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On the Smoothing of a Combinatorial n-Manifold Immersed in the Euclidean (n+1)-Space

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§1. Introduction

By a manifold we shall always mean one which is a separable Hausdorff space. A differentiable manifold will mean one which has a C^∞-structure, defined in terms of a family of allowable coordinate systems [11].

We shall use R^n to denote the n-dimensional euclidean vector space whose points are sequences $(x_1, x_2, \cdots, x_n)$ of real numbers x_i $(i=1, 2, \cdots, n)$ and shall use $|x|$ to denote the norm $\sqrt{(x_1^2+x_2^2+\cdots+x_n^2)}$ of $x=(x_1, x_2, \cdots, x_n)$.

By a *complex* we shall mean a rectilinear, locally finite, simplicial complex in R^q, for some $q \geqq 0$. If K is a complex, $|K|$ denotes the underlying polyhedron of K. If Δ is a (closed) simplex of K, then the *star* $St(\Delta, K)$ of Δ in K is the union of all the simplexes of K which contain Δ. The *link* $L(\Delta, K)$ of Δ in K is the union of all the simplexes of $St(\Delta, K)$ which do not meet Δ. If x is a point in $|K|$, we define $St(x, |K|)$ as $St(\Delta, K)$, where Δ is the simplex of K which contains x in its interior.

Let K, L be complexes in R^q. They are said to be *combinatorially equivalent*, if they have (rectilinear, simplicial) subdivisions which are isomorphic to each other. By a *combinatorial q-cell* (*q-sphere*) we shall mean a polyhedron combinatorially equivalent to a q-simplex (the boundary of a $(q+1)$-simplex).

A complex K is called an (*unbounded*) *combinatorial n-manifold* if the link of each vertex of K is a combinatorial $(n-1)$-sphere. A polyhedron is called a combinatorial n-manifold if it has a simplicial subdivision which is a combinatorial n-manifold.

Let K^n be a combinatorial n-manifold in R^q whose polyhedron $|K|$ has a differentiable structure. Let x be a point of an n-simplex Δ^n in K^n and let (U_x, φ_x) be an allowable C^∞-coordinate system about x where U_x is a neighborhood of x in $|K|$ and φ_x is a homeomorphism $\varphi_x: U_x \to R^n$. If there exists a neighborhood V_x of x in R^q and a map[1] $f: V_x \to U_x$

1) By a map we shall always mean a "continuous" one.

such that $f|\Delta^n \cap V_x$[2] = the identity map and $\varphi_x f\colon V_x \to R^n$ is a map of class C^∞ whose Jacobian matrix has the rank n at every point of $\Delta^n \cap V_x$, then we shall say that *the differentiable structure is compatible with the complex K.* Let a combinatorial manifold M have a differentiable structure. Then we shall say that the *differentiable structure of M is compatible with the combinatorial structure* of M, if some subdivision of M is compatible with the differentiable structure of M.

Let X, Y be topological spaces. A one-one map $f\colon X \to Y$ is called an *imbedding*. A map $f\colon X \to Y$ is called an *immersion* if every point of X has a neighborhood $U \subset X$ such that $f|U$ is an imbedding. A map f from a complex K into R^q is called a semi-linear map if f is linear in every simplex in some subdivision of K.

The purpose of this paper is to prove the following

Theorem. *If a combinatorial n-manifold M^n is immersed semi-linearly into R^{n+1}, then there exists a differentiable structure on M^n compatible with the combinatorial structure of M^n, under the Schoenflies hypothesis up to dimension n. Moreover, for any semi-linear immersion $f\colon M^n \to R^{n+1}$ and a positive continuous function $\varepsilon(p)$ on M^n, there exists a differentiable immersion $g\colon M^n \to R^{n+1}$ with $|f(p) - g(p)| < \varepsilon(p)$.*

The *Schoenflies hypothesis* for dimension n is as follows:

Every combinatorial $(n-1)$-sphere S^{n-1} in R^n is the boundary of a combinatorial n-cell which is the closure of the bounded component of $R^n - S^{n-1}$.

It is well known that the Schoenflies hypothesis has been affirmatively proved for $n \leqq 3$ [1], [3], [5]. Recently the hypothesis was proved by S. Smale [7] for $n \geqq 6$, $n \neq 7$.

We use G_k^n to denote the Grassmann manifold consisting of k-planes in R^{k+n} through the origin. If x, y are non-zero vectors in R^{n+k}, then $\alpha(x, y)$ will denote the angle between them, on the understanding that $0 \leqq \alpha \leqq \pi$. If $P \in G_k^n$, then $\alpha(x, P)$ will denote the angle between x and its orthogonal projection on P, with $\alpha(x, P) = \frac{\pi}{2}$ if x is orthogonal to P. Thus $0 \leqq \alpha(x, P) \leqq \frac{\pi}{2}$. If $P \in G_k^n$, $Q \in G_l^m$, where $n+k = m+l$, $0 < k \leqq l$, $m > 0$, then $\alpha(P, Q)$ will denote $\alpha(P, Q) = \max\{\alpha(x, Q) \mid 0 \neq x \in P\}$. The function α may be regarded as a metric for G_k^n. If $x \in R^{k+n}$, $P \in G_k^n$, then $x + P$ will denote the k-plane consisting of all the vectors $x + y \in R^{n+k}$ for every $y \in P$. If x, y are non-zero vectors in R^n, then $\overleftrightarrow{xy}$ will denote

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2) If $f\colon X \to Y$ is a map and Z is a subset of X, then $f|Z$ will always mean the restriction of f on Z.

$\overleftrightarrow{xy}=\{ty+(1-t)x \mid -\infty<t<+\infty\}$.

A k-plane $P^k \in G_k^n$ is called a *transversal k-plane* to a set X in R^{n+k}, if there exists a positive numbe ε such that

$$\alpha(\overleftrightarrow{xy}, P)>\varepsilon \text{ for } x, y \in X,\ x \neq y,$$

where $\alpha(\overleftrightarrow{xy}, P)$ will mean the angle between the line through 0 parallel to $\overleftrightarrow{xy}$ and P.

Let M^n be an n-manifold and let f be an immersion $f: M^n \to R^{n+k}$. A k-plane $P^k \in G_k^n$ is called a *transversal plane at $p \in M^n$ with respect to f*, if there exists a neighborhood U of p in M^n such that $f|U$ is an imbedding and P^k is transversal to $f(U)$. A map $\varphi: M^n \to G_k^n$ is called a *transverse k-plane field* (*or simply a transverse field*) if $\varphi(p)$ is transversal at p with respect to f for any point $p \in M$. In this case f is also called an *immersion with a transverse field* (or simply a *normal immersion*).

S. S. Cairns [2] and J. H. C. Whitehead [10] proved the following:

Let M^n be a manifold. If there exists a normal imbedding $f: M^n \to R^{n+k}$, then there exists a differentiable structure on M and for any given positive continuous function $\varepsilon(p)$ on M^n there exists a differentiable imbedding $g: M^n \to R^{n+k}$ with $|f(p)-g(p)|<\varepsilon(p)$.

On the other hand, H. Noguchi [6] proved the following:

Let M^n be a compact combinatorial n-manifold without boundary in R^{n+1}. Suppose that the Schoenflies hypothesis is true for dimension $\leqq n$. Then arbitrarily near M there are a combinatorial n-manifold N and an orientation preserving semi-linear homeomorphism onto $\psi: R^{n+1} \to R^{n+1}$ such that N admits a transverse field, and such that $\psi(M)=N$.

We shall generalize the above two theorems from the case of imbedding to that of "immersion". Then we shall obtain our main theorem which gives an answer to the problem of H. Noguchi [6]:

Let a combinatorial n-manifold be mapped into R^{n+1} by a semi-linear mapping f which is a local homeomorphism. Does there exists an analytic n-manifold, and an immersion of it in R^{n+1} which approximates f in some sense?

§ 2. The definitions and the propositions.

The proof of the main theorem is reduced to the proofs of four propositions, the outline of which will be stated in this section with some necessary explanations.

The first step of the proof of the main theorem is to prove the following proposition under the Schoenflies hypothesis up to dimension n.

