\sup\{|\psi(w)|;\ w \in S_\varepsilon\}$ of $\psi \in H^\infty(S_\varepsilon)$. If ϕ is any element of $H^\infty(C_{\varepsilon/2})$, then it follows from the Cauchy integral formula that

$$|\phi'(x)| \leq \|\phi\|_\infty / |x| \sin(\varepsilon/2)\,, \qquad x \in \boldsymbol{R}\setminus\{0\}\ ,$$

and hence

$$(4) \qquad |(\psi\circ\gamma)'(x)| \leq \|\psi\circ\gamma\|_\infty/|x|\sin(\varepsilon/2) = \|\psi\|_\infty/|x|\sin(\varepsilon/2),$$

for each $x\in\boldsymbol{R}\setminus\{0\}$. Defining $(\psi\circ\gamma)(0)$ to be zero, say, it follows from Lemma 1 that the restriction to $\boldsymbol{R}$ of $\psi\circ\gamma$, again denoted by $\psi\circ\gamma$, is a p-multiplier and hence the bounded operator $\psi(L)=(\psi\circ\gamma)(D)$ certainly exists. Noting that $1/\sin(\varepsilon/2)\geq 1$ it follows from (4) that

$$\max\{\|\psi\circ\gamma\|_\infty, \|\xi(\psi\circ\gamma)'(\xi)\|_\infty\} = \|\psi\circ\gamma\|_\infty/\sin(\varepsilon/2) = \|\psi\|_\infty/\sin(\varepsilon/2)$$

and hence, (3) implies the continuity of the mapping $\psi\mapsto\psi(L)=(\psi\circ\gamma)(D)$ from $H^\infty(S_\varepsilon)$ into the space of bounded linear operators on $L^p(\boldsymbol{R})$ equipped with the uniform operator topology. Accordingly, L admits a $H^\infty(S_\varepsilon)$ functional calculus.

It is worth noting that this functional calculus includes the resolvent operators of L. Indeed, if $w\in\boldsymbol{C}\setminus[0,\infty)$, then there exists $u\in\boldsymbol{C}\setminus\boldsymbol{R}$ such that $u^2=w$. Of course, the other square root of w is then $-u$. Let $R_w(z)=(z-w)^{-1}$ for $z\neq w$. Let $\varepsilon\in(0,\pi)$ be any number such that $R_w\in H^\infty(S_\varepsilon)$ in which case $R_w\circ\gamma\in H^\infty(C_{\varepsilon/2})$. It follows from the definition that $R_w(L)=(R_w\circ\gamma)(D)$ since $R_w(x^2)=(x^2-w)^{-1}$, $x\in\boldsymbol{R}$, is a p-multiplier. But, $R_w(x^2)=\psi_1(x)-\psi_2(x)$ for each $x\in\boldsymbol{R}$, where $\psi_1(x)=[2u(x-u)]^{-1}$, $x\in\boldsymbol{R}$, and $\psi_2(x)=[2u(x+u)^{-1}]$, $x\in\boldsymbol{R}$. Lemma 1 implies that both ψ_1 and ψ_2 are p-multipliers and so $R_w(L)=(R_w\circ\gamma)(D)$ $=\psi_1(D)-\psi_2(D)$. But, noting that u and $-u$ are in the resolvent set of D, it is easily checked from the definition of D in terms of the Fourier transform that $\psi_1(D)=(2u)^{-1}(D-uI)^{-1}$ and $\psi_2(D)=(2u)^{-1}(D+uI)^{-1}$. Since D is a closed operator it follows, for each $\lambda\in\rho(D)$, that the range of $D-\lambda I$ on $\mathcal{D}(D)$ is all of $L^p(\boldsymbol{R})$, [7; Theorem 2.16.3], and hence, that the operator $(D-\lambda I)^{-1}$ is everywhere defined. Accordingly, $(D-\lambda I)^{-1}=R(\lambda; D)$ and so the resolvent identities for D imply that

$$\begin{aligned}\psi_1(D)-\psi_2(D) &= R(u;D)R(-u;D) = (D-u)^{-1}(D+u)^{-1}\\ &= (D^2-u^2)^{-1} = (L-w)^{-1}.\end{aligned}$$

But, L is also a closed operator and hence $(L-w)^{-1}=R(w; L)$. It follows that $R_w(L)=(R_w\circ\gamma)(D)=R(w; L)$.

