



Title	Cyclic surgery on genus one knots
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Citation	Osaka Journal of Mathematics. 1997, 34(1), p. 145-150
Version Type	VoR
URL	https://doi.org/10.18910/5624
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CYCLIC SURGERY ON GENUS ONE KNOTS

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(Received May 2, 1996)

0. Introduction

The real projective 3-space, denoted by RP^3 , is identified with the lens space of type (2,1). Then one can ask: when can RP^3 be obtained by Dehn surgery on a knot in the 3-sphere S^3 ? Clearly RP^3 is obtained by Dehn surgery on a trivial knot. However, it is conjectured that no Dehn surgery on a nontrivial knot K in S^3 yields RP^3 (cf. [1,4]). It is known to be true if K is a composite knot [3], a torus knot [9], an alternating knot [10], a satellite knot [1,12,13], or a symmetric knot [1].

In this paper we prove the conjecture for genus one knots.

Theorem 0.1. *Real projective 3-space RP^3 cannot be obtained by Dehn surgery on a genus one knot in S^3 .*

This will be proved by applying the combinatorial techniques developed in [2,5,6,8].

1. Preliminaries

Let K be a genus one knot which is neither a torus knot nor a satellite knot. Let $N(K)$ be a tubular neighborhood of K and let $E(K) = S^3 - \text{int } N(K)$. Suppose that some surgery on K yields RP^3 , that is, $E(K) \cup J = RP^3$ where J is a solid torus. By [2], the surgery coefficient is ± 2 .

Let $P^2 \subset RP^3$ be a projective plane which intersects J in a disjoint union of meridian disks of J . We assume that $|P^2 \cap J|$ is minimal among all projective planes in RP^3 that intersect J in a family of meridian disks of J . Let $p = |P^2 \cap J|$ and $P = P^2 \cap E(K)$. Then P is incompressible in $E(K)$ by the minimality of p . If p is even, then $E(K)$ would contain a closed non-orientable surface by attaching tubes to ∂P . Hence p is odd. Furthermore, if $p = 1$ then K is either a torus knot or a $(2, \pm 1)$ -cable knot. Thus $p \neq 1$.

Let Q be a genus one Seifert surface for K . We may assume that P and Q intersect transversely, and ∂Q intersects each component of ∂P exactly twice. By the incompressibility of P and Q , we can assume that no circle component of

$P \cap Q$ bounds a disk in P or Q .

Let \hat{P} , \hat{Q} be the closed surfaces obtained by capping off the components of ∂P and ∂Q with disks. We can identify \hat{P} with P^2 . We obtain a graph G_P in \hat{P} by taking the disks $\text{cl}(\hat{P} - P)$ as the (fat) vertices of G_P , and the arc components of $P \cap Q$ in P as the edges of G_P . Similarly, we obtain the graph G_Q in \hat{Q} .

Number the components of ∂P , $\{1, 2, \dots, p\}$, in the order in which they appear on $\partial E(K)$. The endpoints of edges of G_Q are labelled by the numbers of the corresponding components of ∂P . Thus around the only vertex v of G_Q , we will consecutively meet the labels $1, 2, \dots, p, 1, 2, \dots, p$ (repeated twice). Since each vertex of G_P has valency two, G_P consists of disjoint cycles.

2. Proof of Theorem 0.1

A *trivial loop* is a length one cycle which bounds a disk face of the graph.

Lemma 2.1. *Neither G_P nor G_Q contains trivial loops.*

Proof. Let e be a trivial loop in G_P , and let D be a regular neighborhood of e in Q . Given the orientation of ∂Q induced by some orientation of D , the points of intersection of ∂Q with the component of ∂P meeting e have opposite signs, a contradiction. If G_Q contains a trivial loop, P would be compressible in $E(K)$, a contradiction.

An edge of G_Q is said to be *level* if its endpoints have the same label.

Lemma 2.2. *G_Q cannot contain two level edges on distinct labels.*

Proof. Let e be a level edge in G_Q with label i . Then e is a loop in G_P based at the vertex V_i corresponding to the component of ∂P with label i . We see that a regular neighborhood of $e \cup V_i$ in \hat{P} is homeomorphic to a Möbius band. Since a projective plane cannot contain two disjoint Möbius bands, we have the conclusion.

A pair of edges $\{e_1, e_2\}$ in G_Q is called an *S-cycle* if it is a Scharlemann cycle of length two. That is, e_1 and e_2 are adjacent parallel edges, and have the same two labels at their endpoints. Note that in this case the two labels are successive (see Figure 1).

Lemma 2.3. *G_Q cannot contain an S-cycle.*

Proof. Let $\{e_1, e_2\}$ be an S-cycle in G_Q with labels $\{i, i+1\}$. Let D be the disk face between e_1 and e_2 . Let H be the annulus in $\partial E(K)$ cobounded by the

components of ∂P with labels i and $i+1$, whose interior is disjoint from P . Set $P' = (\hat{P} - V_i \cup V_{i+1}) \cup H$, where V_i and V_{i+1} are the vertices corresponding to the components of ∂P with labels i and $i+1$, respectively. Then $\text{int } D \cap P' = \emptyset$ and $\partial D \subset P'$ is non-separating in P' . Compressing P' along D gives a new projective plane in RP^3 which intersects J in $p-2$ meridian disks of J . This contradicts the minimality of p .