Proposition 1. *Let M^n be a combinatorial n-manifold and let $f: M^n \to R^{n+1}$ be a semi-linear immersion. Then for any given positive continuous function $\varepsilon(p)$ on M^n there exists a semi-linear normal immersion $g: M^n \to R^{n+1}$ with $|f(p)-g(p)|<\varepsilon(p)$ for every point $p \in M$.*

Let X, Y be metric spaces and let ρ_X, ρ_Y be metrics for X, Y. A map $f: X \to Y$ is called a *Lipschitz map* with respect to ρ_X, ρ_Y if for any point $x \in X$ there exists a neighborhood $U_x \subset X$ of x and a positive number α_x such that

$$\rho_Y(f(x_1), f(x_2)) \leqq \alpha_x \rho_X(x_1, x_2)$$

for all $x_1, x_2 \in U_x$. The map f is called a *regular Lipschitz map* with respect to ρ_X, ρ_Y if for any $x \in X$ there exist a neighborhood $U_x \subset X$ and a pair of positive numbers α_x, β_x such that

$$\beta_x \rho_X(x_1, x_2) \leqq \rho_Y(f(x_1) f(x_2)) \leqq \alpha_x \rho_X(x_1, x_2)$$

for all $x_1, x_2 \in U_x$. Let f be a one-one Lipschitz map of X onto Y. Then $f^{-1}: Y \to X$ is a Lipschitz map if and only if f is regular. In this case f^{-1} is also regular.

Metrics ρ, ρ' for X are called *equivalent* if the identical map of X is a regular Lipschitz map with respect to ρ, ρ'. A collection $\{U_i, \rho_i\}$ of an open covering $\{U_i\}$ of X and a metric ρ_i for U_i is called a *Lipschitz system* of X, if ρ_i, ρ_j is equivalent in $U_i \cap U_j$ for every U_i, U_j. A pair (U, ρ) of a set $U \subset X$ and a metric ρ for U is called an *allowable pair* for a Lipschitz system $\{U_i, \rho_i\}$ of X, if ρ, ρ_i are equivalent in $U_i \cap U$ for every U_i. Two Lipschitz systems $\{U_i, \rho_i\}$, $\{U'_j, \rho'_j\}$ are said to be equivalent if (U'_i, ρ'_i) is an allowable pair for $\{U_j, \rho_j\}$. By a *Lipschitz space* we shall mean a topological space X together with an equivalent class of Lipschitz structures on X. If X, Y are Lipschitz spaces, then a Lipschitz map $f: X \to Y$ (or a regular Lipschitz map $f: X \to Y$) may be defined by the local metrics $\{\rho_i\}$, $\{\rho'_j\}$ of X, Y respectively.

Let M^n be an n-manifold which has a Lipschitz structure. Let U be an open set of M and let $\varphi: U \to R^n$ be a homeomorphism. We define a metric ρ_U for U by

$$\rho_U = |\varphi(p) - \varphi(q)| \qquad \text{for every } p, q \in U.$$

Then (U, ρ_U) is called an allowable pair if (U, ρ_U) is an allowable pair for the Lipschitz structure of M. A manifold M with a Lipschitz structure is called a *Lipschitz manifold*, if M has a set of local coordinate systems which are allowable pairs for the Lipschitz structure of M.

Let $P \in G_k^n$ and let $P^* \in G_n^k$ be the n-plane orthogonal to P. Then the vector space R^{n+k} is the direct sum $R^{n+k} = P + P^*$. Let $(u_1, \cdots, u_n)$,

$(v_1, \cdots, v_k)$ be rectangular cartesian coordinates for P^*, P respectively. If γ is a positive number, let

$$N(P, \gamma) = \{Q \in G_k^n \mid \alpha(Q, P) < \gamma\} .$$

Then a k-plane $Q \in N\left(P ; \frac{\pi}{2}\right)$ is given by a set of equations of the form

$$u_i = \sum_{j=1}^{k} a_{ij} v_j \quad (i = 1, \cdots, n) .$$

Conversely, if $\|a_{ij}\|$ is a given $n \times k$ matrix, then the above equation represents a k-plane in $N\left(P, \frac{\pi}{2}\right)$. Therefore $\rho_P : Q \to \|a_{ij}\|$ is a local coordinate system

$$\rho_P : N\left(P, \frac{\pi}{2}\right) \to R^{nk} .$$

The set of all such coordinate systems, for every $P \in G_k^n$, may be used to define the differentiable structure of G_k^n. On the other hand, the local metric on $N\left(P, \frac{\pi}{2}\right)$ induced by ρ_P may give an allowable Lipschitz structure for the global metric on G_k^n defined by α. Let M^n be an n-manifold and let $f : M^n \to R^{n+k}$ be a normal immersion with a transverse field $\varphi : M^n \to G_k^n$. For any point $p \in M$, there exists a neighborhood $U_p \subset M$ of p and a positive number ε_p such that $f \mid U_p$ is an imbedding with

$$\alpha(\varphi(p), \overleftrightarrow{f(s) f(s')}) > \varepsilon_p$$

for every s, $s \in U_p$, $s \neq s'$.

Let $P_p \in G_n^k$ be an n-plane orthogonal to the k-plane $\varphi(p) = Q_p$. Then the vector space R^{n+k} is the direct sum $R^{n+k} = P_p + Q_p$. Let $(u_p^1, \cdots, u_p^n)$, $(v_p^1, \cdots, v_p^k)$ be rectangular cartesian coordinates for P_p, Q_p. Let π_p be the orthogonal projection of R^{n+k} onto P_p. Since $\pi_p f : U_p \to P_p$ is a homeomorphism, we may introduce a local coordinate system on U_p by $\pi_p f$ with a local metric on U_p defined by $\rho_{U_p}(s, s') = |\pi_p f(s) - \pi_p f(s')|$ for every $s, s' \in U_p$.

Lemma 2.1. *The set of all the local coordinate systems $(U_p, \pi_p f, \rho_{U_p}(s, s'))$ determines a Lipschitz structure on M^n.*

Proof. Let $U_p \cap U_q \neq \phi$ and let $U_r \subset U_p \cap U_q$ be a neighborhood of a point $r \in U_p \cap U_q$. It is sufficient to prove that the homeomorphism $\pi_q \pi_p^{-1} : \pi_p f(U_r) \to \pi_q f(U_r)$ is a regular Lipschitz map.

Let $f(U_r)$ be given by the following equations of the rectangular local coordinate systems (u_p^i, v_p^j) and (u_q^i, v_q^j) respectively

$$v_p^i = g_p^i(u_p^1, \cdots, u_p^n) \qquad i = 1, \cdots, k$$

and

$$v_q^j = g_q^j(u_q^1, \cdots, u_q^n) \qquad j = 1, \cdots, k\,.$$

Let $s, s' \in U_r$. From the fact

$$|v_p' - v_p| = |u_p' - u_p| \cot (\alpha(\overleftrightarrow{f(s')f(s)}, Q)) < |u_p - u_p'| \cot \varepsilon_p \quad .$$

we obtain

$$|f(s') - f(s)| < |u_p' - u_p| \sqrt{1 + \cot^2 \varepsilon_p}\,,$$

where $f(s') = (u_p', v_p')$, $f(s) = (u_p, v_p)$.

On the other hand, from $|u_q' - u_q| \leqq |f(s') - f(s)|$, we obtain

$$|u_q' - u_q| \leqq |u_p' - u_p| \sqrt{1 + \cot^2 \varepsilon_p}.$$

Therefore we obtain

$$\frac{1}{1 + \cot^2 \varepsilon_q} |u_p' - u_p| \leqq |u_q' - u_q| \leqq \sqrt{1 + \cot^2 \varepsilon_p}\, |u_p' - u_p|.$$

Thus $\pi_q \pi_p^{-1}$ is a regular Lipschitz homeomorphism and the lemma is proved.

The second step of the proof of the main theorem is to prove the following proposition.

Proposition 2. *Let $f: M^n \to R^{n+k}$ be a normal immersion with a transverse field $\varphi: M^n \to G_k^n$ and let $\varepsilon(p)$ be a positive continuous function on M^n. Then there exists a Lipschitz map $\psi: M^n \to G_k^n$ with respect to the above mentioned Lipschitz structures on M^n, G_k^n which is a transverse field with respect to f and satisfies*

$$\alpha(\varphi(p), \psi(p)) < \varepsilon(p)\,.$$

ψ may be called a *transverse Lipschitz field* with respect to f.

