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Title	Semisimple normal subgroups of transitive Riemannian isometry groups
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Citation	Osaka Journal of Mathematics. 1982, 19(2), p. 283-286
Version Type	VoR
URL	https://doi.org/10.18910/5630
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Osaka J. Math.
19 (1982), 283-286

SEMISIMPLE NORMAL SUBGROUPS OF TRANSITIVE RIEMANNIAN ISOMETRY GROUPS

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(Received August 1, 1980)

1. Introduction. In this paper we prove the following:

Theorem. *Suppose the connected Lie group A is a product $A=GL$ of a connected subgroup G and a compact subgroup L. Let H be a connected semisimple normal subgroup of G. Then*
(a) *if H is of noncompact type, H is normal in A;*
(b) *if H is compact, then H is contained in a compact semisimple normal subgroup of A.*

Here H "of noncompact type" means all simple connected normal subgroups of H are noncompact.

This theorem is related to the problem of describing the group of all isometries of a connected homogeneous Riemannian manifold M in terms of a given transitive connected subgroup G. Indeed if A is the connected component of the identity in the full isometry group of M, then $A=GL$ where L, the isotropy subgroup of A at a point of M, is compact.

Part (a) of the theorem generalizes and provides a new proof of a result of [1] in which the normality of G in A is established when G itself is semisimple of noncompact type. Following the proof of the theorem, we will note a sufficient condition for equality of the noncompact parts of Levi factors of G and A, generalizing a further result of [1].

2. Recall that all maximal compact subgroups of a connected Lie group A are conjugate under an inner automorphism of A. If $A=GL$ with L compact and if U is a maximal compact subgroup of A, then a conjugate of L lies in U. It is then easily verified that $A=GU$. Thus we are free to replace L by any convenient maximal compact subgroup of A.

A maximal connected semisimple subgroup A_{ss} of A will as usual be called a Levi factor of A. Being semisimple, A_{ss} is a product $A_{ss}=A_{nc}A_c$ of connected normal semisimple subgroups A_{nc} and A_c of noncompact and compact type, respectively. A_{nc} and A_c will be called the noncompact and compact

Technion Preprint series No. MT-474.
1) Rescearch performed as a Lady Davis Fellow at the Technion—Israel Institute of Technology.

parts of A_{ss}. Similar notation will be used for the corresponding Lie algebras.

Proof of the theorem. Choose Levi factors G_{ss} and A_{ss} of G and A with $H\subset G_{ss}\subset A_{ss}$. Denote by $\mathfrak{a}$, $\mathfrak{g}$, $\mathfrak{a}_{ss}$, $\mathfrak{g}_{ss}$, and $\mathfrak{h}$ the Lie algebras of A, G, A_{ss}, G_{ss}, and H, respectively, and by $\mathfrak{a}_r$ and $\mathfrak{g}_r$ the radicals of $\mathfrak{a}$ and $\mathfrak{g}$. As above we write $\mathfrak{a}_{ss}=\mathfrak{a}_{nc}+\mathfrak{a}_c$ and $\mathfrak{g}_{ss}=\mathfrak{g}_{nc}+\mathfrak{g}_c$. Let $\pi_{nc}\colon \mathfrak{a}\to\mathfrak{a}_{nc}$ and $\pi_c\colon \mathfrak{a}\to\mathfrak{a}_c$ be the projections relative to the vector space direct sum $\mathfrak{a}=\mathfrak{a}_{nc}+\mathfrak{a}_c+\mathfrak{a}_r$. Note that $\pi_c(\mathfrak{g}_{nc})=\{0\}$ since $\mathfrak{a}_c$ contains no noncompact semisimple subalgebras, so $\mathfrak{g}_{nc}\subset\mathfrak{a}_{nc}$.