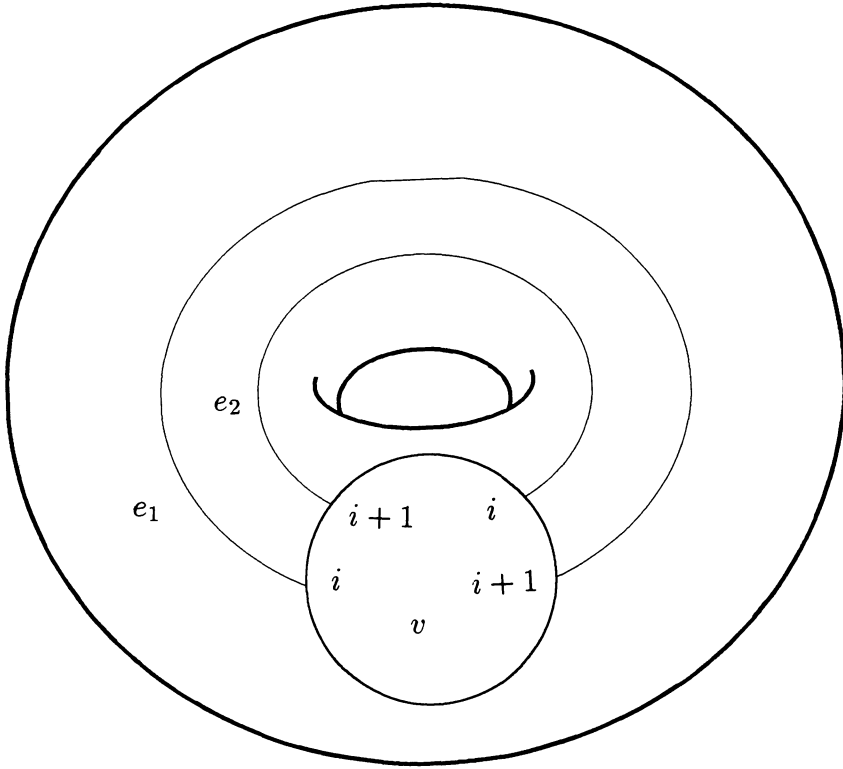


Figure 1

The reduced graph \bar{G}_Q of G_Q is defined to be the graph obtained from G_Q by amalgamating each set of mutually parallel edges of G_Q to a single edge. By Lemma 2.1, \bar{G}_Q consists of essential loops in \hat{Q} . Thus \bar{G}_Q is a subgraph of the graph illustrated in Figure 2 (after a homeomorphism of \hat{Q}).

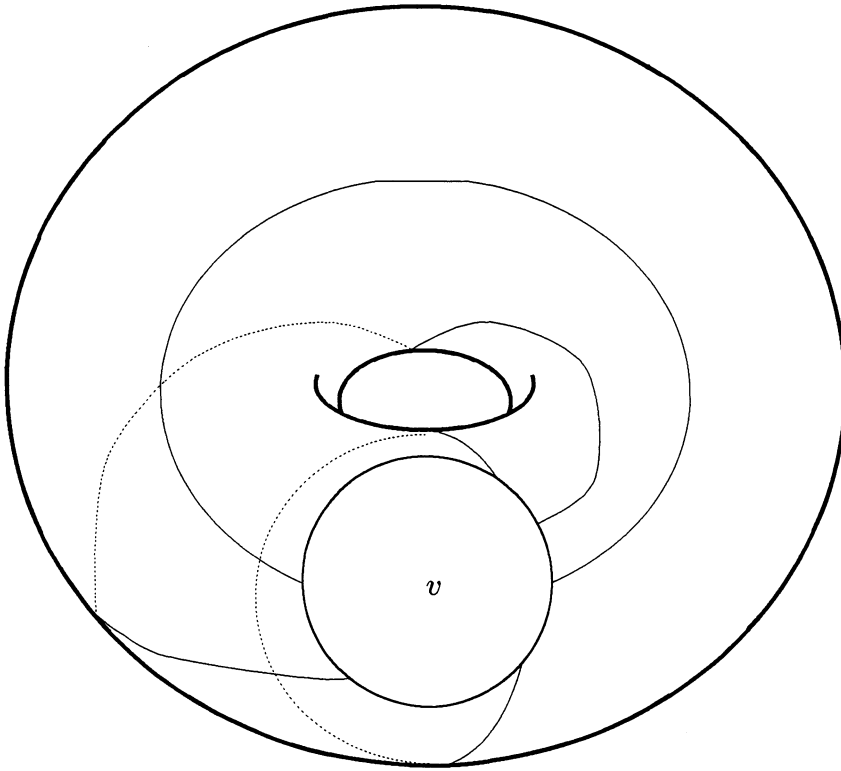


Figure 2

Therefore, the edges in G_Q are partitioned into at most three parallel families of edges. Let U , V , W be the parallel families of edges. We denote by $|U|$ the number of edges in U , etc. Then $|U| + |V| + |W| = p$.

