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CYCLIC SURGERY ON GENUS ONE KNOTS

MASAKAZU TERAGAITO

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0. Introduction

The real projective 3-space, denoted by $\mathbb{RP}^3$, is identified with the lens space of type (2,1). Then one can ask: when can $\mathbb{RP}^3$ be obtained by Dehn surgery on a knot in the 3-sphere $S^3$? Clearly $\mathbb{RP}^3$ is obtained by Dehn surgery on a trivial knot. However, it is conjectured that no Dehn surgery on a nontrivial knot $K$ in $S^3$ yields $\mathbb{RP}^3$ (cf. [1,4]). It is known to be true if $K$ is a composite knot [3], a torus knot [9], an alternating knot [10], a satellite knot [1,12,13], or a symmetric knot [1].

In this paper we prove the conjecture for genus one knots.

Theorem 0.1. Real projective 3-space $\mathbb{RP}^3$ cannot be obtained by Dehn surgery on a genus one knot in $S^3$.

This will be proved by applying the combinatorial techniques developed in [2,5,6,8].

1. Preliminaries

Let $K$ be a genus one knot which is neither a torus knot nor a satellite knot. Let $N(K)$ be a tubular neighborhood of $K$ and let $E(K) = S^3 - \text{int} N(K)$. Suppose that some surgery on $K$ yields $\mathbb{RP}^3$, that is, $E(K) \cup J = \mathbb{RP}^3$ where $J$ is a solid torus. By [2], the surgery coefficient is $\pm 2$.

Let $P^2 \subset \mathbb{RP}^3$ be a projective plane which intersects $J$ in a disjoint union of meridian disks of $J$. We assume that $|P^2 \cap J|$ is minimal among all projective planes in $\mathbb{RP}^3$ that intersect $J$ in a family of meridian disks of $J$. Let $p = |P^2 \cap J|$ and $P = P^2 \cap E(K)$. Then $P$ is incompressible in $E(K)$ by the minimality of $p$. If $p$ is even, then $E(K)$ would contain a closed non-orientable surface by attaching tubes to $\partial P$. Hence $p$ is odd. Furthermore, if $p = 1$ then $K$ is either a torus knot or a $(2, \pm 1)$-cable knot. Thus $p \neq 1$.

Let $Q$ be a genus one Seifert surface for $K$. We may assume that $P$ and $Q$ intersect transversely, and $\partial Q$ intersects each component of $\partial P$ exactly twice. By the incompressibility of $P$ and $Q$, we can assume that no circle component of
$P \cap Q$ bounds a disk in $P$ or $Q$.

Let $\hat{P}$, $\hat{Q}$ be the closed surfaces obtained by capping off the components of $\partial P$ and $\partial Q$ with disks. We can identify $\hat{P}$ with $P^2$. We obtain a graph $G_P$ in $\hat{P}$ by taking the disks $\delta(\hat{P} - P)$ as the (fat) vertices of $G_P$, and the arc components of $P \cap Q$ in $P$ as the edges of $G_P$. Similarly, we obtain the graph $G_Q$ in $\hat{Q}$.

Number the components of $\partial P$, $\{1, 2, \ldots, p\}$, in the order in which they appear on $\partial E(K)$. The endpoints of edges of $G_Q$ are labelled by the numbers of the corresponding components of $\partial P$. Thus around the only vertex $v$ of $G_Q$, we will consecutively meet the labels $1, 2, \ldots, p$, $1, 2, \ldots, p$ (repeated twice). Since each vertex of $G_P$ has valency two, $G_P$ consists of disjoint cycles.

2. Proof of Theorem 0.1

A trivial loop is a length one cycle which bounds a disk face of the graph.

**Lemma 2.1.** Neither $G_P$ nor $G_Q$ contains trivial loops.

Proof. Let $e$ be a trivial loop in $G_P$, and let $D$ be a regular neighborhood of $e$ in $Q$. Given the orientation of $\partial Q$ induced by some orientation of $D$, the points of intersection of $\partial Q$ with the component of $\partial P$ meeting $e$ have opposite signs, a contradiction. If $G_Q$ contains a trivial loop, $P$ would be compressible in $E(K)$, a contradiction.

An edge of $G_Q$ is said to be level if its endpoints have the same label.

**Lemma 2.2.** $G_Q$ cannot contain two level edges on distinct labels.

Proof. Let $e$ be a level edge in $G_Q$ with label $i$. Then $e$ is a loop in $G_P$ based at the vertex $V_i$ corresponding to the component of $\partial P$ with label $i$. We see that a regular neighborhood of $e \cup V_i$ in $\hat{P}$ is homeomorphic to a Möbius band. Since a projective plane cannot contain two disjoint Möbius bands, we have the conclusion.

A pair of edges $\{e_1, e_2\}$ in $G_Q$ is called an S-cycle if it is a Scharlemann cycle of length two. That is, $e_1$ and $e_2$ are adjacent parallel edges, and have the same two labels at their endpoints. Note that in this case the two labels are successive (see Figure 1).

