

Title	On the generalized Novikov first ext group modulo a prime
Author(s)	Ravenel, Douglas C. ; Nakai, Hirofumi
Citation	Osaka Journal of Mathematics. 39(4) P.843-P.865
Issue Date	2002-12
Text Version	publisher
URL	https://doi.org/10.18910/5614
DOI	10.18910/5614
rights	
Note	

Osaka University Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/repo/ouka/all/>

ON THE GENERALIZED NOVIKOV FIRST EXT GROUP MODULO A PRIME

HIROFUMI NAKAI and DOUGLAS C. RAVENEL

(Received March 28, 2001)

1. Introduction

Let BP be the Brown-Peterson spectrum for a fixed prime p , whose homotopy is $BP_* \cong \mathbf{Z}_{(p)}[v_1, v_2, \dots, v_n, \dots]$. In [6] §6.5, the second author has introduced the spectrum $T(m)$, whose BP -homology is

$$BP_*(T(m)) \cong BP_*[t_1, \dots, t_m].$$

This is homotopy equivalent to BP below dimension $2p^{m+1} - 3$.

The Adams-Novikov E_2 -term converging to the homotopy groups of $T(m)$

$$E_2^{*,*}(T(m)) = \text{Ext}_{BP_*(BP)}(BP_*, BP_*(T(m)))$$

is isomorphic by [6, Corollary 7.1.3] to

$$\text{Ext}_{\Gamma(m+1)}(BP_*, BP_*),$$

where

$$\Gamma(m+1) = BP_*(BP)/(t_1, \dots, t_m) \cong BP_*[t_{m+1}, t_{m+2}, \dots].$$

In particular $\Gamma(1) = BP_*(BP)$ by definition. To get the structure of $\text{Ext}_{\Gamma(m+1)}(BP_*, BP_*)$, we will use the chromatic method introduced in [3].

Denote an ideal (p, v_1, \dots, v_{n-1}) of BP_* by I_n , and a comodule

$$v_{n+s}^{-1}BP_*/(p, v_1, \dots, v_{n-1}, v_n^\infty, \dots, v_{n+s-1}^\infty).$$

by M_n^s . Then we can consider the chromatic spectral sequence converging to

$$\text{Ext}_{\Gamma(m+1)}(BP_*, BP_*/I_n)$$

with

$$E_1^{s,t} = \text{Ext}_{\Gamma(m+1)}^t(BP_*, M_n^s).$$

Shimomura calls this Ext group the *general chromatic E_1 -term*.

The limiting case as m approaches infinity is discussed by the second author in [7]. In this paper we will determine the module structure (over an appropriate generalization of $k(1)_*$) of

$$\text{Ext}_{\Gamma(m+1)}^0(BP_*, M_1^1)$$

in Theorem 6.1, which is closely related to the group

$$\text{Ext}_{\Gamma(m+1)}^1(BP_*, BP_*/(p)).$$

The structure of these two groups are described below in Theorems 6.1 and 7.1. Notice that our target $\text{Ext}_{\Gamma(m+1)}^1(BP_*, BP_*/(p))$ is different from the localized object, which is determined in Kamiya-Shimomura [2]. Hereafter we will often abbreviate $\text{Ext}_{\Gamma(m+1)}(BP_*, M)$ by $\text{Ext}_{\Gamma(m+1)}(M)$ for a $\Gamma(m+1)$ -comodule M .

We begin by recalling the analogous result for $m = 0$, which was obtained long ago by Miller-Wilson in [4] (and reformulated in [6] as Theorems 5.2.13, Corollary 5.2.14, and Theorem 5.2.17). Recall that we have the 4-term exact sequence

$$(1.1) \quad 0 \rightarrow BP_*/(p) \rightarrow M_1^0 \rightarrow M_1^1 \rightarrow N_1^2 \rightarrow 0$$

obtained by splicing the two short exact sequences

$$0 \longrightarrow BP_*/(p) \longrightarrow M_1^0 \longrightarrow N_1^1 \longrightarrow 0,$$

and

$$0 \longrightarrow N_1^1 \longrightarrow M_1^1 \longrightarrow N_1^2 \longrightarrow 0.$$

From (1.1) we see that $\text{Ext}_{\Gamma(1)}^1(BP_*/(p))$ is a certain subquotient of

$$(1.2) \quad \text{Ext}_{\Gamma(1)}^1(M_1^0) \oplus \text{Ext}_{\Gamma(1)}^0(M_1^1).$$

For the first summand, we have (for p odd)

$$\text{Ext}_{\Gamma(1)}(M_1^0) = \text{Ext}_{\Gamma(1)}(v_1^{-1}BP_*/(p)) \cong K(1)_* \otimes E(h_{1,0}).$$

In particular we have

$$\text{Ext}_{\Gamma(1)}^1(M_1^0) \cong K(1)_*\{h_{1,0}\}.$$

It turns out that the image of $\text{Ext}_{\Gamma(1)}^1(BP_*/(p))$ into this group is $k(1)_*\{h_{1,0}\}$, which is the v_1 -torsion free component of $\text{Ext}_{\Gamma(1)}^1(BP_*/(p))$.

To describe $\text{Ext}_{\Gamma(1)}^0(M_1^1)$, we recall the elements $x_k \in v_2^{-1}BP_*/(p)$ defined by

$$x_0 = v_2,$$

$$\begin{aligned}
 x_1 &= v_2^p - v_1^p v_2^{-1} v_3, \\
 x_2 &= x_1^p - v_1^{p^2-1} v_2^{p^2-p+1} - v_1^{p^2+p-1} v_2^{p^2-2p} v_3, \\
 \text{and } x_k &= \begin{cases} x_{k-1}^2 & (p = 2) \\ x_{k-1}^p - 2v_1^{(p+1)(p^{k-1}-1)} v_2^{(p-1)p^{k-1}+1} & (p > 2) \end{cases} \quad \text{for } k \geq 3,
 \end{aligned}$$

and integers $a(k)$ defined by

$$\begin{aligned}
 a(0) &= 1, \\
 a(1) &= p, \\
 a(k) &= \begin{cases} 3 \cdot 2^{k-1} & (p = 2) \\ p^k + p^{k-1} - 1 & (p > 2) \end{cases} \quad \text{for } k \geq 2.
 \end{aligned}$$

Then we have

Theorem 1.3 ([4]). *As a $k(1)_*$ -module, $\text{Ext}_{\Gamma(1)}^0(M_1^1)$ is the direct sum of*

- (a) *the cyclic submodules generated by $x_k^s/v_1^{a(k)}$ for $k \geq 0$ and $p \nmid s \in \mathbf{Z}$; and*
- (b) *$K(1)_*/k(1)_*$, generated by $1/v_1^j$ for $j \geq 1$.*

The odd prime case follows from the next proposition ([3, Proposition 5.4]). We refer the reader to the original sources for the case $p = 2$.

Proposition 1.4. *Let p be odd. Modulo $(p, v_1^{1+a(k)})$, the differential*

$$d = \eta_R - \eta_L: v_2^{-1}BP_*/(p) \rightarrow v_2^{-1}BP_*/(p) \otimes_{BP_*} BP_*(BP)$$

on x_k is

$$d(x_k) \equiv \begin{cases} v_1 t_1^p & \text{for } k = 0, \\ v_1^p v_2^{p-1} t_1 & \text{for } k = 1, \\ 2v_1^{a(k)} v_2^{(p-1)p^{j-1}} t_1 & \text{for } k \geq 2. \end{cases}$$

Before Theorem 1.3 was proved, the naive conjecture about $\text{Ext}_{\Gamma(1)}^1(BP_*/(p))$ would have had the exponents $a(k)$ being p^k for all $k \geq 0$. It was clear that

$$\frac{v_2^{sp^k}}{v_1^{p^k}} \in \text{Ext}_{\Gamma(1)}^0(M_1^1),$$

but the existence of “deeper” elements such as

$$\frac{x_2}{v_1^{a(2)}} = \frac{v_2^{p^2} - v_1^{p^2-1} v_2^{p^2-p+1} - v_1^{p^2} v_2^{-p} v_3^p}{v_1^{p^2+p-1}}$$

and

$$\frac{x_3}{v_1^{a(3)}} = \frac{v_2^{p^3} - v_1^{p^3-p} v_2^{p^3-p^2+p} - v_1^{p^3} v_2^{-p^2} v_3^{p^2} - 2v_1^{p^3+p^2-p-1} v_2^{p^3-p^2+1}}{v_1^{p^3+p^2-1}}$$

(and that of $\beta_{sp^2/a(2)}$ and $\beta_{sp^3/a(3)}$ in $\text{Ext}_{\Gamma(1)}^1(BP_*/(p))$ for $s > 1$) came as a surprise, as did the fact that the limiting value (as $k \rightarrow \infty$) of $a(k)/p^k$ is $(p + 1)/p$ (this limit is attained for $p = 2$ but not for odd primes) instead of 1.

