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# Gentzen Method in Modal Calculi, $I^{11}$ 

To Professor Zyoiti Suetuna on his 60th birthday

By Masao Ohnishi and Kazuo Matsumoto

Various decision procedures for modal sentential calculus $S 5$ have been given by W. T. Parry [8], R. Carnap [10], M. Itoh [11], M. Ohnishi-K. Matsumoto [7] and S. Kanger [5]. Among them Gentzentype procedure is only that of S. Kanger. The object of this note is to give an alternative Gentzen-type decision procedure for $S 5$.

## I

Our formulation of Lewis's modal sentential calculus $S 5^{2)}$ is based upon "Sequenzenkalkül $L K$ ", which was constructed by G. Gentzen [3].

Namely :

```
\(\left\{\begin{array}{l}\text { logical symbols : } \\ \cdot(\text { and }), \sim(\text { not }), \quad \vee \text { (or) } \\ \text { rules of inference : }\end{array}\right.\)
\(\left\{\begin{array}{l}\text { structural rules } \\ \quad \text { weakening, contraction, exchange and cut. } \\ \text { logical rules }\end{array}\right.\)
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$(\rightarrow \cdot),(\cdot \rightarrow) ;(\rightarrow \sim),(\sim \rightarrow) ;(\rightarrow \vee),(\vee \rightarrow)$.

Next we add to $L K$ a new logical symbol $\square \square$ (necessary), and we define as follows : if $\alpha$ is a formula, then $\square \alpha$ is also a formula.

New rules for modality are

$$
\frac{\alpha, \Gamma \rightarrow \Theta}{\square \alpha, \Gamma \rightarrow \Theta}(\square \rightarrow), \quad \frac{\square \Gamma \rightarrow \square \Theta, \quad \alpha}{\square \Gamma \rightarrow \square \Theta, \square \alpha}(\rightarrow \square) .
$$

By $\Gamma, \Theta$ we mean a series of formulas as in $L K . \quad \square \Gamma$ (or $\sim \Gamma$ ) means a series of formulas which is formed by prefixing $\square$ (or $\sim$ ) in front of each formula of $\Gamma$.

[^0]Thus established sentential calculus which contains $L K$ is $S 5^{33}$.
M. Wajsberg [9] gave a decision procedure for monadic functional calculus, and W. T. Parry [8] remarked that for an arbitrary formula $\gamma$ in $S 5$ there exists a $\gamma^{*}$ equivalent to $\gamma$ and of degree at most $1^{4)}$, and using this fact he showed the equivalence of $S 5$ and monadic functional calculus, hence gave a decision procedure for $S 5$.

In the following $\gamma^{*}$ always means a formula of degree at most 1 and equivalent to $\gamma$.

When all sequent-formulas of each rule in $S 5$ are of degree at most 1 , we denote this system by $S 5^{*}$.

Theorem 1. $\rightarrow \gamma$ is provable in $S 5$, if and only if $\rightarrow \gamma^{*}$ is provable in S5* $^{*}$.

Lemma 1. Let $\rightarrow \gamma$ be an $S 5$-provable sequent. Then there exists an S5-proof-figure for $\rightarrow \gamma$ which does not include any mix other than $\square$-mix ${ }^{5}$.

The proof of Lemma 1 is carried out by the induction on the rank and grade of mix-formula.

We define a modalized formula (abbr. mf) as follows: (1-3)

1. $\square \alpha$ is an mf .
2. If $\alpha$ is an mf , then so is $\sim \alpha$.
3. If $\alpha$ and $\beta$ are mfs , then so are $\alpha \vee \beta$ and $\alpha \cdot \beta$.

Clearly we have
Lemma $2^{6}$. In the formulation of $S 5$ we can replace $(\rightarrow \square)$ by

$$
\frac{\Gamma^{\prime} \rightarrow \Theta^{\prime}, \alpha}{\Gamma^{\prime} \rightarrow \Theta^{\prime}, \square \alpha}(\rightarrow \square)^{\prime}
$$

where $\Gamma^{\prime}$ and $\Theta^{\prime}$ mean series of $m f s$.