We remark that if $\psi(z)=f(z)/g(z)$ where f and g are polynomials such that $\deg(f)\leq\deg(g)$ and the zeros of g are in the resolvent set $\boldsymbol{C}\setminus[0,\infty)$ of L, then it is natural to define a bounded operator $\tilde{\psi}(L)$ by

$$\tilde{\psi}(L) = \sum_{n=1}^{k}\sum_{j=0}^{m_n} a_{nj}R(w_n; L)^j = \sum_{n=1}^{k}\sum_{j=0}^{m_n} a_{nj}[(L-w_n)^{-1}]^j$$

where $\psi(z)=\sum_{n=1}^{k}\sum_{j=0}^{m_n} a_{nj}(z-w_n)^{-j}$ is the partial fraction decomposition of ψ. Here

$\{w_1, \cdots, w_k\}$ are the zeros of g and, for each $1 \leq n \leq k$, the multiplicity of the zero w_n is m_n. Now if $\varepsilon \in (0, \pi)$ is any number such that $\{w_n\}_{n=1}^{k} \cap \bar{S}_\varepsilon = \emptyset$, then $\psi \in H^\infty(S_\varepsilon)$ and hence there is also the operator $\psi(L)$ defined via (2). It is clear from the previous paragraph that the operators $\tilde{\psi}(L)$ and $\psi(L)$ coincide.

We now outline, briefly, the action of $BV(\boldsymbol{R}^+)$ on L. If $f: \boldsymbol{R} \to \boldsymbol{C}$ is any function, then $V(f)$ denotes the total variation of f. The linear space $BV(\boldsymbol{R})$ consists of all $\boldsymbol{C}$-valued functinos on $\boldsymbol{R}$ which have finite total variation. It is a Banach algebra with respect to pointwise multiplication and norm defined by

$$\|f\|_{BV} = \|f\|_\infty + V(f), \qquad f \in BV(\boldsymbol{R}).$$

Fix $1 < p < \infty$. Then each $m \in BV(\boldsymbol{R})$ is a p-multiplier and the mapping $m \to m(D)$, $m \in BV(\boldsymbol{R})$, is a continuous algebra homomorphism for the uniform operator topology [1; pp. 208–209]. Define $BV(\boldsymbol{R}^+)$ to be the closed subalgebra of $BV(\boldsymbol{R})$ consisting of those functions f such that $f \equiv 0$ in $(-\infty, 0)$. Then, for each $f \in BV(\boldsymbol{R}^+)$, the function $f \circ \gamma$: $x \to f(x^2)$, $x \in \boldsymbol{R}$, belongs to $BV(\boldsymbol{R})$ and $V(f \circ \gamma) \leq 2V(f)$. Accordingly, the map

$$m \mapsto m(L) = (m \circ \gamma)(D), \qquad m \in BV(\boldsymbol{R}^+),$$

is a functional calculus for L. We remark that if $w \in \rho(L) = \boldsymbol{C} \backslash [0, \infty)$, then the restriction to $[0, \infty)$ of $R_w(z) = (z - w)^{-1}$, $z \neq w$, belongs to $BV(\boldsymbol{R}^+)$ since its derivative is an element of $L^1([0, \infty))$. As noted previously, the operator $R_w(L)$, defined to be $(R_w \circ \gamma)(D)$, agrees with the resolvent operator $R(w; L) = (L - wI)^{-1}$.