Using these results one can deduce

Theorem 1.5. *For odd prime p , the group $\text{Ext}_{\Gamma(1)}^1(BP_*/(p))$ is isomorphic to*

$$k(1)_* \{ \beta_{sp^k/j} : s \geq 0, p \nmid s, k \geq 0 \text{ and } 0 < j \leq a_s(k) \} \oplus k(1)_* \{ h_{1,0} \},$$

where $\beta_{sp^k/j}$ is the image of x_k^s/v_1^j under the connecting homomorphism

$$\delta : \text{Ext}_{\Gamma(1)}^0(N_1^1) \rightarrow \text{Ext}_{\Gamma(1)}^1(N_1^0).$$

and $a_s(k) = \begin{cases} p^k & (s = 1) \\ a(k) & (s > 1) \end{cases}.$

Our results (Theorems 6.1 and 7.1 below) have the same form as Theorems 1.3 and 1.5, but with x_k and $a(k)$ replaced by \widehat{x}_k and $\widehat{a}(k)$ defined in (4.1) and (4.3), and with $k(1)_*$ replaced by a bigger ring $v_2^{-1}\widehat{k}(1)_*$ defined in (2.1). The $\widehat{a}(k)$ are the same for all $m > 0$ (except when $m = 1$ and $p = 2$) although the \widehat{x}_k show a slight difference between the cases $m = 1$ and $m > 1$. The case $m = 1$ and $p = 2$ is different and has to be treated separately. For $m > 1$ there are no special conditions for the prime 2. The asymptotic behavior of the exponents is given by

$$\lim_{k \rightarrow \infty} \frac{\widehat{a}(k)}{p^k} = \frac{p^3 + p^2}{p^3 - 1},$$

a slightly larger value than for the case $m = 0$. However for $m > 0$ there are no deeper elements in $\text{Ext}_{\Gamma(m+1)}^1(BP_*/(p))$, i.e., no elements of the form $\widehat{\beta}_{sp^k/j}$ with $p \nmid s$ and $j > p^k$.

We found a new form of periodicity in our statement with no precedent in Theorem 1.3. For example, (except for $p = 2$ and $m = 1$) we have

$$\begin{aligned} \widehat{x}_k - \widehat{x}_{k-1}^p &= -v_1^{p^{k-1}(p+1)} v_2^{p^{k-2}(p^{m+2}-p-1)} \widehat{x}_{k-3}^{p-1} (\widehat{x}_{k-3} - \widehat{x}_{k-4}^p) && \text{for } k \geq 5, \\ \text{and } \widehat{a}(k) &= p^k + p^{k-1} + \widehat{a}(k-3) && \text{for } k \geq 4. \end{aligned}$$

A similar result for the chromatic module M_2^1 is obtained in a joint work with Ippei Ichigi [1]. There we get a similar periodicity with period 4 instead of 3 when $m \geq 5$.

We obtained our result in the summer of 1999. On the other hand, Kamiya-Shimomura [2] told us that they have determined all the structure of $\text{Ext}_{\Gamma(m+1)}^*(M_1^1)$ in the fall of 1999 independently.

We are grateful to the referee for suggesting some corrections to an earlier draft of this paper.

2. Preliminaries

For a $\Gamma(m + 1)$ -comodule M , consider the cobar complex

$$\left\{ C_{\Gamma(m+1)}^n(M), d_n \right\}_{n \geq 0},$$

which is determined by

$$C_{\Gamma(m+1)}^n(M) = \underbrace{\Gamma(m + 1) \otimes_{BP_*} \cdots \otimes_{BP_*} \Gamma(m + 1)}_{n\text{-factors}} \otimes_{BP_*} M,$$

and

$$d_n : C_{\Gamma(m+1)}^n(M) \rightarrow C_{\Gamma(m+1)}^{n+1}(M).$$

Then $\text{Ext}_{\Gamma(m+1)}(M)$ is the cohomology of this cobar complex. By the change-of-rings isomorphism (cf. [6, Theorem 6.1.1]), we have

$$\begin{aligned} \text{Ext}_{\Gamma(m+1)}(M_n^0) &\cong \text{Ext}_{\Gamma(1)}(M_n^0 \otimes_{BP_*} BP_*(T(m))) \\ &\cong \text{Ext}_{\Sigma(n)}(K(n)_*, K(n)_*(T(m))), \end{aligned}$$

where $\Sigma(n) = K(n)_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} K(n)_*$. This object is already known by [6, Corollary 6.5.6].

In order to avoid the excessive appearance of the index m , we will hereafter use the following notations.

$$(2.1) \quad \left\{ \begin{array}{l} \omega = p^m, \\ \widehat{v}_i = v_{m+i}, \\ \widehat{t}_i = t_{m+i}, \\ \widehat{h}_{i,j} = h_{m+i,j}, \\ \widehat{K}(n)_* = K(n)_*[v_{n+1}, \dots, v_{n+m}], \\ \text{and } \widehat{k}(n)_* = k(n)_*[v_{n+1}, \dots, v_{n+m}], \end{array} \right.$$

where $h_{m+i,j}$ is the cocycle represented by t_{m+i}^j .

Theorem 2.2 ([6, Corollary 6.5.6]). *If $n < 2(p - 1)(m + 1)/p$ and $n < m + 2$, then*

$$\text{Ext}_{\Gamma(m+1)}(M_n^0) \cong \widehat{K}(n)_* \otimes E(\widehat{h}_{i,j} : 1 \leq i \leq n, 0 \leq j \leq n - 1).$$

In this paper we will need this result only for $n = 2$, for which it covers the cases $m > 0$ for odd p and $m > 1$ for $p = 2$. For the case $p = 2$ and $m = 1$, we need

Theorem 2.3 ([5]). *If $p = 2$ and $m = 1$, then*

$$\text{Ext}_{\Gamma(2)}(M_2^0) \cong \widehat{K}(2)_* \otimes P(\widehat{h}_{1,0}, \widehat{h}_{1,1}) / (\widehat{h}_{1,1}^2 + v_2^2 \widehat{h}_{1,0}^2) \otimes E(\widehat{h}_{2,0}, \widehat{h}_{2,1}, \rho),$$

where $\rho = \widehat{h}_{3,1} + v_2^5 \widehat{h}_{3,0}$.

This information allow us to determine the structure of $\text{Ext}_{\Gamma(m+1)}(M_1^1)$ using the Bockstein spectral sequence. In fact, we use the following convenient lemma.

Lemma 2.4 (cf. [3, Remark 3.11]). *Assume that there exists a $\widehat{k}(1)_*$ -submodule B^t of $\text{Ext}_{\Gamma(m+1)}^t(M_1^1)$ for each $t < N$, such that the following sequence is exact:*

$$\begin{aligned} 0 &\longrightarrow \text{Ext}_{\Gamma(m+1)}^0(M_2^0) \xrightarrow{1/v_1} B^0 \xrightarrow{v_1} B^0 \xrightarrow{\delta} \cdots \\ \cdots &\xrightarrow{\delta} \text{Ext}_{\Gamma(m+1)}^t(M_2^0) \xrightarrow{1/v_1} B^t \xrightarrow{v_1} B^t \xrightarrow{\delta} \cdots \end{aligned}$$

where δ is a restriction of the coboundary map

$$\delta: \text{Ext}_{\Gamma(m+1)}^t(M_1^1) \rightarrow \text{Ext}_{\Gamma(m+1)}^{t+1}(M_2^0).$$

Then the inclusion $i_t: B^t \rightarrow \text{Ext}_{\Gamma(m+1)}^t(M_1^1)$ is an isomorphism between $\widehat{k}(1)_*$ -modules for each $t < N$.

Proof. Because $\text{Ext}_{\Gamma(m+1)}^t(M_1^1)$ is a v_1 -torsion module, we can filter B^t by

$$B^t(i) = \{x \in B^t : v_1^i x = 0\}$$

and $\text{Ext}_{\Gamma(m+1)}^t(M_1^1)$ by

$$E^t(i) = \{x \in \text{Ext}_{\Gamma(m+1)}^t(M_1^1) : v_1^i x = 0\}.$$

Assume that the inclusion i_k is an isomorphism for $k \leq t - 1$ (the $t = 0$ case is obvious), and consider the following commutative ladder diagram where we abbreviate

$\text{Ext}_{\Gamma(m+1)}^s(M_i^j)$ by $H^s(M_i^j)$.

$$\begin{array}{ccccccccc}
 B^{t-1} & \xrightarrow{\delta} & H^t(M_2^0) & \xrightarrow{1/v_1} & B^t(i) & \xrightarrow{v_1} & B^t(i-1) & \xrightarrow{\delta} & H^t(M_2^0) \\
 i_{t-1} \downarrow \cong & & \parallel & & i_t \downarrow & & i_t \downarrow & & \parallel \\
 H^{t-1}(M_1^1) & \xrightarrow{\delta} & H^t(M_2^0) & \xrightarrow{1/v_1} & E^t(i) & \xrightarrow{v_1} & E^t(i-1) & \xrightarrow{\delta} & H^t(M_2^0)
 \end{array}$$

Using the Five Lemma, we obtain the desired isomorphism $B^t(i) \cong E^t(i)$ ($i \geq 1$) by induction on i . □

In §3 and §4, we will define elements $\widehat{x}_k \in v_2^{-1}BP_*$ for $k \geq 0$ (see (4.1)) satisfying

$$\widehat{x}_k^s \equiv \widehat{v}_2^{sp^k} \pmod{(p, v_1)},$$

and integers $\widehat{a}(k)$ such that each \widehat{x}_k^s/v_1^l is a cocycle of for all $1 \leq l \leq \widehat{a}(k)$.

Using these notations, we can describe the structure of B^0 fitting into the long exact sequence of Lemma 2.4. We have

Lemma 2.5. *For $m > 0$,*

$$B^0 = v_2^{-1}\widehat{k}(1)_* \left\{ \frac{\widehat{x}_k^s}{v_1^{\widehat{a}(k)}} : k \geq 0, s > 0 \text{ and } p \nmid s \right\} \oplus v_2^{-1}\widehat{K}(1)_*/\widehat{k}(1)_*$$

is isomorphic as a $\widehat{k}(1)_*$ -module to $\text{Ext}_{\Gamma(m+1)}^0(M_1^1)$, if the set

$$\left\{ \delta \left(\frac{\widehat{x}_k^s}{v_1^{\widehat{a}(k)}} \right) : k \geq 0, s > 0 \text{ and } p \nmid s \right\} \subset \text{Ext}_{\Gamma(m+1)}^1(M_2^0)$$

is linearly independent over

$$R = \mathbf{Z}/(p)[v_2, v_2^{-1}, v_3, \dots, v_m, v_{m+1}],$$

where δ is the coboundary map in Lemma 2.4.