Let $\mathfrak{g}_{nc}=\mathfrak{k}+\mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$, i.e., $\mathfrak{k}$ is a maximal compactly imbedded subalgebra of $\mathfrak{g}_{nc}$, $[\mathfrak{k},\mathfrak{p}]=\mathfrak{p}$ and $[\mathfrak{p},\mathfrak{p}]=\mathfrak{k}$. By a theorem of Mostow (see [2], pp. 277 and 569), $\mathfrak{a}_{nc}$ has a Cartan decomposition $\mathfrak{a}_{nc}=\mathfrak{k}'+\mathfrak{p}'$ with

$$\mathfrak{k}+\pi_{nc}(\mathfrak{g}_c)\subset\mathfrak{k}' \quad\text{and}\quad \mathfrak{p}\subset\mathfrak{p}'\,. \tag{1}$$

Note that $\mathfrak{k}'+\mathfrak{a}_c$ is a maximal compact subalgebra of $\mathfrak{a}_{ss}$. Hence, letting $\mathfrak{u}$ be any maximal compactly imbedded subalgebra of $\mathfrak{a}$ containing $\mathfrak{k}'+\mathfrak{a}_c$, it follows easily that

$$\mathfrak{u}=(\mathfrak{k}'+\mathfrak{a}_c)+(\mathfrak{u}\cap\mathfrak{a}_r)\,. \tag{2}$$

$\mathfrak{u}\cap\mathfrak{a}_r$ is a solvable ideal of $\mathfrak{u}$ and hence is central in $\mathfrak{u}$. Thus

$$[\mathfrak{u},\mathfrak{u}]\subset\mathfrak{a}_{ss}\,. \tag{3}$$

The connected subgroup U of A with Lie algebra $\mathfrak{u}$ contains a maximal compact subgroup of A, so $A=GU$ and

$$\mathfrak{a}=\mathfrak{g}+\mathfrak{u}\,. \tag{4}$$

We first show that $[\mathfrak{h},\mathfrak{a}_r]=\{0\}$. We may assume that $\mathfrak{h}$ is simple. If $\mathfrak{h}$ is of noncompact type, then $\mathfrak{h}$ is a $\mathfrak{g}_{nc}$-ideal and therefore has a Cartan decomposition $\mathfrak{h}=\mathfrak{k}_0+\mathfrak{p}_0$ with $\mathfrak{k}_0=\mathfrak{k}\cap\mathfrak{h}$ and $\mathfrak{p}_0=\mathfrak{p}\cap\mathfrak{h}$. If $\mathfrak{h}$ is compact, then $\mathfrak{h}\subset\mathfrak{g}_c\subset\mathfrak{u}$ by (1) and (2). Thus in either case, $\mathfrak{h}\cap\mathfrak{u}\neq\{0\}$. In view of (3) and (4) we have

$$[\mathfrak{h}\cap\mathfrak{u},\mathfrak{a}]\subset[\mathfrak{h},\mathfrak{g}]+[\mathfrak{u},\mathfrak{u}]\subset\mathfrak{h}+(\mathfrak{u}\cap\mathfrak{a}_{ss})\subset\mathfrak{a}_{ss}\,. \tag{5}$$

Hence $[\mathfrak{h}\cap\mathfrak{u},\mathfrak{a}_r]=\{0\}$. Since the annihilator in $\mathfrak{g}$ of $\mathfrak{a}_r$ is a $\mathfrak{g}$-ideal, it follows that

$$[\mathfrak{h},\mathfrak{a}_r]=\{0\}\,. \tag{6}$$

Suppose now that $\mathfrak{h}$ is of noncompact type and write $\mathfrak{h}=\mathfrak{k}_0+\mathfrak{p}_0$ as in the paragraph above. By (5),

$$[\mathfrak{k}_0,\mathfrak{a}_{nc}]\subset(\mathfrak{h}+\mathfrak{u})\cap\mathfrak{a}_{nc}=\mathfrak{k}'+\mathfrak{h}=\mathfrak{k}'+\mathfrak{p}_0\,.$$

In particular,

$$[\mathfrak{k}_0,\mathfrak{p}']\subset(\mathfrak{k}'+\mathfrak{p}_0)\cap\mathfrak{p}'=\mathfrak{p}_0\,. \tag{7}$$