3. The non-spectrality of L. At this stage it is natural to inquire whether L admits a functional calculus based on some richer family of functions. Indeed, this is the case for $p = 2$. Suppose that $J \subseteq [0, \infty)$ is an interval. Then $\chi_J \circ \gamma \in BV(\boldsymbol{R}^+)$ is the characteristic function of the set $\{t^{1/2}; t \in J\} \cup \{-t^{1/2}; t \in J\}$ which, with obvious notation, is the union of the two intervals $J^{1/2}$ and $-J^{1/2}$. Accordingly, $\chi_J \circ \gamma = \chi_{J^{1/2}} + \chi_{-J^{1/2}} - \chi_J(0)\chi_{\{0\}}$ and so the operator $\chi_J(L)$ defined via (2) is just $\chi_{J^{1/2}}(D) + \chi_{-J^{1/2}}(D)$; it is a projection commuting with L. Furthermore, the family of projections $\{\chi_J(L);\ J \text{ an interval in } [0, \infty)\}$ is uniformly bounded in $L^p(\boldsymbol{R})$, [11; p. 100]. For the case $p = 2$ this family of projections can be extended so that a projection is assigned to each Borel subset of $[0, \infty)$ and the so extended family forms the resolution of the identity for the self-adjoint operator L. However, if $p \neq 2$, then the state of affairs is quite different as seen by the following

Lemma 2. *Let $\mathfrak{R}^+$ denote the algebra of subsets of $(0, \infty)$ generated by all intervals in $[0, \infty)$, in which case the additive set function $J \to \chi_J(L)$ has a unique extension from the semi-algebra of all intervals in $[0, \infty)$ to $\mathfrak{R}^+$. If $p \in (1, \infty)$, but $p \neq 2$, then the family of projections $\{\chi_E(L); E \in \mathfrak{R}^+\}$ is not uniformly bounded in $L^p(\boldsymbol{R})$.*

Proof. We proceed by contradiction. Suppose then that

$$\sup\{\|\chi_E(L)\|_p \colon E\in\mathcal{R}^+\}<\infty \tag{5}$$

where $\|\cdot\|_p$ denotes the operator norm considered with respect to the Banach space $L^p(\boldsymbol{R})$. Let $\mathcal{R}$ denote the algebra of subsets of $\boldsymbol{R}$ generated by the intervals in $\boldsymbol{R}$ and let $\mathcal{R}_0=\{F\in\mathcal{R};\ F=-F\}$. If $F\in\mathcal{R}_0$, then it is clear that $F^2=\{t^2;\ t\in F\}$ is an element of $\mathcal{R}^+$. The discussion prior to Lemma 2 together with the finite additivity of $E\to\chi_E(D)$, $E\in\mathcal{R}$ and $E\to\chi_E(L)$, $E\in\mathcal{R}^+$ implies that $\chi_{F^2}(L)=\chi_F(D)$. It follows from (5) that

$$\sup\{\|\chi_F(D)\|_p;\ F\in\mathcal{R}_0\}<\infty. \tag{6}$$

Let $F\in\mathcal{R}$. Then $F_-=F\cap(-\infty,0)$ is a finite disjoint union of intervals in $(-\infty,0)$ and $F_+=F\cap[0,\infty)$ is a finite disjoint union of intervals in $[0,\infty)$. Define $F(1)=F_-\cup(-F_-)$ and $F(2)=F_+\cup(-F_+)$. Since both $F(1)$ and $F(2)$ are elements of $\mathcal{R}_0$, it follows from (6), the identities $\chi_{F_-}=\chi_{F(1)}\chi_{(-\infty,0)}$, $\chi_{F_+}=\chi_{F(2)}\chi_{[0,\infty)}$ and $\chi_F=\chi_{F_+}+\chi_{F_-}$ and the finite additivity of $\chi_{(\cdot)}(D)$ that

$$\sup\{\|\chi_F(D)\|_p;\ F\in\mathcal{R}\}<\infty.$$

That this is not the case is well known.

Lemma 2 implies that the family of projections $\{\chi_E(L);\ E\in\mathcal{R}^+\}$ cannot be enlarged to form a spectral measure in $L^p(\boldsymbol{R})$, [5; XVII Lemma 3.3 and Corollary 3.10]. This point suggests that L ought not to be a scalar-type spectral operator. However, to make a precise argument along these lines would require showing that if there were some spectral measure in $L^p(\boldsymbol{R})$, say P, a priori having no connection what-so-ever with the projectors $\chi_J(L)$, for which $L=\int_0^\infty \lambda dP(\lambda)$, then necessarily P arises by extension of the set function $J\mapsto\chi_J(L)$, with domain all intervals J in $[0,\infty)$, to the collection of all Borel sets in $[0,\infty)$. That is, it would have to be established that $P(J)=\chi_J(L)$ for each such interval J. Rather than pursuing this approach directly we prefer a slightly different argument to establish the following result.