Proof. All exactness of the sequence

$$0 \longrightarrow \text{Ext}_{\Gamma(m+1)}^0(M_2^0) \xrightarrow{1/v_1} B^0 \xrightarrow{v_1} B^0 \xrightarrow{\delta} \text{Ext}_{\Gamma(m+1)}^1(M_2^0)$$

is obvious, except $\text{Ker } \delta \subset \text{Im } v_1$. So we need to show only this inclusion. Separate

the R -basis of B^0 into two parts,

$$A = \left\{ \frac{\widehat{x}_k^s}{v_1^{\widehat{a}(k)}} : k \geq 0 \text{ and } p \nmid s > 0 \right\}$$

and

$$B = \left\{ \frac{\widehat{x}_k^l}{v_1^l} : k \geq 0, p \nmid s > 0, \text{ and } 1 \leq l < \widehat{a}(k) \right\} \cup \left\{ \frac{1}{v_1^i} : i > 0 \right\}.$$

Then it is obvious that $\delta(\widehat{x}_\lambda) \neq 0 \in \text{Ext}_{\Gamma(m+1)}^1(M_2^0)$ for $\widehat{x}_\lambda \in A$, but that $\delta(y_\mu) = 0 \in \text{Ext}_{\Gamma(m+1)}^1(M_2^0)$ for $y_\mu \in B$. Thus for any element $z = \sum_\lambda a_\lambda \widehat{x}_\lambda + \sum_\mu b_\mu y_\mu$ of B^0 ($a_\lambda, b_\mu \in R$), we have $\delta(z) = \sum_\lambda a_\lambda \delta(\widehat{x}_\lambda)$. The condition implies that all a_λ are zero when $\delta(z) = 0$, and so $v_1 \sum_\mu b_\mu y_\mu / v_1 = z$. This completes the proof. \square

3. Definition of the elements \widehat{w}_3 and \widehat{w}_4

In this section we will introduce elements \widehat{w}_3 and \widehat{w}_4 in (3.2) to change the bases $\widehat{h}_{i,j}$ ($i = 1, 2$ and $j = 0, 1$) of $\text{Ext}_{\Gamma(m+1)}(M_2^0)$ given in Theorems 2.2 and 2.3. First we recall the right unit η_R on \widehat{v}_j .

Lemma 3.1. *For any prime p and $m \geq 1$, the right unit*

$$\eta_R: BP_* \rightarrow \Gamma(m+1)/(p)$$

on the Hazewinkel generators are

$$\left\{ \begin{array}{l} \eta_R(\widehat{v}_2) = \widehat{v}_2 + v_1 \widehat{t}_1^p - v_1^{p\omega} \widehat{t}_1, \\ \eta_R(\widehat{v}_3) = \widehat{v}_3 + v_2 \widehat{t}_1^{p^2} - v_2^{p\omega} \widehat{t}_1 + v_1 \widehat{t}_2^p - v_1^{p^2\omega} \widehat{t}_2 \\ \quad + v_1 w_1(\widehat{v}_2, v_1 \widehat{t}_1^p, -v_1^{p\omega} \widehat{t}_1) \\ \quad \text{(add } v_1^{4\omega+1} \widehat{t}_1^2 \text{ for } p = 2) \\ \quad \equiv \widehat{v}_3 + v_2 \widehat{t}_1^{p^2} - v_2^{p\omega} \widehat{t}_1 + v_1 \widehat{t}_2^p - v_1^2 \widehat{v}_2^{p-1} \widehat{t}_1^p \pmod{(v_1^3)}, \\ \eta_R(\widehat{v}_4) \equiv \widehat{v}_4 + v_3 \widehat{t}_1^{p^3} - v_3^{p\omega} \widehat{t}_1 + v_2 \widehat{t}_2^{p^2} - v_2^{p^2\omega} \widehat{t}_2 \pmod{(v_1)}. \end{array} \right.$$

where $w_1(-)$ is the first Witt polynomial satisfying

$$w_1(y_1, \dots, y_t, \dots) = \frac{(\sum_t y_t^p) - (\sum_t y_t)^p}{p}.$$

Now let

$$(3.2) \quad \begin{cases} \widehat{w}_3 = v_2^{-1} \widehat{v}_3, \\ \widehat{w}_4 = v_2^{-1} (\widehat{v}_4 - v_3 \widehat{w}_3^p). \end{cases}$$

Using Lemma 3.1, it is easily shown that

Lemma 3.3. *The differentials*

$$d = \eta_R - \eta_L: v_2^{-1}BP_*/(p) \rightarrow v_2^{-1}BP_*/(p) \otimes_{BP_*} \Gamma(m+1)$$

on the above \widehat{w}_k 's are

$$d(\widehat{w}_3) \equiv \widehat{t}_1^{p^2} - v_2^{p\omega-1}\widehat{t}_1 + v_1 v_2^{-1}\widehat{t}_2^p - v_1^2 v_2^{-1} v_2^{p-1} \widehat{t}_1^p \pmod{v_1^3},$$

and
$$d(\widehat{w}_4) \equiv \widehat{t}_2^{p^2} - v_2^{-1} v_3^{p\omega}\widehat{t}_1 + v_2^{p^2\omega-p-1} v_3 \widehat{t}_1^p - v_2^{p^2\omega-1} \widehat{t}_2 \pmod{v_1}.$$

Then we can change the $\widehat{K}(n)_*$ -module basis of Theorems 2.2 and 2.3 using Lemma 3.3. In particular, we have

Corollary 3.4.

$$\begin{aligned} & \text{Ext}_{\Gamma(m+1)}^1(M_2^0) \\ & \cong \begin{cases} \widehat{K}(2)_* \{ \widehat{h}_{1,1}, \widehat{h}_{1,2}, \widehat{h}_{2,2}, \widehat{h}_{2,3} \} & \text{for } p > 2, \text{ or } p = 2 \text{ and } m > 1, \\ \widehat{K}(2)_* \{ \widehat{h}_{1,1}, \widehat{h}_{1,2}, \widehat{h}_{2,2}, \widehat{h}_{2,3}, \rho \} & \text{for } p = 2 \text{ and } m = 1. \end{cases} \end{aligned}$$

When we compute the connecting homomorphism δ of Lemma 2.5, this base-changing method actually works well to determine the structure of $\text{Ext}_{\Gamma(m+1)}^0(M_n^1)$ for a general n . In fact, Kamiya-Shimomura [2] and Shimomura [9] recently determined the structure of $\text{Ext}_{\Gamma(m+1)}^0(M_n^1)$ under some conditions on m and n in a similar way.

4. The elements \widehat{x}_k

In this section, we will define elements $\widehat{x}_k \in v_2^{-1}BP_*$ ($k \geq 0$) to be used in Lemma 2.5 except for $p = 2$ and $m = 1$. The case $p = 2$ and $m = 1$ will be treated in the next section.

Define elements $\widehat{x}_k \in v_2^{-1}BP_*$ ($k \geq 0$) inductively on k by

$$(4.1) \quad \begin{cases} \widehat{x}_0 = \widehat{v}_2, \\ \widehat{x}_1 = \widehat{x}_0^p, \\ \widehat{x}_2 = \widehat{x}_1^p - v_1^{p^2-1} v_2^{\beta+1} \widehat{x}_0 - v_1^{p^2} \widehat{w}_3^p, \\ \widehat{x}_3 = \widehat{x}_2^p, \\ \widehat{x}_4 = \begin{cases} \widehat{x}_3^p + \widehat{y}_1 + \widehat{y}_2 & (m > 1) \\ \widehat{x}_3^p + \widehat{y}_1 + \frac{1}{2} \widehat{y}_3 & (m = 1 \text{ and } p > 2) \end{cases}, \\ \widehat{x}_k = \widehat{x}_{k-1}^p - v_1^{p^{k-1}\alpha} v_2^{p^{k-2}\beta} \widehat{x}_{k-3}^{p-1} (\widehat{x}_{k-3} - \widehat{x}_{k-4}^p) & \text{for } k \geq 5, \end{cases}$$

where $\alpha = p + 1$ and $\beta = p^2\omega - p - 1$, and \widehat{y}_i ($i = 1, 2, 3$) are given by

$$(4.2) \quad \begin{cases} \widehat{y}_1 = -v_1^{p^4+p^3-p^2-p} v_2^{p^2\beta+p} \widehat{x}_2 + v_1^{p^4+p^3-p} v_2^{-p^3-p^2} v_3^{p^3\omega} \widehat{x}_1 \\ \quad - v_1^{p^4+p^3-1} v_2^{(p^2+1)\beta-p^3+1} v_3^{p^2} \widehat{x}_0 + v_1^{p^4+p^3} v_2^{-p^3} \widehat{w}_4^{p^2} \\ \quad - v_1^{p^4+p^3} v_2^{(\beta-p)p^2} v_3^{p^2} \widehat{w}_3^p, \\ \widehat{y}_2 = -v_1^{p^4+p^3-p^2} v_2^{(\beta-p)p^2} v_3^{p^2} \widehat{x}_2, \\ \widehat{y}_3 = \widehat{y}_2 + v_1^{p^4+p^3-1} v_2^{(p^2+1)\beta-p^3+1} \widehat{x}_2 \widehat{x}_0 + v_1^{p^4+p^3} v_2^{(\beta-p)p^2} \widehat{x}_2 \widehat{w}_3^p. \end{cases}$$

Define integers $\widehat{a}(k)$ by

$$(4.3) \quad \widehat{a}(k) = \begin{cases} p^k & \text{for } 0 \leq k \leq 1, \\ p^{k-1}\alpha & \text{for } 2 \leq k \leq 3, \\ p^{k-1}\alpha + \widehat{a}(k-3) & \text{for } k \geq 4. \end{cases}$$