**Lemma 2.3.** $G_Q$ cannot contain an S-cycle.

Proof. Let $\{e_1, e_2\}$ be an S-cycle in $G_Q$ with labels $\{i, i+1\}$. Let $D$ be the disk face between $e_1$ and $e_2$. Let $H$ be the annulus in $\partial E(K)$ cobounded by the
components of $\partial P$ with labels $i$ and $i+1$, whose interior is disjoint from $P$. Set $P'=(\hat{P} - V_i \cup V_{i+1}) \cup H$, where $V_i$ and $V_{i+1}$ are the vertices corresponding to the components of $\partial P$ with labels $i$ and $i+1$, respectively. Then $\text{int} \ D \cap P' = \emptyset$ and $\partial D \subset P$ is non-separating in $P'$. Compressing $P'$ along $D$ gives a new projective plane in $RP^3$ which intersects $J$ in $p-2$ meridian disks of $J$. This contradicts the minimality of $p$.

Figure 1

The reduced graph $\tilde{G}_Q$ of $G_Q$ is defined to be the graph obtained from $G_Q$ by amalgamating each set of mutually parallel edges of $G_Q$ to a single edge. By Lemma 2.1, $\tilde{G}_Q$ consists of essential loops in $\hat{Q}$. Thus $\tilde{G}_Q$ is a subgraph of the graph illustrated in Figure 2 (after a homeomorphism of $\hat{Q}$).
Therefore, the edges in $G_Q$ are partitioned into at most three parallel families of edges. Let $U$, $V$, $W$ be the parallel families of edges. We denote by $|U|$ the number of edges in $U$, etc. Then $|U| + |V| + |W| = p$.

Suppose that $|U| \neq 0$ and $|U|$ is even. Let $e_1, e_2, \ldots, e_{2t}$ be the edges of $U$, numbered consecutively, where $|U| = 2t$. Then $e_1$ and $e_{2t}$ have the same two labels at their endpoints. Therefore, $e_t$ and $e_{t+1}$ form an $S$-cycle. But this contradicts Lemma 2.3. Thus $|U|$ is odd, unless $U = \emptyset$. Similarly for $V$ and $W$.

We now distinguish two cases.

**Case 1.** $G_Q$ consists of at least two parallel families of edges.

We may assume that $U$ and $V$ are non-empty. Then $U$ and $V$ each contain a level edge, since $|U|$ and $|V|$ are odd. But these two level edges have distinct labels, which contradicts Lemma 2.2.
Case 2. \( G_\mathcal{Q} \) consists of one parallel family of edges.

Let \( e_1, e_2, \ldots, e_p \) be the edges in \( G_\mathcal{Q} \), numbered consecutively. We can assume that their endpoints are labelled as shown in Figure 3. Then \( e_i \) and \( e_{p+1-i} \) have the same two labels at their endpoints, for \( 1 \leq i \leq (p-1)/2 \), and \( e_{(p+1)/2} \) is level.

On the other hand, in \( G_\mathcal{P} \), \( e_i \) and \( e_{p+1-i} \) form a length two cycle, if \( i \neq (p+1)/2 \). Note that these cycles bound disks in \( \mathcal{P} \), since \( G_\mathcal{P} \) has a nontrivial loop \( e_{(p-1)/2} \). Hence we can choose an innermost one among the cycles \( \{ e_i e_{p+1-i} \} \), \( i \neq (p+1)/2 \). Let \( \{ e_s e_{p+1-s} \} \) be an innermost cycle in \( G_\mathcal{P} \). Then \( e_s \) and \( e_{p+1-s} \) are parallel in \( G_\mathcal{P} \). Let \( D_1 \subset \mathcal{P} \) be the disk between \( e_s \) and \( e_{p+1-s} \). Let \( D_2 \subset \mathcal{Q} \) be the disk between \( e_s \) and \( e_{p+1-s} \), containing \( e_{(p+1)/2} \). Then \( D_1 \cap D_2 = e_s \cup e_{p+1-s} \). Let \( A = D_1 \cup D_2 \). Then \( A \) is a Möbius band in \( E(K) \). By moving \( \partial A \) slightly into
general position with respect to \( \partial Q \), we see that \( \partial A \) has algebraic (and geometric) intersection number two with \( \partial Q \). Hence \( \partial A \) has slope \( 2/n \) on \( \partial E(K) \) for some \( n \) (cf. [11]). Then the resulting manifold \( M \) obtained by \( (2/n) \)-surgery on \( K \) contains a projective plane, and hence \( M \) is either reducible or \( RP^3 \). In any case, \( |n|=1 \) by [2,7]. But this implies that \( K \) is either a torus knot or a \( (2, \pm 1) \)-cable knot.

This completes the proof of Theorem 0.1.

References


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