$[\mathfrak{u}, \mathfrak{p}_0]=[\mathfrak{k}', \mathfrak{p}_0]$ by (2) and (6), so (4) implies

$$[\mathfrak{a}_{nc}, \mathfrak{p}_0]\subset[\mathfrak{g}, \mathfrak{p}_0]+[\mathfrak{k}', \mathfrak{p}_0]\subset\mathfrak{h}+\mathfrak{p}' .$$

Therefore

$$(8) \qquad [\mathfrak{p}', \mathfrak{p}_0]\subset(\mathfrak{h}+\mathfrak{p}')\cap\mathfrak{k}' = \mathfrak{k}_0 .$$

(7) and (8) together yield $[\mathfrak{p}', \mathfrak{h}]\subset\mathfrak{h}$. Since $\mathfrak{a}_{nc}=[\mathfrak{p}', \mathfrak{p}']+\mathfrak{p}'$, $\mathfrak{h}$ is an $\mathfrak{a}_{nc}$-ideal. (i) now follows from (6).

Next suppose H is compact. Then $\mathfrak{h}\subset\mathfrak{u}\cap\mathfrak{a}_{ss}$ and $[\mathfrak{h}, \mathfrak{a}]\subset\mathfrak{u}$ by (5). Noting that $[\mathfrak{u}\cap\mathfrak{a}_{ss}, \mathfrak{p}']=[\mathfrak{k}', \mathfrak{p}']=\mathfrak{p}'$, we have $[\mathfrak{h}, \mathfrak{p}']\subset\mathfrak{u}\cap\mathfrak{p}'=\{0\}$. Hence $[\mathfrak{h}, \mathfrak{a}_{nc}]=\{0\}$ and $\mathfrak{h}\subset\mathfrak{a}_c$. Let $\mathfrak{h}'$ be the minimal $\mathfrak{a}_c$-ideal containing $\mathfrak{h}$. By (6), $[\mathfrak{h}', \mathfrak{a}_r]=\{0\}$ so $\mathfrak{h}'$ is an $\mathfrak{a}$-ideal. The corresponding connected subgroup H' of A is a compact semisimple normal subgroup containing H.

REMARK. Examples are easily constructed indicating that part (b) of the theorem cannot in general be strengthened. Even when G itself is a semisimple, compact, simply transitive group of isometries of a Riemannian manifold M. Ozeki has shown in [3] that G need not be normal in the full connected isometry group A of M.

Corollary. *Given Lie groups $A=GL$ with A and G connected and L compact, suppose that the noncompact part G_{nc} of a Levi factor of G is normal in G. If no homomorphic image of the radical of G is isomorphic to a transitive group of isometries of a Riemannian symmetric space of the noncompact type, then G_{nc} is the noncompact part of every Levi factor of A (and is normal in A).*

Proof. By the theorem, G_{nc} is normal in A. Since all Levi factors of a connected Lie group are conjugate, G_{nc} lies in every Levi factor of G and A. Let $G_{ss}=G_{nc}G_c$ and $A_{ss}=A_{nc}A_c$ be any Levi factors and let G_r and A_r denote the radicals of G and A. Let

$$\pi\colon\ A\to A/(G_{nc}A_cA_r)$$

be the projection. Note that $\pi(A)$ is semisimple of noncompact type and is trivial if and only if $G_{nc}=A_{nc}$. Modding out the discrete center if necessary, we assume $\pi(A)$ has finite center. Now

$$\pi(A) = \pi(G)\pi(L) = \pi(G_c)\pi(G_r)\pi(L) .$$

As noted previously, we may replace $\pi(L)$ by a maximal compact subgroup U of $\pi(A)$ containing $\pi(G_c)$, so that $\pi(A)=\pi(G_r)U$. Under any left-invariant Riemannian metric, $\pi(A)/U$ is a symmetric space of the noncompact type (see [2], pp. 252–253) on which $\pi(G_r)$ acts transitively by isometries. The corollary follows.

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