Notice that the integers $\widehat{a}(k)$ are equivalently defined inductively on k by

$$(4.4) \quad \widehat{a}(k) = \begin{cases} p\widehat{a}(k-1) & \text{for } 2 < k \equiv 0 \pmod{3}, \\ p\widehat{a}(k-1) + p & \text{for } 2 \leq k \not\equiv 0 \pmod{3}. \end{cases}$$

Lemma 4.5. *Unless $p = 2$ and $m = 1$, the differentials*

$$d = \eta_R - \eta_L: v_2^{-1}BP_*/(p) \rightarrow v_2^{-1}BP_*/(p) \otimes_{BP_*} \Gamma(m+1)$$

on the above \widehat{x}_k 's are

$$\begin{aligned} d(\widehat{x}_0) &\equiv v_1 \widehat{t}_1^p && \text{mod } (v_1^2), \\ d(\widehat{x}_1) &\equiv v_1^{\widehat{a}(1)} \widehat{t}_1^{p^2} && \text{mod } (v_1^{1+\widehat{a}(1)}), \\ d(\widehat{x}_2) &\equiv -v_1^{\widehat{a}(2)} v_2^{-p} \widehat{t}_2^{p^2} && \text{mod } (v_1^{1+\widehat{a}(2)}), \\ d(\widehat{x}_3) &\equiv -v_1^{\widehat{a}(3)} v_2^{-p^2} \widehat{t}_2^{p^3} && \text{mod } (v_1^{1+\widehat{a}(3)}), \\ d(\widehat{x}_k) &\equiv -v_1^{p^{k-1}\alpha} v_2^{p^{k-2}\beta} \widehat{v}_2^{(p-1)p^{k-3}} d(\widehat{x}_{k-3}) && \text{mod } (v_1^{1+\widehat{a}(k)}) \quad \text{for } k \geq 4. \end{aligned}$$

Proof. By Lemma 3.1 we have

$$(4.6) \quad \begin{aligned} d(\widehat{x}_0) &\equiv v_1 \widehat{t}_1^p && \text{mod } (v_1^{p\omega}), \\ d(\widehat{x}_1) &\equiv v_1^p \widehat{t}_1^{p^2} && \text{mod } (v_1^{p^2\omega}). \end{aligned}$$

Moreover, we find that

$$\begin{aligned} d(\widehat{x}_1^p) &\equiv v_1^{p^2} \widehat{t}_1^{p^3} && \text{mod } (v_1^{p^3\omega}), \\ d(-v_1^{p^2} \widehat{w}_3^p) &\equiv -v_1^{p^2} (\widehat{t}_1^{p^3} - v_2^{\beta+1} \widehat{t}_1^p - v_1^{2p} v_2^{-p} \widehat{v}_2^{(p-1)p} \widehat{t}_1^{p^2} + v_1^p v_2^{-p} \widehat{t}_2^{p^2}) \end{aligned}$$

$$\begin{aligned}
 & \text{mod } (v_1^{p^2+3p}), \\
 \text{and } d(-v_1^{p^2-1}v_2^{\beta+1}\widehat{x}_0) & \equiv -v_1^{p^2}v_2^{\beta+1}\widehat{t}_1^p \text{ mod } (v_1^{p^{\omega+p^2-1}}).
 \end{aligned}$$

Summing the above three congruences we obtain

$$\begin{aligned}
 d(\widehat{x}_2) & \equiv -v_1^{p^2+p}v_2^{-p}(\widehat{t}_2^{p^2} - v_1^p\widehat{v}_2^{(p-1)p}\widehat{t}_1^{p^2}) \text{ mod } (v_1^{p^2+2p+2}) \\
 & \equiv -v_1^{\widehat{a}(2)}v_2^{-p}\widehat{t}_2^{p^2} \text{ mod } (v_1^{p^2+2p}), \\
 \text{and } d(\widehat{x}_3) & \equiv -v_1^{\widehat{a}(3)}v_2^{-p^2}\widehat{t}_2^{p^3} \text{ mod } (v_1^{p^3+2p^2}).
 \end{aligned}$$

(4.4) suggests that we should calculate $d(\widehat{x}_k)$ modulo $(v_1^{2+\widehat{a}(k)})$ rather than modulo $(v_1^{1+\widehat{a}(k)})$ when we apply induction on $k \geq 4$. For $k = 4$, we find that modulo $(v_1^{2+\widehat{a}(4)})$

$$(4.7) \quad \left\{ \begin{aligned}
 & d(v_1^{\widehat{a}(4)-p}v_2^{-p^3}\widehat{w}_4^{p^2}) \\
 & \quad \equiv v_1^{\widehat{a}(4)-p}v_2^{-p^3}(\widehat{t}_2^{p^4} - v_2^{-p^2}v_3^{p^3}\omega\widehat{t}_1^{p^2} + v_2^{p^2\beta}v_3^{p^2}\widehat{t}_1^{p^3} - v_2^{(\beta+p)p^2}\widehat{t}_2^{p^2}) \\
 & d(v_1^{\widehat{a}(4)-2p}v_2^{-p^3-p^2}v_3^{p^3}\omega\widehat{x}_1) \\
 & \quad \equiv v_1^{\widehat{a}(4)-p}v_2^{-p^3-p^2}v_3^{p^3}\omega\widehat{t}_1^{p^2} \\
 & d(-v_1^{\widehat{a}(4)-\widehat{a}(2)-p}v_2^{p^2\beta+p}\widehat{x}_2) \\
 & \quad \equiv v_1^{\widehat{a}(4)-p}v_2^{p^2\beta}(\widehat{t}_2^{p^2} - v_1^p\widehat{v}_2^{p^2-p}\widehat{t}_1^{p^2}) \\
 & d(-v_1^{\widehat{a}(4)-p}v_2^{(\beta-p)p^2}v_3^{p^2}\widehat{w}_3^p) \\
 & \quad \equiv -v_1^{\widehat{a}(4)-p}v_2^{(\beta-p)p^2}v_3^{p^2}(\widehat{t}_1^{p^3} - v_2^{\beta+1}\widehat{t}_1^p + v_1^p v_2^{-p}\widehat{t}_2^{p^2}) \\
 & d(-v_1^{\widehat{a}(4)-p-1}v_2^{(p^2+1)\beta-p^3+1}v_3^{p^2}\widehat{x}_0) \\
 & \quad \equiv -v_1^{\widehat{a}(4)-p}v_2^{(p^2+1)\beta-p^3+1}v_3^{p^2}\widehat{t}_1^p.
 \end{aligned} \right.$$

Summing these congruences we obtain

$$\begin{aligned}
 d(\widehat{y}_1) & \equiv v_1^{\widehat{a}(4)-p}v_2^{-p^3}\widehat{t}_2^{p^4} - v_1^{\widehat{a}(4)}v_2^{p^2\beta}(v_2^{-p^3-p}v_3^{p^2}\widehat{t}_2^{p^2} + \widehat{v}_2^{p^2-p}\widehat{t}_1^{p^2}) \\
 & \text{mod } (v_1^{2+\widehat{a}(4)}).
 \end{aligned}$$

On the other hand, we find that modulo $(v_1^{2+\widehat{a}(4)})$

$$d(\widehat{y}_2) \equiv \begin{cases} v_1^{\widehat{a}(4)}v_2^{p^2\beta-p^3-p}v_3^{p^2}\widehat{t}_2^{p^2} & (m \geq 2), \\ -v_1^{p^3\alpha}v_2^{(\beta-p)p^2}v_3^{p^2}(\widehat{t}_1^{p^3} - v_1^p v_2^{-p}\widehat{t}_2^{p^2}) & (m = 1). \end{cases}$$

In the $m \geq 2$ case, we see that

$$\begin{aligned}
 d(\widehat{x}_4) & \equiv -v_1^{\widehat{a}(4)}v_2^{p^2\beta}\widehat{v}_2^{(p-1)p}\widehat{t}_1^{p^2} \\
 & \equiv -v_1^{p^3\alpha}v_2^{p^2\beta}\widehat{v}_2^{(p-1)p}d(\widehat{x}_1) \text{ mod } (v_1^{2+\widehat{a}(4)}).
 \end{aligned}$$

In the $m = 1$ case, we must modify the element \widehat{y}_2 into \widehat{y}_3 as defined in (4.2). We find that

$$\left\{ \begin{aligned} d(v_1^{\widehat{a}(4)-p} v_2^{(\beta-p)p^2} v_3^{p^2} \widehat{w}_3^p) &\equiv v_1^{\widehat{a}(4)-p} v_2^{(\beta-p)p^2} v_3^{p^2} (\widehat{t}_1^{p^3} - v_2^{\beta+1} \widehat{t}_1^p + v_1^p v_2^{-p} \widehat{t}_2^{p^2}) \\ &\quad \text{mod } (v_1^{\widehat{a}(4)+p}), \\ d(v_1^{\widehat{a}(4)-p-1} v_2^{(p^2+1)\beta-p^3+1} v_3^{p^2} \widehat{x}_0) &\equiv v_1^{\widehat{a}(4)-p} v_2^{(p^2+1)\beta-p^3+1} v_3^{p^2} \widehat{t}_1^p \\ &\quad \text{mod } (v_1^{\widehat{a}(4)+p^2-p-1}). \end{aligned} \right.$$

Summing the above congruences we obtain

$$d(\widehat{y}_3) \equiv 2v_1^{\widehat{a}(4)} v_2^{p^2\beta-p^3-p} v_3^{p^2} \widehat{t}_2^{p^2} \quad \text{mod } (v_1^{2+\widehat{a}(4)}).$$

Consequently, we obtain the desired congruence of $d(\widehat{x}_4)$ in $m = 1$ case, too.