Lemma 3. Assuming that all mixes appearing in an S5-proof-figure of $\rightarrow \gamma$ are $\square$-mixes, there exists a proof-figure of $\rightarrow \gamma$, where the degrees of mix-formulas are all at most 1.
3) The following example shows that Gentzen's Hauptsatz does not hold in our $S 5$ :

$$
\begin{aligned}
& \square \alpha \rightarrow \square \alpha \\
&(\rightarrow \sim) \\
& \frac{\rightarrow \square \alpha, \sim \square \alpha}{\rightarrow \square \alpha, \square \sim \square \alpha}(\rightarrow \square) \frac{\alpha \rightarrow \alpha}{\square \alpha \rightarrow \alpha}(\square \rightarrow) \\
& \rightarrow \square \sim \square \alpha, \alpha(\square \alpha)
\end{aligned}
$$

4) For the definition of "a formula of degree $n$ ", see A. R. Anderson [1], p. 203.
5) $\square$-mix means a mix, the outermost symbol of whose mix-formula is $\square$.
6) This Lemma is a formal generalization of R. Feys's formulation [2].

The proof of Lemma 3 can be carried out by the induction on $n$, where $n$ is the maximal degree of $\square$-mix-formula appearing in the proof-figure of $\rightarrow \gamma$. Let $\square \xi$ be an upper-most (in the proof-figure) mix-formula of degree $n$ (2). By the aid of the following Wajsberg's recurring equivalences ${ }^{7}$ : $\square \square \alpha=\square \alpha, \square \sim \square \alpha=\sim \square \alpha, \square(\square \alpha \vee \beta)=$ $\square \alpha \vee \square \beta, \square(\sim \square \alpha \vee \beta)=\sim \square \alpha \vee \square \beta$ and $\square(\alpha \cdot \beta)=\square \alpha \cdot \square \beta$, clearly we can define $\xi^{\dagger}$ with the following properties: 1. $\xi^{\dagger}$ is an mf equivalent to $\square \xi, 2$. $\xi^{\dagger}$ is of degree at most $n-1$, and 3 . in the upper part of the $\square$-mix of $\square \xi$ we can replace $\square \xi$ by $\xi^{\dagger}$ without any $\square$-mix of degree $n$.

Now, Theorem 1 can be proved by Lemmas 1, 2 and 3.
Theorem 2. The Hauptsatz does hold in S5*.
In order to prove Theorem 2 we need the following
Lemma 4. In $S 5^{*}$, if $\square \Gamma \rightarrow \square \Theta, \alpha$ is provable without any mix, then either $\square \mathrm{\Gamma} \rightarrow \square \Theta$ or $\square \mathrm{\Gamma} \rightarrow \alpha$ is provable without any mix, where all formulas of $\mathrm{\Gamma}, \Theta$ and $\alpha$ are of degree $O$ (i.e. LK-formulas).

The proof of Lemma 4 can be easily given by the idea of elimination of formula-bundle of each formula of $\square \Theta$ in the proof-figure of $\square \Gamma \rightarrow \square \Theta, \alpha$.

Now we have only to consider the following cases: 1. When $\rho=2$ and the mix is a $\square$-mix, i.e.