Let f be a normal immersion with a transverse field $\varphi: M^n \to G_k^n$. Let $E(\varphi)$ be the k-plane bundle over M^n which is induced by φ. We may take a point in $E(\varphi)$ to be the pairs (p, x) such that $p \in M$, $x \in \varphi(p)$. Thus $E(\varphi) \subset M \times R^{n+k}$. The projection map $\pi: E(\varphi) \to M^n$ is defined by $\pi(p, x) = p$. Let $\{U_\lambda\}$ be an open covering of M^n such that $f | U_\lambda$ is an imbedding and let $\psi_\lambda: E(\varphi\,;\, U_\lambda) \to R^{n+k} \times R^{n+k}$ be an imbedding defined by $\psi_\lambda(p, x) = (f(p), x)$, where $E(\varphi\,;\, U_\lambda) = \pi^{-1}(U_\lambda) \subset E(\varphi)$. We may define a metric ρ_λ on $E(\varphi\,;\, U_\lambda)$ by

$$\rho_\lambda(p, q) = |\psi_\lambda(p) - \psi_\lambda(q)| \qquad \text{for every } p, q \in E(\varphi; U_\lambda).$$

Then $\{E(\varphi; U_\lambda), \rho_\lambda\}$ defines a Lipschitz structure on $E(\varphi)$. If M is identified with the zero cross section $M_0 = \{(p, 0) \in E(\varphi)\}$ of $E(\varphi)$, then the above mentioned Lipschitz metric induces an equivalent metric with the one introduced in Proposition 2. Whenever we refer to a Lipschitz map to or from M and $E(\varphi)$, it will mean the Lipschitz map with respect to these Lipschitz structures.

Now let f be a normal immersion with a transverse field $\varphi: M^n \to G_k^n$ and let $E(\varphi)$ be the k-plane bundle over M which is induced by φ. We define $\theta: E(\varphi) \to R^{n+k}$ by

$$\theta(p, x) = f(p) + x \qquad \text{for every } p \in M,\ x \in \varphi(p).$$

Then φ is called a *transverse C^∞-field* with respect to φ, if the following conditions (i), (ii) are satisfied.

(i) There exists a positive continuous function $\rho(p)$ on M^n and for any point $p \in M$ there exists a neighborhood $U_p \subset M$ of p such that $\theta: T_\rho(\varphi; U) \to R^{n+k}$ is a regular Lipschitz homeomorphism, where $T_\rho(\varphi; U) = \{(p, x) \mid p \in U,\ x \in \varphi(p),\ |x| < \rho(p)\}$.

Since $T_\rho(\varphi) = \{(p, x) \mid p \in U,\ x \in \varphi(p),\ |x| < \rho(p)\}$ is locally homeomorphically immersed in R^{n+k} by θ we may introduce a differentiable structure in $T_\rho(\varphi)$. Then the second condition is as follows:

(ii) The map $\varphi\pi: T_\rho(\varphi) \to G_k^n$ is of class C^∞.

Now the third step of the proof of the main theorem is to prove the following proposition.

Proposition 3. *Let f be a normal immersion from an n-manifold M^n into R^{n+k} with a transverse Lipschitz field and let $\varepsilon(p)$ be a positive continuous function on M^n. Then there exists a transverse C^∞-field $\psi: M^n \to G_k^n$ with respect to f which satisfies*

$$\alpha(\varphi(p), \psi(p)) < \varepsilon(p) \qquad \textit{for every } p \in M.$$

The final step of the proof of the main theorem is to prove the following proposition.

Proposition 4. *Let M^n be an n-manifold and let $\varepsilon(p)$ be a positive continuous function on M^n. If there exists a normal immersion $f: M^n \to R^{n+k}$ with a transverse C^∞-field, then a differentiable structure may be introduced on M^n and there exists a differentiable immersion $g: M^n \to R^{n+k}$ with $|f(p) - g(p)| < \varepsilon(p)$.*

Moreover, if M^n is a combinatorial manifold and $f: M^n \to R^{n+k}$ is a semi-linear immersion, then the above introduced differentiable structure is compatible with the combinatorial structure of M^n.

The main ideas in this paper will be derived from J. H. C. Whitehead's paper [10] and H. Noguchi's [6].

§ 3. The proof of Proposition 1.

Throughout this section we may assume the Schoenflies hypothesis up to dimension n. Therefore the lemma 3.1 and Proposition 1 are proved under that hypothesis.

Let M^n be a combinatorial n-manifold and let f be a semi-linear immersion of M^n into R^{n+1}. Let K be a subdivision of M such that f is a linear imbedding on each simplex of K. Then f imbeds the star of each simplex of K into R^{n+1}. Let σ_i^q $(i=1, \cdots, n, \cdots)$ be q-simplexes of K^n and let $f(\sigma_i^q)=\Delta_i^q$. Let o_i be an interior point of Δ_i^q and let R_i^{n-q+1} be an $(n-q+1)$-plane through o_i and orthogonal to Δ_i^q. Let ∇_i^{n-q+1} be an $(n-q+1)$-simplex in R_i^{n-q+1}, which contains o_i in its interior such that

$$\nabla_i \cap \partial f(St(\sigma_i, K)) = \phi .$$

Then $\partial\nabla_i \cap f(St(\sigma_i^q))$ is a combinatorial $(n-q-1)$-sphere and separates $\partial\nabla_i$ into two connected $(n-q)$-polyhedra with the common boundary $\partial\nabla_i \cap f(St(\sigma_i^q))$. We denote one of them by B_i which is a combinatorial $(n-q)$-cell under the Schoenflies hypothesis.

Then, according to H. Noguchi ([6], p. 211), $B_i*\Delta_i$[3] and $\partial B_i*\Delta_i$, $B_i*\partial\Delta_i$ are a combinatorial $(n+1)$-cell and combinatorial n-cells respectively which satisfy

$$\partial(B_i*\Delta_i) = (\partial B_i*\Delta_i) \cup (B*_i\partial\Delta_i)$$

and

$$(\partial B_i*\Delta_i) \cap (B_i*\partial\Delta_i) = \partial B_i*\partial\Delta_i .$$

Therefore there exists a semi-linear homeomorphism [4]

$$h_i : \partial B_i*\Delta_i \rightarrow B_i*\partial\Delta_i$$

which is the identity map on the boundary of $\partial B_i*\Delta_i$.

Since $\partial B_i*\Delta_i$ is contained in $f(St(\sigma_i))$ and f is an imbedding on $St(\sigma_i)$, the inverse map f^{-1} of f is defined in $\partial B_i*\Delta_i$. Let $T_i=f^{-1}(\partial B_i*\Delta_i)$.

If the diameter diam. (∇_i) of ∇_i is sufficiently small, then Int. $(T_i) \cap$ Int. $(T_j)=\phi$ for $i \neq j$, where Int. (T_i) is the interior of T_i. Then we define a map $g: M^n \rightarrow R^{n+1}$ by

3) Let X, $Y \subset R^n$, then $X*Y$ will mean the join of X and Y, that is, $X*Y=\{xt+(1-t)y \mid x \in X, y \in Y, 0 \leq t \leq 1\}$.

$$g(p) = f(p) \quad \text{if} \quad p \in M - \bigcup_{i=1}^{\infty} \text{Int.}(T_i)$$
$$= h_i f(p) \quad \text{if} \quad p \in T_i$$

which is a semi-linear immersion $g: M^n \to R^{n+1}$ satisfying

$$|g(p) - f(p)| < \text{diam.}(\Delta_i * \nabla_i) \text{ for every } i .$$

The map $g: M^n \to R^{n+1}$ is called a *deformation* of f with respect to $\{\sigma_i^q | i=1, \cdots, n, \cdots\}$.

Lemma 3.1. *Let a map $f: M^n \to R^{n+1}$ be a semi-linear immersion from a combinatorial n-manifold M^n into R^{n+1} and let $\mathcal{E}(p)$ be a positive continuous function on M. Then there exists a semi-linear immersion $g: M^n \to R^{n+1}$ which has a transverse line with respect to g at every point of M^n and satisfies*

$$|f(p) - g(p)| < \mathcal{E}(p) .$$

Proof. Let K be a subdivision of M such that f is a linear map on every simplex of K. It is clear that an interior point of an n-simplex of K has a transverse line with respect to f. We suppose that f has a transverse line at every point of K except for the points of the q-skeleton K^q of K. Let $\{\sigma_i^q | i=1, \cdots, n, \cdots\}$ be all the q-simplexes of K whose interior point has not any transverse line with respect to f. Let g be a deformation of f with respect to $\{\sigma_i^q | i=1, \cdots, n, \cdots\}$.

According to H. Noguchi ([6], lemma 9) there exists for any point $x \in (B_i * \partial\Delta_i) - \partial\Delta_i$, a neighborhood U_x of x in $g(St(\sigma_i^q))$ and a line $l_x \in G_1^n$ such that l_x is transverse to U_x. Since the restriction $g|St(\sigma_i^q)$ of g on $St(\sigma_i^q)$ is an imbedding, l_x is a transverse line about $p = g^{-1}(x)$ with respect to g. Therefore g has a transverse line at the points of K except for the points of the $(q-1)$-skeleton of K. Let L be a subdivision of K such that each simplex of L is mapped linearly by g. Then g has a transverse field at a point of L except for the points of the $(q-1)$-skeleton L^{q-1} of L. By the n-fold iteration of the above process, we may obtain the required semi-linear immersion $g: M^n \to R^{n+1}$. If we take the diameter of $\Delta^q * \nabla^{n-q+1}$ less than $\frac{1}{n}\mathcal{E}(\Delta^q)$ at every stage of the process, $\mathcal{E}(\Delta^q) = \max\{\mathcal{E}(p) | p \in St(\sigma^q)\}$, then g satisfies the condition $|f(p) - g(p)| < \mathcal{E}(p)$. Thus the lemma is proved.