For $k \geq 5$, assume that

$$d(\widehat{x}_{k-1}) \equiv -v_1^{p^{k-2}\alpha} v_2^{p^{k-3}\beta} \widehat{x}_{k-4}^{p-1} d(\widehat{x}_{k-4}) \quad \text{mod } (v_1^{2+\widehat{a}(k-1)}),$$

and denote $\widehat{x}_k - \widehat{x}_{k-1}^p$ by \widehat{z}_k . By definition (4.1), we note that $\widehat{z}_k = 0$ for $k \equiv 0 \pmod 3$. In case that $k \not\equiv 0$ modulo 3, we have

$$\widehat{z}_k = -v_1^{p^{k-1}\alpha} v_2^{p^{k-2}\beta} \widehat{x}_{k-3}^{p-1} \widehat{z}_{k-3} \quad \text{for } k \geq 5.$$

Notice that \widehat{z}_{k-3} is divided by $v_1^{p^2-1}$ for $k = 5$, by $v_1^{p^{(p+1)(p^2-1)}}$ for $k = 7$, and by $v_1^{p^{k-4}\alpha}$ for $k \geq 8$. On the other hand, by inductive hypothesis we see that $d(\widehat{x}_{k-3}^{p-1})$ is divisible by $v_1^{p^2+p}$ for $k = 5$ and by $v_1^{p^{k-4}\alpha}$ for $k \geq 7$. So we have

$$\begin{aligned} d(\widehat{x}_{k-3}^{p-1} \widehat{z}_{k-3}) &= d(\widehat{x}_{k-3}^{p-1}) \eta_R(\widehat{z}_{k-3}) + \widehat{x}_{k-3}^{p-1} d(\widehat{z}_{k-3}) \\ &\equiv \widehat{x}_{k-3}^{p-1} d(\widehat{z}_{k-3}) \quad \text{mod } (v_1^{2+\widehat{a}(k-3)}). \end{aligned}$$

Therefore the differential on \widehat{z}_k is

$$\begin{aligned} d(\widehat{z}_k) &\equiv -v_1^{p^{k-1}\alpha} v_2^{p^{k-2}\beta} d(\widehat{x}_{k-3}^{p-1} \widehat{z}_{k-3}) \\ &\equiv -v_1^{p^{k-1}\alpha} v_2^{p^{k-2}\beta} \widehat{x}_{k-3}^{p-1} d(\widehat{z}_{k-3}) \quad \text{mod } (v_1^{2+\widehat{a}(k)}). \end{aligned}$$

On the other hand, by inductive hypothesis we have

$$\begin{aligned} d(\widehat{x}_{k-1}^p) &\equiv -v_1^{p^{k-1}\alpha} v_2^{p^{k-2}\beta} \widehat{x}_{k-4}^{p-1} d(\widehat{x}_{k-4}^p) \\ &\equiv -v_1^{p^{k-1}\alpha} v_2^{p^{k-2}\beta} \widehat{x}_{k-3}^{p-1} d(\widehat{x}_{k-4}^p) \quad \text{mod } (v_1^{2+\widehat{a}(k)}). \end{aligned}$$

Summing the above two congruences we obtain

$$d(\widehat{x}_k) \equiv -v_1^{p^{k-1}\alpha} v_2^{p^{k-2}\beta} \widehat{x}_{k-3}^{p-1} d(\widehat{x}_{k-3}) \quad \text{mod } (v_1^{2+\widehat{a}(k)})$$

as desired. □

5. The case $p = 2$ and $m = 1$

In this section we recover some results of Shimomura [8] using the basis obtained in Corollary 3.4.

Define the elements $\widehat{x}_k \in v_2^{-1}BP_*$ in the same fashion as those in (4.1) for $0 \leq k \leq 3$, and

$$(5.1) \quad \begin{cases} \widehat{x}_4 = \widehat{x}_3^2 + \widehat{y}_1 + \widehat{y}_4, \\ \widehat{x}_k = \widehat{x}_{k-1}^2 + v_1^{5 \cdot 2^{k-2}} v_2^{3 \cdot 2^{k-2}} \widehat{x}_{k-2}(\widehat{x}_{k-2} + \widehat{x}_{k-3}^2) \end{cases} \quad \text{for } k \geq 5,$$

where \widehat{y}_4 is

$$\widehat{y}_4 = v_1^{14} v_2^{14} \widehat{x}_3 + v_1^{23} v_2^{25} \widehat{x}_1 + v_1^{25} v_2^8 v_3^8 \widehat{x}_0 + v_1^{25} v_2^{25} \widehat{w}_3 + v_1^{26} v_2^{10} \widehat{w}_4^2.$$

Note that the construction of \widehat{x}_k ($k \geq 4$) in this case is 2-periodic, although it is 3-periodic for the other cases. We are surprised at this difference.

Define integers $\widehat{a}(k)$ by

$$(5.2) \quad \widehat{a}(k) = \begin{cases} 2^k & \text{for } 0 \leq k \leq 1, \\ 3 \cdot 2^{k-1} & \text{for } 2 \leq k \leq 3, \\ 5 \cdot 2^{k-2} + \widehat{a}(k-2) & \text{for } k \geq 4. \end{cases}$$

This gives $\widehat{a}(0) = 1$, $\widehat{a}(1) = 2$, $\widehat{a}(2) = 6$, $\widehat{a}(3) = 12$, $\widehat{a}(4) = 26$, and so on. Notice that the integers $\widehat{a}(k)$ are equivalently defined inductively on k by

$$(5.3) \quad \widehat{a}(k) = \begin{cases} 2\widehat{a}(k-1) & \text{for odd } k, \\ 2\widehat{a}(k-1) + 2 & \text{for even } k. \end{cases}$$

Then we have

Lemma 5.4. *For $p = 2$ and $m = 1$, the differentials*

$$d = \eta_R - \eta_L : v_2^{-1}BP_*/(2) \rightarrow v_2^{-1}BP_*/(2) \otimes_{BP_*} \Gamma(m+1)$$

on the above \widehat{x}_k 's are

$$\begin{aligned} d(\widehat{x}_0) &\equiv v_1 \widehat{t}_1^2 && \text{mod } (v_1^2), \\ d(\widehat{x}_1) &\equiv v_1^{\widehat{a}(2)} \widehat{t}_1^4 && \text{mod } (v_1^{1+\widehat{a}(1)}), \\ d(\widehat{x}_2) &\equiv v_1^{\widehat{a}(2)} v_2^{-2} \widehat{t}_2^4 && \text{mod } (v_1^{1+\widehat{a}(2)}), \end{aligned}$$

$$\begin{aligned}
 d(\widehat{x}_3) &\equiv v_1^{\widehat{a}(3)} v_2^{-4} \widehat{t}_2^8 && \text{mod } (v_1^{1+\widehat{a}(3)}), \\
 d(\widehat{x}_k) &\equiv v_1^{5 \cdot 2^{k-2}} v_2^{3 \cdot 2^{k-2}} \widehat{v}_2^{k-2} d(\widehat{x}_{k-2}) && \text{mod } (v_1^{1+\widehat{a}(k)}) \quad \text{for } k \geq 4.
 \end{aligned}$$

Proof. The $k = 0$ and $k = 1$ cases follow directly from Lemma 3.1 (cf. (4.6)). For $k = 2$ case, we find that

$$\begin{cases}
 d(\widehat{x}_1^2) \equiv v_1^4 \widehat{t}_1^8 & \text{mod } (v_1^{16}), \\
 d(v_1^4 \widehat{w}_3^2) \equiv v_1^4 (\widehat{t}_1^8 + v_2^6 \widehat{t}_1^2 + v_1^2 v_2^{-2} \widehat{t}_2^4 + v_1^4 v_2^{-2} \widehat{v}_2^2 \widehat{t}_1^4) & \text{mod } (v_1^{10}), \\
 d(v_1^3 v_2^6 \widehat{x}_0) \equiv v_1^4 v_2^6 \widehat{t}_1^2 + v_1^7 v_2^6 \widehat{t}_1 & \text{mod } (v_1^9).
 \end{cases}$$

Then we have

$$\begin{aligned}
 d(\widehat{x}_2) &\equiv v_1^6 v_2^{-2} \widehat{t}_2^4 + v_1^7 v_2^6 \widehat{t}_1 + v_1^8 v_2^{-2} v_3^2 \widehat{t}_1^4 && \text{mod } (v_1^9) \\
 &\equiv v_1^6 v_2^{-2} \widehat{t}_2^4 && \text{mod } (v_1^7), \\
 d(\widehat{x}_3) &\equiv v_1^{12} v_2^{-4} \widehat{t}_2^8 && \text{mod } (v_1^{14}).
 \end{aligned}$$

For $k = 4$ case, we obtain the same consequences as in (4.7), but with the third one replaced by

$$d(v_1^{18} v_2^{22} \widehat{x}_2) \equiv v_1^{24} v_2^{20} \widehat{t}_2^4 + v_1^{25} v_2^{28} \widehat{t}_1 + v_1^{26} v_2^{20} v_3^2 \widehat{t}_1^4 \quad \text{mod } (v_1^{27}),$$

and so

$$d(\widehat{y}_1) \equiv v_1^{24} v_2^{-8} \widehat{t}_2^{16} + v_1^{25} v_2^{28} \widehat{t}_1 + v_1^{26} v_2^{10} v_3^4 \widehat{t}_2^4 + v_1^{26} v_2^{20} v_3^2 \widehat{t}_1^4 \quad \text{mod } (v_1^{27}).$$