$$
\left.\frac{\frac{\square \Gamma \rightarrow \square \Theta, \alpha}{\square \Gamma \rightarrow \square \Theta, ~} \square \alpha}{\square \Gamma}(\rightarrow \square) \frac{\alpha, \Sigma \rightarrow \Pi}{\square \alpha, \Sigma \rightarrow \Pi}(\square \rightarrow)\right)(\square \alpha),
$$

we transform this into:

$$
\frac{\square \Gamma \rightarrow \square \Theta, \alpha \quad \alpha, \Sigma \rightarrow \Pi}{\square \Gamma, \Sigma^{*} \rightarrow\left(\square^{\Theta}\right)^{*}, \Pi}(\alpha) .
$$

2. When $\rho>2$ and the left rank $>1$,
2.1

$$
\frac{\frac{\alpha, \Gamma \rightarrow \Theta}{\square \alpha, \Gamma \rightarrow \Theta}(\square \rightarrow)}{\square \alpha, \Gamma, \Sigma^{*} \rightarrow \Theta^{*}, \Pi} \quad \Sigma \rightarrow \Pi \quad(\mathfrak{M}) .
$$

This case is trivial.
2.2

$$
\frac{\frac{\square \Gamma \rightarrow \square \Theta, \alpha}{\square \Gamma \rightarrow \square \Theta, \square \alpha}(\rightarrow \square) \quad \Sigma \rightarrow \Pi}{\square \Gamma, \Sigma^{*} \rightarrow(\square \Theta)^{*},(\square \alpha)^{*}, \Pi}(\square \mathfrak{M}) .
$$

7) See M. Wajsberg [9].

According to Lemma 4 either $\square \Gamma \rightarrow \square \Theta$ or $\square \Gamma \rightarrow \alpha$ is provable. In the former case 2.21 we transform this into:


In the latter case 2.22 we distinguish the following two cases:
2.221. when $\mathfrak{M}=\alpha$,

$$
\frac{\frac{\square \Gamma \rightarrow \alpha}{\square \Gamma \rightarrow \square \alpha}(\rightarrow \square) \quad \Sigma \rightarrow \Pi}{\frac{\square \Gamma, \Sigma^{*} \rightarrow \Pi}{\square \Gamma, \Sigma^{*} \rightarrow(\square \Theta)^{*}, \Pi}}(\square \alpha),
$$

2.222. when $\mathfrak{M} \neq \alpha$,

$$
\frac{\frac{\square \Gamma \rightarrow \alpha}{\square \Gamma \rightarrow \square \alpha}(\rightarrow \square)}{\frac{\square \Gamma \rightarrow \square \Theta, \square \alpha}{\square \Gamma, \Sigma^{*} \rightarrow(\square \Theta)^{*}, \square \alpha, \Pi} \quad} \quad \mathrm{\Sigma} \rightarrow \Pi \text { I }
$$

3. When $\rho>2$, the left rank $=1$ and the right rank $>1$,
3.1

$$
\frac{\Sigma \rightarrow \Pi}{\Sigma, \frac{\alpha, \Gamma \rightarrow \Theta}{\square \alpha, \Gamma \rightarrow \Theta}(\square \rightarrow)} \begin{aligned}
& \Sigma,(\square \alpha)^{*}, \Gamma * \rightarrow \Pi^{*}, \Theta \\
& (\mathfrak{R}),
\end{aligned}
$$

we transform this into: 3.11. in case $\mathfrak{M}=\square \alpha$,
3. 12. in case $\mathfrak{M} \neq \square \alpha$,

$$
\begin{aligned}
& \Sigma \rightarrow \Pi \quad \alpha, I^{\prime} \rightarrow \Theta \\
& \frac{\sum, \alpha^{*}, \Gamma^{*} \rightarrow \Pi^{*}, \Theta}{\left(M, M_{2}\right.} \text { (weakening if necessary) } \\
& \frac{\Sigma, \Gamma^{*} \rightarrow \Pi^{*}, \Theta}{\Sigma, \square \alpha, \Gamma^{*} \rightarrow \Pi^{*}, \Theta}(\square \rightarrow)
\end{aligned}
$$

3. 2

$$
\begin{array}{cl}
\mathrm{\Sigma} \rightarrow \Pi & \begin{array}{l}
\square \Gamma \rightarrow \square \Theta, \quad \alpha \\
\square \Gamma \rightarrow \square \Theta, \square \alpha
\end{array}(\rightarrow \square) \\
\mathrm{\Sigma},(\square \Gamma)^{*} \rightarrow \Pi^{*}, \square \Theta, \square \alpha
\end{array}(\square \mathfrak{M}),
$$

where $\Pi$ contains $\square \mathfrak{M}$, rnd the left rank $=1$. Therefore the non-trivial case is

$$
\frac{\frac{\square \Sigma \rightarrow \square \Pi, \mathfrak{M}}{\square \Sigma \rightarrow \square \Pi, \square \mathfrak{M}}(\rightarrow \square) \quad \frac{\square \Gamma \rightarrow \square \Theta, \alpha}{\square \mathrm{\Gamma} \rightarrow \square \Theta, \square \alpha}(\rightarrow \square)}{\square \Sigma,(\square \mathrm{\Gamma})^{*} \rightarrow \square \Pi, \square \Theta, \square \alpha}(\square \mathfrak{M}) .