The proof of Proposition 1. We take the above constructed semi-linear map $g(p): M^n \to R^{n+1}$ for the semi-linear immersion $f(p): M^n \to R^{n+1}$.

Let K be a complex of M such that each simplex of K is mapped linearly by g and let K^q be the q-skeleton of K.

According to Lemma 3.1, there exist transverse lines with respect to g on K^0 which is denoted by

$$\varphi_0 : K^0 \to G_1^n .$$

Suppose that there exists a map $\varphi_q : K^q \to G_1^n$ such that φ_q is a transverse field with respect to g on K^q. Let σ^{q+1} be a $(q+1)$-simplex of K and let η^q be a q-face of σ^{q+1}. Take points t, s in the interiors of η^q, σ^{q+1} respectively. Since $St(\eta^q)$ contains $St(\sigma^{q+1})$, the totality $T_t(g : M^n)$ of the transverse lines with respect to g at t is contained in the totality $T_s(g ; M^n)$ of the transverse lines with respect to g at s. Therefore $\varphi_q(\partial\sigma^{q+1})$ is contained in $T_s(g ; M)$.

Since $St(\sigma^{q+1})$ is imbedded in R^{n+1} by g, $T_s(g ; T)$ is a contractible set in G_1^n ([9], lemma 3), ([6], Corollary 2 of lemma 2.). Therefore $\varphi_q | \partial\sigma^{q+1} ; \partial\sigma^{q+1} \to T_s(g : M)$ is extended to a map from σ^{q+1} into $T_s(g ; M)$. Thus we may obtain $\varphi_{q+1} : K^{q+1} \to G_1^n$ which is a transverse field with respect to g. Therefore we obtain by induction the required transverse field $\varphi : M^n \to G_1^n$ with respect to g.

§ 4. The proof of Proposition 2.

Since any continuous map between Lipschitz spaces is approximated by a Lipschitz map ([10], Theorem 9.1), there exists a Lipschitz map $\psi(p) : M^n \to G_k^n$ such that $\alpha(\psi(p), \varphi(p)) < \varepsilon(p)$. Now we shall show that there exists a positive continuous function $\rho(p)$ on M such that any map $\psi(p) : M^n \to G_k^n$ with the condition $\alpha(\psi(p), \varphi(p)) < \rho(p)$ is a transverse field with respect to f.

For any point $p \in M$ and a number $0 < \gamma < \frac{\pi}{4}$, there exists a neighborhood U_p' of p in M such that $f | U_p'$ is an imbedding and satisfies

$$\alpha(\overleftrightarrow{f(q')f(q)}, \varphi(p)) < 2\gamma \quad \text{for every} \quad q,\ q' \in U_p' ,\ q \neq q' .$$

Since φ is continuous, there exists a neighborhood U_p of p such that $\bar{U}_p$ is compact and contained in U_p' and $\varphi(\bar{U}_p)$ is contained in $N(\varphi(p), \gamma)$.

Since M^n is a paracompact space, there exist a locally finite open covering $\{U_i\}$, k-planes $\{P_i \in G_k^n\}$ and positive numbers $\left\{0 < \gamma_i < \frac{\pi}{4}\right\}$ which satisfy the following conditions :

(i) $\bar{U}_i$ is a compact set and $f|U_i$ is an imbedding.

(ii) $\alpha(\overleftrightarrow{f(q')f(q)}, P_i) > 2\gamma_i$ for every $q, q' \in U_i$, $q \neq q'$,

(iii) $\varphi(\bar{U}_i) \subset N(P_i, \gamma_i)$.

Let d_i be the distance between $\varphi(\bar{U}_i)$ and $G_k^n - N(P_i, \gamma_i)$. Then d_i is a positive number. Then there exists a positive continuous function $\rho(p)$ on M which satisfies $\rho(p) < d_i$ for any point $p \in \bar{U}_i$ ([10], lemma 5.1).

Now we shall show that $\rho(p)$ is the required positive continuous function on M. Let $\psi(p): M^n \to G_k^n$ be a map which satisfies $\alpha(\varphi(p), \psi(p)) < \rho(p)$ for any $p \in M$. Let us suppose that p is a point of U_i. Because of $p \in U_i$, we may obtain the following:

$$\alpha(\varphi(p), \psi(p)) < \rho(p) < d_i \leqq \alpha(\varphi(p), G_k^n - N(P_i, \gamma_i)) .$$

Therefore $\psi(p)$ is a point of $N(P_i, \gamma_i)$. Take q, q' $(q \neq q')$ in U_i. Then

$$\alpha(\overleftrightarrow{f(q')f(q)}, \psi(p)) + \alpha(\psi(p), P_i) \geqq \alpha(\overleftrightarrow{f(q')f(q)}, P_i) .$$

Therefore we obtain

$$\alpha(\overleftrightarrow{f(q')f(q)}, \psi(p)) \geqq \alpha(\overleftrightarrow{f(q')f(q)}, P_i) - \alpha(\psi(p), P_i) .$$

Since $\alpha(\overleftrightarrow{f(q')f(q)}, P_i) > 2\gamma_i$ and $\alpha(\psi(p), P_i) < \gamma_i$, we obtain

$$\alpha(\overleftrightarrow{f(q')f(q)}, \psi(p)) > \gamma_i \text{ for every points } q, q' \in U_i \; q \neq q' .$$

Since U_i is considered as a neighborhood of p in M^n, $\psi(p)$ is a transverse plane at p with respect to f.

Therefore $\psi(p): M \to G_k^n$ is a transverse field with respect to f. Thus our proposition is proved.

§ 5. The proof of Proposition 3.

Before we proceed to the proof of the proposition, we shall be in need of some lemmas. Let M^n be an n-manifold and let $f: M^n \to R^{n+k}$ be a normal immersion with a transverse field $\varphi: M^n \to G_k^n$. Let $E(\varphi)$ be the k-plane bundle over M^n induced by φ and let a map $\theta: E(\varphi) \to R^{n+k}$ be defined by $\theta(p, x) = f(p) + x$. Let $\rho(p)$ be a positive continuous function on M^n. Then we define $T_\rho(\varphi)$ by

$$T_\rho(\varphi) = \{(p, x) \in E(\varphi) \,|\, p \in M,\ x \in \varphi(p),\ |x| < \rho(p)\} .$$

Then $T_\rho(\varphi)$ is called a *tubular neighborhood* with respect to (f, φ), if for any point p of M^n there exists a neighborhood U_p such that

$\theta | T_\rho(\varphi\,;\, U_p) : T_\rho(\varphi\,;\, U_p) \to R^{n+k}$ is a regular Lipschitz homeomorphism, where

$$T_\rho(\varphi\,;\, U_p) = \{(p, x) \in T_\rho(\varphi) | p \in U_p,\ x \in \varphi(p),\ |x| < \rho(p)\} .$$

Lemma 5.1. *Let M^n be an n-manifold and let $f: M^n \to R^{n+k}$ be a normal immersion with a transverse Lipschitz field $\varphi : M^n \to G_k^n$. Then there exists a positive continuous function $\rho(p)$ on M^n such that $T_\rho(\varphi)$ is a tubular neighborhood with respect to (f, φ).*

Proof. Let $p \in M^n$ and let $0 < \gamma < \frac{\pi}{4}$. Then there exists a neighborhood U_p of p such that $\bar{U}_p$ is compact and $f | U_p$ is an imbedding which satisfies the following:

$$\alpha(\overleftrightarrow{f(q')f(q)}, \varphi(p)) > 2\gamma \quad \text{for every} \quad q', q \in U_p,\ q \neq q' .$$

On the other hand, since φ is a continuous map, we may suppose that U_p satisfies the following:

$$\alpha(\varphi(p), \varphi(q)) > \gamma \quad \text{for every} \quad q \in U_p .$$

Then, from the fact

$$\alpha(\overleftrightarrow{f(q')f(q)}, \varphi(q)) + \alpha(\varphi(q), \varphi(p)) \geqq \alpha(\overleftrightarrow{f(q')f(q)}, P) ,$$

we obtain

$$\begin{aligned} \alpha(\overleftrightarrow{f(q')f(q)}, \varphi(q)) &\geqq \alpha(\overleftrightarrow{f(q')f(q)}, P) - \alpha(\varphi(q), \varphi(p)) \\ &> 2\gamma - \gamma = \gamma . \end{aligned}$$

Since φ is a Lipschitz map and $f | U_p$ is a regular Lipschitz homeomorphism, there exists a positive number λ_p with the condition

$$\alpha(\varphi(q'), \varphi(q)) \leqq \lambda_p |f(q') - f(q)| \quad \text{for every} \quad q, q' \in U_p .$$

Since M^n is a paracompact space, there exists a locally finite open covering $\{U_i\}$ of M^n which satisfies the following:

(i) $\bar{U}_i$ is compact and $f | \bar{U}_i$ is an imbedding,

(ii) $\alpha(\overleftrightarrow{f(q')f(q)}, \varphi(q)) > \gamma$ for every $q, q' \in U_i$, $q \neq q'$,

(iii) $\alpha(\varphi(q), \varphi(q')) \leqq \lambda_i |f(q) - f(q')|$ for some positive number λ_i and for any points $q, q' \in U_i$.