On the other hand, we find that

$$\begin{cases}
 d(v_1^{25} v_2^{25} \widehat{w}_3) \equiv v_1^{25} (v_2^{25} \widehat{t}_1^4 + v_2^{28} \widehat{t}_1) + v_1^{26} v_2^{24} \widehat{t}_2^2, \\
 d(v_1^{23} v_2^{25} \widehat{x}_1) \equiv v_1^{25} v_2^{25} \widehat{t}_1^4, \\
 d(v_1^{26} v_2^{10} \widehat{w}_4^2) \equiv v_1^{26} (v_2^8 v_3^8 \widehat{t}_1^2 + v_2^{10} \widehat{t}_2^8 + v_2^{20} v_3^2 \widehat{t}_1^4 + v_2^{24} \widehat{t}_2^2), \\
 d(v_1^{14} v_2^{14} \widehat{x}_3) \equiv v_1^{26} v_2^{10} \widehat{t}_2^8, \\
 d(v_1^{25} v_2^8 v_3^8 \widehat{x}_0) \equiv v_1^{26} v_2^8 v_3^8 \widehat{t}_1^2
 \end{cases}$$

modulo (v_1^{27}) , so we have

$$d(\widehat{y}_4) \equiv v_1^{25} v_2^{28} \widehat{t}_1 + v_1^{26} v_2^{20} v_3^2 \widehat{t}_1^4 \quad \text{mod } (v_1^{27}).$$

Using the above congruences, we have

$$\begin{aligned}
 d(\widehat{x}_4) &\equiv v_1^{26} v_2^{10} v_3^4 \widehat{t}_2^4 \\
 &\equiv v_1^{20} v_2^{12} v_3^4 d(\widehat{x}_2) && \text{mod } (v_1^{1+\widehat{a}(4)}).
 \end{aligned}$$

(5.3) suggests that we should calculate $d(\widehat{x}_k)$ modulo $(v_1^{2+\widehat{a}(k)})$ rather than modulo $(v_1^{1+\widehat{a}(k)})$ for $k \geq 5$ when we apply induction on k . Denote $\widehat{x}_k + \widehat{x}_{k-1}^2$ by \widehat{z}_k . By definition (5.1) we note that $\widehat{z}_k = 0$ for odd k . In case that k is even, we have

$$\widehat{z}_k = v_1^{5 \cdot 2^{k-2}} v_2^{3 \cdot 2^{k-2}} \widehat{x}_{k-2} \widehat{z}_{k-2} \quad \text{for } k \geq 5.$$

Notice that \widehat{z}_{k-2} is divisible by v_1^4 for $k = 6$ and by $v_1^{5 \cdot 2^{k-4}}$ for $k \geq 8$. On the other hand, by inductive hypothesis $d(\widehat{x}_{k-2})$ is divisible by $v_1^{\widehat{a}(k-2)}$. So we have

$$\begin{aligned} d(\widehat{x}_{k-2} \widehat{z}_{k-2}) &= d(\widehat{x}_{k-2}) \eta_R(\widehat{z}_{k-2}) + \widehat{x}_{k-2} d(\widehat{z}_{k-2}) \\ &\equiv \widehat{x}_{k-2} d(\widehat{z}_{k-2}) \quad \text{mod } \left(v_1^{2+\widehat{a}(k-2)} \right). \end{aligned}$$

Therefore the differential on \widehat{z}_k is

$$\begin{aligned} d(\widehat{z}_k) &\equiv v_1^{5 \cdot 2^{k-2}} v_2^{3 \cdot 2^{k-2}} d(\widehat{x}_{k-2} \widehat{z}_{k-2}) \\ &\equiv v_1^{5 \cdot 2^{k-2}} v_2^{3 \cdot 2^{k-2}} \widehat{x}_{k-2} d(\widehat{z}_{k-2}) \quad \text{mod } \left(v_1^{2+\widehat{a}(k)} \right). \end{aligned}$$

On the other hand, by inductive hypothesis we have

$$d(\widehat{x}_{k-1}^2) \equiv v_1^{5 \cdot 2^{k-2}} v_2^{3 \cdot 2^{k-2}} v_2^{2^{k-2}} d(\widehat{x}_{k-3}^2) \quad \text{mod } \left(v_1^{2+\widehat{a}(k)} \right)$$

because $2(1 + \widehat{a}(4)) = 2 + \widehat{a}(5)$ and $2(2 + \widehat{a}(k - 1)) \geq 2 + \widehat{a}(k)$ for $k \geq 6$. Summing the above two congruences, we obtain

$$d(\widehat{x}_k) \equiv v_1^{5 \cdot 2^{k-2}} v_2^{3 \cdot 2^{k-2}} \widehat{x}_{k-2} d(\widehat{x}_{k-2}) \quad \text{mod } \left(v_1^{2+\widehat{a}(k)} \right).$$

as desired. □

6. The structure of $\text{Ext}_{\Gamma(m+1)}^0(M_1^1)$

Theorem 6.1. *As a $v_2^{-1}\widehat{k}(1)_*$ -module, $\text{Ext}_{\Gamma(m+1)}^0(M_1^1)$ for $m \geq 1$ is the direct sum of*

- (a) *the cyclic submodules generated by $\widehat{x}_k^s/v_1^{\widehat{a}(k)}$ for $k \geq 0, s > 0$ and $p \nmid s$; and*
 - (b) *$v_2^{-1}\widehat{K}(1)_*/\widehat{k}(1)_*$, generated by $1/v_1^j$ for $j \geq 1$,*
- where \widehat{x}_k 's are the elements defined in (4.1) and (5.1).

Proof. First we prove the theorem except for the $p = 2$ and $m = 1$ case. By Lemma 2.5 it suffices to show that the set

$$D = \left\{ \delta \left(\widehat{x}_k^s/v_1^{\widehat{a}(k)} \right) : k \geq 0, s > 0 \text{ and } p \nmid s \right\} \subset \text{Ext}_{\Gamma(m+1)}^1(M_2^0)$$

is linearly independent over

$$R = \mathbf{Z}/(p)[v_2, v_2^{-1}, v_3, \dots, v_m, v_{m+1}].$$

It follows from Corollary 3.4 that $\text{Ext}_{\Gamma(m+1)}^1(M_2^0)$ is the free $\widehat{K}(2)_*$ -module on the four classes represented by

$$\{\widehat{t}_1^p, \widehat{t}_1^{p^2}, \widehat{t}_2^{p^2}, \widehat{t}_2^{p^3}\},$$

so its basis over R is

$$\{\widehat{v}_2^t \widehat{t}_1^p, \widehat{v}_2^t \widehat{t}_1^{p^2}, \widehat{v}_2^t \widehat{t}_2^{p^2}, \widehat{v}_2^t \widehat{t}_2^{p^3} : t \geq 0\}.$$

Now define integers $\widehat{b}(k)$ and $\widehat{c}(k)$ for $k \geq 0$ by

$$\widehat{b}(k) = \begin{cases} 0 & \text{for } 0 \leq k \leq 1, \\ -p^{k-1} & \text{for } 2 \leq k \leq 3, \\ p^{k-2}\beta + \widehat{b}(k-3) & \text{for } k \geq 4, \end{cases}$$

where $\beta = p^2\omega - p - 1$ as before, and

$$\widehat{c}(k) = \begin{cases} 0 & \text{for } 0 \leq k \leq 3, \\ (p-1)p^{k-3} + \widehat{c}(k-3) & \text{for } k \geq 4. \end{cases}$$

Then Lemma 4.5 implies that

$$d(\widehat{x}_k) \equiv \pm v_1^{\widehat{a}(k)} v_2^{\widehat{b}(k)} \widehat{v}_2^{\widehat{c}(k)} \begin{cases} \widehat{t}_1^p & \text{for } k = 0, \\ \widehat{t}_1^{p^2} & \text{for } k > 0 \text{ and } k \equiv 1 \pmod{3}, \\ \widehat{t}_2^{p^2} & \text{for } k > 0 \text{ and } k \equiv 2 \pmod{3}, \\ \widehat{t}_2^{p^3} & \text{for } k > 0 \text{ and } k \equiv 3 \pmod{3} \end{cases}$$

modulo $(v_1^{1+\widehat{a}(k)})$, where $\widehat{a}(k)$ is defined in (4.3). Since

$$d(\widehat{x}_k^s) \equiv s\widehat{x}_k^{s-1} d(\widehat{x}_k) \equiv s\widehat{v}_2^{(s-1)p^k} d(\widehat{x}_k) \pmod{(v_1^{1+\widehat{a}(k)})},$$

it follows that

$$(6.2) \quad \delta \left(\frac{\widehat{x}_k^s}{v_1^{\widehat{a}(k)}} \right) = \pm s v_2^{\widehat{b}(k)} \widehat{v}_2^{(s-1)p^k + \widehat{c}(k)} \begin{cases} \widehat{t}_1^p & \text{for } k = 0, \\ \widehat{t}_1^{p^2} & \text{for } k > 0 & \text{and } k \equiv 1 \pmod{3}, \\ \widehat{t}_2^{p^2} & \text{for } k > 0 & \text{and } k \equiv 2 \pmod{3}, \\ \widehat{t}_2^{p^3} & \text{for } k > 0 & \text{and } k \equiv 3 \pmod{3}. \end{cases}$$

In order to show that these elements $\delta \left(\widehat{x}_k^s / v_1^{\widehat{a}(k)} \right)$ (with $k \geq 0$ and $s > 0$ not divisible by p) are linearly independent over R , it suffices to observe the exponents of \widehat{v}_2 in the right hand side of (6.2).

So we consider the sets $D_0 = \{ \widehat{v}_2^{s-1} : s > 0 \text{ and } p \nmid s \}$ for $k = 0$, and $D_{k_0} = \{ \widehat{v}_2^{(s-1)p^k + \widehat{c}(k)} : k = k_0 + 3k_1, s > 0 \text{ and } p \nmid s \}$ for a fixed k_0 ($1 \leq k_0 \leq 3$). Since the integer $\widehat{c}(k)$ is

$$\widehat{c}(k) = (p - 1)p^{k_0}(1 + p^3 + \dots + p^{3k_1 - 3})$$

for $k = k_0 + 3k_1 \geq 4$ with $1 \leq k_0 \leq 3$, we see

$$(s - 1)p^k + \widehat{c}(k) \equiv sp^k - \frac{p^{k_0}}{1 + p + p^2} \pmod{p^{k+1}}.$$

If $(s - 1)p^k + \widehat{c}(k) = (t - 1)p^l + \widehat{c}(l)$ with $k \equiv l \equiv k_0$ modulo 3, then it follows that $k = l$ and hence $s = t$. Thus all the entries in the sets D_0 and D_{k_0} ($1 \leq k_0 \leq 3$) are disparate, respectively.