Theorem 1. *If $1<p<\infty$ and $p\neq 2$, then L is not a scalar-type spectral operator in $L^p(\boldsymbol{R})$.*

Before indicating a proof we recall more precisely the notion of a scalar-type spectral operator, briefly, a scalar operator. So, let X be a Banach space and $L(X)$ be the space of all continuous linear operators from X into itself. By a spectral measure in X is meant a set function $P\colon\Sigma\to L(X)$, where Σ is a σ-algebra of subsets of some set Ω, such that P is multiplicative (i.e. $P(E\cap F)=P(E)P(F)$ for every $E\in\Sigma$ and $F\in\Sigma$), $P(\Omega)$ is the identity operator I in X and P is countably additive for the strong operator topology in $L(X)$. Given a

$\boldsymbol{C}$-valued, Σ-measurable function on Ω, say ψ, it is possible to define a closed, densely defined operator $P(\psi)$ in X as follows: the domain $\mathcal{D}(P(\psi))$ of $P(\psi)$ consists of those elements $x \in X$ such that ψ is integrable with respect to the X-valued measure $P(\cdot)x$: $E \mapsto P(E)x$, $E \in \Sigma$ (in the usual sense [9]), in which case $P(\psi)x$ is defined to be the element $\int_\Omega \psi(w)dP(w)x$, denoted briefly by $\int_\Omega \psi dPx$. It turns out that $P(\psi) \in L(X)$ if and only if ψ is P-essentially bounded on Ω. A linear operator T in X is said to be a scalar operator if there exists a spectral measure P: $\Sigma \to L(X)$ and a Σ-measurable function ψ such that $T = P(\psi)$. This is the case if and only if there exists a spectral measure Q in X defined on the Borel sets $\mathcal{B}(\sigma(T))$ of $\sigma(T)$ such that $T = Q(\lambda)$. Here λ denotes the identity function in $\boldsymbol{C}$. All of the above definitions and statements concerning scalar operators can be found in [3] and [5].

The idea of the proof of Theorem 1 is as follows. Since iD is the infinitesimal generator of the translation group in $L^p(\boldsymbol{R})$ given by $T_t f = f(t+\cdot)$, $t \in \boldsymbol{R}$, that is, $T_t = e^{itD}$, $t \in \boldsymbol{R}$, it follows from [6; Theorem 2] and [5; XVIII Theorem 2.17] that iD and hence, also D, is *not* a scalar operator if $p \neq 2$. Now, if L were a scalar-operator, then it ought to follow from $L = D^2$ that $D = L^{1/2}$ and hence, D would also be a scalar operator [5; XVIII Theorem 2.17] which is a contradiction. Although this is not quite correct (if it were, then $\sigma(D) = \sigma(L^{1/2})$ would be $[0, \infty)$!) it is the spirit in which the proof will proceed. The difficulty is that D is "not quite" a function of L (see (7)). So, it is necessary to identify the positive square root $L^{1/2}$, of L, more precisely.

Suppose again that $p \in (1, 2)$. Let $H \in L(L^p(\boldsymbol{R}))$ denote the Hilbert transform. That is, H is the operator corresponding to the p-multiplier $\xi \mapsto \operatorname{sgn}(\xi)$, $\xi \in \boldsymbol{R}$. Then $H^2 = I$ and so $\sigma(H) = \{-1, 1\}$. Define a closed operator S in $L^p(\boldsymbol{R})$ with dense domain

$$\mathcal{D}(S) = \{f \in L^p(\boldsymbol{R});\ |\xi| \hat{f}(\xi) = \hat{g}(\xi) \qquad \text{for some } g \in L^p(\boldsymbol{R})\}$$