Now let $z \neq 0$ be a point in $\varphi(q)$ and let $\vartheta = \alpha(\overleftrightarrow{f(q')f(q)}, z)$. Then

$$\vartheta \geqq \alpha(\overleftrightarrow{f(q')f(q)}, \varphi(q)) > \gamma,\ \pi - \vartheta = \alpha(\overleftrightarrow{f(q')f(q)}, -z) > \gamma .$$

Therefore we obtain

$$\begin{aligned}|f(q')-f(q)+z|^2 &= |f(q')-f(q)|^2+|z|^2+2|f(q')-f(q)||z|\cos\vartheta\\ &\geqq (1-|\cos\vartheta|)(|f(q')-f(q)|^2+|z|^2)\\ &\geqq (1-\cos\gamma)(|f(q')-f(q)|^2+|z|^2)\\ &\geqq \sin^2\frac{\gamma}{2}(|f(q')-f(q)|+|z|)^2.\end{aligned}$$

Hence we obtain

$$|f(q')-f(q)+z|\geqq\sin\frac{\gamma}{2}(|f(q')-f(q)|+|z|)\cdots\cdots\cdots\cdots(5.1)$$

Let π_0 be the orthogonal projection from R^{n+k} onto $\varphi(q)$. Let $x'\in\varphi(q')$, $x\in\varphi(q)$. Then

$$\begin{aligned}|x'-\pi_0(x')| &= |x'|\sin\alpha(x',\varphi(q))\leqq|x'|\sin\alpha(\varphi(q'),\varphi(q))\\ &\leqq|x'|\alpha(\varphi(q),\varphi(q'))\leqq\lambda_i|x'||f(q')-f(q)|\end{aligned}$$

and

$$|\pi_0(x')-x|\geqq|x-x'|-|x'-\pi_0(x')|\geqq|x-x'|-\lambda_i|x'||f(q)-f(q')|.$$

From (5.1), we obtain the following

$$\begin{aligned}|\theta(q',x')-\theta(q,x)| &= |f(q')-f(q)+(\pi_0(x')-x)+(x'-\pi_0(x'))|\\ &\geqq|f(q')-f(q)+\pi_0(x')-x|-|x'-\pi_0(x')|\\ &\geqq\left(\sin\frac{\gamma}{2}\right)(|f(q')-f(q)|+|\pi_0(x')-x|)\\ &\quad-\lambda_i|x'||f(q')-f(q)|\\ &\geqq\left(\sin\frac{\gamma}{2}\right)(|f(q')-f(q)|+(|x-x'|)\\ &\quad-\lambda_i|x'||f(q')-f((q)|\left(1+\sin\frac{\gamma}{2}\right).\end{aligned}$$

Hence we obtain

$$|\theta(q',x')-\theta(q,x)|\geqq\frac{1}{2}\sin\frac{\gamma}{2}(|f(q')-f(q)|+|x'-x|)$$

if $|x'|\leqq\frac{1}{2}\sin\frac{\gamma}{2}\Big/\lambda_i\left(1+\sin\frac{\gamma}{2}\right)$.

Therefore θ is a regular Lipschitz homeomorphism on $N(\varphi;U_i)=\left\{(p,x)\in E(\varphi)|p\in U_i,\ x\in\varphi(p),\ |x|<\frac{1}{2}\sin\frac{\gamma}{2}\Big/\lambda_i\left(1+\sin\frac{\gamma}{2}\right)\right\}$. Since $\{U_i\}$ is a locally finite covering and $\bar{U}_i$ is compact, there exists a positive continuous functions $\rho(p)$ on M^n such that

$$\rho(p) < \frac{1}{2}\sin\frac{\gamma}{2}\Big/\lambda_i\Big(1+\sin\frac{\gamma}{2}\Big) \qquad \text{for} \quad p \in \bar{U}_i\,.$$

Then $T_\rho(\varphi)$ is the required tubular neighborhood of M with respect to (f, φ). Thus the lemma is proved.

Now we shall state two lemmas by J. H. C. Whitehead without proof ([10], lemma 9.5, lemma 9.6).

Lemma 5.2. *Let V, W be open sets in R^n such that $\bar{V}$ is compact and is contained in W. Let f: $W \rightarrow R^q$ be a map such that there exists a positive number κ with the following condition:*

$$|g(x')-f(x)| \leqq \kappa|x'-x| \qquad \textit{for any} \quad x, x' \in W\,.$$

Then for any given positive number $\eta > 0$, there exists a differentiable map g: $V \rightarrow R^q$ which satisfies the following:

$$|g(x')-g(x)| \leqq \kappa\sqrt{q}\,|x'-x| \qquad \textit{for every} \quad x', x \in V$$

and

$$|f(x)-g(x)| < \eta \qquad \textit{for every} \quad x \in V\,.$$

Lemma 5.4. *Let U, V, W be open sets in R^n such that $\bar{V}$ is compact and $\bar{U} \subset V$, $\bar{V} \subset W$. Let f: $W \rightarrow R^q$ be a map which is of class C^∞ in an open set $N \subset W$ and satisfies*

$$|f(x')-f(x)| \leqq \kappa|x'-x| \qquad \textit{for every} \quad x, x' \in W$$

and for some positive number κ. Then for any given positive number η, there exists a map h: $W \rightarrow R^q$ which satisfies the following conditions:

(i) *$|h(x)-f(x)| < \eta$ for every $x \in W$ and $h(x)=f(x)$ if $x \in W-V$,*

(ii) *h is of class C^∞ in $U \cup N$*

(iii) *$|h(x')-h(x)| \leqq 4\kappa\sqrt{q}\,|x'-x|$ for every $x, x' \in W$.*

Now we shall proceed to prove Proposition 3.

Let $p \in M$ and $0 < \gamma < \frac{\pi}{4}$. Then there exists a neighborhood $W_p' \subset M$ of p such that $\bar{W}_p'$ is compact and f is an imbedding in $\bar{W}_p'$ and satisfies

$$\alpha(\overleftrightarrow{f(q')f(q)}, \varphi(p)) > 2\gamma \qquad \text{for every} \quad q, q' \in W', \ q \neq q'\,.$$

Let β be a positive number such that

$$0 < \beta < \gamma,\ \beta < \frac{1}{2}\,\{\varepsilon(p)\,|\,p \in \bar{W}_p'\} \quad \text{and} \quad \sqrt{k}\cot 2\gamma < \cot 2\beta\,.$$

Since φ is continuous, we may take a neighborhood W_p of p in W'_p such that $\varphi(\bar{W}_p)\subset N(\varphi(p), \beta)$. Since M is paracompact, there exist a locally finite covering $\{W_i\}$, k-planes $\{P^k\in G^n_k\}$ and positive numbers $\left\{0<\beta_i<\gamma_i<\frac{\pi}{4}\right\}$ which satisfy the following conditions:

(i) $\bar{W}_i$ is compact,

(ii) $\beta_i<\frac{1}{2}\min\{\varepsilon(p)\,|\,p\in\bar{W}_i\}$,

(iii) $\sqrt{k}\cot 2\gamma_i<\cot 2\beta_i$,

(iv) $\frac{1}{2}\alpha(\overleftrightarrow{f(p)f(q)}, P_i)>\gamma_i$ for every $p, q\in W_i$, $p\neq q$,

(v) $\varphi(\bar{W}_i)\subset N(P_i, \beta_i)$,

(vi) $\alpha(\overleftrightarrow{f(p)f(q)}, Q)>\gamma_i$ for every $p, q\in W_i$, $p\neq q$ and every $Q\in N(P_i, \beta_i)$,

(vii) $\alpha(\varphi(p), Q)<2\beta_i<\varepsilon(p)$ for every $p\in W_i$ and every $Q\in N(P_i, \beta_i)$.