$$

e transform this into:

$$
\frac{\square \Sigma \rightarrow \square \Pi, \square \mathfrak{M} \quad \square \Gamma \rightarrow \square \Theta, \alpha}{\square(\square \mathfrak{M})} \begin{aligned}
& \square \mathbf{\Sigma},(\square \Gamma)^{*} \rightarrow \square \Pi, \square \Theta, \alpha \\
& \square \Sigma,(\square \Gamma)^{*} \rightarrow \square \Pi, \square \Theta, \square \alpha
\end{aligned}(\rightarrow \square) .
$$

This completes the proof of Theorem 2.
Theorems 1 and 2 yield a new decision procedure for $S 5$.

## II

W. T. Parry [8] showed that if a formula $\gamma$ of degree 1 is provable in $S 5$, then $\gamma$ is also provable in $S 3$. S. Halldén [4] remarked that if a formula $\gamma$ of degree 1 is provable in any one system of $S 2, S 3, S 4$ and $S 5$, then $\gamma$ is also provable in each of other systems. That is,

Theorem 3. $\rightarrow \gamma^{*}$ is provable in S5, if and only if $\rightarrow \gamma^{*}$ is provable in S2, where $\gamma^{*}$ is of degree 1.

In our former paper [7], we have already obtained the following result: " $\rightarrow \gamma$ is provable in S2 if and only if $p-3 p \rightarrow \gamma^{8}$ is provable in Q2, where $p$ is a sentence-variable".

This result leads to an alternative proof of Theorem 3. That is, we have only to show that $p-3 p \rightarrow \gamma^{*}$ is provable in $Q 2$ if $\rightarrow \gamma^{*}$ is provable in $S 5^{*}$ (see Theorem 1). The essential part of this proof is to show the $Q 2$-admissibility of

$$
\frac{p-3 p, \square \Gamma \rightarrow \square \Theta, \quad \alpha}{p-3 p, \square \Gamma \rightarrow \square \Theta, \square \alpha},
$$

where $\Gamma, \Theta$ and $\alpha$ are all $L K$-formulas. According to an analogous form of Lemma 4 if $p-3 p, \square \Gamma \rightarrow \square \Theta, \alpha$ is provable in $Q 2$, then either $p-3 p$, $\square \Gamma \rightarrow \square \Theta$ or $p-3 p, \square \Gamma \rightarrow \alpha$ is provable in $Q 2$. The former case is clear. In the latter case eliminating all $\square$ 's in the proof of $p-3 p$, $\square \Gamma \rightarrow \alpha$, we obtain a proof of $p>p, \Gamma \rightarrow \alpha$, hence of $p-3 p, \square \Gamma \rightarrow \square \alpha$.
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[^1]
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[^0]:    1) This is a continuation to M. Ohnishi and K. Matsumoto [7].
    2) C. I. Lewis and C. H. Langford [6].
[^1]:    8) $\alpha-\beta \beta$ is the abbreviation of $\square(\sim \alpha \vee \beta)$.