by $Sf = g$, $f \in \mathcal{D}(S)$, where $g \in L^p(\boldsymbol{R})$ satisfies $\hat{g}(\xi) = |\xi| \hat{f}(\xi)$. To see that S is actually closed and densely defined we observe that $-S$ is the infinitesimal generator of a strongly continuous C_0-semigroup, namely the Poisson semigroup given by

$$(P_t f)(w) = t\pi^{-1} \int_{-\infty}^{\infty} f(w-u)(t^2+u^2)^{-1} du\,, \qquad f \in L^p(\boldsymbol{R})\,,$$

for each $t > 0$; see [7; § 21.4], for example. It is clear from the definition of L in terms of the Fourier transform that S is the natural candidate to be called the positive square root of L. Indeed, $S^2 = L$ and, in addition, $\sigma(S) = [0, \infty)$. To see this, we note that if $f \in \mathcal{D}(S)$, then

$$((S-\lambda I)f)^{\wedge}(\xi)=(|\xi|-\lambda)\hat{f}(\xi)\,, \qquad \lambda\in\boldsymbol{C}\,.$$

Since $\xi\mapsto(|\xi|-\lambda)^{-1}$, $\xi\in\boldsymbol{R}$, is a p-multiplier whenever $\lambda\notin[0,\infty)$ (cf. Lemma 1), it is clear that the corresponding operator is the resolvent operator of S at λ. This shows that $\sigma(S)\subseteq[0,\infty)$ and it is not difficult to show equality. If f is a "nice function", then a direct computation shows that

$$(Df)^{\wedge}(\xi)=\xi\hat{f}(\xi)=|\xi|\hat{f}(\xi)\operatorname{sgn}(\xi)=(SHf)^{\wedge}(\xi)=(HSf)^{\wedge}(\xi)\,,$$

a formula which suggests the known equality $D=SH=HS$ [7; § 22.5], written more suggestively as

$$D=HL^{1/2}=L^{1/2}H\,. \tag{7}$$

It is this identity, the correct version of "$D=L^{1/2}$", which will lead to a proof of Theorem 1.

So, suppose that L is a scalar operator. The first aim is to show that S is then also a scalar operator for which the following result is needed. The proof is immediate from the fact that $\sigma(L)=[0,\infty)$ and the estimates (1).

Lemma 3. *If $A=-L$, then $R(\lambda;A)$ exists for* $\operatorname{Re}(\lambda)>0$ *and*

$$\sup\{|\operatorname{Re}(\lambda)|\cdot\|R(\lambda;A)\|;\ \operatorname{Re}(\lambda)>0\}<\infty.$$

It follows from Lemma 3 that

$$-\pi^{-1}\sin(\alpha\pi)\int_0^\infty\lambda^{\alpha-1}(\lambda I+L)^{-1}Lf\,d\lambda\,, \qquad f\in\mathcal{D}(L)\,, \tag{8}$$

is defined for each $0<\alpha<1$ [14; Ch. IX, § 11 Theorem 3]. In the notation of § 11 of Chapter IX in [14] with $A=-L$, if $\hat{A}_\alpha$ is the infinitesimal generator of the holomorphic semigroup $\hat{T}_{\alpha,t}\equiv\hat{T}_t$ defined there, then for each $f\in\mathcal{D}(A)=\mathcal{D}(L)$ the value $\hat{A}_\alpha f$ is equal to (8); see [14; (3) and (4), p. 260]. Noting that $\hat{A}_{1/2}$ is precisely the generator of the Poisson semigroup [14; p. 268], that is, $\hat{A}_{1/2}=-S$, it follows from (8) with $\alpha=1/2$ that

$$Sf=-(-Sf)=\pi^{-1}\int_0^\infty\lambda^{-1/2}(\lambda I+L)^{-1}Lf\,d\lambda\,, \qquad f\in\mathcal{D}(L)\,. \tag{9}$$

In particular, $\mathcal{D}(L)\subseteq\mathcal{D}(S)$.