Now let $\{V_i\}$, $\{U_i\}$ $(i=1, 2, \cdots)$ be open coverings of M such that

$$\bar{U}_i\subset V_i,\quad \bar{V}_i\subset W_i.$$

Then we shall show that there exist Lipschitz maps $\varphi_i: M^n\to G^n_k$ $(i=0, 1, 2, \cdots, n, \cdots)$ which satisfy

(i) $\varphi_0=\varphi$,

(ii) $\varphi_i(\bar{W}_i)\subset N(P_j, \beta_j)$ $i=1, 2, \cdots, j=1, 2, \cdots$

(iii) $\varphi_i(p)=\varphi_{i-1}(p)$ if $p\in M-V_i$

(iv) φ_i is of C^∞-field in some neighborhood $\bar{U}_1\cup\cdots\cup\bar{U}_i=C_i$,

that is to say, there exist a neighborhood $N\subset M$ of C_i and a positive continuous function ρ_i on $N(C_i)$ such that $\varphi_i\pi: T_{\rho_i}(\varphi_i)\to G^n_k$ is of class C^∞, where $T_{\rho_i}(\varphi_i)$ means a tubular neighborhood of $N(C_i)$ with respect to (f, φ_i). If the above mentioned Lipschitz maps φ_i $(i=0, 1, \cdots, n, \cdots)$ are defined, we may prove Proposition 3 as follows.

Since $\{V_i\}$ is locally finite for any point p of M, there exist a neighborhood U_p and an integer h such that $U_p\subset M-V_i$ if $i>h_p$. Define $\psi(p)=\varphi_{h_p}(p)$. Then $\psi(p)$ is the required one. Therefore we shall show the existence of the above mentioned maps $\varphi_i: M^n\to G^n_k$ $(i=0, 1, \cdots, n, \cdots)$ by induction.

Without loss of generality, we may assume that there exists a positive continuous function $\rho(p)$ on M and $T_\rho(\varphi_{q-1})=\{(p, x)\,|\,p\in M,$

$x \in \varphi_{q-1}(p),\ |x| < \rho(p)\}$ is a tubular neighborhood with respect to (f, φ_{q-1}) and $\varphi_{q-1}\pi : T_\rho(\varphi_{q-1};\ U(C_{q-1})) \to G_k^n$ is of class C^∞ and $\theta : T_\rho(\varphi_{q-1}) \to R^{n+k}$ is a regular Lipschitz homeomorphism on $T_\rho(\varphi_{q-1};\ W_i)$, where $T_\rho(\varphi_{q-1};\ U(C_{q-1})) = \{(p, x) \in E(\varphi_{q-1}) \mid p \in N(C_{q-1}),\ x \in \varphi_{q-1}(p),\ |x| < \rho(p)\}$, $T_\rho(\varphi_{q-1};\ W_i) = \{(p, x) \mid p \in W_i,\ x \in \varphi_{q-1}(p),\ |x| < \rho(p)\}$. We shall denote $f(W_q)$, $f(V_q)$ and $f(U_q)$ by $\mathfrak{W}_q$, $\mathfrak{V}_q$ and $\mathfrak{U}_q$ respectively.

According to J. H. C. Whitehead ([10], lemma 10. 2), there exists a neighborhood $\mathfrak{N}(\mathfrak{W}) \subset R^{n+k}$ of $\mathfrak{W}$ such that for every $x \in \mathfrak{N}(\mathfrak{W})$ and every $Q \in N(P_q, \beta_q)$, $x + Q$ intersects $\mathfrak{W}$. We may suppose that $\mathfrak{N}(\mathfrak{W})$ is contained in $\theta T_\rho(\varphi_{q-1};\ W_q)$. Let $\mathfrak{N}(\overline{\mathfrak{V}})$ be a neighborhood of $\overline{\mathfrak{V}}$ in R^{n+k} whose closure is compact and $\overline{\mathfrak{N}}(\overline{\mathfrak{V}}) \subset \mathfrak{W}$.

Let η_1 be the distance between $\overline{\mathfrak{N}}$ and $R^{n+q} - \mathfrak{N}(\overline{\mathfrak{V}})$.

Let δ_q be the metric on $N\left(P_q, \dfrac{\pi}{2}\right)$ which is induced by the map $\rho_{P_q} : N\left(P_q, \dfrac{\pi}{2}\right) \to R^{nk}$. Since $\varphi_{q-1}\pi$ is a Lipschitz map and θ is a regular Lipschitz homeomorphism on $T_\rho(\varphi_{q-1};\ W_q)$, there exists a positive number κ which satisfies

$$\delta_q(\varphi_{q-1}\pi(q'),\ \varphi_{q-1}\pi(q)) \leqq \kappa |\theta(q') - \theta(q)|$$

for every $q', q \in \bar{N}(\bar{V}_q)$, where $N(\bar{V}_q) = \theta^{-1}\mathfrak{N}(\overline{\mathfrak{V}})$.

Since δ_q is an allowable local metric for the global metric α on G_k^n, there exists a positive number b which satisfies $\alpha(Q, R) \leqq b\delta_q(Q, R)$ for every $Q, R \in \bar{N}(P_q, \gamma_q)$. Let $\eta_2 = \sin \dfrac{\beta_q}{2} \Big/ 16b\kappa\sqrt{nk}(1 + \cot 2\beta_q)(1 + \sin(\beta_q/2))$.

Let X, Y be open sets in V_q such that

$$\bar{U}_q \subset X,\ \bar{X} \subset Y,\ \bar{Y} \subset V_q$$

and let $f(X) = \mathfrak{X}$, $f(Y) = \mathfrak{Y}$.

According to J. H. C. Whitehead ([10], lemma 10. 1), there exists a positive number η_3 which satisfies the following conditions. If $h_1 : \mathfrak{V} \to \mathfrak{W}$, $h_2 : \mathfrak{X} \to \mathfrak{W}$ satisfy $|x - h_1(x)| < \eta_3$ for $x \in \mathfrak{V}$, $|x' - h_2(x')| < 4\eta_3$ for $x' \in \mathfrak{X}$ respectively, then $\overline{\mathfrak{Y}} \subset h_1(\mathfrak{V})$, $\overline{\mathfrak{U}} \subset h_2(\mathfrak{X})$ respectively.

Let $\eta_4 = \dfrac{1}{4}$ dist. $(\overline{\mathfrak{Y}}, \overline{\mathfrak{W}} - \mathfrak{V})$, $\eta_5 = \dfrac{1}{4}$ dist. $(f(Y - N(C_{q-1}),\ f(C_{q-1} \cap \bar{W}_q))$ and let $\eta = \min\{\eta_1, \cdots, \eta_5\}$.

Now let P^* be an n-plane which is orthogonal to P_q. Let $\mathfrak{V}^*$, $\mathfrak{W}^*$ and $\mathfrak{U}^*$ be the orthogonal projections of $\mathfrak{V}$, $\mathfrak{W}$ and $\mathfrak{U}$ on P^* respectively. Let (u), (v) be the rectangular coordinates of P^*, P_q respectively. Then $\mathfrak{W}$ is defined by the following equation

$$v = t(u),\quad u \in \mathfrak{W}^*,\quad v \in P_q\,.$$

The map $t: \mathfrak{W}^* \to P_q$ satisfies

$$|t(u')-t(u)| < |u'-u| \cot 2\gamma_q \qquad \text{for any} \quad u', u \in \mathfrak{W}^* .$$

From Lemma 5.2 there exists a differentiable map $g: \mathfrak{V}^* \to P_q$ which satisfies

$$|g(u)-t(u)| < \eta, \ |g(u')-g(u)| < |u'-u| \sqrt{k} \cot 2\gamma_q < |u'-u| \cot 2\beta_q .$$

Let $\mathfrak{V}' = \{(u, g(u)) \mid u \in \mathfrak{V}^*\}$. From the fact $\eta \leqq \eta_1$ we obtain $\mathfrak{V}' \subset \mathfrak{N}(\overline{\mathfrak{V}}) \subset \mathfrak{N}(\mathfrak{W})$. Therefore $\pi_q | \mathfrak{V}' : \mathfrak{V}' \to \mathfrak{W}$ may be defined, where $\pi_q = f\pi\theta^{-1}$: $\theta T(\varphi_{q-1}; W_q) \to \mathfrak{W}$.