In the $p = 2$ and $m = 1$ case our argument is the same subject to the following changes. The integers $\widehat{b}(k)$ and $\widehat{c}(k)$ are defined by

$$\widehat{b}(k) = \begin{cases} 0 & \text{for } 0 \leq k \leq 1, \\ -2^{k-1} & \text{for } 2 \leq k \leq 3, \\ 3 \cdot 2^{k-2} + \widehat{b}(k - 2) & \text{for } k \geq 4, \end{cases}$$

and

$$\widehat{c}(k) = \begin{cases} 0 & \text{for } 0 \leq k \leq 3, \\ 2^{k-2} + \widehat{c}(k - 2) & \text{for } k \geq 4, \end{cases}$$

which is

$$\widehat{c}(k) = \begin{cases} 0 & \text{for } 0 \leq k \leq 3, \\ \frac{4}{3}(2^{k-2} - 1) & \text{for even } k \geq 4, \\ \frac{8}{3}(2^{k-3} - 1) & \text{for odd } k \geq 5. \end{cases}$$

Then (6.2) gets replaced by

$$\delta \left(\frac{\widehat{x}_k^s}{v_1^{\widehat{a}(k)}} \right) = v_2^{\widehat{b}(k)} \widehat{v}_2^{(s-1)p^k + \widehat{c}(k)} \begin{cases} \widehat{t}_1^2 & \text{for } k = 0, \\ \widehat{t}_1^4 & \text{for } k = 1, \\ \widehat{t}_2^4 & \text{for } k > 0 \text{ and } k \equiv 0 \pmod{2}, \\ \widehat{t}_2^8 & \text{for } k > 1 \text{ and } k \equiv 1 \pmod{2}, \end{cases}$$

and we can argue for linear independence as before. □

7. The group $\text{Ext}_{\Gamma(m+1)}^1(BP_*/(p))$

In this section we will use the structure of $\text{Ext}_{\Gamma(m+1)}^0(M_1^1)$ given in Theorem 6.1 to determine the group $\text{Ext}_{\Gamma(m+1)}^1(BP_*/(p))$. As in the case $m = 0$, this group is the direct sum of subquotients of $\text{Ext}_{\Gamma(m+1)}^1(M_1^0)$ and $\text{Ext}_{\Gamma(m+1)}^0(M_1^1)$.

In Lemma 7.2 we will show that the former subquotient has the same form as in the case $m = 0$, i.e., it is $\widehat{k}(1)_*\{\widehat{h}_{1,0}\}$. We will also see that unlike in the classical case, the element $v_1^{-1}\widehat{h}_{1,0}$ supports a nontrivial d_2 in the chromatic spectral sequence.

The summand $v_2^{-1}\widehat{K}(1)_*/\widehat{k}(1)_*$ of $\text{Ext}_{\Gamma(m+1)}^0(M_1^1)$ is the image of

$$d_1 : E_1^{0,0} = \text{Ext}_{\Gamma(m+1)}^0(M_1^0) \longrightarrow E_1^{1,0} = \text{Ext}_{\Gamma(m+1)}^0(M_1^1),$$

so it maps trivially to $\text{Ext}_{\Gamma(m+1)}^1(BP_*/(p))$. The kernel of the map

$$d_1 : E_1^{1,0} = \text{Ext}_{\Gamma(m+1)}^0(M_1^1) \longrightarrow E_1^{2,0} = \text{Ext}_{\Gamma(m+1)}^0(M_1^2),$$

consists of all elements, each of which does not have any monomial with negative v_2 -exponent. We will see in Corollary 7.7 that these are the elements

$$\frac{\widehat{x}_k^s}{v_1^j} \in \text{Ext}_{\Gamma(m+1)}^0(M_1^1) \quad \text{with } k \geq 0, s > 0, p \nmid s, \text{ and } 0 < j \leq p^k.$$

Combining these results we get

Theorem 7.1. *For any prime p and $m \geq 1$, the group*

$$\text{Ext}_{\Gamma(m+1)}^1(BP_*/(p))$$

is isomorphic to

$$\widehat{k}(1)_* \left\{ \widehat{\beta}_{sp^k/j} : s \geq 0, p \nmid s, k \geq 0 \text{ and } 0 < j \leq p^k \right\} \bigoplus \widehat{k}(1)_*\{\widehat{h}_{1,0}\},$$

where $\widehat{\beta}_{sp^k/j}$ is the image of \widehat{x}_k^s/v_1^j under the connecting homomorphism

$$\delta : \text{Ext}_{\Gamma(m+1)}^0(N_1^1) \longrightarrow \text{Ext}_{\Gamma(m+1)}^1(N_1^0).$$

First we consider the subquotient of $\text{Ext}_{\Gamma(m+1)}^1(M_1^0)$.

Lemma 7.2. *For any prime p and $m \geq 1$, the group $E_\infty^{0,1}$ in the chromatic spectral sequence is $\widehat{k}(1)_*\{\widehat{h}_{1,0}\}$. Moreover there is a nontrivial differential in the chromatic spectral sequence,*

$$d_2 \left(v_1^{-1}\widehat{h}_{1,0} \right) = \frac{z}{v_1^{p+1}v_2^{p\omega-1}},$$

where $z = \widehat{v}_2^p - v_1^p v_2^{-1} \widehat{v}_3$.

Proof. We use the chromatic cobar complex

$$\{CC_{\Gamma(m+1)}^n(BP_*/(p)), d_c\}_{n \geq 0}$$

given by

$$CC_{\Gamma(m+1)}^n(BP_*/(p)) = \bigoplus_{s+t=n} C^s(M_1^t),$$

$$d_c = d_e + (-1)^t d_i : C^s(M_1^t) \rightarrow C^s(M_1^{t+1}) \oplus C^{s+1}(M_1^t),$$

where $d_e : C^s(M_1^t) \rightarrow C^s(M_1^{t+1})$ is induced by the composite map $M_1^t \rightarrow N_1^{t+1} \rightarrow M_1^{t+1}$ and $d_i : C^s(M_1^t) \rightarrow C^{s+1}(M_1^t)$ is the differential in the cobar complex (see [6, Definition 5.1.10]).

By Theorem 2.2, we have

$$E_1^{0,1} = \text{Ext}_{\Gamma(m+1)}^1(M_1^0) \cong \widehat{K}(1)_* \{ \widehat{h}_{1,0} \}.$$

The element $\widehat{h}_{1,0}$ is represented by \widehat{t}_1 in the cobar complex and is clearly a permanent cycle in the chromatic spectral sequence. We need to show that $v_1^{-1} \widehat{h}_{1,0}$ does not survive to $E_\infty^{0,1}$. If it does, then the element $\widehat{h}_{1,0} \in \text{Ext}_{\Gamma(m+1)}^1(BP_*/(p))$ is divisible by v_1 and therefore has trivial image under the composite

$$\text{Ext}_{\Gamma(m+1)}^1(BP_*/(p)) \rightarrow \text{Ext}_{\Gamma(m+1)}^1(BP_*/I_2) \rightarrow \text{Ext}_{\Gamma(m+1)}^1(v_2^{-1}BP_*/I_2).$$

The target group was computed in [5], and the element in question is one of its generators.

For the chromatic differential d_2 , we have

$$d(z) \equiv v_1^p v_2^{p\omega-1} \widehat{t}_1 \pmod{(v_1^{p+1})}.$$

It follows that in the chromatic cobar complex $CC_{\Gamma(m+1)}(BP_*/(p))$ the differential

$$d_c : C^1(M_1^0) \oplus C^0(M_1^1) \rightarrow C^2(M_1^0) \oplus C^1(M_1^1) \oplus C^0(M_1^2)$$

satisfies

$$d_c(v_1^{-1} \widehat{t}_1) = \frac{\widehat{t}_1}{v_1} \in C^1(M_1^1),$$

$$d_c\left(\frac{v_2^{1-p\omega} z}{v_1^{p+1}}\right) = -\frac{\widehat{t}_1}{v_1} + \frac{z}{v_1^{p+1} v_2^{p\omega-1}} \in C^1(M_1^1) \oplus C^0(M_1^2),$$

so
$$d_c \left(v_1^{-1} \widehat{t}_1 + \frac{v_2^{1-p\omega} z}{v_1^{p+1}} \right) = \frac{z}{v_1^{p+1} v_2^{p\omega-1}}.$$

In terms of the double complex associated with the chromatic resolution, we have the following picture:

$$\begin{array}{ccccc}
 s = 1 : & v_1^{-1} \widehat{t}_1 & \xrightarrow{d_e} & \frac{\widehat{t}_1}{v_1} & \\
 & & & \uparrow d_i & \\
 s = 0 : & & & \frac{v_2^{1-p\omega} z}{v_1^{p+1}} & \xrightarrow{d_e} \frac{z}{v_1^{p+1} v_2^{p\omega-1}} \\
 & t = 0 & & t = 1 & & t = 2
 \end{array}$$

This means that in the chromatic spectral sequence we have the indicated d_2 . Its target must be nontrivial in E_2 , i.e., it is not in the image under

$$d_1 : E_1^{1,0} = \text{Ext}_{\Gamma(m+1)}^0(M_1^1) \longrightarrow E_1^{2,0} = \text{Ext}_{\Gamma(m+1)}^0(M_1^2).$$

because otherwise $v_1^{-1} \widehat{h}_{1,0}$ would survive to $E_\infty^{0,1}$, contradicting the nondivisibility result above. □

Now we turn to the v_1 -torsion in $\text{Ext}_{\Gamma(m+1)}^1(BP_*/(p))$. Let $\widehat{d}(k)$ be the maximum exponent of v_1 satisfying

$$\widehat{x}_k \equiv \widehat{x}_{k-1}^p \pmod{(p, v_1^{\widehat{d}(k)})}.$$

(if $\widehat{x}_k = \widehat{x}_{k-1}^p$, then we set $\widehat{d}(k) = \infty$.) Thus the integers $\widehat{d}(k)$ ($k \geq 5$) are given inductively by

$$(7.3) \quad \widehat{d}(k) = p^{k-1} \alpha + \widehat{d}(k-3)$$

with $\widehat{d}(2) = p^2 - 1$, $\widehat{d}(3) = \infty$, $\widehat{d}(4) = p^4 + p^3 - p^2 - p$ unless $p = 2$ and $m = 1$, but

$$(7.4) \quad \widehat{d}(k) = 5 \cdot 2^{k-2} + \widehat{d}(k-2)$$

with $\widehat{d}(3) = \infty$, $\widehat{d}(4) = 14$ in the case $p = 2$ and $m = 1$.