Now, by assumption, $L=\int_0^\infty\mu\,dV(\mu)=V(\mu)$ for some spectral measure $V\colon\mathcal{B}([0,\infty))\to L(L^p(\boldsymbol{R}))$. Accordingly, if $f\in\mathcal{D}(L)$, then the functional calculus for scalar operators implies that

$$(\lambda I+L)^{-1}Lf=\int_0^\infty\mu(\lambda+\mu)^{-1}dV(\mu)f\,, \qquad \lambda>0\,.$$

Substituting this expression into (9) and using Fubini's theorem gives

$$Sf=\pi^{-1}\int_0^\infty \mu\Big(\int_0^\infty \lambda^{-1/2}(\mu+\lambda)^{-1}d\lambda\Big)dV(\mu)f= \tag{10}$$
$$=\pi^{-1}\int_0^\infty \mu(\pi\mu^{-1/2})dV(\mu)f=\int_0^\infty \mu^{1/2}dV(\mu)f\,,$$

for each $f\in\mathcal{D}(L)$. To justify the use of Fubini's theorem it must be established that the function $\mu\mapsto\mu^{1/2}$, $\mu\geq 0$, is $V(\cdot)f$-integrable whenever $f\in\mathcal{D}(L)$. But, if $f\in\mathcal{D}(L)=\mathcal{D}(V(\mu))$, then by definition of the operator $V(\mu)$ the identity function μ on $[0, \infty)$ is $V(\cdot)f$-integrable and hence, so is $\mu\mapsto\mu^{1/2}\chi_{[1,\infty)}(\mu)$, $\mu\geq 0$; see [9; Ch. II, § 3 Theorem 1]. Since $\mu\mapsto\mu^{1/2}\chi_{[0,1)}(\mu)$, $\mu\geq 0$, is bounded on $[0, \infty)$ it is also $V(\cdot)f$-integrable [9; Ch. II § 3 Lemma 1] and the desired conclusion follows.

Now, define a set function $P\colon \mathcal{B}([0, \infty))\to L(L^p(\boldsymbol{R}))$ by $P(E)=V(\{\mu\geq 0;\ \mu^{1/2}\in E\})$ for each Borel set $E\subseteq[0, \infty)$. Then P is a spectral measure and $\tilde{S}=P(\lambda)=\int_0^\infty \lambda dP(\lambda)$ is a scalar operator such that

$$\tilde{S}f=\int_0^\infty \lambda dP(\lambda)f=\int_0^\infty \mu^{1/2}dV(\mu)f\,, \qquad f\in\mathcal{D}(P(\lambda))=\mathcal{D}(V(\lambda^{1/2}))\,; \tag{11}$$

see [5; XVIII Theorem 2.17]. In particular, $\sigma(\tilde{S})=[0, \infty)$, [5; XVIII Lemma 2.25]. The argument used above to justify the use of Fubini's theorem in (10) shows that $\mathcal{D}(L)\subseteq\mathcal{D}(\tilde{S})$.

The claim is that $S=\tilde{S}$. The formulae (10) and (11) show that

$$\tilde{S}f=Sf\,, \qquad f\in\mathcal{D}(L)\,. \tag{12}$$

Since $\sigma(S)=[0, \infty)=\sigma(\tilde{S})$, the resolvent sets $\rho(S)$ and $\rho(\tilde{S})$ also coincide. If λ belongs to this common resolvent set, then it follows from (12) that

$$(\tilde{S}-\lambda I)f=(S-\lambda I)f\,, \qquad f\in\mathcal{D}(L)\,.$$

Operate on the left with the bounded resolvent operator $R(\lambda;\tilde{S})$ gives

$$R(\lambda;\tilde{S})(S-\lambda I)f=f\,, \qquad f\in\mathcal{D}(L)\,.$$

But, $f=R(\lambda; S)(S-\lambda I)f$ whenever $f\in\mathcal{D}(L)\subseteq\mathcal{D}(S)$ and it follows that $R(\lambda;\tilde{S})g=R(\lambda; S)g$ for all g in the range of the operator $(S-\lambda I)$ restricted to $\mathcal{D}(L)$. Assume for the moment that the space of all such functions g is dense in $L^p(\boldsymbol{R})$ whenever $\lambda<0$. Then $R(\lambda; S)=R(\lambda;\tilde{S})$ for all $\lambda<0$. Both S and $\tilde{S}$ are closed operators and so $R(\lambda; S)=(S-\lambda I)^{-1}$ and $R(\lambda;\tilde{S})=(\tilde{S}-\lambda I)^{-1}$ for each $\lambda\in\rho(S)=\rho(\tilde{S})$. Accordingly, the equality $R(\lambda; S)=R(\lambda;\tilde{S})$, valid for each $\lambda<0$, implies that