Now we shall show that $\pi_0 = \pi_q | \mathfrak{V}' : \mathfrak{V}' \to \pi_q(\mathfrak{V}')$ is a homeomorphism. Let $x=(u, v)$, $x'=(u', v')$ be points in $\mathfrak{V}'$ and let $x \neq x'$. Since $|g(u')-g(u)| < |u'-u| \cot 2\beta_q$, we obtain

$$\cot \alpha(\overleftrightarrow{x'x}, P_q) = |g(u')-g(u)| / |u'-u| < \cot 2\beta_q ,$$

therefore

$$\cot \alpha(\overleftrightarrow{x'x}, P_q) < \cot 2\beta_q ,$$

hence

$$\alpha(\overleftrightarrow{xx'}, P_q) > 2\beta_q .$$

Let $w \in \mathfrak{W}$. Then

$$\alpha(\overleftrightarrow{x'x}, \varphi_{q-1}f^{-1}(w)) \geqq \alpha(\overleftrightarrow{xx'}, P_q) - \alpha(\varphi_{q-1}f^{-1}(w), P_q) > 2\beta_q - \beta_q = \beta_q .$$

Therefore $\varphi_{q-1}f^{-1}(w)$ intersects $\mathfrak{V}'$ at most at one point. Hence π_q is a homeomorphism from $\mathfrak{V}'$ onto $\pi_q(\mathfrak{V}')$ which is an open set in $\mathfrak{W}$.

Let $h_1: \mathfrak{V} \to \mathfrak{W}$ be defined by

$$h_1(u, t(u)) = \pi_0(u, g(u)) .$$

If $x=(u, f(u)) \in \mathfrak{V}$, then $|x-h_1(x)| \leqq |t(u)-g(u)| < \eta$. From $\eta \leqq \eta_3$, we obtain $\overline{\mathfrak{Y}} \subset h_1(\mathfrak{V}) = \pi_0(\mathfrak{V}')$.

Therefore we may define

$$\mathfrak{X}' = \pi_0^{-1}(\mathfrak{X}), \ \mathfrak{Y}' = \pi_0^{-1}(\mathfrak{Y}) \quad \text{and}$$
$$\mathfrak{N}' = \pi_0^{-1}(f(N(C_{q-1})) \cap \pi_0(\mathfrak{V}')) .$$

Let $\varphi^*: \mathfrak{V}^* \to N\left(P_q, \dfrac{\pi}{2}\right)$ be a map defined by

$$\varphi^*(u) = \varphi_{q-1}f^{-1}\pi_0(u, g(u)) .$$

Let $\mathfrak{X}^*$, $\mathfrak{Y}^*$ and $\mathfrak{N}^*$ be the orthogonal projections of $\mathfrak{X}'$, $\mathfrak{Y}'$ and $\mathfrak{N}'$

into P^* respectively. Since $\varphi_{q-1}\pi$ is differentiable on $T_\rho(\varphi_{q-1}; N(C_{q-1}))$, φ^* is differentiable on $\mathfrak{N}^*$. Let $u, u' \ni \mathfrak{N}^*$ and let $x=(u, g(u))$, $x'=(u', g(u'))$. Then we obtain

$$\begin{aligned}\delta_q(\varphi^*(u'), \varphi^*(u)) &= \delta_q(\varphi_{q-1}f^{-1}\pi_q(x'), \varphi_{q-1}f^{-1}\pi_q(x)) \\ &\leqq \kappa|x'-x| \leqq \kappa(|u'-u|+|g(u')-g(u)|) \\ &\leqq \kappa(1+\cot 2\beta_q)|u'-u|.\end{aligned}$$

Since $\varphi_{q-1}(\bar{W}_q \cap \bar{W}_i) \subset N(P_q, \beta_q) \cap N(P_i, \beta_i)$ $(i=1, 2, \cdots)$ and $\{\bar{W}_i\}$ is locally finite and $\bar{V}_q$ is compact, there exists an integer l_q such that $\bar{V}_q \cap \bar{W}_j = \phi$ if $j > l_q$. Let $\eta' = \{\delta_q(\varphi_{q-1}(\bar{W}_q \cap \bar{W}_j), N\left(P_q, \frac{\pi}{2}\right) - N(P_j, \beta_j))$; $j=1, \cdots, l_q\}$. From Lemma 5.3, there exists a map $\psi^* : \mathfrak{V}^* \to N\left(P_q, \frac{\pi}{2}\right)$ which satisfies the following:

(i) $\delta(\psi^*(u), \varphi^*(u)) < \eta'$ and $\psi^*(u) = \varphi^*(u)$ if $u \in \mathfrak{V}^* - \mathfrak{Y}^*$,

(ii) ψ^* is differentiable in $\mathfrak{N}^* \cup \mathfrak{X}^*$,

(iii) $\delta_q(\psi^*(u'), \psi^*(u)) \leqq 4\kappa(1+\cot 2\beta_q)\sqrt{nk}|u'-u|$.

Let $\psi' : \mathfrak{V}' \to G_k^n$ be defined by

$$\psi'(u, g(u)) = \psi^*(u).$$

Then ψ' is differentiable in $\mathfrak{N}' \cup \mathfrak{X}'$ and $\psi'(\mathfrak{V}') = \psi^*(\mathfrak{V}^*) \subset N(P_q, \beta_q)$ and $\alpha(\psi'(x'), \psi'(x)) = \alpha(\psi^*(u'), \psi^*(u)) \leqq b\,\delta_q(\psi^*(u'), \psi^*(u)) \leqq 4b\kappa(1+\cot 2\beta_q)\sqrt{nk}$ $|u'-u| \leqq 4b\kappa(1+\cot 2\beta_q)\sqrt{nk}|x'-x|$ where $x=(u,g(u))$, $x'=(u', g(u')) \in \mathfrak{V}'$.

Therefore ψ' is a Lipschitz field on $\mathfrak{V}'$ and $T_\rho(\psi') = \{x+y \in R^{n+k} | x \in \mathfrak{V}'$, $y \in \psi'(x)$, $|y| < \sigma\}$ is identified with a tubular neighborhood of $\mathfrak{V}'$ with respect to the identity map: $\mathfrak{V}' \to R^{n+k}$ and ψ', where

$$\sigma = \sin\frac{\beta_q}{2}\Big/ 8b\kappa\sqrt{nk}(1+\cot 2\beta_q)(1+\sin \beta_q/2).$$

Since $\psi'(\mathfrak{V}') \subset N(P_q, \beta_q)$, $\mathfrak{V}' \subset \mathfrak{N}(\mathfrak{W}_q)$, a map $h : \mathfrak{V}' \to \mathfrak{W}$ is defined by $h(x) = (x+\psi'(x)) \cap \mathfrak{W}$.

From $|x-h(x)| \leqq 2|f(u)-g(u)| < 2\eta \leqq 2\eta_2 = \sigma$ for $x=(u, g(u)) \in \mathfrak{V}'$, we obtain $h(\mathfrak{V}') \subset T_\sigma(\varphi')$. Therefore h and $\pi'|h(\mathfrak{V}')$ are the inverse each other, where π' is the projection $\pi' : T_\sigma(\psi') \to \mathfrak{V}'$. Therefore h is a homeomorphism from $\mathfrak{V}'$ onto $h(\mathfrak{V}') = \mathfrak{W} \cap T_\sigma$.

Since $|x-\pi_0^{-1}(x)| < 2\eta$ for $x=(u, t(u)) \in \pi_0(\mathfrak{V}')$, we obtain

$$|x-h\pi_0^{-1}(x)| \leqq |x-\pi_0^{-1}(x)| + |\pi_0^{-1}(x)-h\pi_0(x)| < \eta.$$

Therefore we obtain

$$\overline{\mathfrak{U}} \supset h\pi_0^{-1}(\mathfrak{X}) = h(\mathfrak{X}') \subset h(\overline{\mathfrak{Y}}') = h\pi^{-1}\overline{\mathfrak{Y}} \subset \mathfrak{V}.$$

As $\psi'(x)=\psi^*(x)=\varphi^*(u)=\varphi_{q-1}f\pi_0(x)$ for $x=(u,g(u))\ni\mathfrak{V}'-\mathfrak{Y}'$, we obtain $h(x)=\pi_0(x)$ for $x\in\mathfrak{V}'-\mathfrak{Y}'$.

ψ_q: $h(\mathfrak{V}')\to G_k^n$ is defined by $\psi_q(x)=\psi'\pi'(x)$. Now we define φ_q: $M^n\to G_k^n$ by

$$\begin{aligned}\varphi_q(p) &= \psi_q f(p) \quad \text{if} \quad p\in f^{-1}h(\mathfrak{U}')\\ &= \varphi_{q-1}(p) \quad \text{if} \quad p\in M-f^{-1}h(\mathfrak{Y}).\end{aligned}$$

Then φ_q is a well-defined single-valued map. Since ψ', π' are Lipschitz maps, ψ_q is a Lipschitz map. Since $\overline{\mathfrak{Y}}'\subset\mathfrak{U}'$ and φ_{q-1} is a Lipschitz map, φ_q is a Lipschitz map.