Suppose that $|U| \neq 0$ and $|U|$ is even. Let e_1, e_2, \dots, e_{2t} be the edges of U , numbered consecutively, where $|U| = 2t$. Then e_1 and e_{2t} have the same two labels at their endpoints. Therefore, e_t and e_{t+1} form an S -cycle. But this contradicts Lemma 2.3. Thus $|U|$ is odd, unless $U = \emptyset$. Similarly for V and W .

We now distinguish two cases.

Case 1. G_Q consists of at least two parallel families of edges.

We may assume that U and V are non-empty. Then U and V each contain a level edge, since $|U|$ and $|V|$ are odd. But these two level edges have distinct labels, which contradicts Lemma 2.2.

Case 2. G_Q consists of one parallel family of edges.

Let e_1, e_2, \dots, e_p be the edges in G_Q , numbered consecutively. We can assume that their endpoints are labelled as shown in Figure 3. Then e_i and e_{p+1-i} have the same two labels at their endpoints, for $1 \leq i \leq (p-1)/2$, and $e_{(p+1)/2}$ is level.

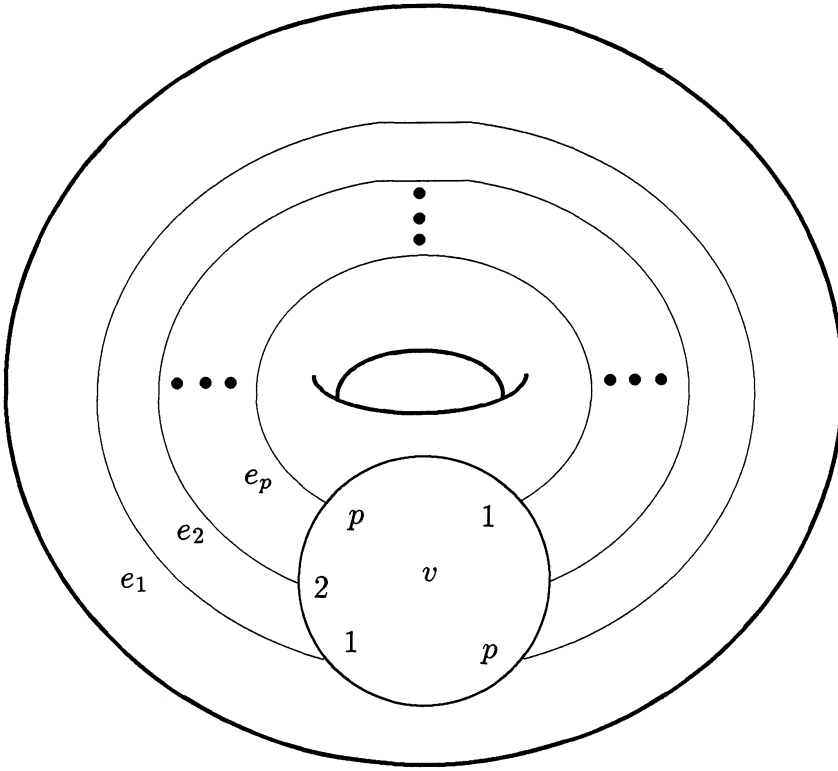


Figure 3

On the other hand, in G_P , e_i and e_{p+1-i} form a length two cycle, if $i \neq (p+1)/2$. Note that these cycles bound disks in \hat{P} , since G_P has a nontrivial loop $e_{(p-1)/2}$. Hence we can choose an innermost one among the cycles $\{e_i, e_{p+1-i}\}$, $i \neq (p+1)/2$. Let $\{e_s, e_{p+1-s}\}$ be an innermost cycle in G_P . Then e_s and e_{p+1-s} are parallel in G_P . Let $D_1 \subset P$ be the disk between e_s and e_{p+1-s} . Let $D_2 \subset Q$ be the disk between e_s and e_{p+1-s} , containing $e_{(p+1)/2}$. Then $D_1 \cap D_2 = e_s \cup e_{p+1-s}$. Let $A = D_1 \cup D_2$. Then A is a Möbius band in $E(K)$. By moving ∂A slightly into

general position with respect to ∂Q , we see that ∂A has algebraic (and geometric) intersection number two with ∂Q . Hence ∂A has slope $2/n$ on $\partial E(K)$ for some n (cf. [11]). Then the resulting manifold M obtained by $(2/n)$ -surgery on K contains a projective plane, and hence M is either reducible or RP^3 . In any case, $|n|=1$ by [2,7]. But this implies that K is either a torus knot or a $(2, \pm 1)$ -cable knot.

This completes the proof of Theorem 0.1.

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