$$\mathcal{D}(S)=\mathrm{Range}(S-\lambda I)^{-1}=\mathrm{Range}(\tilde{S}-\lambda I)^{-1}=\mathcal{D}(\tilde{S})\,.$$

Fix $\lambda<0$. If $f\in\mathcal{D}(S)=\mathcal{D}(\tilde{S})$, then

$$(S-\lambda I)^{-1}(S-\lambda I)f=f=(\tilde{S}-\lambda I)^{-1}(\tilde{S}-\lambda I)f=(S-\lambda I)^{-1}(\tilde{S}-\lambda I)f$$

from which $Sf=\tilde{S}f$ follows by injectivity of $(S-\lambda I)^{-1}$. Accordingly, $S=\tilde{S}$. So, it remains to establish the following

Lemma 4. *Let $\lambda<0$. Then the space of functions $\{(S-\lambda I)f; f\in\mathcal{D}(L)\}$ is dense in $L^p(\boldsymbol{R})$.*

Proof. The aim is to show that the stated space of functions contains the set $\mathcal{D}(S-\lambda I)=\mathcal{D}(S)$ and hence, it will be dense in $L^p(\boldsymbol{R})$. So, if $h\in\mathcal{D}(S-\lambda I)$, then it is to be shown that $h=(S-\lambda I)f$ for some $f\in\mathcal{D}(L)$.

By definition of $\mathcal{D}(S-\lambda I)$ there is $g\in L^p(\boldsymbol{R})$ such that $(|\xi|-\lambda)\hat{h}(\xi)=\hat{g}(\xi)$ and hence, $\hat{h}(\xi)=(|\xi|-\lambda)^{-1}\hat{g}(\xi)=(|\xi|-\lambda)(|\xi|-\lambda)^{-2}\hat{g}(\xi)$. Since $\xi\mapsto(|\xi|-\lambda)^{-2}$ is a p-multiplier (cf. Lemma 1) there is $f\in L^p(\boldsymbol{R})$ such that $(|\xi|-\lambda)^{-2}\hat{g}(\xi)=\hat{f}(\xi)$. In particular, $\hat{h}(\xi)=(|\xi|-\lambda)\hat{f}(\xi)$ and so it remains to show that $f\in\mathcal{D}(L)$. But, $\xi^2\hat{f}(\xi)=\xi^2(|\xi|-\lambda)^{-2}\hat{g}(\xi)$. Since $\xi\mapsto\xi^2(|\xi|-\lambda)^{-2}$ is also a p-multiplier (by Lemma 1 again) there is $\psi\in L^p(\boldsymbol{R})$ such that $\xi^2(|\xi|-\lambda)^{-2}\hat{g}(\xi)=\hat{\psi}(\xi)$ and hence $\xi^2\hat{f}(\xi)=\hat{\psi}(\xi)$. This shows that $f\in\mathcal{D}(L)$ and completes the proof of the lemma.

So, we are at the stage of having established that $S=\tilde{S}=\int_0^\infty\lambda dP(\lambda)$ is a scalar operator if L is a scalar operator.

Now, the Hilbert transform H is equal to Q_1-Q_2 where Q_1 is the projection corresponding to the p-multiplier $\chi_{[0,\infty)}$ and Q_2 is the projection corresponding to the p-multipler $\chi_{(-\infty,0)}$. In particular, $Q_1Q_2=0=Q_2Q_1$ and $Q_1+Q_2=I$. If we define $Q(E)=\chi_E(1)Q_1+\chi_E(-1)Q_2$ for each $E\in\mathcal{B}(\boldsymbol{C})$, then Q is a spectral measure in $L^p(\boldsymbol{R})$ such that $H=\int_{\boldsymbol{C}}\mu dQ(\mu)$. Since H and S commute, it follows that $HP(E)=P(E)H$ for each $E\in\mathcal{B}([0,\infty))$, [5; XVIII Corollary 2.4]. But, H is also a scalar operator, with Q its resolution of the identity, and hence $Q_jP(E)=P(E)Q_i$ for each $j\in\{1,2\}$ and $E\in\mathcal{B}([0,\infty))$, [5; XV Corollary 3.7].