From $h(\mathfrak{Y}')\subset\mathfrak{V}$, we obtain $\varphi_q(p)=\varphi_{q-1}(p)$ if $p\in M-V_q$. Let $p\in\overline{W}_j$. If $p\in M-f^{-1}h(\mathfrak{Y}')$, then $\varphi_q(p)=\varphi_{q-1}(p)\in N(P_j,\beta_j)$ $(j=1,2,\cdots)$. If $p\in f^{-1}h(\mathfrak{Y}')$, then $\varphi_q(p)=\psi^*(u)\in N(P_j,\beta_j)$ $(j=1,\cdots,l_q)$, where $p=f^{-1}h(u,g(u))$.

Therefore we obtain $\varphi_q(\overline{W}_j)\subset N(P_j,\beta_j)$ for every $j>0$. From $f(\overline{W}_q\cap C_{q-1})\cap h\pi_0^{-1}f(\bar{Y}-N(C_{q-1}))=\phi$, we obtain

$$\begin{aligned}C_{q-1}\cap f^{-1}h(\mathfrak{Y}') &= C_{q-1}\cap f^{-1}h\pi_0^{-1}(\overline{\mathfrak{Y}})\subset C_{q-1}\cap f^{-1}h\pi_0^{-1}f(\bar{Y}\cap N(C_{q-1}))\\ &\subset f^{-1}h\pi_0^{-1}f(V_q\cap N(C_{q-1})) = f^{-1}h(\mathfrak{N}').\end{aligned}$$

Therefore we obtain

$$C_{q-1}\subset(N(C_{q-1})-f^{-1}h(\overline{\mathfrak{Y}}'))\cup f^{-1}h(\mathfrak{N}')$$

and φ_q is a C^∞-field in a neighborhood

$$(N(C_{q-1})-f^{-1}h(\overline{\mathfrak{Y}}'))\cup f^{-}h(\mathfrak{N}'\cup\mathfrak{X}') \quad \text{of} \quad C_q = C_{q-1}\cup\bar{U}_q.$$

Thus Proposition 3 is proved.

§ 6. The proof of Proposition 4.

Let $T(\varphi)$ be a tubular neighborhood such that the map $\varphi\pi: T(\varphi)\to G_k^n$ is differentiable. For any point $p\in M$, there exists a neighborhood $W_p\subset M$ of p such that $\theta\,|\,T(\varphi;W_p)$ is a Lipschitz homeomorphism and $\varphi(W_p)$ is contained in $N\left(\varphi(p),\dfrac{\pi}{2}\right)$. Let P_p be an n-plane orthogonal to $\varphi(p)$ through a point $x_p\in\theta T(\varphi;W_p)\cap(\varphi(p)+f(p))$. Then there exists a neighborhood U_p of x_p in P_p so that $U_p\subset\theta T(\varphi;W_p)$. Since θ is a homeomorphism on $T(\varphi;W_p)$, we may define $\pi_p=\theta\pi\theta^{-1}$, $\varphi_p=\varphi f^{-1}$ in $\theta T(\varphi;W_p)$, $f(W_p)$ respectively. As is seen in the previous section, if U_p is sufficiently small, π_p: $U_p\to f(W_p)$ is a homeomorphism. We may introduce a local coordinate system $(f^{-1}\pi_p(U_p),\pi_p^{-1}f)$ in a neighborhood

$f^{-1}\pi_p(U_p)$ of p where π_p^{-1} means the inverse map of $\pi_p|U_p$. Then we shall show that $\{(f^{-1}\pi_p(U_p), \pi_p^{-1}f)\}$ defines a differentiable structure on M.

Suppose that $f^{-1}\pi_p(U_p)\cap f^{-1}\pi_q(U_q)=S\neq\phi$. It is sufficient to prove that $\pi_q^{-1}\pi_p: \pi_p^{-1}f(S)\to\pi_q^{-1}f(S)$ is a differentiable map. Let $x_0\in\pi_p^{-1}f(S)$ and let P be an orthogonal n-plane to $\varphi_p\pi_p(x_0)=Q$. Then $R^{n+k}=P+Q$ and there is a neighborhood $W\subset\pi_p^{-1}f(S)$ of x_0 such that $\varphi_p\pi_p(W)\subset N\left(Q, \frac{\pi}{2}\right)$ and W, $\pi_q^{-1}\pi_p(W)$ are given by the equations

$$v_i=\sum_{j=1}^{n} a_{ij}u_j \qquad i=1, \cdots, k,$$

$$v_i=\sum_{j=1}^{n} b_{ij}u_j \qquad i=1, \cdots, k,$$

respectively, where $(u_i)\in P$, $(v_j)\in Q$.

Since $\varphi_p\pi_p(W)\subset N\left(Q, \frac{\pi}{2}\right)$, the k-plane $\varphi_p\pi_p(x)$, for $x\in W$, is given by the equation

$$u_i=\sum_{j=1}^{k} c_{ij}(\alpha_1, \cdots, \alpha_n)v_j \qquad i=1, \cdots, n,$$

where $(\alpha_1, \cdots, \alpha_n)$ is the coordinates of x in P. Since $\varphi\pi: T(\varphi)\to G_k^n$ is differentiable, the functions $c_{ij}(\alpha_1, \cdots, \alpha_n)$ are differentiable. Since the k-plane $\varphi_p\pi_p(x)+x$ is given by

$$(u)=(\alpha)+||c_{ij}||\,\{(v)-||a_{ij}||(\alpha)\},$$

the (u, v)-coordinates of the point $\pi_q^{-1}\pi_p(x)$ are given by the equations

$$\begin{aligned}(u)&=(\alpha)+||c_{ij}(\alpha)||\,\{(v)-||a_{ij}||(\alpha)\},\\ (v)&=||b_{ij}||(u),\end{aligned}$$

where (α) is the (u)-coordinates of $x\in W$.

Let $\psi(u, \alpha)=(u)-(\alpha)-||c_{ij}(\alpha)||\,\{||b_{ij}||(u)-||a_{ij}||(\alpha)\}$ and let (α_0) be the (u)-coordinates of x_0. Then $||c_{ij}(\alpha_0)||=0$ since $\varphi_p\pi_p(x_0)=Q$, and it follows that $\left\|\frac{\partial\psi(u, \alpha_0)}{\partial u}\right\|$ is the unit matrix. Therefore it follows from the implicit function theorem that $\pi_q^{-1}\pi_p$ is differentiable near x_0 and hence at every point of $\pi_p^{-1}f(S)$.

Next we shall show that the projection $\pi: T(\varphi)\to M$ is differentiable. Let $p_0\in T(\varphi)$ and let $\pi(p_0)=p$. Let P be an n-plane orthogonal to $\varphi(p)=Q$. Then there exist neighborhoods $V\subset x_0+Q$, $U\subset x_0+P$ of $x_0=\theta(p_0)$ such that $\theta'(u+x)=(u, \pi'(u+x))$ is a diffeomorphism on $O'=\{u+x\in R^{n+k}\,|\,u\in U,\ x\in\varphi\pi\theta^{-1}(u),\ \pi'(u+x)\in V\}$, where π' is the orthogonal projection $\pi' : R^{n+k} \to Q + x_0$. Since the following diagram is commutative

$$\begin{array}{ccccc} O = \theta^{-1}(O') & \xrightarrow{\theta} & O' & \xrightarrow{\theta'} & U \times V \\ \downarrow \pi & & f\pi\theta^{-1} \downarrow & & \downarrow \pi' \\ \pi\theta^{-1}(U) & \xrightarrow{f} & f\pi\theta^{-1}(U) & \longrightarrow & U, \end{array}$$

π is differentiable.

Since the cross-section $i(p) = (p, 0) : M \to T_\rho(\varphi)$ of the fibre bundle $T(\varphi)$ is approximated by a differentiable cross-section $h(p) : M \to T(\varphi)$ ([8]), we may define a differentiable map $g : M \to R^{n+k}$ by $g = \theta h$ which satisfies $|g(p) - f(p)| < \rho(p)$, where $\rho(p)$ is the positive continuous function which defines $T_\rho(\varphi)$. Let $0 < \rho(p) < \varepsilon(p)$. Then we may obtain a differentiable map $g : M^n \to R^{n+k}$ such that $|g(p) - f(p)| < \varepsilon(p)$, which is a required one in Proposition 4.

If M is a combinatorial manifold and $f : M^n \to R^{n+k}$ is a semi-linear immersion, then it is obvious that the above introduced differentiable structure is compatible with the combinatorial structure of M^n. Thus Proposition 4 is proved, and the proof of the main theorem is complete.

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