Lemma 7.5. *For any prime p and $m \geq 1$,*

$$\widehat{x}_k \equiv \widehat{x}_2^{p^{k-2}} \pmod{(p, v_1^{p^{k-4} \widehat{d}(4)})}.$$

Furthermore, $\widehat{x}_k \equiv \widehat{x}_4^{2^{k-4}}$ modulo $(2, v_1^{2^{k-6}\widehat{d}(6)})$ in the case $p = 2$ and $m = 1$.

Proof. From (7.3) and (7.4) it follows that $\widehat{d}(k) > p^{k-4}\widehat{d}(4)$ for $k \geq 5$ unless $p = 2$ and $m = 1$, and that $\widehat{d}(k) > 2^{k-6}\widehat{d}(6)$ for $k \geq 7$ in the case $p = 2$ and $m = 1$. Therefore it is obvious that

$$\begin{aligned} \min \{ & \widehat{d}(k), p\widehat{d}(k-1), \dots, p^{k-4}\widehat{d}(4), p^{k-3}\widehat{d}(3) \} \\ & = p^{k-4}\widehat{d}(4) = p^k + p^{k-1} - p^{k-2} - p^{k-3} \end{aligned}$$

unless $p = 2$ and $m = 1$, and

$$\min \{ \widehat{d}(k), 2\widehat{d}(k-1), \dots, 2^{k-6}\widehat{d}(6), 2^{k-5}\widehat{d}(5) \} = 2^{k-6}\widehat{d}(6) = 94 \cdot 2^{k-6}$$

when $p = 2$ and $m = 1$. This completes the proof. □

Lemma 7.6. *Let \widehat{x}_k^s/v_1^j ($j \leq \widehat{a}(k)$) be one of the generators of $\text{Ext}_{\Gamma(m+1)}^0(M_1^1)$. Then the image of this element by the map*

$$\text{Ext}_{\Gamma(m+1)}^0(M_1^1) \rightarrow \text{Ext}_{\Gamma(m+1)}^0(N_1^2)$$

is non-trivial if and only if $k \geq 2$ and $p^k < j \leq \widehat{a}(k)$.

Proof. We may assume that $k \geq 2$. From definition of \widehat{x}_2 , it follows that

$$\widehat{x}_2^{p^{k-2}} \equiv \widehat{v}_2^{p^k} - v_1^{p^k-p^{k-2}} v_2^{\beta p^{k-2}} \widehat{v}_3^{p^{k-2}} + v_1^{p^k} v_2^{-p^{k-1}} \widehat{v}_3^{p^{k-1}} \pmod{(p)}.$$

Then, using the fact that

$$\begin{aligned} 2(p^k - p^{k-2}) & \geq \widehat{a}(k) && \text{for } k = 2 \text{ or } 3 \\ 2(p^k - p^{k-2}) & > p^k \widehat{d}(4) && \text{for } k \geq 4 \end{aligned}$$

and Lemma 7.5 we have

$$\widehat{x}_k^s \equiv \widehat{v}_2^{sp^k} - s\widehat{v}_2^{(s-1)p^k} \left(v_1^{p^k-p^{k-2}} v_2^{\beta p^{k-2}} \widehat{v}_3^{p^{k-2}} - v_1^{p^k} v_2^{-p^{k-1}} \widehat{v}_3^{p^{k-1}} \right)$$

modulo (p, v_1^j) for $k = 2$ and 3 , and modulo $(p, v_1^{p^{k-4}\widehat{d}(4)})$ for $k \geq 4$.

In the right hand side the first and the second terms do not have a negative v_2 -exponent, but the third term in \widehat{x}_k^s/v_1^j is

$$\frac{sv_1^{p^k} v_2^{-p^{k-1}} \widehat{v}_2^{(s-1)p^k} \widehat{v}_3^{p^{k-1}}}{v_1^j}.$$

which may be mapped non-trivially to N_1^2 . Unless $p = 2$ and $m = 1$, we notice that $p^{k-4}\widehat{d}(4) > p^k$. Then we observe that \widehat{x}_k^s/v_1^j is mapped non-trivially to N_1^2 if and only if $j > p^k$ except when $p = 2, m = 1$ and $k \geq 4$.

On the other hand, in the $p = 2$ and $m = 1$ case we find that $\widehat{x}_k \equiv \widehat{x}_4^{2^{k-4}}$ modulo $(v_1^{2^{k-6}\widehat{d}(6)})$ ($k \geq 6$) and

$$\begin{aligned} \widehat{x}_4 &\equiv \widehat{x}_3^2 + v_1^{14}v_2^{14}\widehat{x}_3 \\ &\equiv \widehat{v}_2^{16} + v_1^{12}v_2^{24}\widehat{v}_2^4 + v_1^{14}v_2^{14}\widehat{v}_2^8 + v_1^{16}v_2^{-8}\widehat{v}_3^8 \pmod{(2, v_1^{18})}, \end{aligned}$$

so that

$$\widehat{x}_4^{2^{k-4}} \equiv \widehat{v}_2^{2^k} + v_1^{2^k}v_2^{-2^{k-1}}\widehat{v}_3^{2^{k-1}} + v_1^{3 \cdot 2^{k-2}}v_2^{3 \cdot 2^{k-1}}\widehat{v}_2^{2^{k-2}} + v_1^{7 \cdot 2^{k-3}}v_2^{7 \cdot 2^{k-3}}\widehat{v}_2^{2^{k-1}}$$

modulo $(2, v_1^{9 \cdot 2^{k-3}})$. Notice that $2^{k-6}\widehat{d}(6) > 9 \cdot 2^{k-3} > 2^k$ and that we may ignore the terms except the second one, because the other terms don't have a negative v_2 -exponent. Then we can complete the proof in similar way as the above. \square

Corollary 7.7. *The only elements of $E_1^{1,0}$ which survive to $E_\infty^{1,0}$ are*

$$\frac{\widehat{x}_k^s}{v_1^j} \text{ for } s \geq 0, p \nmid s, k \geq 0 \text{ and } 0 < j \leq p^k.$$

Proof. The summand $v_2^{-1}\widehat{K}(1)_*/\widehat{k}(1)_*$ of $E_1^{1,0}$ is killed by the chromatic differential

$$d_1 : \text{Ext}_{\Gamma(m+1)}^0(M_1^0) \rightarrow \text{Ext}_{\Gamma(m+1)}^0(M_1^1).$$

Joining this result with Lemma 7.6, we have the desired result. \square

References

- [1] I. Ichigi, H. Nakai, and D.C. Ravenel: *The chromatic Ext groups* $\text{Ext}_{\Gamma(m+1)}^0(BP_*, M_2^1)$, *Trans. Amer. Math. Soc.* **354** (2002), 3789–3813.
- [2] Y. Kamiya and K. Shimomura: *The homotopy groups* $\pi_*(L_2V(0) \wedge T(k))$, *Hiroshima Mathematical Journal*, **31** (2001), 391–408.
- [3] H.R. Miller, D.C. Ravenel, and W.S. Wilson: *Periodic phenomena in the Adams-Novikov spectral sequence*, *Annals of Mathematics*, **106** (1977), 469–516.
- [4] H.R. Miller and W.S. Wilson: *On Novikov's Ext¹ modulo an invariant prime ideal*, *Topology*, **15** (1976), 131–141.
- [5] H. Nakai and D.C. Ravenel: *The first cohomology group of the generalized morava stabilizer algebra*, *Proceedings of the American Mathematical Society*, to appear.
- [6] D.C. Ravenel: *Complex Cobordism and Stable Homotopy Groups of Spheres*, Academic Press, New York, 1986. Errata available online at <http://www.math.rochester.edu/u/drav/mu.html>.

- [7] D.C. Ravenel: *The microstable Adams-Novikov spectral sequence*, In D. Arlettaz and K. Hess, editors, *Une dégustation topologique [Topological morsels]: homotopy theory in the Swiss Alps (Arolla, 1999)*, volume 265 of *Contemporary Mathematics*, 193–209, Providence, Rhode Island, 2000. American Mathematical Society.
- [8] K. Shimomura: *The homotopy groups of the L_2 -localized Mahowald spectrum $X(1)$* , *Forum Mathematicum*, **7**, 685–707.
- [9] K. Shimomura: *The homotopy groups $\pi_*(L_n T(m) \wedge V(n-2))$* , to appear.

Hirofumi Nakai
General Education Division,
Oshima National College of Maritime Technology,
Yamaguchi 742-2193,
Japan

Douglas C. Ravenel
Department of Mathematics,
University of Rochester,
Rochester, New York 14627,
U.S.A.