Let $\Omega=[0,\infty)\times\{-1,1\}$ and let Σ denote the Borel subsets of Ω. Define a set function $\Lambda\colon\Sigma\to L(L^p(\boldsymbol{R}))$ by

$$\Lambda(U)=Q_1P(\{t\geq 0;(t,1)\in U\})+Q_2P(\{t\geq 0;(t,-1)\in U\}),\qquad U\in\Sigma.$$

Then it is routine to check that Λ is a spectral measure which may be considered as being defined on all of $\mathcal{B}(\boldsymbol{C})$ with Ω as its support. Let $\psi\colon\Omega\to\boldsymbol{C}$ be the Σ-measurable function defined by $(\lambda,\mu)\mapsto\lambda\mu$ for each $(\lambda,\mu)\in\Omega$. The corresponding scalar operator $\Lambda(\psi)$ that is so induced has domain given by

$$\mathcal{D}(\Lambda(\psi))=\{f\in L^p(\boldsymbol{R});\ \psi\text{ is }\Lambda(\cdot)f\text{-integrable}\}.$$

Since, for each $U\in\Sigma$, we have the identity

$$\Lambda(U)f = Q_1P(\{t\geq 0;\ (t,\ 1)\in U\})f + Q_2P(\{t\geq 0;\ (t,\ -1)\in U\})f$$

whenever $f\in L^p(\boldsymbol{R})$, it is clear that ψ is $\Lambda(\cdot)f$-integrable if and only if the identity function λ, on $[0,\ \infty)$, is $P(\cdot)f$-integrable. Accordingly,

$$\mathcal{D}(\Lambda(\psi)) = \{f\in L^p(\boldsymbol{R});\ \lambda \text{ is } P(\cdot)f\text{-integrable}\} = \mathcal{D}(S)\,,$$

where we have used the fact that $S=P(\lambda)$. But, (7) implies that $\mathcal{D}(S)=\mathcal{D}(D)$. Hence, if $f\in\mathcal{D}(\Lambda(\psi))=\mathcal{D}(D)$, then

$$\begin{aligned}\Lambda(\psi)f &= \int_\Omega \lambda\mu d\Lambda(\lambda,\ \mu)f = Q_1\int_0^\infty \lambda dP(\lambda)f - Q_2\int_0^\infty \lambda dP(\lambda)f = \\ &= Q_1Sf - Q_2Sf = HSf = Df\end{aligned}$$

which shows that $D=\Lambda(\psi)$. Accordingly, D is a scalar operator. This is the desired contradiction and completes the proof of Theorem 1 for the case when $1<p<2$.

For $2<p<\infty$ we proceed via duality. Indeed, noting that the dual operator L^*, of L (when L is considered in $L^p(\boldsymbol{R})$), is just L in $L^q(\boldsymbol{R})$, it sufficies to establish the fact that in a reflexive Banach space X the dual operator T^* of a scalar operator T is a scalar operator in X^*. But, if $T=P(\psi)$ wheie $P\colon \Sigma\rightarrow L(X)$ is a spectral measure and ψ is a Σ-measurable function, then it is an easy consequence of the reflexivity of X and the Orlicz-Pettis lemma that the set function $P^*\colon \Sigma\rightarrow L(X^*)$ defined by $P^*(E)=P(E)^*$, $E\in\Sigma$, is a spectral measure and hence $P^*(\psi)$ is a scalar operator in X^*. It remains only to verify the identity $T^*=P^*(\psi)$. But, this follows from the ieflexivity of X and [5; XVIII Theorem 2.11 (i)]. The proof of Theorem 1 is thereby complete.

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